M2PM2 Progress Test 2

1. Answer each of the following questions, giving your reasoning. You may use any results proved in the lectures.

(a) Is there a surjective homomorphism $\phi : C_{12} \mapsto D_6$?

(b) Is there a surjective homomorphism $\phi : D_{12} \mapsto D_6$?

(c) Is there a surjective homomorphism $\phi : S_4 \mapsto C_3$?

(d) Is there a surjective homomorphism $\phi : A_4 \mapsto C_3$?

(e) Is there a surjective homomorphism $\phi : D_{48} \mapsto S_4$?

(**2 marks for each part**)

Solution

(a) No: if $\phi : C_{12} \mapsto D_6$ is a homom, then $\text{Im}(\phi)$ must be abelian, since for $x, y \in C_{12}$, we have $\phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x)$. So $\text{Im}(\phi)$ cannot be D_6 as this is non-abelian.

(b) Yes: let $N = \langle \rho^3 \rangle$, where ρ is a rotation of order 6 in D_{12}. Then $N \triangleleft D_{12}$, and $D_{12}/N \cong D_6$ as shown in lectures. So the map $\phi(x) = Nx$ for $x \in D_{12}$ is the required homom.

(c) No: suppose $\phi : S_4 \mapsto C_3$ is surjective homom. For any 2-cycle $(i\,j) \in S_4$, $\phi(i\,j)$ is an element of order 1 or 2 in C_3. As C_3 has no element of order 2, $\phi(i\,j) = 1$ for all 2-cycles. But every element of S_4 is a product of 2-cycles, so this implies $\phi(x) = 1$ for all $x \in S_4$, contradiction.

(d) Yes: from a problem on Sheet 4, we know that $V = \{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of A_4. Since A_4/V has order 3 it is isomorphic to C_3, so the map $\phi(x) = Vx$ for $x \in A_4$ is the required homom.

(e) No: suppose $\phi : D_{48} \mapsto S_4$ is surjective homom. Then $N = \text{Ker}(\phi)$ is a normal subgroup of D_{48} of order 2. Since the subgroup generated by a reflection is not normal, it follows that $N = \langle \rho^{12} \rangle$, generated by the rotation of order 2. Hence by the 1st Iso Thoerem, $D_{48}/N \cong S_4$. But this is not true as D_{48}/N has an element $N\rho$ of order 12, whereas S_4 has no such element. Contradiction.