The maximal subgroups of positive dimension in exceptional algebraic groups

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Abstract

In this paper we complete the determination of the maximal subgroups of positive dimension in simple algebraic groups of exceptional type over algebraically closed fields. This follows work of Dynkin, who solved the problem in characteristic zero, and Seitz who did likewise over fields whose characteristic is not too small.

A number of consequences are obtained. It follows from the main theorem that a simple algebraic group over an algebraically closed field has only finitely many conjugacy classes of maximal subgroups of positive dimension. It also follows that the maximal subgroups of sufficiently large order in finite exceptional groups of Lie type are known.

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1 Introduction

Let G be a simple algebraic group of exceptional type G_2, F_4, E_6, E_7 or E_8 over an algebraically closed field K of characteristic p (where we set $p = \infty$ if K has characteristic zero). In this paper we determine the maximal closed subgroups of positive dimension in G. Taken together with the results of [25, 30, 40] on classical groups, this provides a description of all maximal closed subgroups of positive dimension in simple algebraic groups.

We obtain a variety of consequences, including a classification of maximal subgroups of the associated finite groups of Lie type, apart from some subgroups of bounded order.

The analysis of maximal subgroups of exceptional groups has a history stretching back to the fundamental work of Dynkin [11], who determined the maximal connected subgroups of G in the case where K has characteristic zero. The flavour of his result is that apart from parabolic subgroups and reductive subgroups of maximal rank, there are just a few further conjugacy classes of maximal connected subgroups, mostly of rather small dimension compared to $\dim G$. In particular, G has only finitely many conjugacy classes of maximal connected subgroups.

The case of positive characteristic was taken up by Seitz [31], who determined the maximal connected subgroups under some assumptions on p, obtaining conclusions similar to those of Dynkin. If $p > 7$ then all these assumptions are satisfied. This result was extended in [21], where all maximal closed subgroups of positive dimension in G were classified, under similar assumptions on p.

In the years since [31, 21], the importance of removing the characteristic assumptions in these results has become increasingly clear, in view of applications to both finite and algebraic group theory. For example, [24, Theorem 1] shows that any finite quasisimple subgroup $X(q)$ of G, with q a sufficiently large power of p, can be embedded in a closed subgroup of positive dimension in G; this is used to prove that maximal subgroups of finite exceptional groups G_σ (σ a Frobenius morphism) are, with a bounded number of exceptions, of the form X_σ with X a maximal closed subgroup of positive dimension in G (see [24, Corollaries 7,8]).

Here we complete the solution of this problem. We determine all maximal closed subgroups of positive dimension in G in arbitrary characteristic. For the purposes of one of our applications to finite groups of Lie type, we in fact prove a slightly more general result, admitting the presence of field endomorphisms and graph automorphisms of G. Henceforth we simply refer to these as "morphisms of G".

Let G be of adjoint type, and define $\mathrm{Aut}\,(G)$ to be the abstract group generated by inner automorphisms of G, together with graph and field morphisms. In the statement below, by a subgroup of maximal rank we mean a subgroup containing a maximal torus of G, and Sym_k denotes the symmetric group of degree k. Also $\bar{\mathbb{F}}_p$ denotes the algebraic closure of the prime field $\mathbb{F}_p$. Recall also that a Frobenius morphism of G is an endomorphism σ whose fixed point group G_σ is finite.

Here is our main result.

Theorem 1 *Let G_1 be a group satisfying $G \leq G_1 \leq \mathrm{Aut}\,(G)$; in the case where G_1 contains a Frobenius morphism of G, assume that $K = \bar{\mathbb{F}}_p$. Let X be a proper closed connected subgroup of G which is maximal among proper closed connected $N_{G_1}(X)$-invariant subgroups of G. Then one of the following holds:*

(a) *X is either parabolic or reductive of maximal rank;*

(b) *$G = E_7$, $p \neq 2$ and $N_G(X) = (2^2 \times D_4).Sym_3$;*

(c) *$G = E_8$, $p \neq 2, 3, 5$ and $N_G(X) = A_1 \times Sym_5$;*

(d) *X is as in Table 1 below.*

The subgroups X in (b), (c) *and* (d) *exist, are unique up to conjugacy in* $\mathrm{Aut}(G)$*, and are maximal among closed, connected $N_G(X)$-invariant subgroups of G.*

Table 1

G	X simple	X not simple
G_2	$A_1\,(p \geq 7)$	
F_4	$A_1\,(p \geq 13)$, $G_2\,(p = 7)$,	$A_1G_2\,(p \neq 2)$
E_6	$A_2\,(p \neq 2,3)$, $G_2\,(p \neq 7)$, $C_4\,(p \neq 2)$, F_4	A_2G_2
E_7	A_1 (2 classes, $p \geq 17, 19$ resp.), $A_2\,(p \geq 5)$	$A_1A_1\,(p \neq 2,3)$, $A_1G_2\,(p \neq 2)$, A_1F_4, G_2C_3
E_8	A_1 (3 classes, $p \geq 23, 29, 31$ resp.), $B_2\,(p \geq 5)$	$A_1A_2\,(p \neq 2,3)$, $A_1G_2G_2\,(p \neq 2)$, G_2F_4

Theorem 1 determines the maximal subgroups M of any such group G_1 such that $M \cap G$ is closed and has positive dimension: namely, either M contains G, or $M = N_{G_1}(X)$ for X as in (a)-(d) and $MG = G_1$. Below we shall give applications with $G_1 = G$ and with $G_1 = G\langle\sigma\rangle$, where σ is a Frobenius morphism of G.

In Tables 10.1 and 10.2 at the end of the paper we present further information concerning the subgroups X in Table 1:

(1) We give the precise action (as a sum of explicit indecomposable modules) of X on $L(G)$, and also, in the cases $G = F_4, E_6, E_7$, on the module V, where V is the restricted irreducible G-module of high weight $\lambda_4, \lambda_1, \lambda_7$ respectively (of dimension $26 - \delta_{p,3}, 27, 56$). These actions are recorded in Tables 10.1 and 10.2, and proofs can be found in Section 9.

(2) We give the values of $|N_G(X) : X|$; this is always at most 2. In all cases where X has a factor A_2, $N_G(X)$ induces a graph automorphism on this factor, and the only other case where $|N_G(X) : X| = 2$ is that in which $G = E_8$ and $X = A_1G_2G_2$, where $N_G(X)$ has an element interchanging the two G_2 factors. These facts follow from the constructions of the maximal subgroups $A_1A_2, A_1G_2G_2 < E_8$ in [31, p.46, 39], of $A_2G_2 < E_6$ in [31, 3.15], and from [24, 8.1] for $A_2 < E_7$ and $A_2 < E_6$.

The subgroups of G of type (a) in Theorem 1 are well understood. Maximal parabolic subgroups correspond to removing a node of the Dynkin diagram (possibly two nodes if G_1 contains an element involving a graph or graph-field morphism). Subgroups which are reductive of maximal rank are easily determined. They correspond to various subsystems of the root system of G, and we give a complete list (with a proof in Section 8) of those whose normalizers are maximal in G, in Table 10.3 in Section 10. Likewise, Table 10.4 lists the maximal connected subgroups of maximal rank (again with a proof in Section 8).

Application of Theorem 1 with $G_1 = G$ gives a complete determination of the maximal closed subgroups of positive dimension in G - they are just the subgroups $N_G(X)$ for X as in (a)-(d). We state this formally for completeness:

Corollary 2 (i) *The maximal closed subgroups of positive dimension in G are as follows: maximal parabolics; normalizers of reductive subgroups of maximal rank, as listed in Table* 10.3*; the subgroups $(2^2 \times D_4).Sym_3 < E_7$ and $A_1 \times Sym_5 < E_8$ in Theorem* 1(b,c)*; and subgroups $N_G(X)$ with X as* *in Table* 1.

(ii) *The maximal closed connected subgroups of G are as follows: maximal parabolics; maximal closed connected subgroups of maximal rank, as listed in Table* 10.4*; and all subgroups X in Table* 1*, omitting the subgroup $A_1G_2G_2 < E_8$.*

The subgroup $A_1G_2G_2 < E_8$ in Table 1 lies in a subgroup F_4G_2 so is not maximal connected; however its normalizer in E_8 interchanges the two G_2 factors, and indeed $N_{E_8}(X)$ is maximal in E_8.

On glancing at the main results of [21, 31] and comparing them with our Theorem 1, the reader will notice that the conclusions are very similar. Indeed, the only subgroups present in our Table 1 which are not already in [21, 31] are

$$\begin{array}{l} G_2 < E_6 \text{ for } p = 2, 3, \\ A_2 < E_7 \text{ for } p = 5, \\ B_2 < E_8 \text{ for } p = 5. \end{array}$$

Constructions for these maximal subgroups essentially follow along the lines of constructions given in [31], apart from the maximal $G_2 < E_6$ for $p = 2$, for which a new approach is required (see Lemma 6.3.7).

Thus the bulk of our work is concerned with proving that very few maximal subgroups occur in small characteristics (apart from those in conclusion (a) of Theorem 1). We shall discuss below some of the methods we use, but first we present some consequences of our main result.

The next corollary applies to all types of simple algebraic groups, classical and exceptional.

Corollary 3 *If H is a simple algebraic group over an algebraically closed field, then H has only finitely many conjugacy classes of maximal closed subgroups of positive dimension.*

This is immediate from Theorem 1 when H is of exceptional type. For H classical, some argument is required, and is given in Section 8.

Theorem 1 also has significant applications to the study of maximal subgroups of finite exceptional groups of Lie type. Let G be an exceptional adjoint algebraic group over $\bar{\mathbb{F}}_p$, where p is a prime. Let σ be a Frobenius morphism of G such that $L = O^{p'}(G_\sigma)$ is a finite exceptional simple group of Lie type over a finite field $\mathbb{F}_{q_1}$. Finally, let L_1 be a group such that

$L \leq L_1 \leq \mathrm{Aut}(L)$, and let M be a maximal subgroup of L_1 not containing L.

Corollary 4 *There are absolute constants c, d (independent of G, L, L_1) such that if $|M| > c$ then M is explicitly known, and determined up to G_σ-conjugacy, falling into at most $d \log\log q_1$ conjugacy classes of subgroups.*

In order to specify precisely what the "explicitly known" maximal subgroups are in the conclusion of Corollary 4, we need some further discussion.

Our paper [21] contains a "reduction theorem" for maximal subgroups M of L_1 as above: namely, [21, Theorem 2] explicitly determines all maximal subgroups M for which $F^*(M)$ is not simple.

Assume now that $M_0 = F^*(M)$ is simple. Denote by $\mathrm{Lie}(p)$ the set of finite simple groups of Lie type in characteristic p. If $M_0 \notin \mathrm{Lie}(p)$, the possibilities for M_0 up to isomorphism are given in [26] (although the determination of the conjugacy classes of such subgroups is largely an open field at the moment). There is an absolute upper bound (independent of q_1) on the order of these groups. We thus focus on the case where $M_0 \in \mathrm{Lie}(p)$. Say $M_0 = M(q)$, a group of Lie type over $\mathbb{F}_q$, where q is a power of p.

We say that $M(q)$ has the *same type* as G if $M(q) \cong G_\delta^{(\infty)}$ for some Frobenius morphism δ of G (where $G_\delta^{(\infty)}$ denotes the last term in the derived series of G_δ); such maximal subgroups are determined up to G_σ-conjugacy by [22, 5.1]. And we say that M is a *subgroup of L_1 of maximal rank* if $M = N_{L_1}(D_\sigma)$, where D is a σ-stable connected reductive subgroup of maximal rank in G; such maximal subgroups are determined in [19].

We now recall a definition taken from [24]. Let $\Sigma = \Sigma(G)$ be the root system of G, and for a subgroup L of $\mathbb{Z}\Sigma$, let $t(L)$ be the exponent of the torsion subgroup of $\mathbb{Z}\Sigma/L$. For $\alpha, \beta \in \Sigma$, call the element $\alpha - \beta$ of $\mathbb{Z}\Sigma$ a *root difference.* Define

$$t(\Sigma(G)) = \max\ \{t(L) : L \text{ a subgroup of } \mathbb{Z}\Sigma \text{ generated by root differences }\}.$$

R. Lawther has computed the values of $t(\Sigma(G))$ for all exceptional groups except E_8:

G	G_2	F_4	E_6	E_7
$t(\Sigma(G))$	12	68	124	388

The following result is a characteristic-free version of [24, Corollary 7].

Corollary 5 *Let $L = O^{p'}(G_\sigma)$ and $L \leq L_1 \leq Aut(L)$, as above, and let M be a maximal subgroup of L_1 with $F^*(M) = M(q)$, q a power of p. Let $G_\sigma = G(q_1)$. Assume that*

$$\begin{array}{ll} q > t(\Sigma(G)).(2, p-1) & \text{if } M(q) = A_1(q),\ {}^2B_2(q) \text{ or } {}^2G_2(q) \\ q > 9 \text{ and } M(q) \neq A_2^\epsilon(16) & \text{otherwise.} \end{array}$$

Then one of the following holds:

(i) *M is a subgroup of maximal rank;*

(ii) *$M(q)$ has the same type as G;*

(iii) *$q = q_1$ and $M(q) = O^{p'}(X_\sigma)$, where X is a simple maximal connected σ-stable subgroup of G given in the second column of Table 1. The possibilities are as follows (one $Aut(G_\sigma)$-class of subgroups for each group $M(q)$ listed):*

G	$M(q)$
G_2	$A_1(q)\,(p \geq 7)$
F_4	$A_1(q)\,(p \geq 13)$, $G_2(q)\,(p = 7)$
E_6	$A_2^\epsilon(q)\,(\epsilon = \pm,\ p \geq 5)$, $G_2(q)\,(p \neq 7)$,
	$C_4(q)\,(p \neq 2)$, $F_4(q)$
E_7	$A_1(q)\,(2 \text{ classes},\ p \geq 17, 19)$, $A_2^\epsilon(q)\,(\epsilon = \pm,\ p \geq 5)$
E_8	$A_1(q)\,(3 \text{ classes},\ p \geq 23, 29, 31)$, $B_2(q)\,(p \geq 5)$

Corollary 5 can be deduced quickly from Theorem 1 and results in [24], as follows. First, [24, Corollary 7] implies that either conclusion (i) or (ii) holds, or $M(q) = O^{p'}(X_\sigma)$ for some simple maximal closed connected σ-stable subgroup X of G not containing a maximal torus. Applying Theorem 1 with $G_1 = G\langle\sigma\rangle$, it follows that X is as in the second column of Table 1, and hence $M(q)$ is as in conclusion (iii).

Corollary 4 follows from Corollary 5, together with the above discussion: by [22, 5.1], the number of classes of maximal subgroups of the same type as G is bounded above by $d \log\log q_1$ (note that $\log\log q_1$ is an upper bound for the number of maximal subfields of $\mathbb{F}_{q_1}$); the number of classes of maximal subgroups which are parabolic or of maximal rank is bounded by a constant; and by [21, Theorem 2], the number of classes of maximal subgroups M with $F^*(M)$ non-simple is also bounded. By Corollary 5, the remaining maximal subgroups M are either as in Corollary 5(iii) (hence fall into boundedly many classes), or have $F^*(M)$ simple of bounded order. Corollary 4 follows.

We now turn to a general discussion of the proof of Theorem 1. In view of [21, 31], we need only determine maximal subgroups X which are simple and of small rank (at most 4) in low characteristics (at most 7). The precise list of cases to be dealt with is given in Proposition 2.2.1. Given this, the argument begins in similar fashion to that in [31]. Namely, we define a specific 1-dimensional torus T in X, and show that the maximality hypothesis forces T to determine a labelling of the Dynkin diagram of G by 0's and 2's. Such a labelled diagram specifies completely the weights of T in its action on $L(G)$.

The next step is to use this labelling information to determine the possibilities for the composition factors of X on $L(G)$. Every irreducible X-module restricts to T, giving a certain collection of T-weights. The fact that the full list of composition factors of $L(G) \downarrow X$ must determine a collection of T-weights which is compatible with a labelling of the Dynkin diagram with 0's and 2's severely restricts the possibilities for $L(G) \downarrow X$. Indeed, the "Weight Compare Program" used for [31] carries out the above procedure, and prints out a list of possibilities for the composition factors of $L(G) \downarrow X$ corresponding to each possible labelled diagram. For a little more discussion of this, see the remarks following Lemma 2.2.6.

So at this point we have lists of possibilities for the composition factors of $L(G) \downarrow X$. In a couple of cases - namely $X = A_1$ or B_2 with $p = 2$ - these lists are formidably long and not especially useful and we develop special techniques to handle these cases utilizing certain ideals in $L(X)$. In the other cases the information provided by the Weight Compare Program is quite helpful.

We remarked after Theorem 1 that very few new maximal subgroups arise in the small characteristics we are considering here. Thus our aim for the most part is, given a listed possibility for $L(G) \downarrow X$, to obtain a contradiction to the maximality hypothesis on X. If we can force X to have a nontrivial fixed point on $L(G)$, such a contradiction is immediate (see Lemma 2.2.10(iv)). This was the main tool used in [31] to reduce the possibilities for $L(G) \downarrow X$ to a manageable list. However in small characteristics, it is much less easy to force the existence of a fixed point - indeed, it is often impossible. The main reason is that in small characteristics there are usually several indecomposable X-modules of low dimension involving the trivial module; thus, even if $L(G) \downarrow X$ has trivial composition factors, it may be impossible to prove it has a trivial submodule since these indecomposables may be present.

Thus at this point, new methods are required to supplement those of [31].

To our aid comes one of the few advantages of being in small characteristic: given an irreducible X-module $V(\lambda)$ appearing as a composition factor of $L(G) \downarrow X$, there is a reasonable probability of it being *totally twisted* - that is, of the form $V(\mu)^{(q)}$ for some power $q = p^a$ with $a \geq 1$. Indeed, for $p = 2$ or 3 almost all of our listed possibilities for $L(G) \downarrow X$ have at least one nontrivial totally twisted composition factor, and usually there are several such.

The point about totally twisted composition factors is that they are annihilated by $L(X)$. Thus, if we can actually force there to be a totally twisted *submodule* in $L(G) \downarrow X$, it follows that $C_{L(G)}(L(X)) \neq 0$. It turns out that we can indeed force this in many cases (although this may require some effort) and this is the starting point of our new method.

Let $A = C_{L(G)}(L(X))$, and suppose we have shown that $A \neq 0$. An elementary lemma (see 2.3.4) shows that A is contained as a subalgebra in the Lie algebra $L(D)$ of a maximal rank reductive subgroup D lying in a small list of possibilities. A detailed analysis of this subalgebra leads in many cases to the construction of group elements in G which stabilize A (or possibly an ideal of A), yet cannot normalize X, and this contradicts our maximality hypothesis. These group elements are usually obtained by an exponentiation process.

This then is a rough outline of the argument. However, there are a number of cases where the above method does not work - for instance, when we cannot force $A \neq 0$. For these, different and very substantial arguments are required, bringing into play a full panoply of available tools from algebraic group theory (see for example the proofs of Proposition 3.3.3, Proposition 4.1.4, and Proposition 4.2.17).

Notation

Throughout the paper we take G to be a simple adjoint algebraic group of exceptional type over an algebraically closed field K of characteristic p. Fix a maximal torus T_G of G. Let $\Sigma(G)$ denote the root system of G and fix a system of fundamental roots $\Pi(G)$ and B_G the Borel subgroup generated by T_G and all T_G-root subgroups corresponding to fundamental roots. Write $\Pi(G) = \{\alpha_1, \ldots, \alpha_l\}$, and use the ordering of fundamental roots and Dynkin diagrams given in [6, p.250]. On occasion we will make use of the simply connected cover $\hat{G}$ of G. Write $\lambda_1, \ldots, \lambda_l$ for the fundamental dominant weights.

Let $L(G)$ denote the Lie algebra of G, and define

$$L = L(G)'$$

so that $L = L(G)$ unless $(G, p) = (E_6, 3)$ or $(E_7, 2)$, in which case L has codimension 1 in $L(G)$ (see Lemma 2.1.1). We shall use the standard notation e_α $(\alpha \in \Sigma(G))$ for root vectors in $L(G)$, and set $f_\alpha = e_{-\alpha}$. The root subgroup of G corresponding to α will be denoted by $U_\alpha = \{U_\alpha(c) : c \in K\}$.

Let X be a simple algebraic group over K. For a dominant weight λ, let $V_X(\lambda)$ be the rational irreducible KX-module of high weight λ, $W_X(\lambda)$ the Weyl module of high weight λ, and $T_X(\lambda)$ the indecomposable tilting module of high weight λ. Often we use just λ to denote the irreducible module $V_X(\lambda)$. Additional notation for certain indecomposable modules will be given in Section 10.

If $M_1, \ldots, M_r$ are rational KX-modules and $n_1, \ldots, n_r$ positive integers, then the notation

$$(M_1)^{n_1}/\ldots/(M_r)^{n_r}$$

denotes a rational KX-module which has the same composition factors as the direct sum $(M_1)^{n_1} \oplus \ldots \oplus (M_r)^{n_r}$. For example, if $\mu_1, \ldots, \mu_r$ are distinct dominant weights, then $\mu_1^{n_1}/\ldots/\mu_r^{n_r}$ denotes a KX-module which has composition factors $V_X(\mu_i)$ appearing with multiplicity n_i for each i.

Finally,

$$M_1|M_2|\ldots|M_r$$

denotes a rational KX-module V which has a series $0 = V_r < V_{r-1} < \ldots < V_1 < V_0 = V$ of submodules such that $V_{i-1}/V_i \cong M_i$ for $1 \leq i \leq r$.

2 Preliminaries

This chapter contains a number of preliminary results which will be used in our proof of Theorem 1. The first section consists of some lemmas on representation theory. In the second we start the proof proper, by defining a certain 1-dimensional torus T in our subgroup X, and establishing the T-labelling of the Dynkin diagram of G by 0's and 2's referred to in the Introduction. In the third section we prove some general results about the algebra $A = C_L(L(X)')$ which are fundamental to our analysis in later chapters.

2.1 Lemmas from representation theory

Recall that G is an exceptional adjoint algebraic group over K in characteristic p, $L(G)$ is the Lie algebra of G, and $L = L(G)'$, the derived subalgebra.

