12/11/19



To check whether an $n \times n$ matrix over a field $F$ is diagonalisable:

1) Find the eigenvalues $\lambda_i$
2) Find each $a(\lambda_i)$, $g(\lambda_i)$
3) If $g(\lambda_i) = a(\lambda_i)$ $\forall i$, then YES

If ~~not~~ $\exists i$ s.t.

$g(\lambda_i) < a(\lambda_i)$

then No.

## 4. Quotient spaces

Let $V$ be a vector space over $F$, and let $W$ be a subspace.

<u>Defn</u> The <u>quotient space</u> $V/W$ has

<u>vectors</u>: cosets $W + v$ $(v \in V)$

(where $W + v = \{w + v : w \in W\}$).

(A) addition:

$$(W+x)+(W+y) = W+x+y \quad (x, y \in V)$$

(S) scalar mult:

$$\lambda(W+x) = W+\lambda x \quad (x \in V, \lambda \in F)$$

Need to check these operations are well-defined:

addition: this follows from our work on factor groups, since the cosets $W+x$ are the cosets of the normal subgp $(W,+)$ of $(V,+)$.

Scalar mult check:

$$W+x_1 = W+x_2$$
$$\implies x_1 - x_2 \in W$$
$$\implies \lambda(x_1 - x_2) \in W \quad \text{(subspace)}$$
$$\implies \lambda x_1 - \lambda x_2 \in W$$
$$\implies W+\lambda x_1 = W+\lambda x_2$$

**Prop 14.1** With addition and scalar mult defined as in (A) and (S), $V/W$ is a vector space over $F$.

**Pf** As noted, $(V/W, +)$ is an abelian group, so the addition axioms (A1)–(A4) for vector spaces are satisfied. (note: zero vector is the coset $W + 0 = W$).

Now need to verify the scalar mult axioms:

(S1) $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$

(S2) $(\lambda + \mu)v = \lambda v + \mu v$

(S3) $\lambda(\mu v) = (\lambda\mu)v$

(S4) $1v = v$.

**Check (S1):** For $\lambda \in F$ and $W + x_1, W + x_2 \in V/W$,

$$\lambda\big((W+x_1)+(W+x_2)\big)$$
$$\overset{(A)}{=} \lambda(W+x_1+x_2)$$
$$\overset{(S)}{=} W+\lambda(x_1+x_2)$$
$$= W+\lambda x_1+\lambda x_2$$
$$= (W+\lambda x_1)+(W+\lambda x_2)$$
$$= \lambda(W+x_1)+\lambda(W+x_2) \quad \checkmark$$

Ex: check (S2) – (S4).

**Prop 14.2** ~~dim~~ The dimension

$$\dim \frac{V}{W} = \dim V - \dim W.$$

Pf Let $r = \dim W$, and pick a basis

$$w_1, \dots, w_r$$

of $W$.

Extend this to a basis of $V$:

$$w_1, \dots, w_r, v_1, \dots, v_s$$

(so $\dim V = r+s$).

<u>Claim</u> $W+v_1, \ldots, W+v_s$ is a basis of $V/W$.

<u>Pf.</u> <u>Linear indep</u>: suppose

$$\sum_{i=1}^{s} \lambda_i (W+v_i) = W \quad (\text{zero vec. in } V/W)$$

LHS is $W + \sum_{i=1}^{s} \lambda_i v_i$, so this implies

$$\sum_{i=1}^{s} \lambda_i v_i \in W.$$

Hence $\exists$ scalars $\mu_j$ s.t.

$$\sum_{i=1}^{s} \lambda_i v_i = \sum_{j=1}^{r} \mu_j w_j.$$

Since $w_1, \ldots, w_r, v_1, \ldots, v_s$ is a basis of $V$, hence linearly indep, this forces

$$\lambda_i = 0 \quad \forall i.$$

<u>Spanning</u>

Let

$$W+v \in V/W \quad (v \in V).$$

Can write

$$v = \sum_{i=1}^{s} \alpha_i v_i + \sum_{j=1}^{r} \beta_j w_j$$

Then

$$W + v = W + \sum \alpha_i v_i + \sum \beta_j w_j$$

$$= W + \sum \alpha_i v_i$$

(since $\sum \beta_j w_j \in W$)

$$= \sum_{i=1}^{s} \alpha_i (W + v_i)$$

This proves spanning property, hence the Claim.

