M2PM2 Algebra II Problem Sheet 2

1. (a) List the cycle-shapes of elements of A_5, and calculate how many elements there are of each shape. Check that your answers add up to $|A_5|$.

(b) List the cycle-shapes of elements of the alternating group A_8.

(c) Calculate the number of elements of order 2 or 3 in A_7.

(d) Find the number of subgroups of order 5 in A_6.

(e) Find all isomorphism types of subgroups of order 6 in A_6.

2. (a) Write down two permutations in A_9, both of which have order 12 and send $1 \to 6$, $2 \to 3$ and $9 \to 5$.

(b) Let $I = \{1, 2, \ldots, 2n\}$ and let $f : I \to I$ be the permutation sending $i \to 2n + 1 - i$ for all $i \in I$. Find the signature $\mathrm{sgn}(f)$ in terms of n.

(c) Prove that every even permutation in S_n can be expressed as a product of an even number of 2-cycles, and cannot be expressed as a product of an odd number of 2-cycles.

3. True or false:

(a) S_5 has a subgroup which is isomorphic to D_{20}.

(b) A_5 has a subgroup which is isomorphic to D_{10}.

(c) S_5 has a subgroup which is isomorphic to D_{12}.

(d) D_{12} is isomorphic to $D_6 \times C_2$.

4. Let G_1, G_2, G_3 be groups.

(i) Prove that $G_1 \times G_2 \cong G_2 \times G_1$.

(ii) Prove that $(G_1 \times G_2) \times G_3 \cong G_1 \times (G_2 \times G_3)$.

(iii) If $G_1 \cong H_1$ and $G_2 \cong H_2$, show that $G_1 \times G_2 \cong H_1 \times H_2$.

(iv) Show that $G_1 \times G_2$ has a subgroup isomorphic to G_1.

5. (a) Among the groups C_{36}, $C_2 \times C_{18}$, , $C_3 \times C_{12}$, , $C_4 \times C_9$, , $C_6 \times C_6$, which pairs are isomorphic?

(b) Calculate the numbers of abelian groups there are, up to isomorphism, of orders 16, 17 and 18.

6. Find examples of groups G with the following properties:

(i) $|G| = 2^n$ and $x^2 = e$ for all $x \in G$, where n is an arbitrary positive integer

(ii) $|G| > 8$, G is non-abelian, and $x^4 = e$ for all $x \in G$

(iii) G is infinite and non-abelian, and G has a subgroup H such that $|G : H| = 2$ and H is abelian (recall $|G : H|$ is the *index* of H in G, i.e. the number of distinct right cosets of H in G).

7. Let G be a group with the property that $x^2 = e$ for all $x \in G$.

(i) Prove that G is abelian.

(ii) Assuming that G is finite and $|G| > 2$, show that $|G|$ is a multiple of 4.

(iii) Show that if $|G| = 4$, then $G \cong C_2 \times C_2$.

8*. (Harder question) As in Q6(iii), recall that the index $|G : H|$ of a subgroup H of a group G is defined to be the number of distinct right cosets of H in G. Note that G can be finite or infinite, and so can $|G : H|$.

(i) Suppose A, B are subgroups of G with $A \subseteq B \subseteq G$. Prove that $|G : A| = |G : B|\,|B : A|$.

(ii) Suppose that H and K are subgroups of finite index in G (i.e. $|G : H|$ and $|G : K|$ are finite). Prove that $H \cap K$ is also a subgroup of finite index in G.