Lemma 2.1.1 *Either $L(G)$ is an irreducible module for G or one of the following holds:*

(i) *$(G, p) = (E_6, 3)$ or $(E_7, 2), L = L(G)'$ has codimension 1 in $L(G)$, and L is irreducible for G.*

(ii) *$(G, p) = (G_2, 3)$ or $(F_4, 2)$ and $L(G)$ has an ideal I generated by root elements for short roots such that both I and $L(G)/I$ are irreducible for G. Moreover, $L(G)$ is indecomposable.*

Proof Recall that $\hat{G}$ is the simply connected cover of G. Let $\pi : \hat{G} \to G$ be the canonical map and let $\hat{T}_G$ be the preimage of T_G. Then $L(\hat{G})$ has basis $\{e_\alpha, h_{\alpha_i} : \alpha \in \Sigma(G), \alpha_i \in \Pi(G)\}$. If α_0 is the root of highest height, then e_{α_0} is a maximal vector for $\hat{B}_G$, the preimage of B_G, of weight λ_i, where $i = 2, 1, 2, 1, 8$ according as $G = G_2, F_4, E_6, E_7, E_8$ respectively. It follows from [13] that $L(\hat{G})$ is irreducible except for the cases (i),(ii) indicated.

In the exceptional E_6 and E_7 cases in (i), the corresponding Weyl module has a 1-dimensional submodule with irreducible quotient (see [13]), and it is straightforward to find a nonzero central element of $L(\hat{G})$. So in these cases the adjoint representation has an irreducible submodule of codimension 1. It follows from the commutator relations that this submodule is $L = L(G)'$, so that (i) holds.

Now suppose that $(G, p) = (G_2, 3)$ or $(F_4, 2)$. Here we let I be the ideal of $L(G)$ generated by root vectors for short roots. Commutator relations imply that I is a proper ideal and that if β is the highest short root, then

e_β is a maximal vector. It follows from [13] that both I and $L(G)/I$ are irreducible. Finally, a consideration of commutator relations among root vectors in $L(G)$ implies that $L(G)$ is indecomposable. ■

Lemma 2.1.2 *Let $0 \neq l \in L$ and let $C = C_G(l)$.*

(i) If l is semisimple, then C contains a maximal torus of G.

(ii) If l is nilpotent, then $R_u(C) \neq 1$ and hence C is contained in a proper parabolic subgroup of G.

Proof If l is semisimple, then [4, 11.8] implies that l is in the Lie algebra of a maximal torus of G. So in this case C contains a maximal torus and (i) holds.

Now suppose that l is nilpotent. Here [4, 14.26] shows that $l \in L(U)$ where U is a maximal unipotent subgroup of G.

Suppose $R_u(C) = 1$. Now l is centralized by a root subgroup in $Z(U)$, so that $U_C = (U \cap C)^0$ is nontrivial and U_C is contained in a maximal unipotent subgroup of C^0. Our supposition implies C^0 is reductive so there is an element $k \in C^0$ such that $U_C \cap U_C^k = 1$.

Write $k = u_1hwu_2$, where $u_1, u_2 \in U$, h is in a maximal torus of $B = N_G(U)$ and w represents an element of the Weyl group of G. Then $U \cap U^k = U \cap U^{wu_2} = (U \cap U^w)^{u_2}$. Now $l \in L(U) \cap L(U^{wu_2}) = (L(U) \cap L(U^w))^{u_2} = L(U \cap U^w)^{u_2}$, where the last equality holds since U, U^w, and $(U \cap U^w)$ are all products of root groups. Indeed, $U \cap U^w$ is the product of those root subgroups for positive roots which w leaves positive. In particular, $D = Z(U \cap U^w)$ is a connected group (it is invariant under a maximal torus of B) of positive dimension, so $D^{u_2} \leq C(L(U \cap U^w)^{u_2}) \leq C$. But then $D^{u_2} \leq U_C \cap U_C^k = 1$, a contradiction. Therefore, $R_u(C) \neq 1$ and by [5], C is contained in a proper parabolic subgroup of G, giving (ii). ■

The next few lemmas are standard results on representations. Notation is as in the Introduction. Let H be a simply connected simple algebraic group over K.

Lemma 2.1.3 *Let λ be a dominant weight for a maximal torus of H, and write $\lambda = \mu_0 + p\mu_1 + \cdots + p^k\mu_k$, where each μ_i is restricted.*

(i) $V_H(\lambda) \cong V_H(\mu_0) \otimes V_H(\mu_1)^{(p)} \otimes \cdots \otimes V_H(\mu_k)^{(p^k)}$.

(ii) $V_H(\lambda) \downarrow L(H)$ *is a direct sum of irreducible modules each isomorphic to $V(\mu_0)$.*

Proof Part (i) is just the Steinberg Tensor Product Theorem, and (ii) follows from (i) and the fact that the differential of the Frobenius map is 0. ∎

Lemma 2.1.4 ([16, p.207]) *Let V be a rational KH-module. Suppose λ is a maximal dominant weight for which the corresponding weight space of V is nonzero, and $v \in V$ has weight λ. Define*

$$\langle Hv \rangle = \langle h(v) : h \in H \rangle.$$

Then $\langle Hv \rangle$ is an image of the Weyl module $W_H(\lambda)$.

We will use the following consequence of Lemma 2.1.4 on several occasions. Denote by w_0 the longest element of the Weyl group of H. Recall that for weights λ, μ we write $\mu < \lambda$ to mean that $\lambda - \mu$ is a sum of positive roots.

Lemma 2.1.5 *Let V be a rational KH-module, and suppose that H preserves a nondegenerate bilinear form on V. Let v be a weight vector of weight λ, a maximal dominant weight for H.*

(i) *$\langle Hv \rangle$ is an image of the Weyl module $W_H(\lambda)$, and if M is the image of the maximal submodule of $W_H(\lambda)$, then M is a totally singular subspace of V.*

(ii) If $\lambda \neq -w_0(\lambda)$, then $\langle Hv \rangle$ is a singular subspace of V.

(iii) Suppose $w \in V$ is a maximal vector for H having weight δ which is not subdominant to $-w_0(\lambda)$. Then $\langle Hw \rangle \leq M^{\perp}$.

Proof We know that $\langle Hv \rangle$ is an image of $W_H(\lambda)$, and we set M to be the image of the maximal submodule. Let R denote the radical of M with respect to the H-invariant form on V.

(i) Suppose $R < M$ and consider the non-degenerate space $R^{\perp}/R$. If $\delta \neq \lambda$ is the high weight of a composition factor of M, then δ is subdominant to λ (i.e. δ is dominant and $\delta < \lambda$). Hence composition factors of $V/R^{\perp}$ have high weight of the form $-w_0(\delta)$ for δ subdominant to λ. It follows that $v \in R^{\perp}$ and hence $\langle Hv \rangle < R^{\perp}$.

Now M/R is a non-degenerate subspace of $R^{\perp}/R$ so $R^{\perp}/R = (M/R) \perp D$ for some non-degenerate space D. However $\langle Hv \rangle/R$ is indecomposable, a contradiction. This proves (i).

(ii) By (i) we see that M is singular and from the first paragraph of the argument we have $\langle Hv \rangle < M^{\perp}$. So $\langle Hv \rangle/M$ is an irreducible submodule of

$M^{\perp}/M$ of high weight λ. The dominant weight $-w_0(\lambda)$ is the high weight of the dual of $V_H(\lambda)$, so our assumption implies that $\langle Hv\rangle/M$ is singular and hence so is $\langle Hv\rangle$. This establishes (ii).

For (iii) first note that H-composition factors of M each have high weight subdominant to λ. Hence, composition factors of $V/M^{\perp}$ have high weights subdominant to $-w_0(\lambda)$. It follows that $w \in M^{\perp}$ and hence $\langle Hw\rangle \leq M^{\perp}$, as required. ■

Lemma 2.1.6 ([1, 3.9]) *Let $H = SL_2(K)$, and let λ, λ' be dominant weights for H with p-adic expressions $\lambda = \sum p^i\mu_i$ and $\lambda' = \sum p^i\mu_i'$, respectively. Then there is a 2-step indecomposable H-module with composition factors of high weights λ and λ' if and only if there exists k such that $\mu_i = \mu_i'$ for $i \notin \{k, k+1\}, \mu_k + \mu_k' = p-2$, and $\mu_{k+1} - \mu_{k+1}' = \pm 1$.*

We shall require some basic information about *tilting* modules taken from [32, Section 2]. Recall that a rational H-module V is a tilting module if V has a filtration by Weyl modules and also a filtration by dual Weyl modules. For a dominant weight λ, there is a unique indecomposable tilting module $T(\lambda) = T_H(\lambda)$ with highest weight λ, and any tilting module is a direct sum of $T(\lambda)$'s. A direct summand of a tilting module is again a tilting module, and the tensor product of tilting modules is a tilting module.

Now let $H = A_1$, and for a positive integer c, denote by $T(c)$ the unique indecomposable tilting X-module of high weight c. We shall require the structure of certain of these tilting modules. These are given in the next lemma.

Lemma 2.1.7 *Let $H = A_1$.*

(i) *For $0 \leq r \leq p-2$, $T(r+p)$ is uniserial, has dimension $2p$, and has a series*

$$T(r+p) = (p-r-2)|(r+p)|(p-r-2).$$

(ii) *For $0 \leq r \leq p-2$, $T(r+2p)$ has dimension $4p$ and has a series*

$$T(r+2p) = (2p-r-2)|((r+2p) \oplus r)|(2p-r-2).$$

(iii) *The above tilting modules are projective for both unipotent elements of X and nilpotent elements of $L(X)$.*

Proof Part (i) is [32, 2.3(b)], and (ii) follows from the same kind of argument. In each case $T(c)$ can be constructed as a direct summand of a tensor product of restricted irreducible modules including at least one tensor factor of high weight $p-1$. Then (iii) follows as in the proof of [32, 2.3]. ■

2.2 Initial Reductions

In this section we begin the proof of Theorem 1 with a number of lemmas which will be fundamental for what follows.

As in the statement of Theorem 1, let G_1 be a group satisfying $G \le G_1 \le \mathrm{Aut}(G)$; in the case where G_1 contains a Frobenius morphism of G, assume that $K = \bar{\mathbb{F}}_p$. Let X be a proper closed connected subgroup of G which is maximal among proper closed connected $N_{G_1}(X)$-invariant subgroups of G. Write $S = N_{G_1}(X)$, so that $X = (S \cap G)^0$.

Let T_X, T_G be maximal tori of X and G, respectively, with $T_X \le T_G$. We assume that X is not of maximal rank so that the containment is proper. In addition, we set

$$L = L(G)', \quad A = C_L(L(X)').$$

Notice that L is the image of $d\pi$ where $\pi : \hat{G} \to G$ is the canonical map from the simply connected cover $\hat{G}$ of G. Hence L has a basis consisting of root vectors e_α for $\alpha \in \Sigma(G)$, together with some basis of $L(T_G) \cap L$.

The results of [31] and [21] determine X under the assumption that the characteristic p is not too small in cases where X is simple of relatively small rank. Specifically, the following is established.

Proposition 2.2.1 ([31, 21]) *Theorem 1 holds unless X is simple, $C_G(X) = 1$ and X, G, p are as in the following table.*

G	$X = A_1$	$X = A_2$	$X = B_2, G_2$	$X = B_3$	$X = A_3, C_3, B_4$
G_2	$p \le 3$				
F_4	$p \le 3$	$p \le 3$	$p = 2$		
E_6	$p \le 5$	$p \le 3$	$p \le 3$	$p = 2$	
E_7	$p \le 7$	$p \le 5$	$p \le 3$	$p = 2$	
E_8	$p \le 7$	$p \le 5$	$p \le 5$	$p = 2$	$p = 2$

In the table, blank space indicates that there are no cases requiring consideration.

In view of this, we assume throughout that X, G, p are as in the table in Proposition 2.2.1, and that $C_G(X) = 1$.

We next rule out the possibility that S contains special isogenies of G for the cases $(G, p) = (F_4, 2)$ or $(G_2, 3)$ (i.e. morphisms whose fixed point group in G is a finite twisted group of type ${}^2F_4$ or ${}^2G_2$). By settling this early, we avoid repeated technicalities in lemmas to follow.

Lemma 2.2.2 *Assume $(G,p) = (F_4, 2)$ or $(G_2, 3)$ and that Theorem 1 holds for subgroups S not containing special isogenies. Then it is not possible for S to contain special isogenies.*

Proof By way of contradiction assume that $\tau \in S$ is a special isogeny, so that τ^2 induces a field morphism of G, corresponding to an odd power, say q, of p. Then [24, 1.13] shows that τ^2 induces a Frobenius morphism of X with fixed point group of the form $X(q)$. Note that τ induces an involutory automorphism of $X(q)$. As q is an odd power of p, this cannot be a field or graph-field morphism of X. Hence Proposition 2.2.1 implies that $X = B_2$ and $G = F_4$. Here $S = X\langle\tau\rangle$.

Let R be maximal among τ^2-invariant, connected subgroups of G such that $X \leq R$. We are assuming that Theorem 1 holds for the group $G\langle\tau^2\rangle$. So from the statement of the theorem we see that R is either reductive of maximal rank or parabolic. In the first case a consideration of subsystem groups implies $R = D_4, C_4$, or B_4. It follows from [19] that $G = RR^\tau$ with the intersection $R \cap R^\tau$ being of maximal rank. As $R \cap R^\tau$ is S-invariant, this is a contradiction to the maximality of X. Finally, assume R is parabolic. Then so is R^τ. Here too, $R \cap R^\tau$ contains a maximal torus, as can be seen from the Bruhat decomposition. ■

In view of Lemma 2.2.2, from this point forward we assume that special isogenies are not present in S.

The next result is required when Frobenius morphisms are present in S.

Lemma 2.2.3 *Let σ be a Frobenius morphism in $S < G_1$. Then there is a semilinear transformation $\omega : L \to L$ such that the following hold:*

(i) $\mathrm{ad}(g^\sigma)v = \omega \mathrm{ad}(g)\omega^{-1}v$ *for any $g \in G, v \in L$.*

(ii) *$L(X), L(X)'$, and A are all invariant under ω.*

(iii) *If $0 \neq V \leq L$ is ω-stable, then $N_G(V)$ and $C_G(V)$ are σ-stable.*

(iv) *If $0 \neq V \leq L$ is ω-stable, then V has a basis of ω-fixed vectors.*

(v) *ω sends semisimple elements to semisimple elements and nilpotent elements to nilpotent elements.*

Proof A relatively easy proof of (i) follows from representation theory. Consider the representation $G \to G \to GL(L)$, where the first morphism is σ and the second is the adjoint representation. If L is irreducible, then it follows from a weight consideration that this representation is a twist of the adjoint representation. Similarly, for the cases $(G,p) = (G_2,3),(F_4,2)$, where the representation is reducible, although this requires additional arguments (e.g. Lemma 1.3 of [23]). Thus the representation is equivalent to the representation $G \to GL(L) \to GL(L)$, where the first map is the adjoint representation and the second is given by a field automorphism of $GL(L)$. This field automorphism can be realized as conjugation by a semilinear transformation of L corresponding to a q-power map with respect to a certain basis. An application of Lang's theorem shows that we can adjust the semilinear map by an element of $GL(L)$ to obtain a semilinear map ω satisfying (i).

Another argument which yields all parts of the lemma can be obtained from the general theory. As σ is a Frobenius morphism, G is defined over $\mathbb{F}_q$ for some q. That is, the coordinate ring $K[G]$ can be written $K[G] = K \otimes R$, where R is an algebra over $\mathbb{F}_q$ and σ^*, the comorphism of σ, induces the q-power map on R. Letting δ denote the q-power map on K, extended to $K[G]$ by inducing the identity on R, we have $\delta\sigma^* = q$, the q-power map on $K[G]$.

Viewing $L(G)$ as the left invariant derivations of $K[G]$, one calculates that conjugation by δ defines a semilinear morphism, ω, of $L(G)$ which preserves the Lie algebra structure. Hence ω acts on L and (i) holds. As $K = \bar{\mathbb{F}}_q$ and X is σ-stable, X is defined over $\mathbb{F}_q$ and one checks that $L(X)$ is ω-stable, giving (ii).

Let $0 \neq V \leq L$ be ω-stable. Then (iii) follows from (i). To establish (iv) we first claim that V contains a nonzero vector fixed by ω. Now L has a basis $e_1, ..., e_n$ of vectors fixed by ω. Choose $0 \neq v \in V$ with $v = \sum k_i e_i$ with as few nonzero coefficients as possible. We may assume $k_1 \neq 0$ and hence we make take $k_1 = 1$. If all coefficients are in $\mathbb{F}_q$, then v is fixed by ω. Otherwise, $v - \omega v \neq 0$ and has fewer non-zero coefficients than v, a contradiction. Hence the claim holds. Choose $v_1 \neq 0$ fixed by ω. Then $v_1 \in V_\omega$, an $\mathbb{F}_q$-space and we can find a basis $\{v_1, \ldots, v_n\}$ of L_ω. This is also a K-basis of L and we have $V = \langle v_1 \rangle \oplus (V \cap \langle v_2, \ldots, v_n \rangle)$. The latter summand is ω-stable, so (iv) follows from an induction.

Finally, let $v \in V$. From (i) we conclude that $C_G(v)^{\sigma^{-1}} = C_G(\omega^{-1}v)$. Now v is semisimple if and only if its centralizer is reductive, so ω sends semisimple elements to semisimple elements. Also, v is nilpotent if and only if some power of $\mathrm{ad}(v)$ is 0. So (v) follows. ∎

In view of the previous lemma we will sometimes regard S as acting on $L(G)$.

We next define a certain 1-dimensional torus of X that is fundamental for what follows. Fix a system of T_X-invariant root subgroups of X, one for each root in the root system $\Sigma(X)$ of X, and let $\Pi(X)$ be a system of fundamental roots. If $\gamma \in \Sigma(X)^+$ and if $U_\gamma, U_{-\gamma}$ are the corresponding T_X-root subgroups of X, then we let $h_\gamma(c)$ be the image of the matrix $\mathrm{diag}(c, c^{-1})$ under the usual surjection $SL_2 \to \langle U_\gamma, U_{-\gamma}\rangle$.

Definition 2.2.4 For $c \in K^*$ set

$$T(c) = \Pi_{\gamma\in\Sigma(X)^+} h_\gamma(c),$$

and

$$T = \langle T(c) : c \in K^* \rangle.$$

Lemma 2.2.5 (i) *$T(c)e_\alpha = c^2 e_\alpha$ for each $\alpha \in \Pi(X)$.*

(ii) *$T(c)h = h$ for all $h \in L(T_X)$.*

Proof Part (ii) is immediate since $T \le T_X$ and T_X acts trivially on $L(T_X)$. For (i) fix $\alpha \in \Pi(X)$. Then $T(c)e_\alpha = c^r e_\alpha$, where $r = \sum_{\gamma\in\Sigma(X)^+}\langle\alpha,\gamma\rangle$. Let $\Sigma(X)^*$ denote the dual root system consisting of roots $\delta^* = \delta/(\delta,\delta)$, for $\delta \in \Sigma(X)$. Then $r = \Sigma_{\gamma\in\Sigma(X)^+}\langle\gamma^*,\alpha^*\rangle = 2\langle\rho^*,\alpha^*\rangle$, where ρ is the half-sum of positive roots in $\Sigma(X)$. But it is well known that ρ is the sum of all fundamental dominant weights of $\Sigma(X)^*$ and α^* is a fundamental root in $\Sigma(X)^*$. Part (i) follows. ■

Note that since each root in $\Sigma(X)$ is an integral combination of roots in $\Pi(X)$ the previous lemma determines all weights of T on $L(X)$, showing, in particular, that these weights are all even.

We remark that it follows from the definition of T and the previous lemma that $N_S(T)$ covers S/X and contains a representative of the long word w_0 of the Weyl group of X. Indeed Lemma 2.2.5(i) implies that w_0 inverts T, field morphisms send each term to a suitable p-power, and if a graph morphism of X is present in S (cases $X = A_2, A_3$), it can be taken to centralize T.

We next pass to weights of T on L. For $\beta \in \Sigma(G)$, e_β is a weight vector of T and we write

$$T(c)e_\beta = c^{t_\beta} e_\beta,$$

where t_β is an integer.

Lemma 2.2.6 (i) *The T_X-weights on L are each integral combinations of elements of $\Sigma(X)$.*

(ii) *There exists a system of fundamental roots $\Pi(G)$ of $\Sigma(G)$ such that $t_\beta = 0$ or 2 for each $\beta \in \Pi(G)$.*

Proof A weight of T_X on L will be called *integral* if it is a sum of roots in $\Sigma(X)$. If λ is a dominant weight for T_X, then all weights of $V_X(\lambda)$ differ from λ by a sum of roots in $\Sigma(X)$. Hence either all weights of $V_X(\lambda)$ are integral or none are. Moreover, if δ is another dominant weight and if there is a nontrivial extension of $V_X(\lambda)$ by $V_X(\delta)$, then λ and δ are either both integral or neither is integral.

Consequently, we may write $L = I \oplus J$, where both summands are X-invariant, all weights of I are integral and there are no integral weights in J. It follows that I is the sum of $L(T_G) \cap L$ and all root spaces $\langle e_\beta \rangle$ for $\beta \in \Sigma(G)$ such that $\beta \downarrow T_X$ is integral. Let $D = \langle T_G, U_\beta : \beta \downarrow T_X \text{ integral } \rangle$, a maximal rank reductive subgroup of G. By [15, 27.2], D leaves I and J invariant. Note also that the decomposition is preserved by S. Hence X is contained in the full stabilizer in G of the decomposition, a group of maximal rank. This contradicts the maximality of X unless this stabilizer is G. However, by Lemma 2.1.1, L is indecomposable under the action of G (usually irreducible). Now $L(X) \leq I$, so $I \neq 0$ and hence $I = L$. Part (i) follows.

It follows from (i) that t_β is an even integer for each $\beta \in \Sigma(G)$. It is possible to choose a fundamental system $\Pi(G)$ such that $t_\beta \geq 0$ for each $\beta \in \Pi(G)$ (this just amounts to choosing an appropriate fundamental region). Let $H = \langle T_G, U_{\pm\beta} : \beta \in \Pi(G), t_\beta = 0 \text{ or } 2 \rangle$. Then H is a Levi subgroup of G. Since every positive root is a sum of fundamental roots we also have $H = \langle T_G, U_{\pm\beta} : \beta \in \Sigma(G), t_\beta = 0 \text{ or } 2 \rangle$. So the previous lemma shows that $L(X) \leq L(H)$.

