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Prop 11.12 Suppose A is an invertible $n \times n$ matrix. Then $\exists$ elementary matrices $E_1, \dots, E_k$ s.t.

$$A = E_1 \cdots E_k.$$

Pf. By 11.6, we can reduce A to I_n by a sequence of elementary row ops.

By 11.10, the first row op changes A to $F_1 A$, where F_1 is elementary.

The second row op. changes this to $F_2 F_1 A$ (F_2 elem)

Continuing, we see that

$$\vec{I}_n = F_k \cdots F_2 F_1 A.$$

Hence

$$A = F_1^{-1} F_2^{-1} \cdots F_k^{-1}$$

$$= E_1 E_2 \cdots E_k$$

where $E_i = F_i^{-1}$, an elementary matrix (by 11.11(2)).

Eg. Express

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

as a product of elem. matrices.

Ans

$$\underset{\substack{\uparrow \\ A}}{\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}} \xrightarrow{a_2 \to a_2 - a_1} \underset{\substack{\uparrow \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} A}}{\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}}$$

$$\xrightarrow{a_1 \to a_1 + 2a_2} \underset{\substack{\uparrow \\ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} A}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$\xrightarrow{a_2 \to -a_2} \underset{\substack{\uparrow \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} A}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$= F_3 F_2 F_1 A$$

So

$$A = F_1^{-1} F_2^{-1} F_3^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

<u>Prop 11.13</u> If A is $n \times n$ and E is <u>elementary</u>, then

$$\det(EA) = (\det E)(\det A)$$

Pf. Let $A = \begin{pmatrix} \leftarrow a_1 \rightarrow \\ \vdots \\ \leftarrow a_n \rightarrow \end{pmatrix}$.

1) If $E = A_i(\lambda)$, then

$$EA = \begin{pmatrix} a_1 \\ \vdots \\ \lambda a_i \\ \vdots \\ a_n \end{pmatrix}$$

so

$$|EA| \overset{11.5}{=} \lambda |A| \overset{11.11}{=} |E|\,|A|$$

2) If $E = B_{ij}$ then

$$|EA| \overset{11.5}{=} -|A| \overset{11.11}{=} |E|\,|A|$$

3) If $E = C_{ij}(\lambda)$ then

$$|EA| \overset{11.5}{=} |A| \overset{11.11}{=} |E|\,|A|.$$

//

<u>Cor. 11.14</u> Suppose

$$A = E_1 E_2 \cdots E_k$$

where each E_i is elementary. Then

$$|A| = |E_1||E_2| \cdots |E_k|.$$

Pf. Well,

$$|A| = |E_1 E_2 \cdots E_k|$$

$$= |E_1|\,|E_2 \cdots E_k| \quad \text{(by 11.13)}$$

$$= |E_1|\,|E_2|\,|E_3 \cdots E_k| \quad (11.13)$$

Repeating,

$$|A| = |E_1||E_2| \cdots |E_k|. \quad //.$$

Finally 

<u>Proof of Theorem 11.9</u>

$$(\det(AB) = (\det A)(\det B)).$$

Let A, B be $n \times n$ matrices.

1) If $|A| = 0$ or $|B| = 0$,

then $|AB| = 0$.

(Sheet 5, Q3).

2) Now assume

$$|A| \neq 0, \quad |B| \neq 0.$$

So by 11.6, both A and B

are invertible.

By 11.12, $\exists$ elem.

matrices E_i, F_i s.t.

$$A = E_1 \cdots E_k,$$

$$B = F_1 \cdots F_\ell.$$

By 11.14,

$$|A| = |E_1| \cdots |E_k|$$

$$|B| = |F_1| \cdots |F_\ell|.$$

Also

$$AB = (E_1 \cdots E_k)(F_1 \cdots F_\ell)$$

So by 11.14,

$$|AB| = |E_1| \cdots |E_k| \, |F_1| \cdots |F_\ell|$$

$$= |A| \, |B|. \quad //$$

One final result on determinants:

<u>Prop 11.15</u> Let P be an invertible $n \times n$ matrix.

1) $\det(P^{-1}) = \dfrac{1}{\det(P)}$.

2) For any $n \times n$ matrix A,

$$\det(P^{-1}AP) = \det(A).$$

Pf. 1) $\det(P^{-1}) \det(P)$

$$\overset{11.9}{=} \det(P^{-1}P)$$

$$= \det(I) = 1.$$

2) $\det(P^{-1}AP)$

$\overset{11.9}{=} \det(P^{-1})\det(A)\det(P)$

$\overset{(1)}{=} \det(A)$. //

<u>Remark</u> All the theory in this chapter applies to matrices over <u>any</u> field, e.g.

$\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}_p, \ldots$

12. <u>Matrices and linear transformations</u>

Let

V = finite-diml vector space over a field F

$B = \{v_1, \ldots, v_n\}$, basis of V

$T : V \to V$ linear transformation.