By the Claim

$$\dim \frac{V}{W} = s$$

$$= \dim V - \dim W$$

//.

<u>Example</u>

Let $V = \mathbb{R}^3$ and

$$W = \mathrm{Span}(w)$$

where $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Then $\dim \frac{V}{W} = 2$.

To write down a basis, extend $w$ to a basis of $V$, e.g.

$$w, e_1, e_2$$

Then

$$W+e_1, \quad W+e_2$$

is a basis of $V/W$.

## Quotient spaces and linear ~~spaces~~ maps

Let $T: V \to V$ be linear map.
Let $W$ be a subspace of $V$, and assume

$$T(W) \subseteq W$$ 

(i.e. $T(w) \in W \ \forall w \in W$).

**Defn** A subspace $W$ satisfying  is called a <u>$T$-invariant</u> subspace.

We can define two further linear maps:

1) the <u>restriction</u>

$$T_W : W \to W$$

(defined by $T_W(w) = T(w) \ \forall w \in W$).

2) the <u>quotient map</u>

$$\overline{T} : V/W \to V/W$$

defined by

$$\overline{T}(W+v) = W + T(v) \quad \forall v \in V.$$

Check $\overline{T}$ is well-defined:

$$W + v_1 = W + v_2$$

$$\Longrightarrow v_1 - v_2 \in W$$

$$\Longrightarrow T(v_1 - v_2) \in W$$

(as $W$ is $T$-invariant)

$$\Longrightarrow T(v_1) - T(v_2) \in W$$

$$\Longrightarrow W + T(v_1) = W + T(v_2).$$

## Matrices

Take basis

$$B_W = \{w_1, \ldots, w_r\} \text{ of } W$$

Extend to basis

$$B = \{w_1, \ldots, w_r, v_1, \ldots, v_s\} \text{ of } V$$

By 14.2 pf,

$$\overline{B} = \{W + v_1, \ldots, W + v_s\}$$

is a basis of $V/W$.

**Prop 14.3** As above,

let $W$ be a $T$-invariant subspace of $V$, and let ~~bet~~ bases

$$B_W, B, \overline{B}$$

be as above. Let

$$X = [T_W]_{B_W} \quad (r \times r)$$

$$Y = [\overline{T}]_{\overline{B}} \quad (s \times s).$$

Then

$$[T]_B = \begin{array}{c c} & \begin{array}{cc} r & s \end{array} \\ \begin{array}{c} r \\ s \end{array} & \left(\begin{array}{c|c} X & Z \\ \hline 0 & Y \end{array}\right) \end{array}$$

<u>Pf later</u>

<u>Example</u> Let $T: \mathbb{R}^3 \to \mathbb{R}^3$,

$$T(v) = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 0 & 2 \\ 1 & 1 & -2 \end{pmatrix} v$$

Let $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and

$$W = Sp(w)$$

As $T(w) = 0$, $W$ is $T$-invariant, and

$$[T_W]_{B_W} = (0).$$

Extend to basis of $V$:

$$B = \{w, e_1, e_2\}$$

Then

$$T(e_1) = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = w - 3e_2$$

$$T(e_2) = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = w - 3e_1 - e_2.$$

So

$$\overline{T}(W+e_1) = W - 3e_2$$

$$\overline{T}(W+e_2) = W - 3e_1 - e_2.$$

So

$$[\overline{T}]_{\overline{B}} = \begin{pmatrix} 0 & -3 \\ -3 & -1 \end{pmatrix}$$

Finally

$$[T]_{B} = \left(\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 0 & 0 & -3 \\ 0 & -3 & -1 \end{array}\right)$$

14/11/19



**Proof of 143** Recall bases:

$$B_W = \{w_1, \dots, w_r\} \text{ of } W$$

$$B = \{w_1, \dots, w_r, v_1, \dots, v_s\} \text{ of } V$$

$$\overline{B} = \{W + v_1, \dots, W + v_s\} \text{ of } V/W$$

Let

$$T(w_i) = \sum_{j=1}^{r} x_{ji} w_j \quad (1 \leq i \leq r)$$

$$T(v_i) = \sum_{j=1}^{r} z_{ji} w_j + \sum_{j=1}^{s} y_{ji} v_j \quad (1 \leq i \leq s).$$