If $H < G$, then there is a nontrivial torus $Z \leq Z(H)$. Then $Z \leq C_G(L(X))^0 \leq N_G(L(X))^0$. However, Lemma 2.2.3 implies that S normalizes $N_G(L(X))^0$ so the maximality of X forces $X = N_G(L(X))^0 > Z$. However X contains no torus centralizing $L(X)$, a contradiction. It follows that $H = G$ and (ii) holds. ■

Weight Compare Program From now on we assume that $\Pi(G)$ has been chosen to satisfy conclusion (ii) of the previous lemma. Consequently,

X determines a labelling of the Dynkin diagram with all labels either 0 or 2: writing $\Pi(G) = \{\alpha_1, \ldots, \alpha_l\}$ and $t_i = t_{\alpha_i}$, we call

$$t_1 t_2 \ldots t_l$$

the *T-labelling* or *T-labelled diagram* of G.

Such a labelled diagram then determines all T-weights on L and these are bounded by the T-weight of the highest root of $\Sigma(G)$. So in all cases the T-weights are bounded by twice the height of the highest root. By Lemma 2.2.6(i), composition factors of $L(G) \downarrow X$ have weights which are integral combinations of roots, and the composition factors each determine a certain collection of T-weights. The combination of T-weights over all composition factors (including multiplicities) must agree with the list of T-weights determined by the labelled diagram.

In practice we begin with an exceptional group G, a simple group X, and prime p. We then determine all possible composition factors which have T-weights bounded by twice the height of the highest root. We next determine all T-weights of these composition factors. This requires knowing dimensions of weight spaces of irreducible modules in positive characteristic and this can be accomplished using the computer program of [13] or the Sum Formula. Much of the information required is given in tables of [31], but in a few cases supplemental information is required.

The Weight Compare Program simply lists all T-weights corresponding to the various labelled diagrams and then compares these with weights of irreducible modules. The output is a list of compatible composition factors for $L(G) \downarrow X$.

The labelled diagram of T also determines a certain parabolic subgroup of G. In the next lemma we use the notation U_X to indicate the maximal unipotent subgroup of X generated by all T_X-root subgroups corresponding to positive roots.

Lemma 2.2.7 *Let $P = \langle T_G, U_\beta : \beta \in \Sigma(G),\, t_\beta \geq 0 \rangle$.*

(i) *P is a parabolic subgroup of G with Levi factor $L_P = \langle T_G, U_\beta : t_\beta = 0 \rangle$ and unipotent radical $Q = \prod U_\beta$, where the product is taken over all $\beta \in \Sigma(G)$ for which $t_\beta > 0$.*

(ii) *$L_P = C_G(T)$.*

(iii) *$U_X \leq Q$.*

Proof It follows from the commutator relations and action of T that the group Q defined in (i) is a unipotent group normalized by L_P. Also, L_P is generated by a maximal torus and root subgroups corresponding to a closed subsystem of $\Sigma(G)$, from which it follows that L_P is the corresponding (reductive) subsystem subgroup. Let B be a Borel subgroup of L_P containing T_G. Then QB is a connected solvable subgroup with the property that it contains either U_β or $U_{-\beta}$ for each $\beta \in \Sigma(G)$. It follows that QB is a Borel subgroup of G and hence P is a parabolic subgroup of G with unipotent radical Q. This proves (i).

The action of T on T_G-root subgroups of G is determined by its action on root vectors. So the T_G-root subgroups centralized by T are precisely those with $t_\beta = 0$. On the other hand, $C_G(T)$ is a Levi subgroup of G containing T_G, hence generated by root subgroups. It follows that $C_G(T) = L_P$, giving (ii).

To establish (iii) we first claim that $P = N_G(L(Q))$. As $Q \lhd P$, we have $P \leq N_G(L(Q))$. If the containment were proper, the normalizer would contain a root subgroup U_β for which $U_{-\beta} \leq Q$. But then U_β cannot normalize the nilpotent algebra $L(Q)$. This gives the claim.

The T-weights on L are even integers and for each even integer r let L_r denote the subspace of L spanned by all weight vectors of weight r or more. Consider the filtration

$$\cdots \leq L_4 \leq L_2 \leq L_0 \leq L_{-2} \leq L_{-4} \leq \cdots$$

of L. It follows from [15, 27.2] that P stabilizes each term of the filtration and that U_X stabilizes each term, centralizing successive quotients. Also $L_2 = L(Q)$, so that $U_X \leq N_G(L(Q)) = P$. Hence, $U_X \leq \bigcap C_P(L_{2i}/L_{2i+2})$, a normal unipotent subgroup of P. Hence (iii) holds. ■

We shall also require results from [23, Section 6] concerning labellings of arbitrary 1-dimensional tori in G. If J is a 1-dimensional torus, then there is a fundamental system $\Pi(G)$ such that $J(c)e_\beta = c^{l_\beta}e_\beta$ for $\beta \in \Pi(G)$, where the l_β are non-negative integers. Thus J determines a non-negative labelling of $\Pi(G)$ (label β with the integer l_β), and by [23, 6.2] this labelling is unique, up to graph automorphisms of G.

The following result follows from the proof of [23, 6.3].

Lemma 2.2.8 *Let J and J' be 1-dimensional tori in G. Then the following are equivalent:*

(i) *J and J' are conjugate in* $\mathrm{Aut}\, G$*;*

(ii) *J and J' have the same weights on $L(G)$;*

(iii) *J and J' determine the same labelled diagram, up to graph automorphisms.*

We now continue with the analysis of our maximal subgroup X.

Lemma 2.2.9 *Let λ be a dominant weight of T_X. Then each T-weight of $V_X(p\lambda)$ is a multiple of $2p$ provided $(X,p) \neq (A_1,2),(B_2,2),(C_3,2)$.*

Proof The T_X-weights of $V_X(p\lambda)$ have the form $p\gamma$, where γ is λ minus a sum of roots. So by Lemma 2.2.6, it is only necessary to show that, with the exceptions in the statement, $p\lambda \downarrow T$ is a multiple of $2p$.

Now λ is a sum of fundamental weights. By the assumption after Proposition 2.2.1 we have $X = A_1$, A_2, A_3, B_2, B_3, C_3, or B_4. In the following table we express the fundamental weights λ_i in terms of fundamental roots in $\Pi(X)$. We use the notation $\sum c_i\alpha_i = (c_1, c_2, \ldots)$.

$A_1 : \lambda_1 = \frac{1}{2}(1)$.
$A_2 : \lambda_1 = \frac{1}{3}(2,1), \lambda_2 = \frac{1}{3}(1,2)$.
$A_3 : \lambda_1 = \frac{1}{4}(3,2,1), \lambda_2 = \frac{1}{2}(1,2,1), \lambda_3 = \frac{1}{4}(1,2,3)$.
$B_2 : \lambda_1 = (1,1), \lambda_2 = \frac{1}{2}(1,2)$.
$B_3 : \lambda_1 = (1,1,1), \lambda_2 = (1,2,2), \lambda_3 = \frac{1}{2}(1,2,3)$.
$C_3 : \lambda_1 = \frac{1}{2}(2,2,1), \lambda_2 = (1,2,1), \lambda_3 = \frac{1}{2}(2,4,3)$.
$B_4 : \lambda_1 = (1,1,1,1), \lambda_2 = (1,2,2,2), \lambda_3 = (1,2,3,3), \lambda_4 = \frac{1}{2}(1,2,3,4)$.

From Lemma 2.2.6 and the expressions above we can immediately find the T-weights of the fundamental weights λ_i, and we see that these are all even except when $X = A_1, A_3, B_2$ or C_3. Note that denominators in these cases are powers of 2.

For the exceptional cases note that Lemma 2.2.6(i) shows that X is of adjoint type, hence $V_X(p\lambda)$ is a representation of the adjoint group, so that $p\lambda$ is a sum of roots. If p is odd, then by the above $p\lambda$ is a sum of roots if and only if λ is, and so in this case we have all T-weights a multiple of $2p$, as required.

Finally, assume $p = 2$. In view of the exceptions in the statement of the lemma, we need only consider $X = A_3$. Write $\lambda = a\lambda_1 + b\lambda_2 + c\lambda_3$. The T-weight of λ is $3a + 4b + 3c$. Use the above to express 2λ in terms of roots. Then the coefficient of α_3 is $\frac{1}{2}(a + 2b + 3c)$. So for 2λ to be a sum of roots, this number must be an integer and hence $a + c$ is even. This implies that the T-weight of λ is even and hence the T-weight of 2λ is a multiple of $2p = 4$, as required. ■

The next lemma gives basic information about the centralizer of X and its action on $L(G)$.

Lemma 2.2.10 (i) $C_S(X) = 1$.

(ii) $C_G(X) = C_G(L(X)') = 1$.

(iii) *If* $0 \neq V < L$ *and* V *is* S*-invariant, then* $X = N_G(V)^0$ *and* $C_G(V) = 1$.

(iv) $C_L(X) = 0$.

(v) X *is of adjoint type.*

Proof The equality $C_G(X) = 1$ is an assumption we made following Proposition 2.2.1. Hence $C_S(X)$ consists of Frobenius morphisms of G and possibly an element in the coset of a graph automorphism, if $G = E_6$. Centralizers of Frobenius morphisms are finite by definition, so if $C_S(X) > 1$, then it is generated by an involution, say τ, in the coset of a graph automorphism of $G = E_6$. But then $C_G(\tau)$ has dimension 52 or 36 (see [9, 2.7] for $p > 2$, and [2, Section 19] for $p = 2$), which is greater than $\dim X$. This is a contradiction as S normalizes $C_S(X) = \langle\tau\rangle$ and hence also normalizes its centralizer in G. This proves (i). By Lemma 2.2.3, $C_G(L(X)')$ is S-invariant. Maximality of X implies that this centralizer is finite, hence centralized by X. So the second part of (ii) follows from the first part.

Given a subspace $V < L$ as in (iii), its stabilizer contains X and is S-invariant by Lemma 2.2.3(iii). Hence $X = N_G(V)^0$ by maximality. Then $C_G(V)$ is finite, hence centralized by X. So (iii) follows from (ii).

By Lemma 2.2.3(i), $J = C_L(X)$ is S-invariant. Assume $J \neq 0$. Of course X acts trivially on this space so we consider the action of S/X on J. If we define $A(G)$ to be the group generated by inner and graph automorphisms of G, then $N_{A(G)}(X) = X\langle\tau\rangle$, where τ is the identity or an involutory outer automorphism of X. In either case S acts on J_1, an eigenspace of τ on J. Now $S/X\langle\tau\rangle$ is generated by the image of a Frobenius morphism. If σ is such a morphism, then by Lemma 2.2.3(iv), σ has a fixed point on J_1. From the Jordan decomposition and Lemma 2.2.3(v) we see that S normalizes a 1-space $\langle e\rangle < L$ with e either semisimple or nilpotent. Maximality implies that $X = C_G(e)^0$. However, by Lemma 2.1.2, $C_G(e)^0$ is either reductive of maximal rank or has a nontrivial unipotent radical, according to e being semisimple or unipotent. In either case we have a contradiction to maximality. Hence $J = 0$, completing the proof of (iv).

Finally, (v) follows from Lemma 2.2.6(i). ∎

Lemma 2.2.11 *Assume that $G = E_6$ and S contains an element in the coset of a graph automorphism of G. Then $X = A_2$.*

Proof Suppose that $\tau \in S$ is in the coset of a graph automorphism of G. If τ induces an inner automorphism of X, then $\tau x \in C_S(X)$ for some $x \in X$, which contradicts Lemma 2.2.10(i). Therefore, τ induces a graph automorphism of X, so the assertion follows from the assumption made after Proposition 2.2.1. ∎

Lemma 2.2.12 (i) *If e is a long root element of $L(G)$, then any subspace of L containing e is normalized by the corresponding root subgroup of G.*

(ii) *If V is an S-invariant subspace of $L(G)$, then V does not contain a long root element of $L(G)$.*

Proof (i) Let $U = U_\gamma$ ($\gamma \in \Sigma(G)$) be the long root subgroup of G with $L(U) = \langle e \rangle$, and let J be the corresponding subgroup $\langle U_{\pm\gamma} \rangle \cong SL_2$. Then J has composition factors $2/1^a/0^b$ on $L(G)$.

First assume that $p \neq 2$. Then $L \downarrow J$ is completely reducible and it follows that elements of U induce elements of the form $1 + c\,\mathrm{ad}(e) + \frac{1}{2}c^2(\mathrm{ad}(e))^2$ on L. So any subspace of L invariant under $\mathrm{ad}(e)$ is also invariant under the action of U.

When $p = 2$, the situation is a little more complicated. Here $L(J) \cong sl_2$, which is indecomposable for J with a trivial submodule. Since L is self-dual it follows that $L \downarrow J$ is a tilting module, so that the restriction is a direct sum of $T_J(2)$ (which can be realized as gl_2), together with irreducibles of weights 0 and 1. Therefore, in the action on $L/\langle e \rangle$, elements of U induce $1 + c\,\mathrm{ad}(e)$. Since we are only considering subspaces that contain e the result follows.

(ii) Suppose V contains a long root element of $L(G)$. Then (i) implies that V is normalized by a long root subgroup of G. But then Lemma 2.2.10 implies that X contains a long root subgroup of G, and [22, 2.1] yields the precise embedding of X in G. Combining this with the possibilities listed in Proposition 2.2.1 we see that $C_G(X)$ has positive dimension, contradicting Lemma 2.2.10. ∎

When $G = E_6$ or E_7, we sometimes consider the irreducible 27- and 56-dimensional modules $V_{27} = V_{E_6}(\lambda_1)$ and $V_{56} = V_{E_7}(\lambda_7)$ for the simply connected cover $\hat{G}$ of G. The following lemma will be useful in this regard.

Lemma 2.2.13 *Assume that $G = E_6$ or E_7 and let $V = V_{27}$ or V_{56}. Let $\hat{X}$ be the connected preimage of X in $\hat{G}$. If $G = E_6$, assume that S does not contain elements in the coset of a graph or graph-field morphism of G. Then $C_V(\hat{X}) = 0$.*

Proof Write $SG = G\langle\sigma\rangle$ with $\sigma \in S$ a field morphism. Then $S = X\langle\tau, \sigma\rangle$, where $\tau \in G$ is either the identity or an involution. Let $W = C_V(\hat{X})$ and assume this is nonzero. If $\tau = 1$, set $W_1 = W$. If $\tau \neq 1$, we have one of two possible situations. In the first case $W = W_1 \oplus W_2$, corresponding to the eigenspace decomposition of W under the action of τ, and we assume $W_1 \neq 0$. Here we note that τ acts as an involution or possibly an element of order 4 squaring to -1 in the E_7 case. In the second case $p = 2, \tau$ is an involution, and we let W_1 denote the fixed points of τ.

There is a semilinear map ω satisfying 2.2.3(i) for vectors $v \in V$. Suppose ω fixes the subspace W_1. As in 2.2.3(iv) we can then choose a vector $w \in W$ such that $\langle w\rangle$ is fixed by both ω and τ. But then S stabilizes $N_G(\langle w\rangle)$ and so maximality implies $X = N_G(\langle w\rangle)^o$. However, the dimension of this stabilizer is at least $78 - 26 = 52$ or $133 - 55 = 78$, according as $G = E_6$ or E_7, and this contradicts Proposition 2.2.1.

Now suppose ω does not stabilize W_1 and W_2. Here $G = E_7$ and τ induces an element of order 4 with ω interchanging the spaces. So ω^2 leaves W_1 and W_2 invariant. Hence, we can choose a 2-space, say M, stabilized by both ω and τ which intersects each W_i in a ω^2-invariant 1-space. Then $N_G(M)$ is S-invariant. If $0 \neq v \in M$, then $N_G(M) \geq N_G(\langle v\rangle) \cap N_G(M/\langle v\rangle)$. Arguing as in the above paragraph we see that this intersection has dimension at least $78 - 54 = 24$, so this contradicts Proposition 2.2.1.

■

2.3 Subalgebras of L

Continue with the notation of the previous section, so that X is a maximal S-invariant simple connected subgroup of G. Recall that $L = L(G)'$ and $A = C_L(L(X)')$.

Many of our later arguments will be based on the fact that in low characteristic we are often able to show that A is nonzero. While this is not immediately conclusive, the study of this subalgebra of $L(G)$ plays a fundamental role in our analysis. In this section we establish several basic results concerning the subalgebra A.

Write $A(G)$ for the group generated by inner and graph automorphisms of G. Then $\mathrm{Aut}(G)/A(G)$ is cyclic and is generated by a Frobenius morphism of G. We have $S = (S \cap A(G))\langle\sigma\rangle$, where $\sigma = 1$ or a Frobenius morphism of G, and $S \cap A(G) = X\langle\tau\rangle$, where τ induces a trivial or an involutory graph automorphism of X. When considering actions of S on L we write $S = X\langle\tau, \omega\rangle$, where ω is the semilinear transformation of L provided by Lemma 2.2.3.

Let R be the subalgebra of A generated by all nilpotent elements. Then R is S-invariant. Note that all T-weight vectors of A for nonzero weights are contained in R, hence A/R affords a trivial X-module. Hence, Lemma 2.2.10(iv) shows that if $A \neq 0$ then also $R \neq 0$.

We begin by recording the following consequence of Lemma 2.2.10(iii).

Lemma 2.3.1 *If E is any S-invariant subalgebra of L, then $N_G(E)^0 = X$ and $C_G(E) = 1$. In particular, if $R \neq 0$ then $N_G(R)^0 = X$ and $C_G(R) = 1$.*

Lemma 2.3.2 *Suppose $A \neq 0$, and let $E \leq A$ be an X-invariant subalgebra and J a minimal ideal in E. Then either J is X-invariant or X leaves invariant an abelian ideal of E containing J.*

Proof For $x, y \in X$, xJ and yJ are both ideals in E, so that $[xJ, yJ] \leq xJ \cap yJ$. Minimality of J implies that this commutator is trivial if $xJ \neq yJ$. Now $N_X(J)$ is closed (work in GL, noting that subspace stabilizers are closed). So either X normalizes J, or J has infinitely many conjugates under the action of X.

Suppose the latter holds and set $B = \Sigma_{x \in X} xJ$, an X-invariant ideal of R. As above, intersections of distinct summands are trivial, so we may write $B = x_1J \oplus \cdots \oplus x_kJ$, for suitable $x_1, ..., x_k \in X$.

Choose $x \in X$ with $xJ \notin \{x_1J, ..., x_kJ\}$. Then as above xJ commutes with each summand in B. As $xJ \leq B$, xJ is abelian. Therefore J, and hence also B, is abelian, proving the lemma. ∎

Corollary 2.3.3 *Assume $A \neq 0$ and let I be minimal among X-invariant subalgebras of A. Then I is abelian or simple.*

Proof Let J be a minimal ideal of I. Set $R = E$ in Lemma 2.3.2 and conclude that either J is X-invariant or it is contained in an abelian X-invariant ideal of I. Minimality of I shows that either $J = I$ or I is abelian. Hence, I is simple or abelian. ■

The next lemma is fundamental to our study of the embedding of the subalgebra A in L.

Lemma 2.3.4 *Suppose* $(X,p) \neq (A_1, 2), (B_2, 2), (C_3, 2)$ *and* $A \neq 0$. *Then* $A \leq L(D)$, *where*

$$D = \langle T_G, U_\alpha : \alpha \in \Sigma(G),\ e_\alpha \textit{ has } T\textit{-weight a multiple of } 2p \rangle$$

is a semisimple maximal rank subgroup of G *with* $Z(D) = 1$. *For* $p > 2$, *the possibilities for* D *are as follows:*

G	p	D
E_8	3	A_2E_6, A_8, A_2^4
	5	A_4A_4
E_7	3	A_2A_5
E_6	3	A_2^3
F_4	3	A_2^2
G_2	3	A_2

In particular, p *is not a good prime for* G.

Proof (i) Let $\Delta = \{\alpha \in \Sigma(G) : t_\alpha \equiv 0 \bmod 2p\}$ and set $D = \langle T_G, U_{\pm\beta} : \beta \in \Delta \rangle$. Then Δ is a closed subsystem of $\Sigma(G)$ and D is a reductive group of maximal rank. Moreover, $D \neq G$ as $L(X) \not\leq L(D)$.

Let V be an X-composition factor of A. Write $V = V_0 \otimes V_1^{(p)} \otimes \cdots \otimes V_k^{(p^k)}$, with each V_i restricted. Then $L(X)'$ acts trivially on each tensor factor $V_i^{(p^i)}$ for $i > 0$, and hence $V \downarrow L(X)'$ is homogeneous of type V_0. On the other hand, since V_0 is restricted, $L(X)'$ is irreducible on V_0. As $L(X)'$ is trivial on V we conclude that $V_0 = 0$. So Lemma 2.2.9 shows that all T-weights of V are multiples of $2p$. It follows that $A \leq L(D)$.

We have $Z(D) \leq C_G(L(D)) \leq C_G(A)$. Lemma 2.2.10(iii) gives $C_G(A) = 1$, and hence it follows that $Z(D) = 1$. When $p > 2$ the listed possibilities for D are the only reductive subgroups of maximal rank having trivial center.

■

Lemma 2.3.5 *Assume that $0 \neq A \leq L(D)$ as in Lemma 2.3.4, and that p is odd and no weight of T on $L(X)$ is divisible by p. If V is a nonzero S-invariant subspace of A, then $N_D(V)^0 = T_X$.*

Proof By Lemma 2.2.10(iii) we have $N_G(V)^0 = X$. Hence $N_D(V) \leq D \cap N_G(X)$. Now D is generated by T_G, together with root groups U_α for which e_α has T-weight a multiple of $2p$. Hence it follows from the hypothesis on T-weights that $(D \cap N_G(X))^0 = T_X$, as required. ■

Observe that by definition D is $N_S(T)$-invariant. If $A \neq 0$ and D is as in Lemma 2.3.4, then from the action of T on $L(D)$ we obtain a labelling of the Dynkin diagram of D corresponding to weights of T on root vectors in a basis of fundamental roots. Detailed information regarding these labellings will be obtained in the course of later arguments.

We will use the following terminology. Say $D = EF$, a product of two commuting semisimple subgroups. We have $R \leq A \leq L(D)$ and $R \leq L(E) + L(F)$. By "the projection of R to $L(E)$" we mean the image of R in the Lie algebra $(L(E) + L(F))/L(F)$.

We will study these projections in some detail, particularly when $X = A_1$. Most of the relevant lemmas will be carried out in context, but for $p = 3$ certain lemmas are required for the analysis of both $X = A_1$ and $X = A_2$. We present these lemmas next.

