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Finally:

<u>Claim 3</u> C_3 is <u>not</u> a homom. image of D_{12}

<u>Pf.</u> Suppose $\exists$ a surjective homom.

$$\phi : D_{12} \twoheadrightarrow C_3.$$

Consider a reflection $\sigma \in D_{12}$.

We know $\phi(\sigma)$ is an element of C_3 and has order 1 or 2 (by 5.1). As C_3 has no elt. of order 2,

$$\phi(\sigma) = 1.$$

As every elt. of D_{12} is a product of reflections,

$$\phi(x) = 1 \quad \forall x \in D_{12}$$

ie. ϕ is the trivial homom, not surjective $\times\!\!\!\times$.

So the homom. images of D_{12} are

$D_{12}, C_1,$

$C_2, C_2 \times C_2, D_6.$

10. Simple groups

Defn A group G is simple if its only normal subgps. are the trivial ones $\{e\}$, G.

Eg.1) C_p, p prime is simple.

These are the only abelian simple groups (Sh 4, Q4).

__Eg__(2) The alternating group A_5 is simple (Project on Thursday).

[In fact, A_n is simple $\forall n \geq 5$.]

__The point of simple groups__

__Defn__ We say N is a __maximal normal subgp__ of G if

1) $N \lhd G$, $N \neq G$
2) if $N \subseteq M \lhd G$ then $M = N$ or G.

(i.e. N is contained in no __larger__ normal subgp of G).

__Eg.__ $A_3 \lhd S_3$, A_3 is a max normal subgp of S_3.

<u>Prop 10.1</u> If N is a maximal normal subgp of G, then the factor group G/N is simple

<u>Pf.</u> Sheet 4, Q5.

Eg. A_3 max normal in S_3

$$\frac{S_3}{A_3} \cong C_2 \text{ simple.}$$

<u>Composition series</u>

Let G be a finite group.

Choose a maximal normal subgp

$$G_1 \lhd G.$$

Then G/G_1 is a simple gp.

Now choose a maximal normal subgp

$$G_2 \lhd G_1.$$

By Prop. 10.1, $\frac{G_1}{G_2}$ is simple.

Repeat: get a series of max. normal subgps:

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \dots \triangleright G_k = \{e\}.$$

where each factor group $\frac{G_i}{G_{i+1}}$ is a simple group.

Write $S_i = G_i / G_{i+1}$.

We call such a series a <u>composition series</u> of G, and we call the simple groups $S_0, \dots, S_{k-1}$ ~~the~~ <u>composition factors</u> of G.

Eg 1) $\underline{G = S_3}$:

Composition series:

$$S_3 \triangleright A_3 \triangleright \{e\}$$

Comp factors

$$C_2, C_3$$

2) $\underline{G = C_6}$

Composition series ($\exists$ two of them):

$$C_6 \triangleright C_3 \triangleright \{e\}$$

and

$$C_6 \triangleright C_2 \triangleright \{e\}$$

Comp. factors C_2, C_3.

3) $\underline{G = S_4}$

Comp. series

$$S_4 \triangleright A_4 \triangleright \underset{\substack{\uparrow \\ \text{(Sh4 Q2)}}}{V} \triangleright C_2 \triangleright \{e\}.$$

Comp factors

$$C_2, C_3, C_2, C_2$$

4) $\underline{G = S_5}$

Comp series

$$S_5 \triangleright A_5 \triangleright \{e\}$$

Comp factors C_2, A_5.

Fact Although G can have several different composition series, the sequence of composition factors is always the same, possibly in a different order.

In this sense, every finite group G is "built" out of a sequence of simple groups.

Classification of finite simple groups

This is one of the most famous theorems

in the world.
It was proved between 1950 and 2004 by ~500 authors, 15,000 pages, ...

The theorem says that every finite simple group is in the following list:

1) C_p, p prime

2) A_n, $n \geq 5$

3) 16 further infinite families of finite matrix groups

4) 26 <u>sporadic</u> simple groups

$M_{11}, \dots\dots\dots, M$

(M is the <u>Monster</u> group, order ~ 10^{54}, a group of 196883×196883 matrices)

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Some clarifications:

Sheets: Tuesday lec.

Solutions: Friday lec.

Test material: previous Friday lec.

PART B Linear Algebra

11. Determinants

Recall the determinant of a 3×3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$$
$$- a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}$$
$$+ a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Each term is a product of 3 entries, one from each row and column, hence defines a permutation of $\{1,2,3\}$ sending $i \to j$ if a_{ij} is present:

Term	Perm. π	Sign
$a_{11}a_{22}a_{33}$	e	$+$
$a_{11}a_{23}a_{32}$	(23)	$-$
$a_{12}a_{21}a_{33}$	(12)	$-$
$a_{12}a_{23}a_{31}$	(123)	$+$
$a_{13}a_{21}a_{32}$	(132)	$+$
$a_{13}a_{22}a_{31}$	(13)	$-$

<u>Note</u> Each sign is equal to the signature $s(\pi)$, and all 6 perms. of S_3 occur.

