22/10/19

Class on Thursday:

Project will be on

$$GL(n, \mathbb{Z}_p) \quad (p \text{ prime})$$

Key fact

Prop 6.2 If $\phi : G \to H$ is a homomorphism then

$$\ker(\phi) \lhd G.$$

Pf. Write

$$K = \ker(\phi).$$

Then K is a subgp of G by 5.3.

For $g \in G$ and $x \in K$,

$$\phi(g^{-1}xg) = \phi(g^{-1})\phi(x)\phi(g)$$

$$= \phi(g^{-1})\, e_H\, \phi(g)$$

(as $x \in K = \ker(\phi)$)

$$= \phi(g)^{-1}\phi(g)$$

$$= e_H.$$

Therefore $g^{-1}xg \in \mathrm{Ker}(\phi) = K.$

This shows

$$g^{-1}Kg \subseteq K \quad \forall g \in G.$$

So $K \trianglelefteq G$ by 6.1. //

Eg. 1) Signature homom

$$s : S_n \to C_2$$

has kernel A_n. So $A_n \trianglelefteq S_n$ (as we showed in a previous eg.).

2) Homom. $\phi : D_{2n} \to C_2$

$$\phi(\rho^i\sigma^j) = (-1)^j.$$

Here

$$\mathrm{Ker}(\phi) = \langle \rho \rangle,$$

the rotation subgp. So

$$\langle \rho \rangle \trianglelefteq D_{2n}.$$

3) Here's a different homom. $\phi : D_8 \longrightarrow C_2$:

$$\phi(\rho^i \sigma^j) = (-1)^i \quad \forall i, j$$

(Check this *is* a homom — see Sheet 3, Q5 : Warning: it is *not* a homom. for D_{10} !).

The kernel

$$\mathrm{Ker}(\phi) = \{e, \rho^2, \sigma, \rho^2\sigma\}$$
$$\cong C_2 \times C_2 .$$

7. Factor groups

Let G be a group with subgroup N.

Recall that we have right cosets Nx for $x \in G$ and G is the union of disjoint right cosets of N.

The number of distinct right cosets of N in G

is denoted

$$|G:N|.$$

If G is finite, $|G:N| = \dfrac{|G|}{|N|}$.

Eg. $|S_n : A_n| = 2$

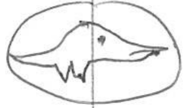
$|\mathbb{Z} : 2\mathbb{Z}| = 2$

$|\mathbb{Q} : \mathbb{Z}| = \infty$ (Ex).
(addition)

Idea for defining a "factor group" G/N:

elements: all right cosets Nx $(x \in G)$

multiplication: a natural definition would be

 $(Nx)(Ny) = Nxy$ $\forall x, y \in G$.

Problem Need to check this mult.  is

<u>well-defined</u>, i.e.

 $\left.\begin{array}{l} Nx = Nx' \\ Ny = Ny' \end{array}\right\} \Longrightarrow Nxy = Nx'y'$

~~Whe~~ For <u>most</u> subgroups N of G, this is <u>false</u>:

<u>Eg.</u> $G = S_3$, $N = \langle (12) \rangle$

<u>Ex.</u> Find elts x, y, x', y' for which the implication  does <u>not</u> hold

<u>But</u> if N is a <u>normal</u> subgp, then  holds:

<u>Prop 7.1</u> Suppose $N \triangleleft G$. Then for $x, y, x', y' \in G$,

$$\left.\begin{array}{l} Nx = Nx' \\ Ny = Ny' \end{array}\right\} \Longrightarrow Nxy = Nx'y'$$

---

and hence the mult. of cosets  is well-defined.

For the proof: define a <u>left coset</u> of N in G:

$$xN = \{xn : n \in N\}$$

(for $x \in G$).

Usually $xN \neq Nx$, but:

<u>Lemma 7.2</u> If $N \lhd G$ then

$$xN = Nx \quad \forall x \in G.$$

<u>Pf.</u> Let $x \in G$, $n \in N$.