Then

$$\overline{T}(W + v_i) \overset{\text{defn}}{=} W + T(v_i) = W + \sum_{j=1}^{s} y_{ji} v_j = \sum_{j=1}^{s} y_{ji} (W + v_j)$$

Hence

$$[T_W]_{B_W} = (x_{ij}) = X \quad (r \times r)$$

$$[\overline{T}]_{\overline{B}} = (y_{ij}) = Y \quad (s \times s)$$

and

$$[T]_B = \begin{array}{c|c} & \begin{array}{cc} \overset{r}{\longleftrightarrow} & \overset{s}{\longleftrightarrow} \end{array} \\ \begin{array}{c} r \updownarrow \\ s \updownarrow \end{array} & \left(\begin{array}{c|c} X & Z \\ \hline O & Y \end{array}\right) \end{array}.$$

## Corollary 14.4

As in Prop 14.3, let $T: V \to V$, let $W$ be a $T$-invariant subspace of $V$ and let $T_W$, $\overline{T}$ be the restriction and quotient maps. If $c_T(x)$ is the characteristic poly. of $T$, then

$$c_T(x) = c_{T_W}(x) \cdot c_{\overline{T}}(x).$$

Pf. ~~By~~ As in 14.3,

$$[T]_B = \left(\begin{array}{c|c} X & Z \\ \hline 0 & Y \end{array}\right)$$

where

$$X = [T_W]_{B_W}, \quad Y = [\bar{T}]_{\bar{B}}.$$

Then

$$c_T(x) = \det \left(\begin{array}{c|c} xI_r - X & -Z \\ \hline 0 & xI_s - Y \end{array}\right)$$

$$= \det(xI_r - X) \cdot \det(xI_s - Y)$$

(Sheet 5, Q4)

$$= c_{T_W}(x)\, c_{\bar{T}}(x). \quad //$$

## 15. <u>Triangularisation</u>

Recall: an <u>upper triangular</u> $n \times n$ matrix is of the form

$$\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

(i.e. $a_{ij} = 0$ if $i > j$).

Similarly for <u>lower triangular</u>.

Some nice properties:

<u>Prop 15.1</u> Let $A, B$ be upper triangular:

$$A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad B = \begin{pmatrix} \mu_1 & & * \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix}$$

1) Char poly of $A$ is

$$\prod_{i=1}^{n} (x - \lambda_i)$$

and evalues are $\lambda_1, \dots, \lambda_n$.

2) $AB$ is also upper triangular, and

$$AB = \begin{pmatrix} \lambda_1\mu_1 & & * \\ & \ddots & \\ 0 & & \lambda_n\mu_n \end{pmatrix}.$$

As we know, many matrices cannot be diagonalised ☹

However ☺!

<u>Theorem 15.2</u> Let $V$ be a (f.d.) vector space over $\mathbb{C}$, and $T: V \to V$ a linear map. Then $\exists$ basis $B$ of $V$ s.t.

$$[T]_B \text{ is upper triangular}$$

<u>Cor. 15.3</u> If $A$ is an $n \times n$ matrix over $\mathbb{C}$, then $\exists$ invertible $P$ s.t. $P^{-1}AP$ is upper triangular.

<u>Note</u> 15.2 is false for ~~vect~~ vector spaces over $\mathbb{R}$ (e.g.) — e.g. for $T: \mathbb{R}^2 \to \mathbb{R}^2$ with char. poly. $x^2+1$.

<u>Pf of 15.2</u>

By induction on $n = \dim V$.

True for $n = 1$.

Assume statement is true for vector spaces of dimension $\leq n-1$.

Now let

$$n = \dim V, \quad T : V \to V.$$

As $V$ is over $\mathbb{C}$, $T$ has an evalue $\lambda \in \mathbb{C}$ (Prop 13.1), and an evector $w_1 \in V$ s.t.

$$T(w_1) = \lambda w_1.$$

Let

$$W = \mathrm{Span}(w_1).$$

Then $W$ is a <u>$T$-invariant</u> subspace.