Hence for 3×3 matrix $A = (a_{ij})$, determinant

$$|A| = \sum_{\pi \in S_3} s(\pi)\, a_{1,\pi(1)}\, a_{2,\pi(2)}\, a_{3,\pi(3)}$$

Generalisation to $n \times n$ matrices:

<u>Defn</u> For $n\times n$ matrix $A = (a_{ij})$, the <u>determinant</u> of A is

$$\sum_{\pi \in S_n} s(\pi)\, a_{1,\pi(1)} \cdots a_{n,\pi(n)}.$$

Written as $\det(A)$ or $|A|$.

<u>Eg</u> 1) For $n = 2$,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$S_2 = \{e, (12)\}$, so

$$\det(A) = \underset{\substack{\uparrow \\ e}}{a_{11} a_{22}} - \underset{\substack{\uparrow \\ (12)}}{a_{12} a_{21}}.$$

2) Suppose A is an <u>upper triangular</u> matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & 0 & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}.$$

The only permutation π in S_n ~~which co~~ for which

$a_{1,\pi(1)} \cdots a_{n,\pi(n)}$ is not 0

is $\pi = e$. Therefore

$$|A| = a_{11} a_{22} \cdots a_{nn}$$

Eg $|I_n| = 1$.

<u>Aim</u> To prove the basic properties of determinants:

(1) A is invertible iff $|A| \neq 0$.

(2) Determinant is multiplicative, i.e.

$$\det(AB) = (\det A)(\det B).$$

Key to (1) is to work out the effect ~~A~~ on the determinant of doing elementary row operations.

<u>Prop 11.1</u> If A^T is the transpose of the $n \times n$ matrix A, then

$$|A^T| = |A|.$$

Pf. Let $A^T = (b_{ij})$, so $b_{ij} = a_{ji}$. Then

$$|A^T| = \sum_{\pi \in S_n} s(\pi)\, b_{1,\pi(1)} \cdots b_{n,\pi(n)}$$

$$= \sum_{\pi \in S_n} s(\pi)\, a_{\pi(1),1} \cdots a_{\pi(n),n}$$

Let $\sigma = \pi^{-1}$. Then

$$a_{\pi(1),1} \cdots a_{\pi(n),n}$$

$$= a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}.$$

Also

$$s(\sigma) = s(\pi) \quad \text{by } 3.1.$$

Hence

$$|A^T| = \sum_{\sigma \in S_n} s(\sigma)\, a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

$$= |A|. \qquad //$$

<u>Consequence</u> Any result about dets involving row ops has an analogue involving column ops.

<u>Prop 11.2</u> Let $1 \leq i \leq n$. Then det is a linear function of the i^{th} row, i.e.

$$\det\begin{pmatrix} \leftarrow a_1 \rightarrow \\ \vdots \\ \leftarrow \lambda a_i \rightarrow \\ \vdots \\ a_n \end{pmatrix} = \lambda \det\begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix}$$

and

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_i + b_i \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ b_i \\ \vdots \\ a_n \end{pmatrix}$$

(where a_i, b_i are row vectors).

<u>Pf.</u> Each term

$$a_{1,\pi(1)} \cdots a_{n,\pi(n)}$$

in the defn. of determinant

has exactly one entry from the i^{th} row, which is $\lambda a_{i,\pi(i)}$ or $a_{i,\pi(i)} + b_{i,\pi(i)}$. //.

<u>Prop 11.3</u> If B is obtained from A by swapping two rows (or two columns), then

$$|B| = -|A|.$$

~~Pf~~ We prove this for columns (then it follows ~~for~~ for rows by 11.1).

Say cols. r and s are swapped. Let

$$\tau = (rs).$$

Then $B = (b_{ij})$, where

$$b_{ij} = a_{i,\tau(j)}.$$

Hence

$$|B| = \sum_{\pi} s(\pi)\, b_{1,\pi(1)} \cdots b_{n,\pi(n)}$$

$$= \sum_{\pi} s(\pi)\, a_{1,\tau\pi(1)} \cdots a_{n,\tau\pi(n)}$$

$$= \sum_{\pi \in S_n} -s(\tau\pi)\, a_{1,\tau\pi(1)} \cdots a_{n,\tau\pi(n)}$$

(as $s(\tau\pi) = -s(\pi)$ by 3.1)

$$= -|A|.$$

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<u>Prop 11.4</u> If A has two equal rows, then $|A| = 0$.