Then as $N \lhd G$,

$$x^{-1}nx \in N$$

$$\Rightarrow nx \in xN$$

and so $Nx \subseteq xN$.

Also

$$xnx^{-1} \in N$$

$$\Rightarrow xn \in Nx$$

and so $xN \subseteq Nx$. //

<u>Pf of Prop 7.1</u>

Let $N \lhd G$ and assume

$$Nx = Nx', \quad Ny = Ny'.$$

Then

$$\begin{aligned} Nxy &= Nx'y \quad (\text{as } Nx = Nx') \\ &= x'Ny \quad (\text{Lemma 7.2}) \\ &= x'Ny' \quad (\text{as } Ny = Ny') \\ &= Nx'y' \quad (\text{Lemma 7.2}) \end{aligned}$$

<u>Conclusion</u> If $N \lhd G$, then we have a <u>well-defined</u> mult. of the right cosets of N:

$$(Nx)(Ny) = Nxy \quad \forall x, y \in G$$

<u>Theorem 7.3</u> Suppose $N \lhd G$. Define

$$\frac{G}{N} = \text{set of all right cosets } Nx \ (x \in G)$$

and define mult. on $\frac{G}{N}$ as in . Then $\frac{G}{N}$ is a group, called the <u>factor group</u> (or <u>quotient group</u>) of G by N.

<u>Pf.</u> <u>Closure</u> Clear.

<u>Assoc.</u> $(Nx\,Ny)\,Nz$

$= Nxy\,Nz$

$= N(xy)z$

$= Nx(yz)$ (assoc. in G)

$= Nx\,Nyz$

$= Nx\,(Ny\,Nz)$.

<u>Identity</u> is the right coset

$$Ne \quad (= N)$$

since

$$Nx\,Ne = Nx = Ne\,Nx \quad \forall x \in G$$

<u>Inverse</u> of Nx is Nx^{-1}

since

$$Nx\,Nx^{-1} = Nxx^{-1} = Ne.$$

<u>Note</u> If G is finite, then

$$|G/N| = \frac{|G|}{|N|}.$$

If G is infinite, $\frac{G}{N}$ may be finite or infinite.

Examples of factor groups

1) We know

$$A_n \lhd S_n.$$

Factor group $\frac{S_n}{A_n}$ has exactly two elts:

$$A_n, \quad A_n(12)$$

So $\frac{S_n}{A_n} \cong C_2$. Note

elt. A_n = identity of $\frac{S_n}{A_n}$

elt. $A_n(12)$ has order 2, since

$$\left(A_n(12)\right)^2 = A_n(12)\, A_n(12)$$
$$= A_n (12)^2$$
$$= A_n.$$

2) Any group G has trivial normal subgroups

$$\{e\} \text{ and } G.$$

Factor groups:

$$\frac{G}{G} \cong C_1$$

and

$$\frac{G}{\{e\}} \cong G$$

via isomorphism

$$\{e\}g \longrightarrow g \qquad (g \in G).$$

3) Let $G = D_{12}$

$$= \{\rho^i \sigma^j : 0 \leq i \leq 5,\ j = 0 \text{ or } 1\}.$$

This has some normal subgroups consisting of rotations:

$$N_1 = \langle \rho \rangle \cong C_6$$

$$N_2 = \langle \rho^2 \rangle \cong C_3$$

$$N_3 = \langle \rho^3 \rangle \cong C_2$$

(see Sheet 3 qu).

