M2PM2 Algebra II   Project 9

In this project we study a new ring and find its maximal ideals. Define

$$R = \text{ set of continuous functions } f : [0,1] \to \mathbb{R}.$$

1. Show that with the usual definitions of addition and multiplication of functions, R is a commutative ring with 1.

2. Let $\gamma \in [0,1]$, and define $M_\gamma = \{f \in R : f(\gamma) = 0\}$. Show that

(a) M_γ is an ideal of R.

(b) M_γ is a maximal ideal.

3. In this part you will prove that every maximal ideal of R is equal to M_γ for some γ.

(a) Let I be an ideal of R, and assume that for any $x \in [0,1]$, $\exists f \in I$ such that $f(x) \neq 0$. Deduce that $I = R$ in the following steps.

(1) Let $x \in [0,1]$. Show that there is an open neighbourhood N_x of x, and a function $f \in I$, such that $f \neq 0$ on N_x.

(2) The open neighbourhoods N_x ($x \in [0,1]$) cover $[0,1]$. As $[0,1]$ is *compact*, there is a finite set of these neighbourhoods $N_{x_1}, \dots N_{x_k}$ that also cover $[0,1]$.

(3) For $1 \leq i \leq k$, let $f_i \in I$ be a function that is nonzero on N_{x_i}. Let $f = f_1^2 + \cdots + f_k^2$. Show that $f \in I$.

(4) Deduce that $I = R$.

(b) Using part (a), prove that every maximal ideal of R is equal to M_γ for some γ.

4. Decide whether or not M_γ is a principal ideal.

Solution 1. Routine!

2. (a) Routine!

(b) Suppose $M_\gamma \subset J \subseteq R$, where J is an ideal. Choose $f \in J$ with $f(\gamma) \neq 0$. Let $g \in R$, and choose a scalar λ such that $(g - \lambda f)(\gamma) = 0$. Then $g - \lambda f \in M_\gamma \subset J$, hence $g \in J$. Therefore $J = R$. This shows that M_γ is maximal.

3. (a) (1) Routine!

(2) This is just a given fact.

(3) As I is an ideal each $f_i^2 \in I$, so $f \in I$.

(4) $f \neq 0$ on $[0,1]$, so $1/f \in R$. So I contains a unit, hence $I = R$.

(b) Let M be a maximal ideal. By (a), $\exists \gamma \in [0,1]$ such that $M \subseteq M_\gamma$. Therefore $M = M_\gamma$ as M is maximal.

4. Suppose M_γ is prinicpal, say $M_\gamma = fR$. If $\exists \alpha \neq \gamma$ such that $f(\alpha) = 0$, then $g(\alpha) = 0$ for all $g \in fR = M_\gamma$, which is clearly false. Hence $f(x)$ is nonzero for all $x \neq \gamma$. Now define $h : [0,1] \to \mathbb{R}$ by

$$h(x) = \begin{cases} f(x), & x \leq \gamma \\ -f(x), & x > \gamma \end{cases}$$

As f is continuous and $f(\gamma) = 0$, it follows that h is also continuous and so $h \in M_\gamma$. Hence $\exists g \in R$ such that $h = fg$. But this implies that $g(x) = 1$ for $x < \gamma$ and $g(x) = -1$ for $x > \gamma$. So g cannot be continuous, which is a contradiction. Hence M_γ is non-principal.