M2PM2 Algebra II Problem Sheet 3

1. Which of the following functions ϕ are homomorphisms? For those which are homomorphisms, find Im ϕ and Ker ϕ. (*Notation*: $[x]_n$ stands for the residue class of x modulo n, and $\mathbb{R}_{>0}$ stands for the set of positive real numbers.)

$$\phi : C_{12} \to C_{12} \text{ defined by } \phi(x) = x^3 \;\; \forall x \in C_{12}$$
$$\phi : S_4 \to S_4 \text{ defined by } \phi(x) = x^3 \;\; \forall x \in S_4$$
$$\phi : (\mathbb{Z}, +) \to (\mathbb{Z}_n, +) \text{ defined by } \phi(x) = [x]_n \;\; \forall x \in \mathbb{Z}$$
$$\phi : (\mathbb{R}_{>0}, \times) \to (\mathbb{R}_{>0}, \times) \text{ defined by } \phi(x) = \sqrt{x} \;\; \forall x \in \mathbb{R}_{>0}$$
$$\phi : (\mathbb{Z}_6, +) \to (\mathbb{Z}_7, +) \text{ defined by } \phi([x]_6) = [x]_7 \;\; \forall [x]_6 \in \mathbb{Z}_6$$
$$\phi : (\mathbb{Z}_6, +) \to (\mathbb{Z}_7^*, \times) \text{ defined by } \phi([x]_6) = [2^x]_7 \;\; \forall [x]_6 \in \mathbb{Z}_6$$

2. Let $\phi : G \to H$ be a homomorphism. Show that ϕ is injective iff $\mathrm{Ker}(\phi) = \{e\}$.

3. Let G be a group. True or false (give a proof or a counterexample):

(i) If G is abelian and $r \in \mathbb{Z}$, then the function $\phi : G \to G$ defined by $\phi(x) = x^r \;\; \forall x \in G$ is a homomorphism.

(ii) If $\phi : G \to G$ defined by $\phi(x) = x^2 \;\; \forall x \in G$ is a homomorphism, then G is abelian.

(iii) If $\phi : G \to G$ defined by $\phi(x) = x^4 \;\; \forall x \in G$ is a homomorphism, then G is abelian.

(iv) If $\phi : G \to G$ defined by $\phi(x) = x^{-1} \;\; \forall x \in G$ is a homomorphism, then G is abelian.

4. Let G be a group, and suppose M and N are normal subgroups of G.

(i) Show that $M \cap N \lhd G$.

(ii) Prove that $G/(M \cap N)$ is isomorphic to a subgroup of the direct product $G/M \times G/N$. Give examples to show that $G/(M \cap N)$ may or may not be isomorphic to $G/M \times G/N$.

(iii) Let $MN = \{mn : m \in M, n \in N\}$. Show that $MN \lhd G$. (Note: don't forget to show MN is a subgroup first.)

(iv) Prove that $MN/N \cong M/(M \cap N)$.

5. Adopt the usual notation $D_{2n} = \{e, \rho, \ldots, \rho^{n-1}, \sigma\, \rho\sigma, \ldots, \rho^{n-1}\sigma\}$, and assume $n \geq 3$.

(i) Let $i \in \mathbb{N}$. Show that $\langle \rho^i \rangle \lhd D_{2n}$ but $\langle \rho^i \sigma \rangle \not\lhd D_{2n}$.

(ii) Define a map $\phi : D_{2n} \to C_2$ by $\phi(\rho^r \sigma^s) = (-1)^r$ for $0 \leq r \leq n-1$, $0 \leq s \leq 1$. For which values of n is ϕ a homomorphism?

6. Let p is a prime number greater than 2.

(a) Prove that the dihedral group D_{2p} has exactly three different normal subgroups.

(b) Find all groups H (up to isomorphism) such that there is a surjective homomorphism from D_{2p} onto H.

7. Giving reasons, decide whether there exists a surjective homomorphism

(i) from C_{12} onto C_4 (ii) from C_{12} onto $C_2 \times C_2$
(iii) from D_8 onto C_4 (iv) from D_8 onto $C_2 \times C_2$
(v) from S_n onto C_4.

8. True or false:

(a) If G is an abelian group, and N is a subgroup of G, then $N \lhd G$ and the factor group G/N is abelian.

(b) If a group G has a normal subgroup N such that both N and G/N are abelian, then G is abelian.

(c) If a group G has subgroups M, N such that $M \lhd N$ and $N \lhd G$, then $M \lhd G$.

9. Let G, H be groups. Show that $G \times H$ has a normal subgroup G_0 isomorphic to G, and that the factor group $(G \times H)/G_0$ is isomorphic to H.

10*. More on groups of small orders:

(a) Prove that all groups of order 9 are abelian.

(b) Let p be an odd prime. Show that the only groups of order $2p$ are C_{2p} and D_{2p} (up to isomorphism).