Lemma 2.3.6 *Assume $p = 3$ and $0 \neq A \leq L(D)$, where D has a factor $E = A_2 \cong SL_3$. Assume that there is no such factor with T-labelling 00, and that if all A_2 factors of D are G-conjugate, then one has different labels from the others. Then D has an $N_S(T)$-invariant A_2 factor, and for any such factor R projects faithfully to $L(A_2)$.*

Proof We first observe the existence of an $N_S(T)$-invariant A_2 factor of D. This is clear from Lemma 2.3.4 unless D is a product of A_2's. If $G = F_4$ and $D = A_2A_2$, then the two simple factors are not conjugate, so both are $N_S(T)$-invariant. Otherwise, all factors are G-conjugate and the assertion is clear from our hypothesis.

So let E be an $N_S(T)$-invariant factor A_2 of D. Suppose the projection of R to $L(E)$ has a nontrivial kernel and choose a minimal ideal J in this kernel. Then Lemma 2.3.2 implies that either J is X-invariant or else $J \leq B$, an abelian X-invariant ideal of R. In the former case, J and all its images under $N_S(T)$ are centralized by E, so the sum of these images contradicts

Lemma 2.2.10(iii). So we may assume the latter holds and that B projects nontrivially to E. As in the definition of R we may assume B is generated by nilpotent elements, hence so is its projection to $L(E)$.

Suppose $e \in L(E) = sl_3$ is a root element which centralizes the projection of B. It follows that for $c \in K^*$, ce centralizes B and $u = 1 + ce \in E = SL_3$ is a unipotent element centralizing B. Once again this contradicts Lemma 2.2.10(iii). So we can assume there is no such root element.

We now consider possible T-labellings of E. There is an element $s \in N_X(T) \leq N_S(T)$ which inverts T. So s normalizes E, interchanging positive and negative T-weight spaces of B and their projections to $L(E)$. Let the T-labelling of E be ab, with $a \geq b$.

Suppose first that 0 is the only T-weight in the projection of B to $L(E)$. The projection of B to $L(E)$ is generated by nilpotent elements so $b = 0$ and $C_{L(E)}(T) = L(A_1T_1)$, the Lie algebra of a Levi subgroup. But all nilpotent elements in this subalgebra are root elements, a contradiction.

So we now assume the projection of B to $L(E)$ has a nilpotent element for a nonzero T-weight, which we may assume to be positive. If $a > b > 0$, then all T-weight spaces in $L(E)$ for positive weights are 1-dimensional and generated by root elements, so this does not occur. Suppose the labelling is $a0$. Then the T-weight space of $L(E)$ for weight a is the Lie algebra of the unipotent radical of a maximal parabolic, of which all nonzero vectors are root elements. This is again a contradiction.

Finally, assume the labelling is aa. If $2a$ is a T-weight of the projection of B we again get a root element. So assume that this is not the case. Then the projection of B has a nilpotent element of weight a. The problem here is that there are both root elements and also regular nilpotent elements of this weight in $L(E)$. However, applying s we see that there is also a nilpotent element of weight $-a$ centralizing the projection of B, whereas the centralizer of a regular nilpotent element contains no such element. So again we have a root element, which yields a final contradiction. ∎

The next lemma concerns a special but important case where $D = A_8 < E_8$, which will be needed when $X = A_1$.

Lemma 2.3.7 *Assume $p = 3, X = A_1$, $G = E_8$ and $0 \neq A \leq L(D)$ with $D = A_8$. Then there does not exist an S-invariant subalgebra C of A such that $C \downarrow X = (2^{(3)})^r$ with $r > 1$.*

Proof Suppose false and let C be a minimal such algebra. Let E, H, F

be the weight spaces of C corresponding to T-weights $6, 0, -6$, respectively. Weight considerations imply that both E and F are abelian and $[EF] \leq H$..

Write $N_X(T) = T\langle s\rangle$ where s inverts T and acts on C, inducing -1 on H, while interchanging E and F. If $h_1, h_2 \in H$, then $[h_1, h_2] \in H$, while s fixes this commutator. It follows that H is also abelian.

We first claim that $L(D) = L(D)' \oplus Z(L(D))$ and $L(D)'$ is simple. Indeed, there is a morphism $\gamma : SL_9 \to D$ and since $L(G) \downarrow D = L(D) \oplus \wedge^3 W \oplus \wedge^3 W^*$ we see that $Z(sl_9)$ is in the kernel of $d\gamma$. This shows that $d\gamma(sl_9)$ is a simple subalgebra of $L(D)$ of codimension 1. On the other hand D is unique up to G-conjugacy and if we take D to have root system with basis $\{\alpha_1, \alpha_3, \ldots \alpha_8, -\delta\}$, where δ is the highest root, then we see that $0 \neq z = h_1 - h_3 + h_2$ centralizes $L(D)$. This establishes the claim. It follows that $L(D)'$ contains all nilpotent elements of $L(D)$ and is the image of sl_9 under $d\gamma$.

We will consider the preimage, say $\hat{C}$, of C in sl_9 and adopt obvious notation. As a vector space, $\hat{C} = \hat{E} \oplus \hat{H} \oplus \hat{F}$, with each summand a weight space for $\hat{T}$, the connected component of the preimage of T in SL_9.

We proceed in several steps.

Step 1. C does not contain a nontrivial abelian subalgebra, say F, which is S-invariant.

For suppose otherwise and choose an X-invariant irreducible subspace of F with basis $\{e, h, f\}$. Since C is homogeneous under the action of X and since $N_S(T)$ leaves the 0-weight space invariant, we can choose the subspace such that $\langle h\rangle$ is stabilized by $N_S(T)$. Let $\hat{e}, \hat{h}, \hat{f}$ be preimages in sl_9. Then $[\hat{h}, \hat{e}] = z \in Z(sl_9)$. On the other hand, the commutator must have weight 6 (with respect to the connected preimage of T). Hence, $[\hat{h}, \hat{e}] = 0$. Similarly, $[\hat{h}, \hat{f}] = 0$ and, of course, $[\hat{h}, \hat{h}] = 0$. Now consider the element $cI \pm \hat{h}$ for c a scalar. For suitable c, this element has determinant 1 and its image in D centralizes $\langle e, h, f\rangle$. Now consider the sum of all conjugates of $\langle e, h, f\rangle$ by elements ω^j, a subspace of F. (Here, as before, ω is the semilinear map on L corresponding to σ provided by Lemma 2.2.3, where $S = X\langle\sigma\rangle$.) As all the images are abelian and contain h, this is an S-invariant subspace of $L(D)$ centralized by a non-identity element of D. This contradicts Lemma 2.2.10, establishing the claim.

Step 2. There does not exist a 3-dimensional S-invariant ideal of C.

For suppose otherwise and let I be such an ideal with basis e, h, f consisting of weight vectors of weights $6, 0, -6$. By Step 1, I is not abelian, so I

is simple by Corollary 2.3.3. In particular I is generated as an algebra by e and f. Now $[e,C] = I \cap [e,C] \leq I \cap (E+H) = \langle e,h\rangle$, so that $C_C(e)$ has dimension at least $\dim(C)-2$. Similarly for f. Then $C_C(I) = C_C(e) \cap C_C(f)$ has dimension at least $\dim(C) - 4 > 0$. Also, $C_C(I)$ is X-invariant and ω-invariant. It follows that $C = I \oplus C_C(I)$. Let $0 \neq h_o \in C_C(I)$ be a weight vector for weight 0. As in Step 1 we see that preimages of h_o in sl_9 commute with preimages of e and f, and hence with the preimage of I. So the argument of the previous paragraph implies $C_G(I) > 1$, a contradiction.

Step 3. $C = C' = \langle E, F\rangle$.

By minimality of C and Step 1 we see that either $C = C'$ or $C' = \langle e,h,f\rangle$, an irreducible X-module. The latter contradicts Step 2, so $C = C'$. Now $\langle E, F\rangle$ is an ideal of C with abelian quotient. So $C = C' = \langle E,F\rangle$, as required.

For $h \in H$, write $h = h_s + h_n$ with h_s semisimple and h_n nilpotent. Define $H_{ss} = \{h_s : h \in H\}$, the subspace of semisimple parts of elements of H.

Step 4. There exists an ω-invariant decomposition $E = E_1 \oplus \cdots \oplus E_k$, where each E_i is H_{ss}-invariant with kernel containing a hyperplane, and such that distinct summands have distinct kernels.

To see this, first note that elements of H_{ss} have the form $f(h)$ for $h \in H$, where f is a polynomial (here we are viewing elements of D as images of elements of sl_9). Hence elements of H_{ss} commute and normalize E and F. Note that H_{ss}, E, F are all ω-invariant. It follows from Lemma 2.2.3(iv) that each of these spaces has a basis of fixed points under ω.

Now $(H_{ss})_\omega$ acts on E_ω (a vector space over $\mathbb{F}_q$). If J is an irreducible summand, then KJ decomposes under the action of $(H_{ss})_\omega$ as a sum of weight spaces, where the weights are conjugate under $\langle\omega\rangle$ so that the various weights are q-powers of each other. In particular, the weights all have the same kernel which contains a hyperplane in $(H_{ss})_\omega$.

It follows from the above paragraph that there is an ω-invariant decomposition $E = E_1 \oplus \cdots \oplus E_k$ where each E_i is $(H_{ss})_\omega$-invariant with kernel containing a hyperplane. Moreover, distinct summands have distinct kernels. Taking K-spaces we see each E_i is invariant under H_{ss} and Step 4 follows.

Step 5. There is an ω-invariant decompostion $C = C_1 \oplus \ldots \oplus C_k$, where each C_i is an ideal. In particular, distinct summands commute.

First note that from the uniqueness of the Jordan decomposition we see that s also induces -1 on H_{ss}, and so it follows that setting $F_i = E_i^s$, we have $F = F_1 \oplus \cdots \oplus F_k$, where for each i, the kernel of F_i and E_i agree. For $i \neq j$, $[E_i, F_j] \leq H$ and taking brackets with an element of H in the kernel of one action but not the other, we find that $[E_i, F_j] = 0$.

Now set $C_i = \langle E_i, F_i \rangle$. Then for each i, C_i is an ω - invariant ideal of C and these ideals commute pairwise. A dependence relation among the ideals implies $Z(C)$ is nontrivial, a contradiction. Hence, $C = C_1 \oplus \cdots \oplus C_k$.

Step 6. We claim $k = 1$.

Let J be a minimal ideal in C_i, hence a minimal ideal of C. Suppose J is not X-invariant. Then Lemma 2.3.2 implies J is abelian and $I = \sum_{x \in X, i \geq 0} x\omega^i J$ is a sum of minimal abelian ideals. Distinct summands commute so that I is also abelian. As this sum is invariant under X and ω, this contradicts Step 1. Hence, X leaves invariant minimal ideals, $J_i \leq C_i$ for each i. Choose an irreducible X-submodule, $\langle e_1, h_1, f_1 \rangle$ in J_1. An earlier argument shows that, $\hat{h}_1$, a preimage of h_1, commutes with both $\hat{E}_2$ and $\hat{F}_2$ and hence centralizes $\hat{C}_2$. Hence $\sum_{i \geq 0} \omega^i J_2$ is invariant under X and ω and is centralized by nonidentity element of D, arising from $\hat{h}_1$. This contradicts Lemma 2.2.10. Therefore $k = 1$, as claimed.

Set $H_n = \{h_n : h \in H\}$. If $H_n = 0$, then H is diagonalizable and induces scalars on E and F. By our supposition, $\dim H > 1$, so there is an element $0 \neq h \in H$ which centralizes E and F. But then $h \in Z(C) = 0$, a contradiction. Hence $H_n \neq 0$. Now H_n induces a nilpotent algebra on E, so $C_E(H_n) \neq 0$.

If H_n centralizes E, then conjugating by s, we find that H_n centralizes $\langle E, F \rangle = C$. As in Step 1 this implies the existence of a nontrivial (unipotent) element of D centralizing C, against Lemma 2.2.10. Set $E_o = [E, H_n]$, so that $0 < E_o < E$. As H is abelian we see that $[E_o, H] \leq E_o$.

We next argue that $[E_o, F] \leq H_n$. Let $e_o \in E_o, e \in E$, and $f \in F$. Write $[e_o, f] = h_n + h_{ss}$. Then $[e, [e_o, f]] = [e, h_n + h_{ss}]$. On the other hand $[e, [e_o, f]] = [e_o, [e, f]] \in [e_o, H] \leq E_o$. This shows that $[E, h_n + h_{ss}] \leq E_o$. As $[E, h_n] \leq E_o < E$, this implies h_{ss} centralizes E/E_o. Hence h_{ss} is in the kernel of the action on $E = E_1$ and $F = F_1$ and so h_{ss} centralizes C. Arguing as in Step 1 we get $h_{ss} = 0$.

Using the above paragraph and conjugation by s we see that the subspace $E_o \oplus (H_n \cap C) \oplus E_o^s$ is a proper ideal of C and is S-invariant. Let I be a minimal ideal contained within this ideal. Then the argument at the start of

Step 5 shows that I is X-invariant. Adding the translates of I by powers of ω we obtain a proper subalgebra, invariant under both X and ω. Minimality of C implies that I is invariant under ω and by Step 1, $I \cong 2^{(3)}$ is a simple algebra. But this contradicts Step 2. ■

The analysis of the embedding $A \leq L(D)$ yields information regarding certain nilpotent elements. The following lemmas show that in special situations the nilpotent elements can be exponentiated to yield unipotent elements of G.

Lemma 2.3.8 *Assume $p = 5$ and $0 \neq A \leq L(D)$, where $D = A_4A_4 < E_8$. Let $e = e_1 + e_2 \in L(A_4A_4)$, where each e_i is a nilpotent element of the corresponding factor sl_5 with $e_i^3 = 0$ (as a matrix in sl_5). Then there is a nontrivial unipotent element in D which leaves invariant each* $\mathrm{ad}(e)$*-invariant subspace of $L(D)'$.*

Proof Suppose $\mathrm{ad}(e)$ fixes a subspace W of $L(D)$. Consider the natural surjective homomorphism $\pi : SL_5 \times SL_5 \to D$. As in the previous lemma, the differential $d\pi$ is a surjective map $sl_5 \oplus sl_5 \to L(D)'$, which has codimension 1 in $L(D)$. Let e be the image of $\hat{e} = \hat{e}_1 + \hat{e}_2$, where $\hat{e}_i \in sl_5$ is a nilpotent element. Since $d\pi$ is an isomorphism when restricted to a maximal nilpotent subalgebra, we have $\hat{e}_i^3 = 0$ for $i = 1, 2$.

Fix $i \in \{1, 2\}$, and set $\hat{u}_i = exp(\hat{e}_i) = 1 + \hat{e}_i + \frac{1}{2}\hat{e}_i^2$. Let $\alpha_i = \mathrm{ad}\hat{e}_i$. Then for $w_i \in sl_5$ we have:

$$\begin{aligned}
&\alpha_i(w_i) = \hat{e}_i w_i - w_i \hat{e}_i,\\
&\alpha_i^2(w_i) = [\hat{e}_i, [\hat{e}_i, w_i]] = \hat{e}_i^2 w_i - 2\hat{e}_i w_i \hat{e}_i + w_i \hat{e}_i^2,\\
&\alpha_i^3(w_i) = \hat{e}_i^3 w_i - 3\hat{e}_i^2 w_i \hat{e}_i + 3\hat{e}_i w_i \hat{e}_i^2 - w_i \hat{e}_i^3 = -3(\hat{e}_i^2 w_i \hat{e}_i - \hat{e}_i w_i \hat{e}_i^2),\\
&\alpha_i^4(w_i) = -\hat{e}_i^2 w_i \hat{e}_i^2.
\end{aligned}$$

Then

$$\begin{aligned}
\hat{u}_i^{-1}(w_i)\hat{u}_i = &\ (1 - \hat{e}_i + \hat{e}_i^2/2)(w_i)(1 + \hat{e}_i + \hat{e}_i^2/2)\\
= &\ w_i - \hat{e}_i w_i + \hat{e}_i^2 w_i/2 + w_i \hat{e}_i - \hat{e}_i w_i \hat{e}_i + \hat{e}_i^2 w_i \hat{e}_i/2 + w_i \hat{e}_i^2/2\\
&\ -\hat{e}_i w_i \hat{e}_i^2/2 + \hat{e}_i^2 w_i \hat{e}_i^2/4\\
= &\ w_i - \alpha_i(w_i) + \alpha_i^2(w_i)/2 - \alpha_i^3(w_i)/6 - \alpha_i^4(w_i).
\end{aligned}$$

Now let $w \in W$. Setting $\hat{u} = \hat{u}_1\hat{u}_2$ and $\hat{w} = w_1 + w_2$, we have

$$\hat{u}^{-1}\hat{w}\hat{u} = \hat{w} - \mathrm{ad}(\hat{e})(\hat{w}) + \mathrm{ad}(\hat{e})^2(\hat{w})/2 - \mathrm{ad}(\hat{e})^3(\hat{w})/6 - \mathrm{ad}(\hat{e})^4(\hat{w}).$$

The result follows by taking images in D and $L(D)$ under the above morphism π and its differential.

■

A similar but easier argument yields the following result.

Lemma 2.3.9 *Assume $p = 3$ and $0 \neq A \leq D$, where all factors of D are of type A_k. Suppose $e \in L(D)$ and that the projection of e to each factor has square 0 (as a matrix in $L(A_k)$). Then there is a nontrivial unipotent element of D which leaves invariant each* $\mathrm{ad}(e)$*- invariant subspace of $L(D)'$.*

The following is a more specialized variant of Lemma 2.3.8.

Lemma 2.3.10 *Assume $p = 5$ and $M < sl_5$, both T-invariant subalgebras of L. Assume that all T-weights of M are $10, 0$ or -10 and that all T-weights of sl_5 are at most 40. If $e \in M$ has weight 10 and $e^4 = 0$, then $exp(e) = 1 + e + e^2/2 + e^3/6 \in N_{SL_5}(M)$.*

Proof First verify that within SL_5 we have $exp(e)exp(-e) = 1$. Now fix an element $m \in M$. We check that $exp(e)(m)exp(-e)$ is equal to

$$\begin{aligned}&(m + em + (e^2/2)m + (e^3/6)m) + (-me - eme - (e^2/2)me - (e^3/6)me)+\\&(m(e^2/2) + em(e^2/2) + (e^2/2)m(e^2/2) + (e^3/6)m(e^2/2))+\\&(-m(e^3/6) - em(e^3/6) - (e^2/2)m(e^3/6) - (e^3/6)m(e^3/6)).\end{aligned}$$

Next, set $\alpha = \mathrm{ad}(e)$ and check that

$$\begin{aligned}\alpha(m) &= em - me,\\ \alpha^2(m) &= [e, [e, m]] = e^2m - 2eme + me^2,\\ \alpha^3(m) &= e^3m - 3e^2me + 3eme^2 - me^3,\\ \alpha^4(m) &= e^4m - 4e^3me + 6e^2me^2 - 4eme^3 + me^4.\end{aligned}$$

It follows that

$$(*)\ \ exp(e)(m)exp(-e) = \begin{array}{l} m + \alpha(m) + \alpha^2(m)/2 + \alpha^3(m)/6 + \alpha^4(m)/24 \\ +(1/12)(e^3me^2 - e^2me^3) + (1/36)(e^3me^3). \end{array}$$

Now $\alpha^3(m) \in M$ has T-weight at least 20, so by hypothesis $\alpha^3(m) = 0$. Then $0 = e(\alpha^3(m))e = 3(e^2me^3 - e^3me^2)$ so that $0 = e^3me^2 - e^2me^3$. Also, as an element of sl_5, e^3me^3 has weight at least 50, so by hypothesis this element is also 0. It now follows from $(*)$ that $exp(e)(m)exp(-e) \in M$, as required. ■

3 Maximal subgroups of type A_1

In this section we prove our main theorem, Theorem 1 of the Introduction, in the case where the subgroup X is of type A_1. Recall that G is an exceptional adjoint algebraic group, and G_1 is a group satisfying $G \leq G_1 \leq \mathrm{Aut}(G)$. Naturally, we consider only the small characteristic cases required by Proposition 2.2.1.

Theorem 3.1 *Suppose that X is maximal among proper closed connected $N_{G_1}(X)$-invariant subgroups of G. Assume further that*

(i) $C_G(X) = 1$, *and*

(ii) $p \leq 7$ *if* $G = E_7, E_8$; $p \leq 5$ *if* $G = E_6$; *and* $p \leq 3$ *if* $G = F_4, G_2$.

Then X is not of type A_1.

Let X, p be as in the hypothesis of the theorem, with $X = A_1$. Write $S = N_{G_1}(X)$.

Then Lemma 2.2.10 shows that $C_S(X) = 1$, whence $S = X\langle\sigma\rangle$, where either $\sigma = 1$ or σ is a Frobenius morphism of G. Moreover, it follows from Lemma 2.2.2 that σ is not an exceptional isogeny of F_4 or G_2 in case $p = 2, 3$, respectively.

Since $X = A_1$, the torus T defined in Definition 2.2.4 is a maximal torus of X. We have $N_X(T) = T\langle s\rangle$, where s inverts T. Let T_G be a maximal torus of G containing T. Recall that $\Sigma(G), \Pi(G)$ denote the root system and a fundamental system of G relative to T_G.

We shall prove Theorem 3.1 in sections, one for each value of p. The case where $p = 2$ is somewhat less technical than other cases, so we treat this case first.

We shall need a little notation concerning A_1-modules. The irreducible KA_1-module of high weight r is denoted by $V(r)$ or just by r, and the corresponding Weyl module by $W(r)$. Recall also from the Introduction that the notation $r/s/t/\ldots$ denotes an A_1-module with composition factors $r, s, t, \ldots$, while the notation $V = V_1|V_2|\ldots V_k$ denotes an A_1-module V having a series with successive factors $V_1, V_2, \ldots, V_k$.

3.1 The case $p = 2$

In this section we establish Theorem 3.1 in the case $p = 2$. Assume then that $p = 2$ and $X = A_1$ is maximal S-invariant, as in the theorem.

By Lemma 2.2.10(v), X is of adjoint type. Hence we can write $L(X) = \langle e, h, f\rangle$, where e, h, f are vectors of T-weights $2, 0, -2$ respectively, and $[e, f] = 0$. In particular, if we define

$$I = \langle e, f\rangle$$

then I is an ideal of $L(X)$. Note that $I = L(X)'$.