Let

$$B_W = \{w_1\}, \text{ basis of } W.$$

Note

$$[T_W]_{B_W} = (\lambda). \quad (1 \times 1)$$

Consider quotient space $V/W$ of dim $n-1$. As in §14, we have quotient map

$$\overline{T} : V/W \to V/W$$

defined by

$$\overline{T}(W+v) = W + T(v). \quad (v \in V)$$

By induction hypothesis,

$\exists$ basis of $V/W$ :

$$\overline{B} = \{W+v_2, \dots, W+v_n\}$$

such that the matrix

$$[\overline{T}]_{\overline{B}} \text{ is upper triangular.}$$

Let

$$Y = [\overline{T}]_{\overline{B}}.$$

Then we have basis of $V$

$$B = \{w_1, v_2, \dots, v_n\}$$

and by Prop 14.3,

$$[T]_B = \left(\begin{array}{c|c} [T_W]_{B_W} & Z \\ \hline O & [\overline{T}]_{\overline{B}} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \lambda & Z \\ \hline O & Y \end{array}\right)$$

$$= \left(\begin{array}{c|c} \lambda & Z \\ \hline & \begin{array}{cc} \ddots & * \\ O & \end{array} \end{array}\right)$$

which is upper triangular!

This completes the proof by induction. //

The proof gives us a procedure to find a basis $\mathcal{B}$ which triangularises $T$:

1) Find an evector $w_1$ : $W = \mathrm{Sp}(w_1)$
2) Consider quotient space $V/W$ and $\overline{T} : V/W \to V/W$.
Find an evector $W + w_2$ of $\overline{T}$.
Let $W' = \mathrm{Sp}(w_1, w_2)$, $T$-invariant.
3) Repeat: find an evector $W' + w_3$ for $\overline{\overline{T}} : V/W' \to V/W'$.

Continue: the required basis $\mathcal{B} = \{w_1, w_2, w_3, \dots\}$.

Eg. $T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(v) = Av$, where

$$A = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}$$

<u>Steps</u> 1) Char. poly of $A$ is $(x-1)^3$

2) Evector $w_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$W = \mathrm{Sp}(w_1)$.

3) Basis

$B' = \{w_1, e_2, e_3\}$ of $\mathbb{R}^3$

$\bar{B} = \{W+e_2, W+e_3\}$ of $V/W$.

Find

$$[T]_{B'} = \begin{array}{c} \begin{matrix} w_1 & e_2 & e_3 \end{matrix} \\ \left(\begin{array}{c|cc} 1 & * & * \\ \hline 0 & 2 & 1 \\ 0 & -1 & 0 \end{array}\right) \end{array}$$

So $[\bar{T}]_{\bar{B}} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

4) Evector of $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$

(in $V/W$):

$W + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $(= W + e_2 - e_3)$

(So $w_2 = e_2 - e_3$).

5) Final basis

$B = \{w_1, w_2, e_3\}$

Check:

$$[T]_B = \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix}.$$

15/11/19

## 16. Cayley-Hamilton Theorem.

Recall: if $T: V \to V$ linear, and $p(x)$ is a poly, then

$$p(T) = a_n T^n + \dots + a_1 T + a_0 I$$

where $p(x) = a_n x^n + \dots + a_1 x + a_0$.

Similarly can define $p(A)$ for a square matrix $A$.

**Theorem 16.1 (Cayley-Hamilton)**

1) If $A$ is an $n \times n$ matrix over $\mathbb{C}$, with characteristic poly $p(x)$, then

$$p(A) = 0.$$

2) If $T: V \to V$ is a linear map with char. poly. $p(x)$, then

$$p(T) = 0.$$

## Remarks

1) The result is ~~obv~~ very easy in the case where $A$ is diagonal: Let $A = \begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{pmatrix}$.

Then the evalues are $\lambda_i$, the roots of the char. poly. $p(x)$. So

$$p(A) = \begin{pmatrix} p(\lambda_1) & & O \\ & \ddots & \\ O & & p(\lambda_n) \end{pmatrix} = O.$$

2) A direct proof for $2\times 2$ matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The char. poly. is

$$p(x) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = x^2 - (a+d)x + ad - bc$$

To prove Cayley-Hamilton, need to show

$$p(A) = A^2 - (a+d)A + (ad-bc)I = O.$$

Yuk! Need better idea....

3) Cayley-Hamilton is true for $n \times n$ matrices over <u>any</u> field (not just $\mathbb{C}$) — see Project 5.