We'll study the factor groups

$$\frac{G}{N_i} \quad (i = 1, 2, 3).$$

a) $\dfrac{G}{N_1} = \dfrac{D_{12}}{\langle \rho \rangle}$ has order 2, with elements

$$\langle \rho \rangle, \ \langle \rho \rangle \sigma.$$

So $\dfrac{D_{12}}{\langle \rho \rangle} \cong C_2$.

b) $\dfrac{G}{N_2} = \dfrac{D_{12}}{\langle \rho^2 \rangle}$

has order 4, elts

$$N_2, \ N_2\rho, \ N_2\sigma, \ N_2\rho\sigma.$$

By Thm 5.1, we know

$$\frac{G}{N_2} \cong C_4 \text{ or } C_2 \times C_2$$

Which?

To decide, we compute the order of each element:

$$(N_2\rho)^2 = (N_2\rho)(N_2\rho) = N_2\rho^2 = N_2 \quad (\text{as } \rho^2 \in N_2)$$

Hence

$$o(N_2\rho) = 2.$$

Similarly

$$(N_2\sigma)^2 = N_2\sigma^2 = N_2$$

$$(N_2\rho\sigma)^2 = N_2$$

So all non-identity elts. of $\frac{G}{N_2}$ have order 2, therefore

$$\frac{D_{12}}{\langle\rho^2\rangle} \cong C_2 \times C_2$$

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c) $\dfrac{G}{N_3} = \dfrac{D_{12}}{\langle \rho^3 \rangle}$

has order 6, elts

$N_3, N_3\rho, N_3\rho^2, N_3\sigma, N_3\rho\sigma, N_3\rho^2\sigma$

By Theorem 5.1,

$$\frac{G}{N_3} \cong C_6 \text{ or } D_6$$

Which?

Compute the order of the elts: these are

$1, 3, 3, 2, 2, 2$

Hence

$$\frac{D_{12}}{\langle \rho^3 \rangle} \cong D_6.$$

[Alternative pf: if

$x = N_3\rho$

$y = N_3\sigma$

check that

$$o(x) = 3, \quad o(y) = 2$$

and $\quad yx = x^{-1}y.$

~~Thus~~ Also

$$\frac{G}{N_3} = \{e, x, x^2, y, xy, x^2y\}$$

from list .

Therefore $\dfrac{G}{N_3} \cong D_6$ by

Lemma 4.4.].

8. The First Isomorphism Theorem

In the last few chapters we've introduced theory of:

- homoms $\phi : G \to H$
- Image & kernel of ϕ
- Normal subgps, fact that $\ker(\phi) \lhd G$
- Factor groups $\dfrac{G}{N}$

The First Iso Theorem connects all of these concepts:

<u>Theorem 8.1</u> Let $\phi : G \to H$ be a homomorphism. Then the factor group

$$\frac{G}{\mathrm{Ker}(\phi)} \cong \mathrm{Im}(\phi).$$

<u>Corollary 8.2</u> If G is a <u>finite</u> group and $\phi : G \to H$ is a homom, then

$$|G| = |\mathrm{Im}(\phi)| \cdot |\mathrm{Ker}(\phi)|$$

(a "rank-nullity" theorem for groups).

<u>Proof of 8.1</u>

Let $\phi : G \to H$ and let

$$K = \ker(\phi) \trianglelefteq G.$$

[Aim: to define an isomorphism

$$\frac{G}{K} \longrightarrow \operatorname{Im}(\phi)$$

Natural to try the map

$$Kx \longrightarrow \phi(x) \quad \forall x \in G.$$

First we need to check this is a well-defined map.].

<u>Step 1</u> If $Kx = Ky$ then $\phi(x) = \phi(y)$ $(x, y \in G)$

Pf. $Kx = Ky \implies xy^{-1} \in K$

$$\implies \phi(xy^{-1}) = e$$

$$\implies \phi(x)\,\phi(y^{-1}) = e$$

$$\implies \phi(x)\,\phi(y)^{-1} = e$$

$$\implies \phi(x) = \phi(y).$$

By Step 1, we have a well-defined function

$$\alpha : \frac{G}{K} \longrightarrow \mathrm{Im}(\phi) \text{ defined by}$$

$$\alpha(Kx) = \phi(x) \quad \forall x \in G.$$

<u>Step 2</u> α is an isomorphism.