<u>Pf</u> Swapping the two equal rows does not change A, but changes the sign of $|A|$, by 11.3. Hence

$$|A| = -|A|$$

so $|A| = 0$. //

Effect of an elementary row op on determinant:

<u>Prop 11.5</u> Suppose B is obtained from A by doing an elementary row op.

i) If two rows of A are swapped to get B, then

$$|B| = -|A|.$$

2) If a row of A is multiplied by a scalar $\lambda \neq 0$ to get B, then

$$|B| = \lambda |A|.$$

3) If a scalar multiple of one row of A is added to another row to get B, then

$$|B| = |A|.$$

4) $|A| = 0 \Leftrightarrow |B| = 0.$

Pf. 1) is 11.3.

2) follows from 11.2.

3) Let

$$A = \begin{pmatrix} \leftarrow a_1 \rightarrow \\ \vdots \\ \leftarrow a_n \rightarrow \end{pmatrix}$$

and

$$B = \begin{pmatrix} a_1 \\ \vdots \\ a_i + \lambda a_j \\ \vdots \\ a_n \end{pmatrix}.$$

By 11.2,

$$|B| = \det \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix}$$

$$+ \lambda \det \begin{array}{cl} \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow i\text{th row} \\ \\ \leftarrow j\text{th row} \\ \\ \\ \end{array} \end{array}$$

Second term is 0, by 11.4.

Hence $|B| = |A|$.

4) Follows from (1), (2) & (3). //

First aim:

Theorem 11.6 Let A be an $n \times n$ matrix, over any field F.

The following four statements are equivalent to each other:

(1) $|A| \neq 0$

(2) A is invertible

(3) System of linear eqns

$$Ax = 0 \quad (x \in F^n)$$

has unique solution $x = \underline{0}$.

(4) A can be reduced to I_n by elementary row ops.

Pf. We know from MIGLA that

$$(2) \Leftrightarrow (3) \Leftrightarrow (4).$$

<u>$(1) \Rightarrow (4)$</u>

Assume $|A| \neq 0$.

Now A can be reduced to an echelon form matrix A' using elementary row ops.

By 11.5 (4),

$$|A'| \neq 0.$$

Hence

$$A' = \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & \ddots & \\ & & 1 \end{pmatrix}$$

Further row ops. reduce A' to I_n, so (4) holds.

<u>(4) $\Rightarrow$ (1)</u>

Assume A can be reduced to I_n by elementary row ops. As $|I_n| = 1$, 11.5(4) implies $|A| \neq 0$, giving (1). //

From the equivalence of (1) and (3) of 11.6, we get

<u>Corollary 11.7</u> Let A be $n \times n$. If the system

$$Ax = 0 \quad (x \in F^n)$$

has a <u>nonzero</u> solution $x \neq \underset{\sim}{0}$, then

$$|A| = 0.$$

Expansion of dets

Let $A = (a_{ij})$ be an $n \times n$ matrix. The *ij-minor* A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and j^{th} col of A.

Eg. For 3×3 det,

$$|A| = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}|.$$

Prop 11.8 ~~The~~ For each i, there is an expansion of $|A|$ by the i^{th} row, namely

$$(-1)^{i-1}|A| = a_{i1}|A_{i1}| - a_{i2}|A_{i2}| + \dots + (-1)^{n-1} a_{in}|A_{in}|.$$

Pf. Sheet 5 qn. ☺

Multiplicativity

Theorem 11.9 If A, B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B).$$

Our approach is via ~~of~~ "elementary matrices".

Defn An *elementary* $n \times n$ matrix is one of the following:

1) $$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} = A_i(\lambda) \quad (\lambda \neq 0)$$

2) $$\begin{array}{c} \\ i \to \\ \\ j \to \\ \\ \end{array} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} = B_{ij}$$

(this is I_n with rows i & j swapped)

3) $$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \lambda & \\ & & & \ddots & \\ & & & & 1 \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} = C_{ij}(\lambda)$$

(λ in ij-entry, $i \neq j$)

Eg. The 2×2 elementary matrices are

$$\begin{pmatrix} \lambda & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & \lambda \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \lambda \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ \lambda & 1 \end{pmatrix}$$

<u>Prop 11.10</u> Let A be $n \times n$. An elementary row op. on A changes it to EA, where E is elementary.

Pf. Multiplying A by $A_i(\lambda)$ multiplies i^{th} row by λ; by B_{ij} swaps rows i & j; by $C_{ij}(\lambda)$ adds $\lambda \times$ row j to row i. //

Prop 11.11

1) ~~Def~~ Determinants of elem. matrices:

$$|A_i(\lambda)| = \lambda$$

$$|B_{ij}| = -1$$

$$|C_{ij}(\lambda)| = 1.$$

2) Inverses:

$$A_i(\lambda)^{-1} = A_i(\lambda^{-1})$$

$$B_{ij}^{-1} = B_{ij}$$

$$C_{ij}(\lambda)^{-1} = C_{ij}(-\lambda)$$

In particular, the inverse of an elementary matrix is elementary.