---

<u>Proof of 16.1</u>

Let $A$ be $n \times n$ over $\mathbb{C}$, with characteristic poly. $p(x)$.

By Theorem 15.3, $\exists P$ s.t. $P^{-1}AP$ is upper triangular.

Let $B = P^{-1}AP$, and

$$B = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

By 12.1,

$$\text{char. poly of } B = p(x) = \prod_{i=1}^{n} (x - \lambda_i).$$

Also 12.1,

$$p(B) = P^{-1} p(A) P.$$

We show that $p(B) = O$

(hence $p(A) = O$, as required),

Now

$$p(B) = (B - \lambda_1 I) \cdots\cdots (B - \lambda_n I)$$

$$= \begin{pmatrix} 0 & & * \\ & \lambda_2 - \lambda_1 & \\ O & & \ddots \\ & & & \lambda_n - \lambda_1 \end{pmatrix} \cdots \begin{pmatrix} \lambda_1 - \lambda_n & & * \\ & \ddots & \\ O & & \lambda_{n-1} - \lambda_n \\ & & & 0 \end{pmatrix}$$

Let

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n.$$

Then

$$(B - \lambda_n I) v = \begin{pmatrix} \lambda_1 - \lambda_n & & * \\ 0 & \ddots & \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} x_1' \\ \vdots \\ x_{n-1}' \\ 0 \end{pmatrix}$$

Next,

$$(B - \lambda_{n-1} I)(B - \lambda_n I) v$$

$$= \begin{pmatrix} \lambda_1-\lambda_{n-1} & & * \\ 0 & \ddots & \\ \vdots & & \\ 0 & \cdots & 0 & * \\ 0 & \cdots & 0 & \lambda_n-\lambda_{n-1} \end{pmatrix} \begin{pmatrix} x_1' \\ \vdots \\ x_{n-1}' \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_1'' \\ \vdots \\ x_{n-2}'' \\ 0 \\ 0 \end{pmatrix}$$

Continuing, see that eventually

$$(B-\lambda_2 I)\dots(B-\lambda_n I)v = \begin{pmatrix} x_1'''''' \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Finally,

$$p(B)v = (B-\lambda_1 I)\dots(B-\lambda_n I)v$$

$$= \begin{pmatrix} 0 & & * \\ 0 & \lambda_2-\lambda_1 & \\ \vdots & & \ddots \\ 0 & \cdots & 0 & \lambda_n-\lambda_1 \end{pmatrix} \begin{pmatrix} x_1''''' \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence

$$p(B)v = 0 \quad \forall v \in \mathbb{C}^n$$

This implies

$$p(B) = 0. \quad //$$

## 17. Invariants of matrices

**Prop 17.1** If $A$ and $B$ are similar $n \times n$ matrices over field $F$, then they have

1) same det, char. poly, evalues and alg. multiplicites
2) same geometric multiplicities
3) same rank and nullity
4) same trace.

(where trace $tr(A) = \sum_{i=1}^{n} a_{ii}$).

Pf. 1) See 12.1.

2) Let $V = F^n$ and $T: V \to V$ the linear map

$$T(v) = Av \quad (v \in F^n).$$

Let $B = P^{-1}AP$.

If

$$E = \{e_1, \dots, e_n\}$$

(standard basis) and

$$F = \{Pe_1, \dots, Pe_n\}$$

then

$$[T]_E = A, \quad [T]_F = B.$$

For an evalue $\lambda$ of $A$,

$$\dim \ker (T - \lambda I)$$
$$= \dim \ker (A - \lambda I)$$
$$= \dim \ker (B - \lambda I)$$

Hence geom. mult. of $\lambda$ for $A$ is same as for $B$.

3) Follows from (2), as nullity is geom. mult. $g(0)$.

4) The char. poly of $A$ is

$$\det (x I_n - A)$$

$$= \begin{vmatrix} x - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & x - a_{22} & \cdots & -a_{2n} \\ \vdots & & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & x - a_{nn} \end{vmatrix}$$

$$= x^n - x^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})$$
$$+ \cdots\cdots$$

$$= x^n - \operatorname{tr}(A)\, x^{n-1} + \cdots.$$

§ ~~The~~ If $B$ is similar

to $A$, it has the same char poly, hence

$$\mathrm{tr}(B) = \mathrm{tr}(A). \quad //.$$