<u>Pf.</u> <u>α a homom</u>:

$$\alpha((Kx)(Ky)) = \alpha(Kxy)$$
$$= \phi(xy)$$
$$= \phi(x)\phi(y)$$
$$= \alpha(Kx)\alpha(Ky) \checkmark$$

<u>α surjective</u>

$$h \in \mathrm{Im}(\phi) \Longrightarrow h = \phi(x)$$
for some $x \in G$

$$\Longrightarrow h = \alpha(Kx)$$

Hence α is surjective.

α injective

$$\alpha(Kx) = \alpha(Ky)$$
$$\Longrightarrow \phi(x) = \phi(y)$$
$$\Longrightarrow \phi(x)\phi(y)^{-1} = e$$
$$\Longrightarrow \phi(xy^{-1}) = e$$
$$\Longrightarrow xy^{-1} \in \ker\phi = K$$
$$\Longrightarrow Kx = Ky.$$

Hence we've shown

$$\alpha : \frac{G}{K} \to \mathrm{Im}(\phi)$$

is an isomorphism, proving Theorem 8.1 //

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Examples

1) Consider the signature homom.

$$s : S_n \to C_2$$

We know

$$\mathrm{Ker}(s) = A_n$$

$$\mathrm{Im}(s) = C_2$$

Hence Thm 8.1 gives

$$S_n / A_n \cong C_2 .$$

2) Consider homom

$$\phi : (\mathbb{R}, +) \to (\mathbb{C}^*, \times)$$

defined by

$$\phi(x) = e^{2\pi i x} \quad (x \in \mathbb{R})$$

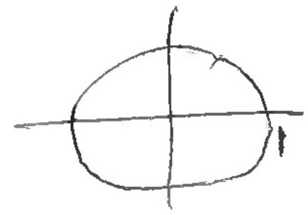
Here

$$\mathrm{Ker}(\phi) = \mathbb{Z}$$

$$\mathrm{Im}(\phi) = \{ z \in \mathbb{C} : |z| = 1 \}$$

$$= T,$$

the <u>circle group</u>



So Theorem 8.1 gives

$$\frac{\mathbb{R}}{\mathbb{Z}} \cong T$$

3) Let's go back to a previous qn about homoms:

<u>Qn</u> Does $\exists$ a nontrivial homom. $\phi : S_3 \to C_3$?

<u>Ans</u> No.

Here's a more systematic approach.

Suppose $\exists$ nontrivial hom. $\phi : S_3 \to C_3$.

Then $\mathrm{Im}(\phi)$ must be C_3.

So if $N = \ker(\phi)$, then by Thm 8.1

- $N \triangleleft S_3$
- $\frac{S_3}{N} \cong \mathrm{Im}(\phi) = C_3$

Hence N is a normal subgroup of S_3 of order 2.

But S_3 has <u>no</u> normal subgps of order 2 $\times\!\!\times$

This method applies generally to the question: given groups G, H, does $\exists$ a surjective homom.

$$\phi : G \twoheadrightarrow H \,?$$

9. Normal subgroups and homomorphisms

Given a homom $\phi : G \to H$ we get a normal subgp of G, $\mathrm{Ker}(\phi)$.

Homoms		Normal subgps
ϕ	$\longrightarrow$	$\mathrm{Ker}\,\phi$

Can we go in the other direction:

$$\phi \overset{?}{\longleftarrow} N$$

Yes we can:

