M2PM2 Algebra II Problem Sheet 8

1. Let V be a finite-dimensional vector space over a field F, and $T : V \to V$ a linear map. Let $m(x)$ be a minimal polynomial for T (i.e. a monic polynomial in $F[x]$ of smallest possible degree such that $m(T) = 0$).

(i) Prove that $m(x)$ is unique (ie. if $m_1(x), m_2(x)$ are min polys for T, then $m_1 = m_2$).

(ii) Prove that if $p(x)$ is a polynomial over F such that $p(T) = 0$, then $m(x)$ divides $p(x)$.

2. (a) Let A be a square matrix over $\mathbb{C}$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Prove that the minimal polynomial of A is $\prod_{i-1}^{k}(x - \lambda_i)^{r_i}$, where r_i is the size of the largest λ_i-block in the JCF of A.

(b) List all the possible JCFs for a matrix that has characteristic polynomial $(x+1)^5(x+2)^3(x-2)^4$ and minimal polynomial $(x+1)^3(x+2)(x-2)^3$.

3. Calculate the minimal polynomials of the matrices $\begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$.

4. Let A be an $n \times n$ matrix over $\mathbb{C}$.

(i) Prove that A is diagonalisable iff its minimal poly has no repeated roots.

(ii) Suppose that $A^k = I$ for some positive integer k. Prove that A is diagonalisable.

6. Let V be a vector space, and V_1, V_2 subspaces of V. Prove that the following are equivalent:

(1) $V = V_1 \oplus V_2$,

(2) $\dim V = \dim V_1 + \dim V_2$ and $V_1 \cap V_2 = \{0\}$.

7. Let V be a vector space over a field F, and $T : V \to V$ a linear map. Suppose $f(x)$, $g(x) \in F[x]$ are coprime polynomials (i.e. their hcf is 1) such that $f(T)g(T) = 0$. Prove that

$$V = \mathrm{Ker}(f(T)) \oplus \mathrm{Ker}(g(T)).$$

(*Hint*: there are polys $s(x), t(x)$ such that $sf + tg = 1$, hence $s(T)f(T) + t(T)g(T) = I_V$, the identity linear map on V. Apply both sides of this equation to a vector $v \in V$.)

8. Deduce Proposition 20.4 of lectures from Q7 using induction on k: if $T : V \to V$ has characteristic poly $\prod_{i=1}^{k}(x - \lambda_i)^{a_i}$, where $\lambda_1, \ldots, \lambda_k$ are the distinct evalues, then $V = V_1 \oplus \cdots \oplus V_k$, where $V_i = \mathrm{Ker}(T - \lambda_i I)^{a_i}$.

9. (a) Let $A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$, and define $T : V \to V$ by $T(v) = Av$, where $V = \mathbb{C}^3$. Find a Jordan basis of V, i.e. a basis B such that $[T]_B$ is a JCF matrix.

(b) Let V be the vector space of polynomials over $\mathbb{C}$ of degree at most 5, and define linear maps S and $T : V \to V$ by

$$S(p(x)) = p'(x), \quad T(p(x)) = p''(x) \quad \text{for all } p(x) \in V.$$

Find Jordan bases of V for S and T.

(c) Let $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, and define $T : V \to V$ by $T(v) = Av$, where $V = \mathbb{C}^4$. Find subspaces V_1, V_2 such that $V = V_1 \oplus V_2$, each V_i is T-invariant, and T_{V_i} has only one eigenvalue. Hence find a Jordan basis of V, i.e. a basis B such that $[T]_B$ is a JCF matrix.