M2PM2 Algebra II Problem Sheet 4

1. (a) Prove that if there is a surjective homomorphism from S_n onto C_r, then r must be 1 or 2.

(b) Prove that if N is a normal subgroup of S_n such that the factor group S_n/N is cyclic, then $N = S_n$ or A_n.

2. Define V to be the following set of permutations in S_4:

$$V = \{e, (12)(34), (13)(24), (14)(23)\}$$

(so V consists of the identity and all permutations of cycle-shape (2,2)).

(i) Show that V is a subgroup of S_4.

(ii) Show that for any $g \in S_4$ and $v \in V \setminus \{e\}$, the element $g^{-1}vg$ has order 2 and is an even permutation. Deduce that $V \lhd S_4$.

(iii) The factor group S_4/V has order 6, so by lecs is isomorphic to C_6 or D_6. Which?

3. Find all the homomorphic images of the following groups:

(i) D_8 (ii) D_{24} (iii) S_4.

Hint: Try to do this without finding all the normal subgroups!

4. Prove that the only abelian simple groups are C_p with p prime.

5. (a) Let $N \lhd G$, and let H be a subgroup of the factor group G/N. Prove that there is a subgroup K of G such that $N \subseteq K$ and $H = K/N$.

(b) In (a), show that $H \lhd G/N$ iff $K \lhd G$.

(c) Deduce that if N is a *maximal* normal subgroup of G, then the factor group G/N is simple.

6. (a) Find a composition series for the group D_{24}.

(b)* Find a composition series for the group $GL(2, 3)$. (Use Project 2!)

7. For a group G, we say that elements $x, y \in G$ are *conjugate* in G if there exists $g \in G$ such that $y = g^{-1}xg$.

(i) Define the relation $\sim$ on G by $x \sim y$ iff x, y are conjugate in G. Prove that $\sim$ is an equivalence relation.

(ii) The equivalence classes of $\sim$ are called the *conjugacy classes* of G. Prove that a subgroup H of G is normal iff it is a union of conjugacy classes.

(iii) Find the conjugacy classes of D_8. Hence find all the normal subgroups of D_8.

(iv)* Show that two elements of S_n are conjugate iff they have the same cycle-shape. Hence find all the normal subgroups of S_4 and S_5.