M2PM2 Algebra II
Problem Sheet 7

1. Show that if A_1, A_2 are square matrices, then $A_2 \oplus A_1$ is similar to $A_1 \oplus A_2$.

2. Suppose that λ is an eigenvalue of a block-diagonal matrix $A = A_1 \oplus \cdots \oplus A_k$. Prove that $\dim E_\lambda(A) = \sum_1^k \dim E_\lambda(A_i)$, where $E_\lambda(A)$ and $E_\lambda(A_i)$ are the λ-eigenspaces of A and A_i.

3. (i) Write down all the possible Jordan Canonical Forms having characteristic polynomial $x(x+1+i)^2(x-3)^3$.

(ii) Calculate the number of non-similar Jordan Canonical Forms having characteristic polynomial $x^3(x-1)^6$.

4. Find the JCFs of the following matrices:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & i & 2 \\ 0 & 0 & 0 & 0 & 0 & i \end{pmatrix}$$

5. Among the following matrices, which pairs are similar?

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

6. Let A_n be the $n \times n$ matrix having all of its entries equal to 1. Find the JCF of A_n.

7. Let $J_n(\lambda)$ be a Jordan block. Prove that the matrix $J = J_n(\lambda) - \lambda I$ is similar to its transpose. Deduce that $J_n(\lambda)$ is similar to its transpose.

8. Prove that every square matrix over $\mathbb{C}$ is similar to its transpose.

9. (i) Show that if $\lambda \neq 0$, then $J_n(\lambda)^2$ is similar to $J_n(\lambda^2)$.

(ii) Using the JCF theorem and part (i), prove that every invertible matrix A over $\mathbb{C}$ has a square root (i.e. show $\exists B$ such that $B^2 = A$).