M2PM2 Algebra II Problem Sheet 5

1. Calculate the determinant of the matrix $\begin{pmatrix} 6 & 2 & 1 & 0 & 5 \\ 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & 2 & -2 & 3 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{pmatrix}$.

2. For a real number α define

$$A(\alpha) = \begin{pmatrix} 1 & \alpha & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 2 & \alpha & 1 & -1 \\ -1 & \alpha & 1 & 1 \end{pmatrix}$$

(a) Find the determinant of $A(\alpha)$.

(b) Find a value α_0 of α such that the system $A(\alpha_0)x = 0$ has a nonzero solution for $x \in \mathbb{R}^4$.

(c) Prove that when $\alpha < \alpha_0$, there is no real 4×4 matrix B such that $B^2 = A(\alpha)$.

3. Let A, B be $n \times n$ matrices. Prove that if $|A| = 0$ or $|B| = 0$, then $|AB| = 0$.
Note: you may NOT assume the result $|AB| = |A|\,|B|$ from lecs (this qn is supposed to be part of the proof of that result). But you MAY assume the result in lecs that says a matrix is invertible iff it has nonzero determinant.

4. Prove the rule for the expansion of a determinant by the i^{th} row stated in lectures.

5. Let B_n be the $n \times n$ matrix

$$\begin{pmatrix} -2 & 4 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 4 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}$$

(a) Prove that if $n \geq 4$, then $|B_n| = 8|B_{n-3}|$.

(b) Prove that $|B_n| = 0$ if $n = 3k - 1$ (where k is a positive integer).

(c) Find $|B_n|$ if $n = 3k$ or $3k + 1$.

7. Let $A = \begin{pmatrix} B & C \\ \mathbf{0} & D \end{pmatrix}$, where B is $s \times s$, D is $t \times t$, C is $s \times t$, and $\mathbf{0}$ is the $t \times s$ zero matrix. Prove that $\det(A) = \det(B)\det(D)$.

8. Express the matrices $\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 3 & 8 & 7 \end{pmatrix}$ as products of elementary matrices.

8. For a prime p, let $M_n(\mathbb{Z}_p)$ denote the set of $n \times n$ matrices over $\mathbb{Z}_p$.

(a) Let $A = \begin{pmatrix} 2 & 1 \\ 5 & 6 \end{pmatrix} \in M_2(\mathbb{Z}_p)$. For which primes p is A invertible? For which p is A diagonalisable over $\mathbb{Z}_p$?

(b) Let $p = 3$ and let $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & \alpha \end{pmatrix} \in M_3(\mathbb{Z}_3)$. For which values of $\alpha \in \mathbb{Z}_3$ is B diagonalisable over $\mathbb{Z}_3$?

(c)* Let $C = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \in M_2(\mathbb{Z}_p)$. For which primes is C diagonalisable over $\mathbb{Z}_p$?

9. For $n \times n$ matrices A, B, write $A \sim_1 B$ to mean that B can be obtained from A by a sequence of elementary row operations; and $A \sim_2 B$ to mean that A and B are similar.

(a) Prove that $\sim_1$ and $\sim_2$ are equivalence relations.

(b) Is either of these relations contained in the other? (i.e. does $A \sim_1 B \Rightarrow A \sim_2 B$, or vice versa?)