In addition we let δ denote the root of highest height in $\Sigma(G)$, and e_δ the corresponding T_G-root vector in $L(G)$.

By Lemma 2.2.6, the torus T determines a labelling of the Dynkin diagram of G by 0's and 2's. Let $P = QL_P$ denote the parabolic subgroup described in Lemma 2.2.7, with unipotent radical Q and Levi subgroup L_P. Then $L_P = C_G(T)$. If $l \in L(X)$ and $v \in L(G)$ it will be convenient to write lv rather than $[l, v]$.

The first lemma records some immediate consequences of Lemma 2.2.10.

Lemma 3.1.1 (i) $C_G(X) = 1$.
(ii) $C_L(X) = 0$.
(iii) $C_G(I) = 1$.

The case $G = G_2$ requires a different argument from the other cases, and we begin by ruling out this case.

Lemma 3.1.2 $G \neq G_2$.

Proof Suppose $G = G_2$, and consider the action of X on the 6-dimensional symplectic module $V = V_G(\lambda_1)$. By [19], G_2 is transitive on singular 1-spaces with point stabilizer being parabolic. If S contains Frobenius morphisms, then these are field morphisms and as in Lemma 2.2.13 we conclude that $C_V(X) = 0$. By Steinberg's tensor product theorem, irreducible KX-modules have dimension a power of 2. It follows that X acts irreducibly on a 2-space in V.

Now X induces an adjoint group in its action on each composition factor, from which we see that nontrivial composition factors are each nontrivial twists of the usual module. Using the facts that there are no nontrivial extensions among such modules and that $C_V(X) = 0$, we conclude that $V \downarrow X$ is completely reducible. But then $L(X)$ induces the identity on V, which is not possible. ∎

The key tool in establishing Theorem 3.1 (for $p = 2$) is the following Proposition.

Proposition 3.1.3 *Assume that $G = E_6, E_7$ or E_8, and that*

$$fe_\delta = c_1 e_\alpha + c_2 e_\beta,$$

where c_1, c_2 are scalars, and either $c_2 = 0$, or $c_1, c_2 \neq 0$ and α, β are orthogonal roots in $\Sigma(G)$. Then $C_G(I) > 1$.

We make the following remark for later use in the case $X = B_2$ (handled in Chapter 5). In the case $G = E_8$, maximality is not used in the proof of Proposition 3.1.3. It is used for E_6 and E_7, but only to rule out a case where δ has T-weight 2; here, 2 is the largest T-weight and this could not occur if $X = B_2$.

The proof of Proposition 3.1.3 follows from two key lemmas.

Lemma 3.1.4 *Assume the hypotheses of Proposition* 3.1.3. *Then $fe_\delta \in C_{L(G)}(I)$.*

Proof For notational reasons it will be convenient to work with $G = E_8$. The other cases are similar and changes required for these cases will be noted in the course of the proof. For α a root we regard the corresponding fundamental subgroup $J_\alpha \cong SL_2$ and with this identification regard $U_\alpha(c) = I + ce_\alpha$.

As $[ef] = 0$, we have $e(fe_\delta) = f(ee_\delta)$. Since e is in the Lie algebra of the maximal unipotent group corresponding to the system of positive roots and since δ is the root of highest height, we have $ee_\delta = 0$. So the main issue here is to show that $f(fe_\delta) = 0$.

Let $V = \{V(c) : c \in K\}$ be the T-invariant 1-dimensional unipotent group of X having Lie algebra $\langle f \rangle$. By the argument of Lemma 2.2.7(i), V is contained in the product of T_G-root groups of G corresponding to negative roots. Write

$$V(1) = U_{-\beta_1}(b_1) \dots U_{-\beta_k}(b_k) U_{-\gamma_1}(d_1) \dots U_{-\gamma_s}(d_s),$$

where all $\beta_i, \gamma_i \in \Sigma(G)^+$, each β_i has T-weight 2 and each γ_j has T-weight greater than 2. By definition, $T = \{T(d) : d \in K^*\}$, where $T(d)$ denotes the image in X of the diagonal maxtrix of SL_2 having eigenvalues d, d^{-1}. Conjugating the above expression for $V(1)$ by $T(c^{1/2})$, we obtain

$$V(c) = U_{-\beta_1}(b_1c)\dots U_{-\beta_k}(b_kc)U_{-\gamma_1}(d_1c^{a_1})\dots U_{-\gamma_s}(d_sc^{a_s}),$$

where each a_j is a positive integer at least 2. Adjusting f by a scalar multiple, it follows that we may write

$$f = b_1e_{-\beta_1} + \dots + b_ke_{-\beta_k}.$$

We claim that $\beta_j \neq \delta$ for all j. To see this note that β_j has T-weight 2, whereas this is not the case for δ since the expression for δ in terms of fundamental roots has all coefficients at least 2. (For $G = E_6$ or E_7, δ could have T-weight 2 if the labelling had just one 2 and this was over either α_1 or α_6 in the E_6 case and over α_7 in the E_7 case. But in these cases $\dim C_G(T) > \dim G/2$, and since X is generated by two conjugates of T we conclude that $C_G(X)$ has positive dimension, contradicting Lemma 3.1.1(i).) This proves the claim.

Consequently we have

$$fe_\delta = \sum b_ie_{\delta-\beta_i},$$

where the sum ranges over those i for which $\delta - \beta_i$ is a root. In the case where $G = E_8$ this condition forces each β_i appearing in the sum to have coefficient of α_8 equal to 1.

Now by hypothesis we have $fe_\delta = c_1e_\alpha + c_2e_\beta$. We will proceed under the assumption that $c_1, c_2 \neq 0$. The changes required for the other case are obvious. Write

$$\alpha = \delta - \beta_{i_0},$$
$$\beta = \delta - \beta_{i_1}.$$

Then

$$ffe_\delta = \sum c_1b_ie_{\alpha-\beta_i} + \sum c_2b_je_{\beta-\beta_j} \qquad (*)$$

where the sums range over i, j such that $\alpha-\beta_i, \beta-\beta_j$, respectively, are roots. Also, it is conceivable that there is a situation where $\alpha = \beta_i$ or $\beta = \beta_j$, in which case h_α or h_β would appear in the expression for ffe_δ.

Now $W(E_7)$ is transitive on roots with α_8-coefficient equal to 1, and fixes δ. Since $\delta - \alpha_8$ is a root, so is $\delta - \beta_i$ for all roots β_i with α_8-coefficient equal to 1. Therefore β_{i_0} and β_{i_1} are the only roots in $\{\beta_1, \dots, \beta_k\}$ with α_8-coefficient nonzero.

We first consider those terms in ffe_δ with α_8-coefficient equal to 0. This part of the expression has the following form:

$$b_{i_0}b_{i_1}e_{\delta-\beta_{i_0}-\beta_{i_1}} + b_{i_1}b_{i_0}e_{\delta-\beta_{i_1}-\beta_{i_0}} + b_{i_0}^2 e_{\delta-2\beta_{i_0}} + b_{i_1}^2 e_{\delta-2\beta_{i_1}}.$$

As $p = 2$, the first two terms add to 0. Since δ has at least one odd coefficient when expressed as the sum of fundamental roots, it cannot be twice a positive root. Suppose $\delta - 2\beta_{i_0}$ is a root. Then this root has α_8-coefficient equal to 0. Such roots are in $\Sigma(E_7)$, and all of these are conjugate. Hence, $\delta - 2\beta_{i_0}$ is $W(E_7)$-conjugate to α_4. Therefore, there is a root γ such that $\delta - 2\gamma = \alpha_4$. However, the α_4-coefficient of δ is even, so this is a contradiction. Essentially the same argument works for $G = E_6$ or E_7. Of course the above comments apply equally to β_{i_1}. We have shown that the only relevant terms in ffe_δ involve roots of the form $\alpha-\beta_i$ and $\beta-\beta_j$, roots having α_8-coefficient 1. These positive roots are all conjugate under $W(E_7)$. Fix i. There is an element $w \in W(E_7)$ such that $(\alpha-\beta_i)^w = \delta - \alpha_8$. Hence, $(\delta - \beta_{i_0} - \beta_i)^w = \delta - \alpha_8$ and so $(\beta_{i_0} + \beta_i)^w = \alpha_8$. Hence $\beta_{i_0} + \beta_i$ is a positive root. Similarly for $\beta_{i_1} + \beta_j$.

Since $p = 2$ we have $V(c)^2 = 1$. This is an equation in Q and we consider the image of this equation in the class two group Q/Q_0, where $Q_0 = [[Q,Q],Q]$. We have

$$V(c) = U_{-\beta_1}(b_1c)\dots U_{-\beta_k}(b_kc)U_{-\gamma_1}(c_1c^2)\dots U_{-\gamma_t}(c_tc^2) \text{ (modulo } Q_o),$$

where $\gamma_1, ..., \gamma_t$ are the roots of T-weight 4 and the corresponding root elements are in the center of Q/Q_0. Order the β_i such that β_{i_0} is first and β_{i_1} is second. If we square this expression for $V(c)$ and rearrange, we obtain terms of the form $U_{-\beta_i-\beta_j}(a)$, which arise from the expression $U_{-\beta_i}(c)U_{-\beta_j}(d) = U_{-\beta_j}(d)U_{-\beta_i}(c)U_{-\beta_i-\beta_j}(cd)$. Consider those terms where $\beta_i + \beta_j$ has α_8-coefficient equal to 1. This contribution to $(V(c))^2$ (modulo Q_0) has the form

$$\prod U_{-\beta_{i_0}-\beta_i}(c_1b_ic^2) \cdot \prod U_{-\beta_{i_1}-\beta_j}(c_2b_jc^2),$$

where the first product is over those i such that $\beta_{i_0} + \beta_i$ is a root and the second over those j such that $\beta_{i_1} + \beta_j$ is a root.

Now $V(c)^2 = 1$, so the above expression must also be 1. Fix an i appearing in the first product. If there exists a j for which $\beta_{i_0} + \beta_i = \beta_{i_1} + \beta_j$, then we must have $c_1b_i + c_2b_j = 0$. For this i and j we have then have $\alpha - \beta_i = \beta - \beta_j$ and the corresponding contribution to (*) is 0. On the other hand, if there is no such j, then necessarily $c_1b_i = 0$ and the coefficient of

$e_{\alpha-\beta_i}$ in (*) is 0. This accounts for all terms in the first product. There may remain terms in the second product, but if so, as above, they correspond to j for which $c_2 b_j = 0$.

We have now accounted for all terms in (*) and this completes the proof of the lemma. ∎

The next step is to pass from the centralizer in $L(G)$ of $\langle e, f\rangle$ to the centralizer in G.

Lemma 3.1.5 *Assume the hypotheses of Proposition* 3.1.3*. Then there is an element $g \in G$ for which $C_{L(G)}(fe_\delta) = L(C_G(g))$.*

Proof We will give the argument for $G = E_8$. The cases of E_6 and E_7 are entirely similar.

First consider the case where $fe_\delta = c_1 e_\alpha$. If $c_1 = 0$, then $e_\delta \in C_{L(G)}(I)$; and if $c_1 \neq 0$ then Lemma 3.1.4 gives $e_\alpha \in C_{L(G)}(I)$. So in any case there is a root γ such that $e_\gamma \in C_{L(G)}(I)$.

We next compute the dimension of $C_{L(G)}(e_\gamma)$. Take $A_1E_7 < E_8$ with $e_\gamma \in L(A_1)$. Here and in the following all maximal rank groups are taken to contain T_G. By [23, 2.1], we have $L(E_8) \downarrow A_1E_7 = L(A_1E_7) \oplus (V_2 \otimes V_{56})$, where V_2 is a usual 2-dimensional module for A_1 and V_{56} is the 56-dimensional irreducible module $V(\lambda_7)$ for E_7. It is clear that $C_{V_2 \otimes V_{56}}(e_\gamma)$ has dimension 56. For the other term, write $L(A_1) = \langle e_\gamma, h_\gamma, f_\gamma\rangle$, where $[e_\gamma, f_\gamma] = h_\gamma$. As $p = 2$, h_γ is in the center of $L(A_1)$, and so $\langle e_\gamma, h_\gamma\rangle$ is an ideal of $L(A_1E_7)$. As $T_X < A_1E_7$, there is an element in the Lie algebra of a maximal torus of $L(A_1E_7)$ which normalizes but does not centralize e_γ. It follows that $\dim(C_{L(A_1E_7)}(e_\gamma)) = 136 - 2 = 134$. Hence $\dim(C_{L(G)}(e_\gamma)) = 134 + 56 = 190$.

Let U_γ be a root subgroup of G having Lie algebra $\langle e_\gamma\rangle$. Then $C_G(U_\gamma) = P'$, where P is a parabolic subgroup of G having Levi factor E_7T_1. It follows that $\dim(C_G(U_\gamma)) = 190$, so that $C_{L(G)}(e_\gamma) = L(C_G(U_\gamma))$. Since $C_G(U_\gamma) = C_G(u)$ for $1 \neq u \in U_\gamma$, this establishes the lemma in the case where $fe_\delta = c_1 e_\alpha$.

Now consider the other case, which is similar but slightly more complicated. Here $fe_\delta = c_1 e_\alpha + c_2 e_\beta$ with $c_1, c_2 \neq 0$ and α, β perpendicular roots. Here we set $g = U_\alpha(c_1)U_\beta(c_2)$ and compare the dimensions of $C_{L(G)}(fe_\delta)$ and $L(C_G(g))$.

Consider the subsystem group $A_1A_1D_6$ of E_8, where we take the A_1

subgroups to correspond to α and β. We can embed $A_1A_1D_6 < A_1E_7$. As above, $L(E_8) \downarrow A_1E_7 = L(A_1E_7) \oplus (V_2 \otimes V_{56})$. Using this together with the information in [23, 2.1,2.3] we conclude that

$$L(E_8) \downarrow A_1A_1D_6 = L(A_1A_1D_6) \oplus (0 \otimes 1 \otimes \lambda_5) \oplus (1 \otimes 0 \otimes \lambda_6) \oplus (1 \otimes 1 \otimes \lambda_1).$$

Here, λ_5, λ_6 denote the two 32-dimensional spin modules for D_6, and 1 denotes the usual 2-dimensional A_1-module. One checks that the decomposition is actually a direct decomposition by looking at the action of $Z(L(A_1A_1)) = \langle h_\alpha, h_\beta \rangle$.

Write $l = c_1e_\alpha + c_2e_\beta$. Then $\dim(C_{0\otimes 1}(l)) = 1 = \dim(C_{1\otimes 0}(l))$ and $\dim(C_{1\otimes 1}(l)) = 2$. Hence

$$\dim(C_{L(G)}(l)) = \dim(C_{L(A_1A_1D_6)}(l)) + 32 + 32 + 24.$$

We next consider the first term of this sum.

Now $\langle e_\alpha, h_\alpha \rangle$ is an ideal of one of the A_1 factors and similarly for β. Also $[e_\alpha, f_\alpha] = h_\alpha$ and $[e_\beta, f_\beta] = h_\beta$. From the containment $A_1A_1 < A_2A_2$ (the latter invariant under T_G), we see that there exist elements $t_\alpha, t_\beta \in L(T_G)$, such that $[t_\alpha, e_\alpha] = e_\alpha$, $[t_\alpha, e_\beta] = 0$, $[t_\beta, e_\beta] = e_\beta$, and $[t_\beta, e_\alpha] = 0$.

It follows from the above remarks that $[l, L(A_1A_1D_6)]$ is a 4-space, so that $\dim(C_{L(A_1A_1D_6}(l)) = 72 - 4 = 68$. Hence, $\dim(C_{L(G)}(l)) = 68 + 32 + 32 + 24 = 156$.

From [2, Section 17] we have $C_G(U_\alpha(c_1)U_\beta(c_2)) = U_{78}B_6$, where U_{78} denotes a 78-dimensional unipotent group. Moreover, as in [2] we have $C_G(U_\alpha(c_1)U_\beta(c_2)) = C_G(U_\alpha(c_1c)U_\beta(c_2c))$ for all $c \neq 0$. Therefore we have $C_G(g) \leq C_G(l)$. Taking Lie algebras we have $L(C_G(g)) \leq L(C_G(l)) \leq C_{L(G)}(l)$ and from the dimension considerations above, we have equality, which proves the result. ∎

Proposition 3.1.3 is immediate from Lemmas 3.1.4 and 3.1.5.

Proof of Theorem 3.1 for $p = 2$

The Theorem will follow from Lemma 3.1.1 and Proposition 3.1.3, provided we can verify the hypotheses of the Proposition. In this section we analyse when these hypotheses are satisfied and study the cases where they are not.

Since L_P centralizes T, to establish the hypotheses of Proposition 3.1.3 we can replace f by an L_P-conjugate of f. We will give a detailed argument in the case of E_8. The E_7 and E_6 cases are very similar and we will present details for only the less obvious configurations. Write $L'_P = L_1 \ldots L_s$, a product of simple groups L_i. For $G = E_8$ we take j maximal such that the fundamental root α_j is in $\Pi(L_P)$, and order the simple factors of L_P so that $\alpha_j \in \Pi(L_s)$. As in the proof of Lemma 3.1.4 we write $f = b_1 e_{-\beta_1} + \ldots + b_k e_{-\beta_k}$.

We begin with an easy lemma indicating one way in which the hypotheses of Proposition 3.1.3 are satisfied.

Lemma 3.1.6 *Suppose that $G = E_8$, that at most two β_i have nonzero coefficient of α_8, and that if there are two such β_i then these roots are orthogonal. Then the hypotheses of Proposition* 3.1.3 *are satisfied.*

Proof Maintaining the notation of Lemma 3.1.4, suppose the roots are β_{i_0} and β_{i_1}. It was seen in the proof of 3.1.4 that these roots each have coefficient of α_8 equal to 1. Then $fe_\delta = c_1 e_\alpha + c_2 e_\beta$, where $\alpha = \delta - \beta_{i_0}$ and $\beta = \delta - \beta_{i_1}$. We then have $\langle \alpha, \beta \rangle = \langle \delta - \beta_{i_0}, \delta - \beta_{i_1} \rangle = \langle \delta, \delta \rangle - \langle \beta_{i_0}, \delta \rangle - \langle \delta, \beta_{i_1} \rangle + \langle \beta_{i_0}, \beta_{i_1} \rangle = 2 - 1 - 1 - 0 = 0$, as required. ∎

The next lemma provides a restriction on composition factors, which will be used at several points in the proof.

Lemma 3.1.7 *Let V be a self-dual module for $X = A_1$ for which $C_V(X) = 0$. Suppose the X-composition factors on V are $6^x/4^y/2^z/0^w$, where we indicate just the high weights of composition factors and their multiplicities. Then $w \leq 2y$.*

Proof First observe that from [1] we see that the relevant Weyl modules have the following structure, where in each case the module is uniserial: $W(2) = 2|0, W(4) = 4|0|2$, and $W(6) = 6|4|0$.

If $v \in V$ is a T-weight vector of weight 6, then by Lemma 2.1.4, $\langle Xv \rangle$ is an image of $W(6)$. Consider the sum, say V_6, of all cyclic modules of this form. By assumption there are no fixed points and one can argue by consideration of the socle of this module that $V_6 = (6|4)^a \oplus 6^{x-a}$. Now, factor out V_6 and repeat with high weight 4 vectors in V/V_6 to generate $V_4/V_6 = (4|0|2)^b \oplus (4|0)^c \oplus 4^{y-a-b-c}$. As there do not exist trivial submodules, we necessarily have $c \leq a$.

Now consider high weight 2 vectors in V/V_4. As V is self dual, there are no trivial quotient modules, so these generate V, and we have $V/V_4 = (2|0)^d \oplus 2^{z-b-d}$.

Consider the preimage of the fixed point space of V/V_4 in V/V_6. The fixed point space of this preimage can have dimension at most a, as otherwise there would be a fixed point in V. Since the fixed point space of V_4/V_6 has dimension c, at most $a - c$ trivial modules in V/V_4 can pull past the high weight 4 composition factors in V_4/V_6, of which there are $y - a$. Therefore, we have the inequality $d \leq (y - a) + (a - c)$. Consequently, $c + d \leq y$.

Now $w = b + c + d$, and by the previous paragraph $b \geq w - y$. On the other hand, $b \leq y$, so we obtain $w \leq 2y$, as required. ∎

Lemma 3.1.8 *Let* $G = E_8$. *Then Theorem* 3.1 *(*$p = 2$*) holds if* $\alpha_8 \notin \Pi(L_P)$.

Proof Recall that f has T-weight -2, hence is a linear combination of terms of the form $e_{-\beta}$ where β is a positive root which involves just one fundamental root of T-weight 2. In the situation of this lemma, the only roots β which can contribute to fe_δ are those with nonzero coefficient of α_8.

First suppose $j < 7$. Then α_8 is orthogonal to the root system of L_P, and all fundamental roots α_i for $i > j$ are labelled by 2. Here, $e_{-\beta_c} e_\delta = 0$ unless $\beta_c = \alpha_8$, so $fe_\delta = ae_{\delta - \alpha_8}$ for some scalar a, and we immediately have the hypotheses of Proposition 3.1.3, hence the Theorem.

Next suppose that $j = 7$ and $L_s = A_r$ for some r. The space S spanned by all root vectors of T-weight 2 and having α_8-coefficient 1 is a natural $(r + 1)$-dimensional module for A_r. So, replacing f by an L_s-conjugate, we can assume that there is at most one β_i with nonzero coefficient of α_8. With this conjugation we again have the hypotheses of Proposition 3.1.3. Similarly, if $L_s = D_6$, then S is the natural 12-dimensional orthogonal space for L_s, and so L_s has two orbits of nonzero vectors on S, represented by a root vector and the sum of two root vectors for orthogonal roots. Hence by the proof of Lemma 3.1.6 we have the hypotheses of Proposition 3.1.3 and hence the Theorem holds.

There is only one further case to consider here, where $L_P' = E_7$. Here $\dim C_G(T) = 134$. Now X is generated by two conjugates of T, so that $\dim C_G(X) \geq 134 + 134 - 248 = 20$. Of course, this implies $C_G(X) \neq 1$, a contradiction, so we again have Theorem 3.1 ∎

Lemma 3.1.9 *Assume* $G = E_8$. *Then Theorem* 3.1 *(*$p = 2$*) holds if* $j = 8$ *and there is no fundamental node adjacent to both* $\Pi(L_s)$ *and another* $\Pi(L_r)$.

