M2PM2 Algebra II Problem Sheet 9

In this sheet, all rings are assumed to be commutative rings with 1.

1. Write down all the irreducible polynomials in $\mathbb{Z}_2[x]$ of degree at most 4.

2. Let $d \in \mathbb{Z}$ be a non-square integer, and define $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} : a, b, \in \mathbb{Z}\}$. Prove that $\mathbb{Q}(\sqrt{d})$ is a subfield of $\mathbb{C}$.

3. Which of the following maps ϕ are homomorphisms of rings:

(i) $\phi : \mathbb{Z}_6 \to \mathbb{Z}_3$, where $\phi([x]_6) = [x]_3$

(ii) $\phi : \mathbb{Z} \to \mathbb{Z}$, where $\phi(x) = 2x$

(iii) $\phi : \mathbb{Q}[x] \to \mathbb{Q}[x]$, where $\phi(a_n x^n + \cdots + a_1 x + a_0) = a_n + \cdots + a_1 + a_0$

(iv) $\phi : \mathbb{Q} \to \mathbb{Q}$, where $\phi(x) = |x|$

(v) $\phi : \mathbb{Z}[i] \to \mathbb{Z}_5$, where $\phi(a + bi) = [a - 2b]$ (here $a, b \in \mathbb{Z}$, $[a - 2b] \in \mathbb{Z}_5$).

4. True or false:

(i) If I is an ideal of a ring R containing a unit, then $I = R$.

(ii) If F is a field and $\phi : F \to F$ is a homomorphism, then ϕ is either the zero map or the identity map.

(iii) If F_1, F_2 are fields and $\phi : F_1 \to F_2$ is a homomorphism, then ϕ is either the zero map or an isomorphism.

(iv) The rings $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{3}]$ are isomorphic.

(v) The quotient ring $\mathbb{R}[x]/(x^2 + 1)$ is a field isomorphic to $\mathbb{C}$.

(vi) The quotient ring $\mathbb{Z}_{11}[x]/(x^2 + x + 4)$ is a field.

(iii) $\mathbb{Z}[x]$ is a PID.

5. Let I, J be ideals of a ring R. Prove the following.

(a) The subset $I + J = \{i + j : i \in I, j \in J\}$ is an ideal of R.

(b) If IJ is defined to be the set of sums of finitely many terms ij with $i \in I, j \in J$, then IJ is an ideal of R and is contained in $I \cap J$.

6. Prove that $\mathbb{Z}[\sqrt{-d}]$ is not a PID for any odd integer $d \geq 3$.

7. Let R be the polynomial ring $\mathbb{Z}_3[x]$.

(i) Is $x^3 - x^2 - 1$ an irreducible element of R?

(ii) Let I be the principal ideal $(x^3 - x^2 - 1)R$. Is R/I an integral domain?

(iii) How many elements are there in the quotient ring R/I?

(iv) Let J be the principal ideal x^3R. Are the quotient rings R/I and R/J isomorphic?

TURN OVER FOR MORE!!

8. (i) For any prime p, prove the existence of a field of order p^2.

(ii)* For any prime p, prove the existence of a field of order p^3.

9. Let $F_4 = \mathbb{Z}_2[x]/(x^2+x+1)$, a field of order 4, and $F_8 = \mathbb{Z}_2[x]/(x^3+x+1)$, a field of order 8.

(i) Show that every quadratic over $\mathbb{Z}_2$ has a root in F_4.

(ii) Show that every irreducible cubic over $\mathbb{Z}_2$ has a root in F_8. What about quadratics?

10. Find all solutions $x, y \in \mathbb{Z}$ of the following Diophantine equations:

(i) $y^3 - x^2 = 0$

(ii) $y^3 - x^2 + x = 0$

(iii) $y^3 - 4x^2 - 4x = 5$ (*Hint:* Use the fact that $\mathbb{Z}[i]$ is a UFD.)

(vi)* $y^3 - x^2 = 11$ (*Hint:* Use the fact that $\mathbb{Z}[\eta]$ is a UFD, where $\eta = \frac{1}{2}(1+\sqrt{-11})$.)

Bonus Question 11. Have an excellent vacation!! Don't work too hard – well, at least have a break on Boxing Day!