M2P2 Algebra II Problem Sheet 6

1. For each of the following linear maps $T : V \to V$, find the eigenvalues, and for each eigenvalue λ find its algebraic and geometric multiplicities, and determine whether T is diagonalisable.

(i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (-x_1 + x_2 - x_3,\ -4x_2 + 6x_3,\ -3x_2 + 5x_3)$.

(ii) V is the vector space of polynomials of degree at most 3 over $\mathbb{R}$, and $T(p(x)) = p(1+x) - p'(1-x)$ for all $p(x) \in V$.

(iii) V is the vector space of all 2×2 matrices over $\mathbb{R}$, and $T(A) = MA$ for all $A \in V$, where $M = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

(iv) V the vector space of polynomials over $\mathbb{R}$ of degree at most 2, and $T(p(x)) = x(2p(x+1) - p(x) - p(x-1))$ for all $p(x) \in V$.

(v) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(v) = Av$, where $A = \begin{pmatrix} -1 & a & b \\ 0 & 1 & c \\ 0 & 0 & -1 \end{pmatrix}$. (Answer will depend on a, b, c.)

2. Let V be a vector space with a subspace W, and let X be a subspace of the quotient space V/W. Prove that there is a subspace Y of V such that $W \subseteq Y$ and $X = Y/W$.

3. Let A be the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

where the a_i are scalars. Prove that the characteristic polynomial of A is $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. (*Hint:* Try induction.)

4. Let $n = \dim V$, and suppose $T : V \to V$ is a linear map with the property that the only T-invariant subspaces of V are 0 and V.

(i) Let $0 \neq v \in V$. Prove that $\{v, T(v), \ldots, T^{n-1}(v)\}$ is a basis of V.

(ii) Deduce there exists a basis B of V such that $[T]_B$ is equal to a matrix A as in Q3.

5. For each of the following matrices A, find an invertible matrix P over $\mathbb{C}$ such that $P^{-1}AP$ is upper triangular:

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$$

6. Let A be an $n \times n$ matrix over $\mathbb{C}$, and suppose that the only eigenvalue of A is 0. Prove that $A^n = 0$.

7. In this question you can use Q3 and Cayley-Hamilton.

(a) Find a 3×3 matrix which has characteristic polynomial $x^3 - 7x^2 + 2x - 3$.

(b) Find a 3×3 matrix A such that $A^3 - 2A^2 = I$.

(c) Find a 4×4 invertible matrix B such that $B^{-1} = B^3 + I$.

(d) Find a 5×5 invertible matrix B such that $B^{-1} = B^3 + I$.

(e) Find a real 4×4 matrix C such that $C^2 + C + I = 0$.

(f) For each n find an $n \times n$ matrix D such that $C^n = I$ but $C \neq I$.

8. Let A be an arbitrary $n \times n$ matrix. Which of the following quantities are invariants of A (i.e. are the same for any matrix which is similar to A) ?

(i) $\text{rank}(A^3 - I)$ (ii) $\text{trace}(A + A^5)$

(iii) $c_1(A)$, the sum of the entries in the first column of A

(iv) $\text{rank}(A - A^T)$ (v) $\text{trace}(2A - A^T)$.