Proof First assume $L_s = A_i$ with $i \leq 5$. Suppose α_l is a fundamental root not in $\Pi(L_P)$ but adjoining $\Pi(L_s)$. By hypothesis there is only one root of T-weight 2 with nonzero coefficient of both α_8 and α_l, namely the sum of the roots from α_l to α_8. Moreover, l is unique unless $i = 5$, in which case there are two possible choices for l. In any case Lemma 3.1.6 implies that the hypothesis of Proposition 3.1.3 holds, and so we have the Theorem.

The remaining cases are where $L_s = A_6, A_7$ or D_7. For the last two cases we will apply Lemma 3.1.7. The non-negative T-weights on $L(G)$ are as follows: $0^{64}, 2^{56}, 4^{28}, 6^8$ if $L_s = A_7$; and $0^{92}, 2^{64}, 4^{14}$ if $L_s = D_7$. Hence the composition factors of $L(G) \downarrow X$ are $6^8/4^{28}/2^{48}/0^{64}$ and $4^{14}/2^{64}/0^{92}$, respectively. In both cases we contradict Lemma 3.1.7.

So this leaves us with the case $L_s = A_6$. There are two ways in which this can occur; either $\Pi(L_P) = \{\alpha_3, ..., \alpha_8\}$ or $\Pi(L_P) = \{\alpha_2, \alpha_4, ..., \alpha_8\}$. In the second case consider all roots with α_8-coefficient 1 and α_3-coefficient 1. The span of the corresponding root vectors is a natural module for an A_5 Levi factor of L_s. Hence, conjugating f by an element of L_s, if necessary, we can obtain the hypothesis of Proposition 3.1.3 with fe_δ a multiple of a root vector. The first case is similar. Here the weight -2 subspace of $L(G)$ is the sum of two irreducible modules for $L_s = A_6$, a natural module and the wedge-square of this module. For the latter module we argue as above that we can conjugate f by an element of $A_5 < L_s$ so that there is at most one β_i for which the coefficient of α_8 and α_2 are both nonzero. Indeed, after the conjugation we can take this $\beta_i = \alpha_2 + \alpha_4 + \alpha_5 + \ldots + \alpha_8$. Moreover, the summand affording the natural module has only one basis vector with nontrivial coefficient of both α_1 and α_8, namely $\alpha_1 + \alpha_3 + \alpha_4 + \ldots + \alpha_8$. Once again Lemma 3.1.6 implies we have the hypothesis of Proposition 3.1.3. This completes the proof of the lemma. ∎

Lemma 3.1.10 *Theorem* 3.1 *holds for* $G = E_8, p = 2$.

Proof The configurations not covered by the previous two lemmas are those where $\alpha_8 \in \Pi(L_s)$ and there exists another simple factor, say L_r, of L_P, such that some fundamental root α_k is adjacent to both $\Pi(L_r)$ and $\Pi(L_s)$.

First suppose $k \geq 5$. Consider the roots β of T-weight 2 which involve both α_k and α_8. Then $W(L_r)$ is transitive on such roots. Define J to be the sum of the corresponding root spaces of $L(G)$. Note that L_r is the only component of L_P acting nontrivially on J. There are several possibilities for the action of L_r on J. If L_r is of type A or D and this action is a natural module for L_r, then we once again have the hypotheses of Proposition 3.1.3 and hence the Theorem holds.

Next consider the case where $(k, L_r) = (7, E_6)$. Here the T-labelling is 00000020, and we compute that $L(G) \downarrow X = 6^2/4^{27}/2^{52}/0^{82}$, giving a contradiction by Lemma 3.1.7.

The remaining cases are as follows: $(k, L_r) = (6, D_5)$ or $(5, A_4)$. Here, J is a spin module, or the wedge-square of the natural module, respectively.

In both of these cases we consider the action of L_r on J. To understand the orbit structure we work in the simple algebraic groups E_6, D_5 and consider a maximal parabolic subgroup of type D_5, A_4 respectively. The action of a Levi subgroup on the unipotent radical is the action of L_r on J in each case, and orbit representatives are given by [29]. From [29] we conclude that there are in each case exactly two orbits on nonzero vectors, represented by a root vector and the sum of two root vectors for orthogonal roots. This settles each of these cases.

Now assume $k < 5$, where the possibilities are as follows:

$$\begin{array}{ll} k = 4: & L = A_4A_2A_1, A_4A_1A_1, A_4A_2, \text{ or } A_4A_1 \\ k = 3: & L = A_6A_1 \text{ or } A_5A_1. \end{array}$$

In each case we look at the space J spanned by all root vectors of T-weight -2 having nonzero coefficient of α_8 and α_k. When $k = 4$, the factors of L_P other than L_s act on J as a natural module or a tensor product of two natural modules. In each case, there are just two orbits on nonzero vectors, with representatives given by a root vector or the sum of multiples of two root vectors for orthogonal roots.

Finally, suppose that $k = 3$. When $L_P = A_6A_1$ one checks that J is the tensor product of natural modules for the A_1 factor and a Levi subgroup A_5 of the A_6 factor. So here again, we have the hypothesis of Proposition 3.1.3 after conjugating by an element of L_P. Now consider the case $L_P = A_1A_5$. Here J is the sum of the natural module for the A_1 factor and a trivial module corresponding to the root $\alpha_2 + \alpha_4 + \alpha_5 + ... + \alpha_8$. Conjugating by an element of L_P we once again obtain the hypotheses of Proposition 3.1.3. ■

At this point we have established Theorem 3.1 for $G = E_8\,(p = 2)$ and we now discuss the other types.

Lemma 3.1.11 *Theorem* 3.1 *holds if* $G = E_7$, $p = 2$.

Proof The argument here is very similar to the arguments of the last few lemmas. Proceeding as in those lemmas there are a number of easy cases which are dealt with just as before. The remaining ones are as follows: $L'_P = D_6, E_6, A_2A_3A_1, D_5, A_1A_5, A_4A_2, A_6, D_5A_1$. Note that α_1 is the only fundamental root not orthogonal to δ.

In the first two cases $\dim C_G(T) > \frac{1}{2}\dim G$. Since X is generated by two conjugates of T it follows that $C_G(X)$ has positive dimension, which contradicts Lemma 2.2.10. In the last five cases, a consideration of T-weights implies that X has the following composition factors on $L(G)$:

$$6/4^{16}/2^{25}/0^{47};\;\; 6^2/4^{15}/2^{28}/0^{39};\;\; 6^5/4^{15}/2^{25}/0^{33};\;\; 4^7/2^{35}/0^{49};\;\; 4^{10}/2^{32}/0^{49},$$

respectively. These all contradict Lemma 3.1.7.

In the third case $\Pi(A_2) = \{\alpha_1, \alpha_3\}$. Here we must consider roots β_i of T-weight 2 with nontrivial coefficient of α_1 and α_4. Note that $W(A_1A_3)$ is transitive on such roots, and the sum of the corresponding root spaces affords a tensor product of natural modules for A_1 and A_3. As in previous cases this yields the hypotheses of Proposition 3.1.3. ■

Lemma 3.1.12 *Theorem* 3.1 *holds if* $G = E_6$, $p = 2$.

Proof Here too, the argument is similar to previous ones. This time, α_2 is the only fundamental root not orthogonal to δ. The only cases requiring special consideration are where $L'_P = D_5$, $A_2A_1A_1$, D_4, A_5, A_1A_4, $A_1A_2A_2$, A_4, A_4. The last two entries are due to the two distinct types of A_4 Levi factor. In the first case, $\dim C_G(T) = 46 > \frac{1}{2}\dim G$, which gives a contradiction as in previous cases. In the last six cases a consideration of T-weights shows that X has the following composition factors on $L(G)$:

$$4^8/2^{16}/0^{30};\;\; 4/2^{20}/0^{36};\;\; 4^5/2^{20}/0^{28};\;\; 6^2/4^9/2^{16}/0^{20};$$

$$6/4^{10}/2^{14}/0^{26};\; \text{or}\;\; 6^5/4^{10}/2^6/0^{26}.$$

In each of these case we contradict Lemma 3.1.7.

This leaves the second case. Here the only difficulty is where $\Pi(A_2) = \{\alpha_2, \alpha_4\}$ and in this case we aim to verify the hypotheses of Proposition 3.1.3. Consider those roots β_i appearing within the expression for f for which β_i has nonzero coefficient of α_2. Under the action of the Levi factor L_P, the sum of the corresponding root spaces is the sum of two natural modules for the A_1 factors of L_P. Hence, conjugating by an element of L_P, we can assume that at most two β_i satisfy this condition. If there are two, then they are automatically orthogonal so Lemma 3.1.6 shows that the hypotheses of Proposition 3.1.3 hold and hence Theorem 3.1 holds. ■

It remains to consider the case $G = F_4$. We have separated this case from the others as there are certain degeneracies in the commutations in characteristic 2 which complicate matters. We will continue with the same sorts of arguments, but paying attention to possible difficult issues. Note that $L(G)$ has two nontrivial G-composition factors, namely the two 26-dimensional restricted modules for F_4 of high weights λ_1 and λ_4.

Lemma 3.1.13 *X does not have a fixed point on either G-composition factor of $L(G)$.*

Proof Let V denote one of the 26-dimensional modules for $G = F_4$ and suppose that $C_V(X) \neq 0$. Recall from Lemma 2.2.2 that $S = X\langle\sigma\rangle$, where σ is either the identity or a field morphism of G. In the latter case, the argument of Lemma 2.2.3 shows that there is a semilinear map ω satisfying (i), (iii), and (iv) of Lemma 2.2.3. Hence, ω stabilizes $C_V(X)$, and hence by Lemma 2.2.3, ω fixes a nonzero vector $v \in C_V(X)$. Then the stabilizer G_v is S-invariant, and has dimension at least $52 - 26 = 26$, which contradicts the maximality of X. ■

Now $G = F_4$ admits a graph morphism, and the image of X under this morphism is another maximal subgroup of type A_1. Consequently, it will suffice to consider just half of the potential labelled diagrams.

Lemma 3.1.14 *L_P cannot have semisimple rank* 3.

Proof Suppose otherwise. By the above remarks we need only consider the cases where $L_P' = B_3$ or A_1A_2, where in the second case we take the A_1 root system to consist of long roots. Consider the 26-dimensional G-module V, where $V = V(\lambda_1)$ in the first case, and $V = V(\lambda_4)$ in the second. Starting from the labelled diagram, compute the T-weights, and then the X-composition factors on V. The result is $4^6/0^{14}$ in the first case, and $4^3/2^6/0^8$ in the second. Now Lemma 3.1.7 gives a fixed point in each case. ■

Lemma 3.1.15 *It is not the case that $e, f \in C_{L(F_4)}(e_\gamma)$, for a long root γ.*

Proof We first claim that $C_{L(F_4)}(e_\gamma) = C_{L(F_4)}(U_\gamma)$, where U_γ is the T_G-root subgroup corresponding to γ. Notice that this claim, together with Lemma 3.1.1, will establish the Lemma.

In the course of the proof of Lemma 3.1.5 we showed that $C_{L(E_6)}(e_\gamma) = L(C_{E_6}(U_\gamma))$. Also, $L(C_{E_6}(U_\gamma)) \leq C_{L(E_6)}(U_\gamma) \leq C_{L(E_6)}(e_\gamma)$. It follows that $C_{L(E_6)}(e_\gamma) = C_{L(E_6)}(U_\gamma)$, and intersecting with $L(F_4)$ we have $C_{L(F_4)}(e_\gamma) = C_{L(F_4)}(U_\gamma)$, as required. ■

Lemma 3.1.16 *The T-labelling of the Dynkin diagram of G cannot be* $22{*}{*}$ *or* 2020*.*

Proof Assume false. As before we write

$$f = b_1 e_{-\beta_1} + ... + b_k e_{-\beta_k},$$

where each β_i has T-weight 2. If the coefficient of α_1 is zero for each β_i, then $fe_\delta = 0$. Since $ee_\delta = 0$, as well, this contradicts the previous lemma. So we may assume that some β_i has nonzero coefficient of α_1.

If the T-labelling is $22{*}{*}$, and if β_i has nonzero coefficient of α_1, then $\beta_i = \alpha_1$. And if the T-labelling is 2020, then conjugating by an element of the A_1 factor of L_P corresponding to α_2, we may suppose that just one β_i has nonzero coefficient of α_1 and $\beta_i = \alpha_1$. Reordering, if necessary, we can take $i = 1$. Hence, in either case $fe_\delta = b_i e_{\delta-\alpha_1}$.

Suppose β_i has α_2-coefficient equal to 0 for each $i > 1$. Then $e_{-\beta_i} fe_\delta = 0$ for all i. Hence $ffe_\delta = 0$. Also, $efe_\delta = fee_\delta = 0$. But this contradicts the previous lemma. Therefore, for some $i > 1, \beta_i$ has nonzero coefficient of α_2.

At this point we can argue as in the proof of Lemma 3.1.4. The expression for f came from a corresponding expression for $V(c)$, which involved root groups for all roots β_i together with some of T-weight larger than 2. However, in view of the previous discussion we can now check that $V(c)^2 \neq 1$, a contradiction. ■

Lemma 3.1.17 *Theorem* 3.1 *holds if $G = F_4$, $p = 2$.*

Proof In view of earlier comments regarding the graph automorphism and Lemmas 3.1.14 and 3.1.16, it will suffice to consider the cases where $L_P' = B_2$ or A_1A_1, where in the latter case the A_1 factors correspond to α_1, α_4.

Consider the last case. Conjugating by an element of the A_1-factor corresponding to α_1 we may assume that no β_i has nonzero coefficient of α_1. But then $fe_\delta = 0$, contradicting Lemma 3.1.15.

So this leaves the case where $L_P' = B_2$. From a consideration of the weights on long roots we see that if $V = V_G(\lambda_1)$ then $V \downarrow X = 8/6^4/4/0^6$.

The Weyl module $W_X(8)$ is uniserial with the following structure (see [1]): $8|0|4|6$. Consider a weight vector $v \in V$ of weight 8 and the corresponding cyclic module $\langle Xv \rangle$, which is an image of this Weyl module. The maximal submodule of $\langle Xv \rangle$ is singular under the bilinear form on V (see Lemma 2.1.5). However, the multiplicity in $V \downarrow X$ of the irreducible module of high weight 4 is just 1, so this irreducible cannot occur within $\langle Xv \rangle$. By Lemma 3.1.13, X has no nonzero fixed points on V. Consequently, $\langle Xv \rangle$ must be irreducible and non-degenerate. Applying Lemma 3.1.7 to the perpendicular space of $\langle Xv \rangle$ within V, we contradict Lemma 3.1.13, completing the proof. ■

3.2 A_1-modules

In this subsection we present some preliminary results concerning A_1-modules which will be used to settle the cases with p odd.

We begin with a definition taken from [31, p.55]. Let $X = A_1$, and $V(r)$ the irreducible KX-module of high weight r. Write $r = \sum_0^t r_i p^i$ with $0 \le r_i \le p-1$ for all i, so that by Steinberg's tensor product theorem (see Lemma 2.1.3), $V(r) \cong V(r_0) \otimes V(r_1)^{(p)} \otimes \ldots \otimes V(r_t)^{(p^t)}$. We say that $V(r)$ has *p-type zero* if $r_0 = 0$ or $p-2$.

Lemma 3.2.1 *Let M be a finite-dimensional rational KX-module. Then*

$$M = M_X(0) \oplus N,$$

an X-invariant decomposition, where every composition factor of $M_X(0)$ has p-type zero, and no composition factor of N has p-type zero. If M is self-dual then so are $M_X(0)$ and N.

Proof This is [31, 4.2]. ■

If M is an X-module, T is a maximal torus of X, and r is a non-negative integer, let M_r be the T-weight space in M corresponding to the weight r.

Lemma 3.2.2 *Let M be a finite-dimensional self-dual X-module in characteristic $p \neq 2$, with highest weight r. Define $Y = \langle Xv : v \in M_r \rangle$, and let Z be the intersection of all maximal X-submodules of Y. Then Z is totally singular, and Y/Z is a non-degenerate subspace of $Z^{\perp}/Z$ isomorphic to $V(r)^{n_r}$, where n_r is the multiplicity of $V(r)$ as a composition factor of M.*

Proof By Lemma 2.1.4, for $v \in M_r$, $\langle Xv \rangle$ is an image of the Weyl module $W_X(r)$. Let E_v be the maximal submodule of $\langle Xv \rangle$, so that the composition factors of E_v are each of high weight strictly less than r. By Lemma 2.1.5, E_v is totally singular. Composition factors of $M/(E_v)^{\perp}$ also have high weight less than r so $M_r < (E_v)^{\perp}$ and hence $Y \leq (E_v)^{\perp}$.

Let $E < Y$ be the sum of the spaces E_v, as v ranges over M_r. Then $Y/E \cong V(r)^{n_r}$ and by the above paragraph E is totally singular. It follows that $E = Z$.

Finally, consider $Y/Z \leq Z^{\perp}/Z$. By definition of Y, the quotient space has all composition factors of high weight strictly less than r. This implies that Y/Z is non-degenerate. ■

Now we return to our maximal S-invariant subgroup $X = A_1 < G$, with maximal torus T, satisfying the hypotheses of Theorem 3.1. Assume p is odd.

Recall the notation $L = L(G)'$. For $r \geq 0$, let n_r be the multiplicity of $V(r)$ as a composition factor of $L \downarrow X$. By Lemma 2.2.6, T gives a labelling of the Dynkin diagram of G with 0's and 2's. Hence, if $n_r \neq 0$ then $r \leq 2ht(\delta)$, twice the height of the highest root δ in $\Sigma(G)$. As in Lemma 3.2.1 we have

$$L \downarrow X = L_X(0) \oplus N$$

where none of the composition factors of N have p-type zero.

Since p is odd and X is of adjoint type, the Lie algebra $L(X)$ is simple and we can write $L(X) = \langle e, h, f \rangle$ with

$$[e, f] = h,\ [h, e] = 2e,\ [h, f] = -2f.$$

Recall the notation

$$A = C_L(L(X)).$$

The next lemma, though elementary, is of fundamental importance throughout our proof.

Lemma 3.2.3 *Suppose the highest T-weight in $L_X(0)$ is a multiple of p, say kp. Then either $n_{kp-2} \geq 2n_{kp}$, or there is a composition factor kp in $A = C_L(L(X))$.*

Proof Let $v \in L$ be a maximal vector of weight kp. Then $ev = 0$. If also $fv = 0$, then $v \in C_L(L(X))$, which therefore has a composition factor kp. So assume $fv \neq 0$. Then fv has weight $kp - 2$, and as $\langle Xv \rangle$ is $L(X)$-invariant, $\langle Xv \rangle$ has $kp-2$ as a composition factor. We deduce from Lemma 2.1.5 that $\langle Xv : v \in L_{kp} \rangle$ has singular subspace with n_{kp} composition factors $kp - 2$, and it follows that $n_{kp-2} \geq 2n_{kp}$. ■

The same argument yields the following:

Lemma 3.2.4 *Let V be an X-module of highest weight kp, and for each s let m_s be the multiplicity of $V(s)$ as a composition factor of V.*

(i) *If V is self-dual, then either $m_{kp-2} \geq 2m_{kp}$, or there is a composition factor kp in $C_V(L(X))$.*

(ii) *In any case, either $n_{kp-2} \geq n_{kp}$, or $C_V(L(X))$ has a composition factor kp.*

The next two lemmas are relevant to the cases $p = 5$ or 7.

Lemma 3.2.5 *Suppose $p = 5$ or 7.*

(i) *We have*

$$W(2p-2) = (2p-2)|0, \;\; W(2p) = 2p|(2p-2),$$
$$W(4p-2) = (4p-2)|2p, \;\; W(4p) = 4p|(4p-2).$$

(ii) *If $p = 7$ then $n_0 < n_{2p-2}$, and if $p = 5$ then $n_0 < n_{2p-2} + n_{p(2p-2)}$.*

(iii) *If $n_{2p} > 0$ and $n_{2p} \geq n_{2p-2} + n_{4p-2} + n_{2p^2-4p}$, then A contains a submodule $2p$.*

Proof (i) is immediate since the indicated extensions exist by Lemma 2.1.6, and the dimensions sum to that of the Weyl module.

(ii) For $p = 7$, the only irreducible module of high weight at most $2ht(\delta)$ which extends the trivial module is $2p - 2$. Hence if $n_0 \geq n_{2p-2}$ then $L \downarrow X$ must have a trivial submodule or quotient. Since L is self-dual, this implies that $C_L(X) \neq 0$, a contradiction. Hence $n_0 < n_{2p-2}$. The $p = 5$ argument is the same, only here the irreducible $p(2p-2)$ is also in the range and extends 0.

(iii) The only irreducibles in the required range which extend $2p$ are $2p-2, 4p-2$, and $2p^2-4p$ (only for $p=5$) so the conclusion follows as above. ∎

Lemma 3.2.6 *Assume that $p = 5$ or 7.*

(i) *Suppose the highest $L_X(0)$-weight is $2p$ or less. Then $n_{2p-2} \geq 2n_0$.*

(ii) *Suppose the highest $L_X(0)$-weight is $4p-2$ or less. Then either $n_{2p-2} \geq 2n_{2p}$, or A contains a submodule $2p$.*

(iii) *Suppose the highest $L_X(0)$-weight is $4p$ and $n_{4p} = 1$. Then either $n_{2p-2} \geq 2n_{2p} - 2$, or A contains a submodule $2p$ or $4p$.*

Proof (i) We work in the X-module $M = L_X(0)$. If $Y = \langle Xv : v \in M_{2p}\rangle$, then by Lemmas 3.2.2 and 3.2.5, Y has radical $Z \cong (2p-2)^l$. Let $V = Z^\perp/Z$ and write $V = (Y/Z) \perp E$. Then $Y/Z = (2p)^{n_{2p}}$. Set $W = \langle Xv : v \in E_{2p-2}\rangle$. If $2p-2|0$ appears as a submodule of W, then in the preimage of this (i.e. $2p-2|0|Z$), generating with a suitable vector of weight $2p-2$ gives a submodule 0 of $L \downarrow X$, a contradiction. Therefore $Z^\perp/Z = (2p-2)^{n_{2p-2}-2l} \oplus 0^{n_0}$. Since L has no submodule 0, we have $n_0 \leq l$. Hence $n_0 \leq l \leq \frac{1}{2}n_{2p-2}$.