Let

$$T(v_i) = \sum_{j=1}^{n} a_{ji} v_j$$

The <u>matrix of T</u> w.r.t $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = (a_{ij}) \qquad (n \times n)$$

If $S : V \to V$ is another linear trans., can ~~#~~ compose S and T to get

$$ST : V \to V$$

and

$$[ST]_{\mathcal{B}} = [S]_{\mathcal{B}} [T]_{\mathcal{B}}.$$

More generally: for a polynomial

$$p(x) = a_r x^r + \dots + a_1 x + a_0 \in F[x],$$

~~det~~ define

$$p(T) : V \to V$$

by

$$p(T) = a_r T^r + \dots + a_1 T + a_0 1_V$$

($1_V : V \to V$ the identity map)

And for an $n \times n$ matrix A,

$$p(A) = a_r A^r + \dots + a_1 A + a_0 I_n.$$

Applying  , if $A = [T]_B$,

$$[p(T)]_B = p(A).$$

Change of basis

Let

$$B = \{v_1, \dots, v_n\}$$

$$B' = \{w_1, \dots, w_n\}$$

are two bases of V,

and

$$w_i = \sum_{j=1}^{n} p_{ji} v_j$$

the $P = (p_{ij})$ is the

<u>change of basis matrix</u>.

It is invertible, and

$$[T]_{B'} = P^{-1} [T]_B P.$$

<u>Defn</u> Two $n \times n$ matrices A, B are <u>similar</u> if $\exists$ invertible P s.t. $B = P^{-1} A P$.

<u>Remark</u> 1) The relation

$$A \sim B \text{ iff } \underset{\text{similar}}{A, B \text{ are}}$$

is an equivalence relation.

2) $[T]_B$ and $[T]_{B'}$ are similar.

Prop 12.1 Suppose A, B are similar matrices. Then

1) $\det A = \det B$
2) A and B have the same characteristic poly.
3) For any polynomial $p(x)$, the matrices $p(A)$ and $p(B)$ are similar.

Here:

Defn The characteristic poly. of an $n \times n$ matrix A is the polynomial

$$\det(xI_n - A).$$

Pf of 12.1

1) is 11.15 (2).

2) Let

$$B = P^{-1}AP.$$

Then

$$\text{char poly of } B = \det(xI - B)$$

$$= \det(xI - P^{-1}AP)$$

$$= \det(P^{-1}(xI)P - P^{-1}AP)$$

$$= \det(P^{-1}(xI - A)P)$$

$$= \det(xI - A) \quad \text{(by (1))}$$

$$= \text{char poly of } A.$$

(3) Let $B = P^{-1}AP$.

Then

$$B^2 = (P^{-1}AP)(P^{-1}AP)$$

$$= P^{-1}A^2P$$

and ~~Simp~~ ~~Simp~~ similarly

$$B^r = P^{-1}A^rP \quad (\text{any } r \in \mathbb{N})$$

and similarly for any poly $p(x)$,

$$p(B) = P^{-1}p(A)P.$$

//

<u>Defn</u> Let $T: V \to V$ be a linear transformation. The <u>determinant</u> of T, $\det(T)$, is defined to be $\det([T]_B)$, where B is a basis of V. (By 12.1(1), this determinant does not depend on the choice of B.) The <u>char. poly</u> of T is the poly. $\det(xI - [T]_B)$.

13. <u>Eigenvalues</u>

Recall: if $T: V \to V$ linear map, an <u>eigenvector</u> of T is a vector $v \in V$ s.t.

- $v \neq 0$
- $T(v) = \lambda v$, some $\lambda \in F$,

and λ is an <u>eigenvalue</u> of T.

Prop 13.1 $\lambda \in F$ is an eigenvalue of T iff λ is a root of the characteristic poly. of T.

Pf. λ is an eigenvalue of T iff the equation

$$(T - \lambda I)v = 0$$

has a nonzero solution $v \neq 0_V$. By 11.7, this holds iff

$$\det(T - \lambda I) = 0. \quad //$$

Cor 13.2 If V is a f.d. vector space over $\mathbb{C}$, and $T : V \to V$ is linear map, then T has an eigenvalue $\lambda \in \mathbb{C}$.

Pf. The char. poly of T has a root $\lambda \in \mathbb{C}$, by the Fundamental Thm. of Algebra. //

__Remark__ This may not be true for other fields.

Eg. $F = \mathbb{Z}_3$, $V = F^2$ and

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}.$$

W.r.t. basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$,

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

char. poly. x^2+1.

This has no root in $\mathbb{Z}_3$.

__Defn__ Let λ be an evalue of $T: V \to V$.

The __λ-eigenspace__ of T is

$$E_\lambda = \{v \in V : T(v) = \lambda v\}$$
$$= \ker(T - \lambda I),$$

a subspace of V.

