M2PM2 Algebra II Problem Sheet 1

1. Decide whether each of the following statements is true or false. Throughout, G is a group.

1. If we can find elements g, h in G such that $gh = hg$ then G is abelian.
2. If G is not cyclic then G is not abelian.
3. If G is not abelian then G is not cyclic.
4. If G is infinite then no non-identity element of G has finite order.
5. If every element of G has finite order, then G is finite.
6. If H is a subgroup of G and $H \cong G$, then $G = H$.
7. If $G = D_{2n}$ then every element of G has order $1, 2$ or n.
8. If $G = S_n$ then no element of G has order greater than n.
9. If the order of every non-identity element of G is a prime number then G is cyclic.
10. If $G = \langle g \rangle$ is an infinite cyclic group, then g and g^{-1} are the only generators of G.
11. The cyclic group C_{12} contains precisely two elements g such that $C_{12} = \langle g \rangle$.
12. If $G = GL(2, \mathbb{R})$, then some non-identity elements of G have finite order and some have infinite order.
13. $(\mathbb{Z}_{17}^*, \times)$ is a cyclic group.
14. Every group of order 4 is abelian.
15. Up to isomorphism, there are precisely two groups of order 4.
16. Up to isomorphism, there are only finitely many groups of order 1000000.
17. Up to isomorphism, there are only finitely many finite groups.

2. Let D_{2n} be the dihedral group of order $2n$ consisting of the rotations $e, \rho, \rho^2, \ldots \rho^{n-1}$ and reflections $\sigma_1, \ldots \sigma_n$ of a regular n-gon. Write $\sigma = \sigma_1$.

(a) Prove that the right coset $\langle \rho \rangle \sigma$ consists of all the reflections.

(b) Deduce that every element of D_{2n} has the form $\rho^i \sigma^j$ for some i, j.

(c) Show that $\sigma \rho = \rho^{-1} \sigma$.

(d) Using (c) show that $\sigma \rho^i = \rho^{-i} \sigma$ for any i, and express an arbitrary product $(\rho^a \sigma^b)(\rho^c \sigma^d)$ in the form $\rho^i \sigma^j$.

(e) Find (in terms of n) the number of elements of order 2 in D_{2n}.

3. Prove that isomorphism of groups is an equivalence relation (i.e. prove that the relation $\sim$ defined by $G \sim H \Leftrightarrow G \cong H$ is an equivalence relation).

4. (i) Prove that the cyclic group C_n has a unique subgroup of each order dividing n.

(ii) Find all the finite subgroups of $(\mathbb{R}^*, \times)$.

(iii) Prove that every finite subgroup of $(\mathbb{C}^*, \times)$ is cyclic.

5. Which pairs among the following nine groups are isomorphic?

$(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{Q}^*, \times)$, $(\mathbb{R}^*, \times)$,
$(\mathbb{Q}_{>0}, \times)$ (the positive rationals under mult.),
$(\mathbb{R}_{>0}, \times)$ (the positive reals under mult.),
the cyclic subgroup $\langle \pi \rangle$ of $(\mathbb{R}^*, \times)$,
the group $(\mathbb{Q} \setminus \{-1\}, *)$, where $a * b = ab + a + b \ \forall a, b \in \mathbb{Q} \setminus \{-1\}$.

6. Let $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, two matrices in $GL(2, \mathbb{C})$.

(a) Show that $A^4 = B^4 = I$, $A^2 = B^2$ and $BA = A^3 B$.

(b) Show that the set $Q_8 = \{A^r B^s : r, s \in \mathbb{Z}\}$ consists of exactly 8 matrices. Write them down.

(c) Prove that Q_8 is a subgroup of $GL(2, \mathbb{C})$.

(d) Determine whether or not Q_8 is isomorphic to D_8.