(ii) Suppose A does not contain a submodule $2p$. If $v \in L$ is a maximal vector of weight $4p-2$ then by Lemma 2.1.4, $\langle Xv\rangle$ is an image of $W(4p-2)$. By Lemma 3.2.5(i) this image must be an irreducible $4p-2$. Thus the spaces $\langle Xv\rangle$ for all such vectors v generate a non-degenerate submodule Y of L containing all the $4p-2$ composition factors. Then $Y^\perp$ is X-invariant and has highest $L_X(0)$-weight $2p$ or less. The conclusion now follows from the argument of part (i).

(iii) Suppose A contains no $2p$ or $4p$ submodule. As in (ii), if v is a maximal vector of weight $4p$ then $\langle Xv\rangle = W(4p) = 4p|(4p-2)$. Let Y be the (singular) submodule $4p-2$, and work now in $Z = Y^\perp/Y$. If $w \in Z$ is a maximal vector of weight $4p-2$, then $\langle Xw\rangle = 4p-2$ or $4p-2|2p$. But the latter cannot occur, for if it did, generating with a weight $4p-2$ vector in the preimage of $\langle Xw\rangle$ in $Y^\perp$ would yield a submodule $2p$. Therefore Z has a non-degenerate submodule containing all its $4p-2$ composition factors (and no others). Now under the action of X the maximal vectors in Z of weight $2p$ must generate $2p|2p-2$ or $2p$, and at most one of the latter can occur (as $Y^\perp$ has no submodule $2p$). Hence Z has a submodule $(2p-2)^{n_{2p}-1}$ which is totally singular, whence $n_{2p-2} \geq 2n_{2p} - 2$. ∎

If we have $A = C_L(L(X))$ nonzero, then by Lemma 2.3.4 we know that $A \leq L(D)$, where D is a semisimple subgroup of maximal rank in G as defined in 2.3.4. The next lemma provides additional information under these circumstances.

Lemma 3.2.7 *Suppose p is odd and $0 \neq A \leq L(D)$ as in Lemma 2.3.4. If V is a nonzero S-invariant subspace of A, then $N_D(V) \leq N_X(T) = T\langle s\rangle$. In particular, $N_D(V)^0 = T$ and $N_D(V)$ contains no non-identity unipotent elements.*

Proof By Lemma 2.3.5 we have $N_D(V)^0 = T$ and $N_D(V) \leq N_X(T)$. As p is odd, $N_X(T)$ contains no unipotent elements, and the result follows. ∎

The next result gets rid of a particular possibility for the labelling of T.

Lemma 3.2.8 *If $G = F_4$, E_7 or E_8 and $p \neq 2$, then T is not a regular torus (i.e. the T-labelling is not the one with all labels 2).*

Proof Suppose T is regular, so $C_G(T) = T_G$. Let $l = 4, 7$ or 8 be the rank of G and let $\Pi(G) = \{\alpha_1, \ldots, \alpha_l\}$. Now $N_X(T) = T\langle s\rangle$ where s is an involution inverting T. The longest element w_0 of the Weyl group $W(G)$ inverts T, and hence we may take $s = n_{w_0}$, an element of $N_G(T_G)$ mapping to w_0. Note that in the cases under consideration w_0 sends each fundamental root to its negative.

We can write $L(X) = \langle e, h, f\rangle$, where $f = e^s$. Working in a root system relative to T_G, write $e = \sum_1^l c_i e_{\alpha_i}$. Then $f = e^s = \sum \pm c_i e_{-\alpha_i}$. If some $c_i = 0$, then there is a rank 1 torus T_1 centralizing e and f, hence centralizing $L(X)$, contrary to Lemma 2.2.10(iii). Hence $c_i \neq 0$ for all i; that is, e is regular nilpotent in $L(G)$.

Now let U be a 1-dimensional unipotent subgroup of X normalized by T, and embed $UT < B_G = U_G T_G$, a Borel subgroup of G, with $U < U_G, T < T_G$. Then $U_G/U_G' = \prod_1^l \bar{U}_{\alpha_i}$, a commuting product (where $\bar{U}_{\alpha_i}$ denotes the image of U_{α_i}). Pick $1 \neq u \in U$, and say u maps to $\prod \bar{U}_{\alpha_i}(d_i)$ (where $d_i \in K$).

If some $d_j = 0$, then $\langle u^T\rangle = U$ maps to $\prod_{i\neq j} \bar{U}_{\alpha_i}$, and hence U is contained in the unipotent radical of the minimal parabolic subgroup P of G corresponding to the root α_j. But this means that $e \in L(U) \subseteq L(R_u(P))$, whereas e is regular, a contradiction.

Hence $d_j \neq 0$ for all j. This implies that u is a regular unipotent element of G. But for $p \leq 7$, regular unipotent elements have order greater than p (see [42, 0.4]), so this is impossible. ∎

3.3 The case $p = 7$

In this subsection we prove Theorem 3.1 for $p = 7$.

Assume the hypotheses of Theorem 3.1, with $p = 7$. Then $G = E_7$ or E_8, and $X = A_1$ is a maximal S-invariant subgroup of G.

As in the previous section, let $L = L(G)$ and denote by n_r the multiplicity of $V(r)$ as a composition factor of $L \downarrow X$. Since $p = 7$ is a good prime for G, Lemma 2.3.4 gives

$$A = C_L(L(X)) = 0.$$

From Section 3.2 we have the following inequalities among the multiplicities n_r:

(a) if the highest $L_X(0)$-weight is $7k$ for some k, then $n_{7k-2} \geq 2n_{7k}$ (Lemma 3.2.3)

(b) $n_0 < n_{12}$ (Lemma 3.2.5(ii))

(c) either $n_{14} = 0$ or $n_{14} < n_{12} + n_{26}$ (Lemma 3.2.5(iii))

(d) if the highest $L_X(0)$-weight is 14 or less (resp. 26 or less) then $n_{12} \geq 2n_0$ (resp. $n_{12} \geq 2n_{14}$) (Lemma 3.2.6)

(e) if the highest $L_X(0)$-weight is 28 and $n_{28} = 1$, then $n_{12} \geq 2n_{14} - 2$ (Lemma 3.2.6(iii)).

(f) T is not a regular torus in G (Lemma 3.2.8)

As discussed in the Introduction and after Lemma 2.2.6, the Weight Compare Program lists all possibilities for the composition factors of $L \downarrow X$ which are compatible with the fact that there is a T-labelling of $\Pi(G)$ with 0's and 2's. Combining this with the restrictions (a)-(f) above, we obtain the following.

Lemma 3.3.1 *The possibilities for the multiplicities of the composition factors of $L \downarrow X$ are as follows:*

G	*Case*	$L \downarrow X$	T*-labelling*
E_7	(1)	$10^3/8^3/6^5/4^7/2^9$	0002002
	(2)	$14/12^2/10^4/8^2/6^3/4^4/2^8/0$	2002002
E_8	(3)	$10^4/8^6/6^{10}/4^{16}/2^{14}$	00002000
	(4)	$18^2/16/14^3/12^6/10^4/8^5/6^5/4^4/2^6/0^3$	00020020

Lemma 3.3.2 *Cases* (1), (2) *and* (3) *of Lemma* 3.3.1 *do not occur.*

Proof First consider $G = E_7$, and assume case (1) or (2) holds.

In case (1) we have $L_X(0) = 0$, so by [31, 4.6], X is conjugate to any A_1 in G whose torus determines the same labelled diagram as T. By [31, p.65], if we take an A_1 lying in a subsystem subgroup A_2A_5, projecting to a regular A_1 in each factor (with no field twist), then this A_1 also has labelled diagram 0002002, and hence is conjugate to X. Therefore $X < A_2A_5$, centralizing $Z(A_2A_5)$, a group of order 3, contrary to the fact that $C_G(X) = 1$ (by the hypothesis of Theorem 3.1).

Now consider case (2). By [31, p.65], a 1-dimensional torus T' which lies in a subgroup $A_1A_1B_4$ of a subsystem subgroup A_1D_6, projecting to a regular torus in each factor has the same weights on L as does T. It then follows from Lemma 2.2.8 that T and T' are conjugate in G. So T lies in a subgroup $A_1A_1B_4 < A_1D_6$. Now if V_{56} denotes the 56-dimensional E_7-module $V(\lambda_7)$, then by [23, 2.3],

$$V_{56} \downarrow A_1D_6 = 1 \otimes \lambda_1/0 \otimes \lambda_5,$$

and hence $V_{56} \downarrow A_1A_1B_4 = 1 \otimes 2 \otimes 0/1 \otimes 0 \otimes \lambda_1/0 \otimes 1 \otimes \lambda_4$. Hence we see that the T-weights on V_{56} are $11, 9^3, 7^4, 5^5, 3^7, 1^8$ and their negatives. It follows that the composition factors of X (or rather its preimage in simply connected E_7) on V_{56} are $11/9^2/7/5^2/3^4/1^2$. Since the only composition factor extending the 2-dimensional module of high weight 1 is $11 = 4 \otimes 1^{(7)}$, we deduce that X stabilizes a 2-space in V_{56}. The variety of all 2-spaces in V_{56} has dimension 108, so the stabilizer Y of this 2-space has dimension at least $\dim G - 108 = 25$. Note that $X < Y^0$. However this is not an immediate contradiction to the maximality of X, as Y^0 may not be S-invariant.

Suppose first that X lies in no parabolic subgroup of G. Let M be a maximal connected subgroup of G containing Y^0. Then M is not reductive of maximal rank (as $C_G(M) \leq C_G(X) = 1$), and M is not parabolic. As $\dim M \geq 25$, it follows from [31, Theorem 1] that $M = A_1F_4$ or G_2C_3. Again, Y^0 lies in no parabolic or maximal rank subgroup of M. Considering the other maximal subgroups of M in turn, we deduce that $Y^0 = M$. But neither A_1F_4 nor G_2C_3 stabilizes a 2-space in V_{56} (see [23, 2.5]).

Thus X lies in a parabolic subgroup P of G. As $L \downarrow X$ has just one trivial composition factor, P is a maximal parabolic. Write $P = QR$, where $Q = R_u(P)$ and R is a Levi subgroup. Recall that $S = X\langle\sigma\rangle$, with σ either trivial or a Frobenius morphism of G. If P is σ-stable then $X < P$ is not maximal S-invariant, a contradiction. Therefore $P \neq P^\sigma$. Now P^σ is G-conjugate to P, and by the Bruhat decomposition we have $P \cap P^\sigma = P \cap P^g = (P \cap P^w)^x$ for some $g \in G$, $x \in P$ and $w \in N_G(T_G)$, where T_G

is a maximal torus in $P \cap P^w$. In the notation of [7, Section 2.8], we have $P = P_J$ and $R = L_J$, where $J = \Pi(G) \backslash \{j\}$ for some j.

Now by [7, 2.8.7] we have $P \cap P^w = UL_K$, where U is a unipotent group and $K = J \cap w(J)$. Since X lies in a conjugate of $P \cap P^w$ and has only one trivial composition factor on $L(G)$, it follows that $K = J$, whence $w \in N_G(L_J) = N_G(R)$. If $T_1 = Z(R)$ then $N_G(T_1)/C_G(T_1) \cong Z_2$, generated by the action of w_0, the longest element of the Weyl group $W(G)$. As $C_G(T_1) = R$, we therefore have $N_G(R) = R\langle w_0 \rangle$, and hence $P \cap P^w = P \cap P^{w_0} = R$. This means that $X \leq R^x$. But then X is centralized by the torus T_1^x, contradicting the fact that $C_G(X) = 1$.

Finally, consider case (3) of Lemma 3.3.1. Here $G = E_8$. In this case we have $L_X(0) = 0$, and using [31, 4.6 and p.65] as in the proof of case (1) above, we see that X is contained in a subsystem subgroup D_4D_4, centralizing $Z(D_4D_4) = 2^2$, again a contradiction to $C_G(X) = 1$. ■

The last possibility (4) in Lemma 3.3.1 is very much more complicated to handle. We state this as a separate proposition, and deal with it in a series of lemmas.

Proposition 3.3.3 *There does not exist a maximal S-invariant subgroup $X = A_1$ in $G = E_8$ ($p = 7$) such that*

$$L(G) \downarrow X = 18^2/16/14^3/12^6/10^4/8^5/6^5/4^4/2^6/0^3.$$

Assume the proposition is false, and let X be such a maximal A_1. The first goal in the proof is to determine the precise action of X on $L(G)$ as a direct sum of explicit indecomposables.

First decompose $L(G)$ into blocks, according to (possible) nontrivial extensions of irreducible X-modules, given by Lemma 2.1.6: this gives

$$L(G) \downarrow X = L_X(0) \oplus L_X(2) \oplus L_X(4) \oplus L_X(6),$$

where

$L_X(0)$ has composition factors $14^3/12^6/0^3$,
$L_X(2)$ has composition factors $16/10^4/2^6$,
$L_X(4)$ has composition factors $18^2/8^5/4^4$, and
$L_X(6) = 6^5$.

Each of $L_X(0), L_X(2), L_X(4), L_X(6)$ is self-dual.

We shall describe the structures of these summands in terms of tilting modules. Recall that $T(c)$ denotes the indecomposable tilting X-module of high weight c. The structure of these modules is given in Lemma 2.1.7.

Lemma 3.3.4 *We have* $L_X(0) = T(14)^3$.

Proof If $v \in L_X(0)$ is a weight vector of weight 14, then as there are no irreducible submodules of high weight 14 (by Lemma 2.3.4) we have $\langle Xv \rangle = W(14) = 14|12$. Generating by all such vectors we get $W(14)^3$. Factoring out this submodule the quotient must be of the form $W(12)^3 = (12|0)^3$, as otherwise there would be a trivial quotient module, hence a fixed point.

We have shown that $L_X(0)$ has as filtration by Weyl modules. Since it is self-dual it also has a filtration by dual Weyl modules. Therefore $L_X(0)$ is a tilting module and thus the direct sum of indecomposable tilting modules of the form $T(c)$. As the high weight is 14 and $T(14) = 12|(14+0)|12$ by Lemma 2.1.7, we have the assertion. ■

Lemma 3.3.5 $L_X(2)$ *has one of the following structures:*

(i) $16 \oplus (T(10))^i \oplus (W(10) \oplus W(10)^*)^j \oplus 2^{6-2i-2j} \oplus 10^{4-i-2j}$, *where* $0 \leq i \leq 3,\ 0 \leq j \leq 2$.

(ii) $T(16) \oplus T(10)^i \oplus (W(10) \oplus W(10)^*)^j \oplus 2^{5-2i-2j} \oplus 10^{2-i-2j}$, *where* $0 \leq i \leq 2,\ 0 \leq j \leq 1$.

Proof Recall that $L_X(2) = 16/10^4/2^6$. First assume that there is an irreducible submodule of high weight 16. Since $L_X(2)$ is self-dual, so is this summand and we work within the orthogonal complement. Here we can apply [32, 2.4] to get (i).

Now assume that there does not exist an irreducible submodule of high weight 16. If $v \in L_X(2)$ is a T-weight vector of weight 16, then $\langle Xv \rangle = Z = 16|10$. Then $L_X(2)/Z = 10^3/2^6$ and we can choose a T-weight vector here of weight 10 for which the corresponding X-module has a quotient 10/16. It follows from [12] that $\mathrm{Ext}^1_X(W(16), W(10))$ has dimension 1, with an extension given by $T(16)$. We claim that $T(16)$ is a submodule of $L_X(2)$: for otherwise, there would exist a uniserial module of shape $10|16|10$; and then, working in the direct sum of this module and $W(10) = 10/2$, we could construct an extension of $W(16)$ by $W(10)$ with 2 as a submodule, contradicting the above information on Ext^1. This proves the claim.

By construction the $T(16)$ submodule is nondegenerate. Taking an orthogonal complement we obtain (ii) using [32, 2.4].

■

In the next lemma we consider $L_X(4)$. In one part of the lemma we use the notation $M(18)$ to refer to the maximal submodule of $T(18)$. Thus $M(18) = (18 \oplus 4)|8$.

Lemma 3.3.6 *$L_X(4)$ has one of the following structures:*

(i) $18^2 \oplus T(8)^i \oplus (W(8) \oplus W(8)^*)^j \oplus 8^{5-i-2j} \oplus 4^{4-2i-2j}$, *where* $0 \leq i \leq 2,\ 0 \leq j \leq 2$.

(ii) $18 \oplus T(18) \oplus T(8)^i \oplus (W(8) \oplus W(8)^*)^j \oplus 8^{3-i-2j} \oplus 4^{3-2i-2j}$, *where* $0 \leq i \leq 1,\ 0 \leq j \leq 1$.

(iii) $(W(18) \oplus W(18)^*) \oplus T(8)^i \oplus (W(8) \oplus W(8)^*)^j \oplus 8^{3-i-2j} \oplus 4^{4-2i-2j}$, *where* $0 \leq i \leq 2,\ 0 \leq j \leq 1$.

(iv) $M(18) \oplus M(18)^* \oplus T(8)^i \oplus (W(8) \oplus W(8)^*)^j \oplus 8^{3-i-2j} \oplus 4^{2-2i-2j}$, *where* $0 \leq i \leq 1,\ 0 \leq j \leq 1$.

(v) $T(18)^2 \oplus T(8)^i \oplus 8^{1-i} \oplus 4^{2-2i}$, *where* $0 \leq i \leq 1$.

Proof Recall that $L_X(4) = 18^2/8^5/4^4$. If there exists a submodule of the form 18^2, then this splits off and using [32, 2.4] we get (i).

Now suppose there is a single 18 submodule which splits off. Working in a complement, the argument given in the proof of Lemma 3.3.5(ii) yields (ii).

Next suppose that there is single 18 submodule, which does not split off. Suppose, in addition, that there exists an indecomposable submodule of the form $8|18 = W(18)^*$. On the other hand generating by a high weight vector of weight 18 we must get a submodule $W(18)$. Hence we have submodule $W(18) \oplus W(18)^*$. There is a complement to this submodule and we get (iii) by an application of [32, 2.4].

Now suppose that there is a single 18 submodule, say Z, which does not split off, but no submodule $W(18)^*$. The nonsplitting condition implies that there is a vector v of weight 8 such that $\langle Xv \rangle > Z$. Now $\langle Xv \rangle / Z$ is an image of $W(8)$, so the nonexistence of a $W(18)^*$ submodule implies that $J = \langle Xv \rangle = 8|(18 \oplus 4) \cong M(18)^*$. By duality, $L_X(4)$ has a submodule R such that $L_X(4)/R \cong (18 \oplus 4)|8$. Also, $J \leq R$.

By assumption and duality there is no $W(18)$ quotient. Hence an 8 submodule of R which pulls past the 4 quotient of $L_X(4)/R$, also pulls past the 18 quotient. Do this as many times as possible, lifting 4 submodules upwards. Taking a maximal vector in $L_X(4) - Z$ of weight 18, the corresponding cyclic module generates $M \cong W(18)$, so the 8 submodule cannot block the 18 submodule of $L_X(4)$. There is a submodule $D > J$ such that $D/J = (18 \oplus 4)/8$ and $D = (18 \oplus 4 \oplus 8)|(18 \oplus 4 \oplus 8)$.

Now there is an 4 submodule of D/M which does not correspond to an 4 submodule of $L_X(4)$. It is therefore blocked by the 8 submodule of M and we obtain a submodule of the form $M(18)$. This module added to J yields a submodule $M(18) \oplus M(18)^*$. It is easy to see that this submodule is nondegenerate. Splitting it off and applying [32, 2.4], we obtain (iv).

Finally, we consider the case where there is no 18 submodule. Here we can proceed as in part (ii) of the preceding lemma to split off $T(18)^2$ and then obtain (v). ∎

Now let u be a non-identity unipotent element of X. Our next aim is to identify the conjugacy class of u in G. Recall that J_r denotes a unipotent Jordan block of dimension r.

Lemma 3.3.7 *The following give the Jordan blocks of u in its action on certain X-modules.*

(i) $T(14) \downarrow u = T(16) \downarrow u = T(18) \downarrow u = J_7^4$.

(ii) $T(10) \downarrow u = T(8) \downarrow u = J_7^2$.

(iii) $(W(10) \oplus W(10)^*) \downarrow u = J_7^2 \oplus J_4^2$.

(iv) $(W(8) \oplus W(8)^*) \downarrow u = J_7^2 \oplus J_2^2$.

(v) $(M(18) \oplus M(18)^*) \downarrow u = J_7^4 \oplus J_6^2 \oplus J_4^2$.

Proof Parts (i) and (ii) are immediate from Lemma 2.1.7. For the remaining parts first note that if $E \downarrow \langle u \rangle$ is a direct sum of J_7's and if $e \in E$, then there exists a J_7 summand of E containing e. From here the Jordan decomposition of x on $E/\langle e \rangle$ is transparent. Now $W(10) = T(10)/2$ and $W(8) = T(8)/4$. Further we note that u has the same Jordan form on a module and its dual. Together these facts yield (iii) and (iv). For (v) consider $M(18)^* = T(18)$. Here u acts on 8 as $J_3 \oplus J_1$. Factor out J_3 and get Jordan decomposition $(J_7)^3 \oplus J_4$ on the quotient space. By construction, the image of the J_1 is not contained in the J_4 summand and it follows that we may assume that this image is contained in one of the J_7 summands. At this point, factoring out the image of J_1 yields the assertion. ∎

We now consider the possible Jordan forms of u on $L(G)$. First, $L_X(0) = T(14)^3$ and $L_X(6) = 6^5$, together contributing J_7^{17}. The only J_2-blocks occur within $W(8)$ and $W(8)^*$, each contributing a single J_2. So the total contribution is J_2^k, with $k = 0, 2$ or 4. The only trivial Jordan blocks occur within 8 and 16 submodules, each of which contribute a single J_1. Hence the total contribution is at most J_1^6.

The remarks of the previous paragraph, together with [18, Table 9], which lists the Jordan block structure of all unipotent classes of G in their action on $L(G)$, restrict the possibilities for the G-class of the unipotent element $u \in X$ to the following.

Lemma 3.3.8 *One of the following holds:*

(i) $u = A_4A_2, L(G) \downarrow u = J_7^{19} \oplus J_5^{11} \oplus J_3^{18} \oplus J_1^6$.

(ii) $u = D_6(a_2), L(G) \downarrow u = J_7^{29} \oplus J_5^4 \oplus J_4^4 \oplus J_3 \oplus J_1^6$.