Eg. V = vector space of polys. of degree ≤ 2 over F.

Define $T: V \to V$ by

$$T(p(x)) = p(1-x)$$

We work out the eigenspaces of T.

Ans W.r.t. basis $B = \{1, x, x^2\}$,

$$[T]_B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

char. poly $(x-1)^2(x+1)$, evalues $1, -1$.

Eigenspace $E_1 = \ker(T - I)$

Solve

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix} v = 0$$

Solution: $\begin{pmatrix} a \\ b \\ -b \end{pmatrix}$ $(a, b \in F)$

So

$$E_1 = \{a + bx - bx^2 : a, b \in F\}$$

a 2-dim'l eigenspace.

<u>Eigenspace</u> $E_{-1} = \{c - 2cx : c \in F\}$, 1-dim.

<u>Diagonalisation</u>

<u>Defn</u> Linear map $T: V \to V$ is <u>diagonalisable</u> if $\exists$ basis B of V consisting of eigenvectors of T (so $[T]_B$ is a diagonal matrix).

<u>Eg.</u> In above example

basis of E_1 : $\{1, x - x^2\}$

basis of E_{-1} : $\{1 - 2x\}$.

Since

$\{1, x - x^2, 1 - 2x\} = B$

is linearly indep, it is a basis of V, so T is diagonalisable, and

$$[T]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Powerful result:

Proposition 13.2 Let $T: V \to V$ be a linear map. Suppose $v_1, \dots, v_k$ are evectors of T corresponding to distinct evalues $\lambda_1, \dots, \lambda_k$ (ie. $T(v_i) = \lambda_i v_i$ for $1 \le i \le k$, and $\lambda_1, \dots, \lambda_k$ are all distinct).

Then $\{v_1, \dots, v_k\}$ is a linearly independent set.

Pf. Induction on k.

~~The~~ Let $P(k)$ be the statement of the prop. Then $P(1)$ is true, (as $v_1 \neq 0$).

Assume $P(k-1)$ true.

Let $v_1, \dots, v_k$ be as in the statement. Suppose

(1) $\alpha_1 v_1 + \dots + \alpha_k v_k = 0$

$(\alpha_i \in F)$.

Apply T to both sides: get

$${}_2 \quad \lambda_1\alpha_1 v_1 + \dots + \lambda_k \alpha_k v_k = 0$$

Then $\lambda_k \times {}_1 - {}_2$ gives

$$\alpha_1(\lambda_k - \lambda_1)v_1 + \dots + \alpha_{k-1}(\lambda_k - \lambda_{k-1})v_{k-1} = 0.$$

By $P(k-1)$, all these coeffs are 0, i.e.

$$\alpha_i(\lambda_k - \lambda_i) = 0,$$

for $1 \leq i \leq k-1$. As $\lambda_1, \dots, \lambda_k$ are all distinct, $\lambda_k - \lambda_i \neq 0 \ \forall i$.

Hence

$$\alpha_i = 0 \quad \text{for } 1 \leq i \leq k-1.$$

Then by ${}_1$,

$$\alpha_k v_k = 0$$

So also $\alpha_k = 0$.

Therefore $P(k)$ is true, and the prop. follows by induction. //

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Corollary 13.3 Let $\dim V = n$ and $T: V \to V$. Suppose the characteristic poly. of T has n distinct roots. Then T is diagonalisable.

Pf. Let $\lambda_1, \dots, \lambda_n$ be the roots (all distinct), ~~and~~ with corresponding evectors $v_1, \dots, v_n$.

By 13.2, $v_1, \dots, v_n$ are linearly indep, hence ~~from~~ form a basis of V. //

Eg. Upper triangular matrix

$$A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Char. poly is
$$(x-\lambda_1)\cdots(x-\lambda_n).$$

If $\lambda_1, \dots, \lambda_n$ are all distinct then A is diagonalisable (by 13.3).

If not, A may or may not be diagable

Eg. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

diag.   not diagble

Multiplicities

Defn $T : V \to V$ linear map with char. poly. $p(x)$.

Let λ be a root of $p(x)$ (ie an evalue of T).

Suppose

$$p(x) = (x-\lambda)^{a(\lambda)} \, q(x)$$

where λ is not a root of $q(x)$. We call $a(\lambda)$

the <u>algebraic multiplicity</u> of λ.

The <u>geometric multiplicity</u> of λ is

$$g(\lambda) = \dim E_\lambda,$$

the dimension of the eigenspace E_λ.