Prop 9.1 Let $N \triangleleft G$. Let $H = \frac{G}{N}$ and define $\phi : G \to H$ by

$$\phi(x) = Nx \quad \forall x \in G.$$

Then ϕ is a surjective homom, & $\mathrm{Ker}(\phi) = N$

<u>Pf</u> <u>ϕ is a homom:</u>

$$\phi(xy) = Nxy = (Nx)(Ny) = \phi(x)\,\phi(y).$$

<u>Kernel</u>:

$$x \in \ker(\phi) \iff \phi(x) = N \quad \left(\text{identity of } \frac{G}{N}\right)$$

$$\iff Nx = N$$

$$\iff x \in N.$$

Therefore $\ker(\phi) = N$. //

Eg. Let $G = D_{12}$

$= \{e, \rho, \dots, \rho^5, \sigma, \rho\sigma, \dots, \rho^5\sigma\}$

From previous example we have normal subgroups

$$N_1 = \langle \rho \rangle$$

$$N_2 = \langle \rho^2 \rangle$$

$$N_3 = \langle \rho^3 \rangle$$

We showed

$$\frac{D_{12}}{N_1} \cong C_2$$

$$\frac{D_{12}}{N_2} \cong C_2 \times C_2$$

$$\frac{D_{12}}{N_3} \cong D_6.$$

Hence by Prop 9.1, there are surjective homoms.

$$\phi_1 : D_{12} \twoheadrightarrow C_2$$

$$\phi_2 : D_{12} \twoheadrightarrow C_2 \times C_2$$

$$\phi_3 : D_{12} \twoheadrightarrow D_6.$$

<u>Final remark</u>

Given a group G, we can find all ~~groups~~ <u>homomorphic images</u> of G, i.e. all groups H for which $\exists$ surjective homom. $\phi : G \to H$.

This can be done as follows:

A) Find all the normal subgps. of G

B) The possibilities for H are precisely the factor groups $\frac{G}{N}$ for $N \lhd G$.

<u>Eg 1)</u> $\underline{G = S_3}$.

We found all the

normal subgps of G:

$\{e\}, \; A_3, \; S_3.$

So the homom. images of S_3

are $\frac{S_3}{\{e\}}, \frac{S_3}{A_3}, \frac{S_3}{S_3}$, i.e.

$S_3, \; C_2, \; C_1$

2) <u>The homom. images of D_{12}</u>

We already know the following homom images:

$D_{12}, C_1,$

$C_2, \; C_2 \times C_2, \; D_6.$

Let H be a homom. image of D_{12}, i.e. $\exists$ surjective homom.

$$\phi : D_{12} \twoheadrightarrow H.$$

By Thm 8.1, $|H|$ divides 12.

Ignoring the possibilities in the list , the possibilities for H are

$$C_6, C_4, C_3$$

Are these homom. images of D_{12}?

<u>Claim 1</u> C_6 is <u>not</u> a homom. image

<u>Pf</u>. Suppose $\exists$ surj.

$$\phi : D_{12} \twoheadrightarrow C_6.$$

Then if $K = \ker[\phi]$,

- $K \trianglelefteq D_{12}$
- $|K| = 2$
- $\frac{D_{12}}{K} \cong C_6$.

The subgroups of D_{12} of order 2 are of the form $\langle x \rangle$, $o(x) = 2$, so they are

$\langle \rho^3 \rangle$, $\langle \rho^i \sigma \rangle$ $(i = 0, \ldots, 5)$.

Of these,

$\langle \rho^3 \rangle = N_3$ is normal

but

$\langle \rho^i \sigma \rangle \ntrianglelefteq D_{12}$

(Sheet 3, Q5).

Hence $K = \ker(\phi) = N_3$

But then $\frac{D_{12}}{K} \cong D_6$ ※

<u>Claim 2</u> C_4 is not a homom. image of D_{12}.

Pf. Same argument: suppose $\exists\, \phi : D_{12} \twoheadrightarrow C_4$, with kernel K.

Then $K \triangleleft D_{12}$ and $|K| = 3$.

The only subgp of order 3 in D_{12} is

$$\langle \rho^2 \rangle = N_2.$$

Hence $K = N_2$, but then $\frac{D_{12}}{K} \cong C_2 \times C_2$ (not C_4). ✗

Still to cover: is C_3 a homom image of D_{12}? ....