<u>Eg</u>. From a previous eg. if

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$a(1) = 2, \quad a(-1) = 1$$
$$g(1) = 2, \quad g(-1) = 1$$

Now let

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Then

$$a(1) = 2, \quad a(-1) = 1$$

$$g(1) = 1, \quad g(-1) = 1$$

Since

$$\dim E_1 = \overset{\text{nullity}}{\cancel{\ }} \underset{}{\overset{B-I}{\begin{pmatrix} 0 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}}}$$

$$= 1$$

<u>Prop 13.4</u> If λ is an evalue of $T: V \to V$, then

$$g(\lambda) \leq a(\lambda).$$

Pf Let

$$r = g(\lambda) = \dim E_\lambda$$

and $v_1, \dots, v_r$ a basis of E_λ.

Extend to a basis of V:

$$\mathcal{B} = \{v_1, \dots, v_r, w_1, \dots, w_s\}.$$

We work out the matrix $[T]_{\mathcal{B}}$:

$$T(v_1) = \lambda v_1$$

$$\vdots$$

$$T(v_r) = \lambda v_r$$

and

$$T(w_i) = \sum_{j=1}^{r} a_{ji} v_j + \sum_{j=1}^{s} b_{ji} w_j$$

Hence

$$[T]_B = \begin{array}{c} \begin{matrix} v_1 & \cdots & v_r & w_1 & \cdots & w_s \end{matrix} \\ \left(\begin{array}{ccc|c} \lambda & & 0 & \\ & \ddots & & A \\ 0 & & \lambda & \\ \hline & 0 & & B \end{array}\right) \end{array}$$

(where $A = (a_{ij})$ $(r \times s)$
$B = (b_{ij})$ $(s \times s)$).

The char. poly of $[T]_B$ is

$$\det \left(\begin{array}{ccc|c} x-\lambda & & 0 & \\ & \ddots & & -A \\ 0 & & x-\lambda & \\ \hline & 0 & & xI_s - B \end{array}\right)$$

By Sheet 5, Q5 (?), this is equal to

$$(x-\lambda)^r g(x)$$

where

$$g(x) = \det(xI_s - B)$$

Hence $a(\lambda) \geq r = g(\lambda)$. //

Criterion for diagonalisation:

<u>Theorem 13.5</u> Let $\dim V = n$, $T: V \to V$ linear map. Let the char. poly. of T be

$$p(x) = \prod_{i=1}^{r} (x-\lambda_i)^{a(\lambda_i)}$$

where $\lambda_1, \ldots, \lambda_r$ are the distinct evalues (so $\sum_{i=1}^{r} a(\lambda_i) = n$).

The following three statements are equivalent to each other:

(1) T is diagonalisable

(2) $\sum_{i=1}^{r} g(\lambda_i) = n$

(3) $g(\lambda_i) = a(\lambda_i) \ \forall i$.

<u>Pf. (1) $\Rightarrow$ (2)</u>

Assume (1), so V has a basis B consisting of evectors of T.

Each vector in B belongs

to some eigenspace E_{λ_i}, so

$$\sum_{1}^{r} g(\lambda_i) \geq |B| = n.$$

By 13.4,

$$\sum_{1}^{r} g(\lambda_i) \leq \sum_{1}^{r} a(\lambda_i) = n$$

Therefore $\sum g(\lambda_i) = n$.

<u>(2) $\Leftrightarrow$ (3)</u> This is clear, since

$$\sum g(\lambda_i) = n \Leftrightarrow \sum g(\lambda_i) = \sum a(\lambda_i)$$

$$\Leftrightarrow g(\lambda_i) = a(\lambda_i) \ \forall i$$

(by 13.4).

<u>(2) $\Rightarrow$ (1)</u>

Assume (2), so

$$\sum_{1}^{r} g(\lambda_i) = n.$$

~~Let~~ For each i, let B_i be a basis of E_{λ_i} and let

$$B = \bigcup_{i=1}^{r} B_i.$$

Observe that the sets

$B_1, \dots, B_r$

are disjoint, hence

$$|B| = n.$$

<u>Claim</u> B is a basis of V.

<u>Pf.</u> We show B is linearly indep. Suppose

$$\sum_{b \in B_1} \alpha_b b + \dots \sum_{b \in B_r} \gamma_b b = 0$$

Let

$$v_1 = \sum_{b \in B_1} \alpha_b b \qquad (\alpha_b, \dots, \gamma_b \in F)$$

$$\vdots$$

$$v_r = \sum_{b \in B_r} \gamma_b b$$

Then

$$v_i \in E_{\lambda_i}.$$

and

$$v_1 + \dots + v_r = 0$$

As $\lambda_1, \dots, \lambda_r$ are distinct, Prop 13.2 implies

$$v_i = 0 \quad \forall i.$$

~~As~~ Therefore

$$0 = v_1 = \sum_{b \in B_1} \alpha_b b$$

Hence all the coeffs $\alpha_b = 0$,

and similarly for the other

coeffs in .

Hence $\mathcal{B}$ is linearly indep,

proving the Claim.

Therefore T is diagonalisable,

and (1) holds. //