(iii) $u = E_7(a_5), L(G) \downarrow u = J_7^{29} \oplus J_6^2 \oplus J_5 \oplus J_4^4 \oplus J_3^3 \oplus J_1^3$.

(iv) $u = E_8(a_7), L(G) \downarrow u = J_7^{30} \oplus J_5^4 \oplus J_3^6$.

(v) $u = A_6, L(G) \downarrow u = J_7^{35} \oplus J_1^3$.

(vi) $u = A_6A_1, L(G) \downarrow u = J_7^{35} \oplus J_3$.

(vii) $u = A_4A_3, L(G) \downarrow u = J_7^{24} \oplus J_6^2 \oplus J_5^3 \oplus J_4^6 \oplus J_3^6 \oplus J_2^4 \oplus J_1^3$.

(viii) $u = A_5A_1, L(G) \downarrow u = J_7^{25} \oplus J_6^6 \oplus J_5^4 \oplus J_3 \oplus J_2^4 \oplus J_1^6$.

(ix) $u = D_5(a_1)A_2, L(G) \downarrow u = J_7^{28} \oplus J_5^3 \oplus J_4^2 \oplus J_3^6 \oplus J_2^4 \oplus J_1^3$.

(x) $u = E_6(a_3)A_1, L(G) \downarrow u = J_7^{28} \oplus J_6^2 \oplus J_5^3 \oplus J_4^2 \oplus J_3^2 \oplus J_2^4 \oplus J_1^3$.

In the next lemma we use the possibilities for the blocks $L_X(2)$ and $L_X(4)$ given by Lemmas 3.3.5 and 3.3.6 to identify the class of u in G.

Lemma 3.3.9 *If u is a non-identity unipotent element of X, then u lies in the class A_6A_1 in G, and $L(G) \downarrow u = J_7^{35} \oplus J_3$.*

Proof Suppose J_2^4 occurs in the decomposition $L(G) \downarrow u$. This can only arise from a summand $(W(8) \oplus W(8)^*)^2$ in $L_X(4)$. So Lemma 3.3.6 implies that $L_X(4) \downarrow u = J_7^6 \oplus J_5^2 \oplus J_3^3 \oplus J_2^4 \oplus J_1$. This immediately rules out possibilities (viii) and (x) of Lemma 3.3.8. Cases (vii) and (ix) require an additional J_3^3 to come from other blocks. This implies that either $i = 0, j = 2$ in 3.3.8(i) or $i = 0, j = 1$ in 3.3.8(ii). Then $L_X(2) \downarrow u = J_7^4 \oplus J_5 \oplus J_4^4 \oplus J_3^3 \oplus J_1$ or $J_7^6 \oplus J_4^2 \oplus J_3^3$. Neither of these yield a sufficient number of blocks J_1.

For the remaining possibilities there are no J_2's, and hence $W(8) \oplus W(8)^*$ does not occur. Note that J_4 only appears in $W(10) \oplus W(10)^*$ and in $M(18) \oplus M(18)^*$. The latter involves J_6. For $D_6(a_2)$ the former must appear twice. Then $L_X(2) \downarrow u = J_7^4 \oplus J_5 \oplus J_4^4 \oplus J_3^3 \oplus J_1$ and the J_3 contribution gives a contradiction. In the $E_7(a_5)$ case, we need $L_X(4) \downarrow u = J_7^8 \oplus J_6^2 \oplus J_1^2$ so the latter must occur and we have a contradiction from the J_5 contribution.

The A_6 case is easy to rule out due to the absence of a J_3 block. Indeed, we see from Lemma 3.3.8 that in each case such a block exists. Next consider $E_8(a_7)$, where there is no J_1 or J_2 block. It follows that neither 8 nor $W(8) \oplus W(8)^*$ can occur in $L_X(4)$. We find that $L_X(4) = T(18)^2 \oplus T(8)$ which restricts to u as J_7^{10}. We must then have $L_X(2) \downarrow u = J_7^3 \oplus J_5^4 \oplus J_3^6$, whereas 3.3.5 implies that $L_X(2)$ contributes an even number of J_7's.

Finally, consider A_4A_2. Here we need to account for J_5^{11} and this is possible only if $L_X(2) = 16 \oplus 2^6 \oplus 10^4$ and $L_X(4) = 18^2 \oplus 8^5 \oplus 4^4$. Checking the action of e we see that $L(G) \downarrow e = J_7^{17} \oplus J_5^{10} \oplus J_4^8 \oplus J_3^9 \oplus J_2^{10}$. It follows that $\dim C_{L(G)}(e) = \dim C_G(e) = 54$. Consider the possibilities for the class of e. There is a Levi subgroup R such that e is distinguished in $L(R)$ (see [7, p.164]).

If $R = G$, then e is in the Richardson orbit of the Lie algebra of the unipotent radical of a distinguished parabolic of G (see [7, p.137]). Hence $C_G(e)$ has the same dimension as the centralizer of the corresponding distinguished unipotent element, namely the dimension of the Levi factor of the parabolic. But such unipotent elements all have centralizer dimension less than 54. So the Levi subgroup R must be proper. Now e is trivial on $L(Z(R))$ and yet there are no trivial Jordan blocks in the decomposition of $L(G)$. This is only possible if $R = A_6A_1T_1$, so that $L(T_1) < L(A_6)$ (as $p = 7$). However, one checks that regular nilpotent elements in A_6A_1 have Jordan decomposition $J_7^{35} \oplus J_3$. This contradiction completes the proof. ■

The class of u immediately determines the action of X on $L(G)$.

Lemma 3.3.10 *We have*

$$L(G) \downarrow X = T(14)^3 \oplus T(18)^2 \oplus T(8) \oplus T(16) \oplus T(10)^2 \oplus 6^5 \oplus 2.$$

We next determine the class of a nonzero nilpotent element $e \in L(X)$.

Lemma 3.3.11 *e has type A_6A_1.*

Proof From the preceding lemma and the fact that e is projective on all the tilting modules we see that $L(G) \downarrow e = J_7^{35} \oplus J_3$. As we saw at the end of the previous result, this is only possible if e has type A_6A_1. ■

Recall that $T < T_G$, a maximal torus of G, and the root system of G relative to T_G is $\Sigma(G)$, with fundamental system $\Pi(G) = \{\alpha_1, \ldots, \alpha_8\}$. If

δ is the highest root, then $\{\alpha_1, \alpha_3, \ldots, \alpha_8, -\delta\}$ is a fundamental system for a subsystem of type A_8; let E be the corresponding subsystem subgroup A_8. Define Y to be a subgroup A_1 of E, embedded via an indecomposable representation $4|1 \otimes 1^{(7)} \cong W(8)^*$ on $V_{A_8}(\omega_1)$. Ultimately we shall show that X is G-conjugate to Y. Write $L(Y) = \langle e', h', f' \rangle$ with $[e', f'] = h'$, $[e', h'] = 2e'$, $[f', h'] = -2f'$.

Lemma 3.3.12 *Replacing X by a suitable G-conjugate, we may assume the following:*

(i) $T < T_G < E = A_8$ *where the T-labelling of A_8 has all labels* 2.

(ii) $e = e' = e_{\alpha_1} + 2e_{\alpha_3} + 3e_{\alpha_4} + 4e_{\alpha_5} + 5e_{\alpha_6} + 6e_{\alpha_7} + f_\delta$.

(iii) $f' = f_{\alpha_1} + 6f_{\alpha_4} + 5f_{\alpha_5} + 4f_{\alpha_6} + 3f_{\alpha_7} + 2f_{\alpha_8} + e_\delta$.

(iv) $h = h'$.

Proof We can view the Y-module $W(8)^*$ as the space of homogeneous polynomials of degree 8 in two variables x, y, with the natural action of $Y = PSL_2$. A maximal torus of Y is then a regular torus of A_8 and one checks that this torus determines the same labelling with respect to G as does T.

Let $U_1 = \{U_1(c) : c \in K\}$ be a 1-dimensional unipotent subgroup of Y. Using the usual basis $x^8, x^7y, x^6y^2, \ldots, y^8$ for the space of homogeneous polynomials, and taking $U_1(c)$ to send $x \to x$, $y \to cx + y$, we find that the matrix form of $U_1(c)$ on $W(8)^*$ is (recalling $p = 7$)

$$\begin{pmatrix} 1 &&&&&&&& \\ c & 1 &&&&&&& \\ c^2 & 2c & 1 &&&&&& \\ c^3 & 3c^2 & 3c & 1 &&&&& \\ c^4 & 4c^3 & 6c^2 & 4c & 1 &&&& \\ c^5 & 5c^4 & 3c^3 & 3c^2 & 5c & 1 &&& \\ c^6 & 6c^5 & c^4 & 6c^3 & c^2 & 6c & 1 && \\ c^7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ c^8 & c^7 & 0 & 0 & 0 & 0 & 0 & c & 1 \end{pmatrix}$$

It is immediate that the Lie algebra of U_1 is generated by a nilpotent element e' of type A_6A_1, and e' is as in (ii). Replacing X by a G-conjugate we may assume that $e' = e$. Similarly, f' is as in (iii).

We claim $C_G(e) = U_{33}A_1$, where U_{33} is a unipotent group of dimension 33. As p is good, $\dim C_G(e) = \dim C_{L(G)}(e)$, and from the Jordan block decomposition we see that the latter dimension is 36. Next note that $C_G(e)$ contains no torus of rank 2: for otherwise, say $T_2 \leq C_G(e)$, $C_G(T_2)$ is a Levi factor of G, and e would necessarily have a trivial Jordan block on the Lie algebra of this Levi factor, a contradiction. By [39, III,3.12] we have $C_G(e) = C_G(v)$ for some unipotent element $v \in G$. A check of unipotent element centralizers [28] shows that the only such centralizer of dimension 36 is $U_{33}A_1$, proving the claim.

Now $T < N_G(\langle e \rangle)$, and so T acts on $C_G(e)$. Take a subgoup A_6A_1 lying in A_8, corresponding to the subsystem with base $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, -\delta$. Now $e \in L(A_6A_1T_1)$, and the center of this Levi subgroup is a 1-dimensional torus $T_0 < C_G(e)$. Also there is a 1-dimensional torus $T_1 < A_6A_1$ inducing weight 2 on e. One checks that

$$T_0(c) = h_{\alpha_1}(c^4)h_{\alpha_2}(c^7)h_{\alpha_3}(c^8)h_{\alpha_4}(c^{12})h_{\alpha_5}(c^9)h_{\alpha_6}(c^6)h_{\alpha_7}(c^3)$$

and

$$T_1(c) = h_{\alpha_1}(c^6)h_{\alpha_3}(c^{10})h_{\alpha_4}(c^{12})h_{\alpha_5}(c^{12})h_{\alpha_6}(c^{10})h_{\alpha_7}(c^6)h_{-\delta}(c).$$

Conjugating in $C(e)$ we may assume $T < T_0T_1$ and write $T(c) = T_0(c^i)T_1(c^j)$. From the action of T on e we get $j = 1$. As the largest T-weight is 18 we see that $i = \pm 3$. Now there is an element of the Weyl group $W(G)$ inverting T_0 and inducing an involutory graph automorphism of A_8. Adjusting by an inner automorphism of A_6A_1 we may assume this element centralizes e and T_1. Conjugating by such an element we may now assume $T(c) = T_0(c^{-3})T_1(c)$. Similarly, if T_Y is the maximal torus of Y normalizing $\langle e \rangle$ we may conjugate within $C_G(e)$ to get $T(c) = T_Y(c)$. Hence we may take $h = h'$, proving (iv). ∎

From the proof of the last lemma we have $T(c) = T_0(c^{-3})T_1(c)$, from which we obtain the following.

Lemma 3.3.13 *Relative to the usual base $\alpha_1, \ldots, \alpha_8$, $G = E_8$ has T-labelling $2(-18)222222$.*

The aim now is to conjugate from f' to f using an element of $C_G(e) \cap C_G(T)$. Lemma 3.3.15 below determines the latter intersection.

Notice that $f' - f \in C_{L(G)}(e)$ and has T-weight -2. The next lemma determines the dimension of certain weight spaces of $C_{L(G)}(e)$.

Lemma 3.3.14 *The T-weight spaces of $C_{L(G)}(e)$ corresponding to weights 0 and -2 each have dimension 3 and lie in $L_X(0) = T(14)^3$.*

Proof Consider the composition factors of $L(G) \downarrow X$ and look for those containing fixed points of e of T-weight 0 and -2. These only occur within composition factors 0 and 12, respectively, both of which must occur within $L_X(0) = T(14)^3$. One checks that the Jordan blocks of $T(14)$ under the action of e are as follows, where each block has length 7 with basis having the given T-weights:

$$\begin{array}{l}(-12,-10,-8,-6,-4,-2,0),\\(-14,-12,-10,-8,-6,-4,-2),\\(0,2,4,6,8,10,12),\\(2,4,6,8,10,12,14).\end{array}$$

The result follows. ■

Lemma 3.3.15 *$C = C_G(e) \cap C_G(T)$ is a 3-dimensional group of the form $T_0U_1U_2$, where U_1, U_2 are commuting unipotent groups, normalized by the torus T_0. For suitable choices of signs these groups are given explicitly as follows:*

(i) $T_0 = \{h_{\alpha_1}(c^4)h_{\alpha_2}(c^7)h_{\alpha_3}(c^8)h_{\alpha_4}(c^{12})h_{\alpha_5}(c^9)h_{\alpha_6}(c^6)h_{\alpha_7}(c^3) : c \in K^*\}$

(ii) $U_1 = \{U_{11222110}(c)U_{11232100}(\pm 3c)U_{11122210}(\pm c) : c \in K\}$

(iii) $U_2 = \{U_{-11122111}(c)U_{-11221111}(\pm 2c)U_{-01122211}(\pm 3c) : c \in K\}$.

Proof One first calculates using Lemma 3.3.12(ii) that for suitable choices of signs these groups U_1, U_2 do in fact lie in C. Now $C_G(e) = U_{33}A_1$ is connected and T normalizes $C_G(e)$. As T has connected centralizer in A_1 as well as U_{33}, we conclude that C is connected. The previous lemma shows that the T-weight space in $C_{L(G)}(e)$ for weight 0 has dimension 3. Hence C has dimension at most 3. From the Bruhat decomposition we see that $T_0U_1U_2$ has dimension 3. Also, a direct check shows that U_1 and U_2 commute, with both normalized by T_0 and affording T-weights $2, -1$, respectively. It follows from this and the structure of $C_G(e)$ that C must have a unipotent radical of dimension 2. This gives the result. ■

Note that $f' - f$ lies in $C_{L(G)}(e)$ and has T-weight -2. Let J denote the full weight space of $C_{L(G)}(e)$ corresponding to weight -2. Lemma 3.3.14 shows that $\dim J = 3$.

The group C acts on the coset $f' + J$, and we shall show that under this action there are precisely two orbits. To this end we consider the map $\phi : C \to J$ given by $c \to f' - (f')^c$.

Lemma 3.3.16 (i) $\phi(T_0) = \{kf_{\alpha_8} : k \neq 2\}$.

(ii) $\phi(U_1) = \langle j_1 \rangle$, *where* $j_1 = ae_{01122210} + be_{11122110} + ce_{11221110} + de_{11222100}$ *for some scalars* a, b, c, d.

(iii) *Set* $J_0 = \langle f_{\alpha_8} \rangle$ *and* $J_1 = \langle j_1 \rangle$. *Then* $\phi(T_0U_1) = \tilde{J_0} + J_1$, *where* $\tilde{J_0} = J_0 \backslash \{2f_{\alpha_8}\}$.

Proof For (i) note that by construction T_0 centralizes $f_{\alpha_1}, f_{\alpha_4}, f_{\alpha_5}, f_{\alpha_6}, f_{\alpha_7}$ and e_δ. However, T_0 induces nontrivial scalars on f_{α_8}. The assertion follows as the coefficient of f_{α_8} in f' is 2.

Part (ii) is checked by straightforward computation. Moreover viewing U_1 as K^+ we see that $\phi \downarrow U_1$ is linear.

Finally, (iii) follows follows from (i) and (ii), noting that U_1 fixes f_{α_8}. ■

We have now identified a 2-space of J, namely $J_0 + J_1$. The action of U_2 is a little more complicated as, unlike U_1, the map to J is not linear.

Lemma 3.3.17 *We have* $\phi(U_2(c)) = cl_1 + c^2 l_2 + c^3 l_3$, *where*

$$l_1 = rf_{11122211} + sf_{11222111} + tf_{0112221} + me_{123433221} + ne_{12244321} + qe_{22343221}$$
$$l_2 = ue_{11122110} + ve_{11221110} + we_{01122210},$$
$$l_3 = xf_{\alpha_8},$$

for suitable constants $r, s, \ldots, x$.

Proof This is a direct computation. The quadratic and cubic terms arise from conjugating e_δ (which appears in f') by U_2. ■

Lemma 3.3.18 *We have* $J = J_0 + J_1 + J_2$, *where* $J_2 = \langle l_1 \rangle$.

Proof Consider the image of U_2 in $L(G)/(J_0 + J_1)$. We have $l_3 \in J_0$ and clearly l_1 is not contained in $J_0 + J_1$. Also the image must span a 1-space as it is contained in $J/(J_0 + J_1)$. The only possibility is that $l_2 \in J_0 + J_1$ (forcing $d = 0$ in 3.3.16(ii)) and $J = J_0 + J_1 + \langle l_1 \rangle$. Indeed, otherwise the existence of linear and quadratic coefficients forces the image to span a 2-space. ■

We can now establish the key result.

Lemma 3.3.19 *We have $f' + J = (f')^{T_o U_1 U_2} \cup (f' - 2f_{\alpha_8})^{T_o U_1 U_2}$. Moreover, $f' - 2f_{\alpha_8}$ is of type $A_5 A_1$.*

Proof Arguing as in the proof of Lemma 3.3.16(i) we first note that $f' + J_0 = f'^{T_0} \cup \{f' - 2f_{\alpha_8}\}$. Next note that $U_1 < C_G(f_{\alpha_8})$, so that Lemma 3.3.16 implies $f' + J_0 + J_1 = (f' + J_0)^{U_1} = f'^{T_0 U_1} \cup \{f' - 2f_{\alpha_8}\}^{U_1}$.

As U_2 fixes f_{α_8}, it is easily checked that $J_0 + J_1$ is U_2-invariant. Choose $x \in f' + J$ and write $x = f' + j_0 + j_1 + j_2$ with obvious notation. From 3.3.16 and 3.3.17 we see that there are elements $u_2 \in U_2, a_0 \in J_0$, and $a_1 \in J_1$ such that $(f' + a_o + a_1)^{u_2} = x$. Hence $f' + J = (f' + J_0 + J_1)^{U_2} = (f'^{T_0 U_1} \cup \{f' - 2f_{\alpha_8}\}^{U_1})^{U_2} = f'^{T_0 U_1 U_2} \cup \{f' - 2f_{\alpha_8}\}^{U_1 U_2}$.

Finally we note that $f' - 2f_{\alpha_8} = f_{\alpha_1} + 6f_{\alpha_4} + 5f_{\alpha_5} + 4f_{\alpha_6} + 3f_{\alpha_7} - e_\delta$. This element is clearly a regular nilpotent element of an $A_5 A_1$ subsystem. ■

We can now complete the proof of Proposition 3.3.3. Lemma 3.3.19 shows that $L(X)$ is conjugate to $L(Y)$. But then $L(X) < L(A_8)$ and so is centralized by an element of order 3 in G. This contradicts Lemma 2.2.10(ii).

The proof of Theorem 3.1 for $p = 7$ is now complete.

3.4 The case $p = 5$

In this section we prove Theorem 3.1 assuming $p = 5$. Let $X = A_1$ be maximal S-invariant in G, as in the hypothesis of the theorem. We have $G = E_6, E_7$ or E_8.

We begin with E_6 and E_7, which are relatively easy to handle.

Lemma 3.4.1 *G is not E_6 or E_7.*

Proof Assume $G = E_6$ or E_7. Then $p = 5$ is good, so $A = C_L(L(X)) = 0$ by Lemma 2.3.4. Using the Weight Compare Program, together with Lemmas 3.2.3 - 3.2.8, we check that the only possibilities for $L \downarrow X$ are as follows:

(1) $G = E_6$, $L \downarrow X = 10^2/8^4/6^2/4^3/2^5/0^2$, T-labelling 200202

(2) $G = E_6$, $L \downarrow X = 22/16^2/14/12/10/8^2/2/0$, T-labelling 222222

(3) $G = E_7$, $L \downarrow X = 10^3/8^6/6^5/4^4/2^{11}/0^3$, T-labelling 0002002.

Consider first case (1). Here [31, p.65] shows that there is a 1-dimensional torus $T' < A_1A_5$, projecting to a torus in a regular A_1 in each factor and having the same weights on L as T. Then Lemma 2.2.8 shows that T and T' determine the same labelled diagram of G and hence are conjugate. Thus, $T < A_1A_5$. Consider the 27-dimensional E_6-module $V_{27} = V(\lambda_1)$. By [23, 2.3], $V_{27} \downarrow A_1A_5 = 1 \otimes \lambda_1/0 \otimes \lambda_4$, whence we see that the T-weights on V_{27} are $8, 6^2, 4^4, 2^4, 0^5$ and their negatives. It follows that $V_{27} \downarrow X = 8/6/4^2/2/0^2$. Of these composition factors, only $8 = 3 \otimes 1^{(5)}$ extends 0, and hence we see that X fixes a 1-space in V_{27}. Let M be the stabilizer of this 1-space, so $X < M$ and $\dim M \geq \dim G - 26 = 52$.

If σ does not lie in the coset of a graph-field morphism of G, then Lemmas 2.2.11 and 2.2.13 give a contradiction. Thus σ lies in the coset of a graph-field morphism.

It is well known that E_6 has precisely three orbits on the 1-spaces of V_{27} (for example, this follows from [29]). The stabilizers of 1-spaces in the three orbits are P_1, F_4 and a subgroup $U_{16}B_4T_1$ lying in P_6 (see for example [8]). Thus M is one of these stabilizers.

Suppose first that $M \leq P_1$. Then $X < P_1 \cap P_1^\sigma = P_1 \cap P_6^g$ for some $g \in G$. Now the number of (P_1, P_6)-double cosets in G is equal to the number of $(W(D_5), W(D_5))$-double cosets in $W(G)$; this number is 3, since the action of $W(G)$ on the cosets of $W(D_5)$ is the action of $O_6^-(2)$ on singular points, which is rank 3. Thus, up to G-conjugacy there are three possibilities for $P_1 \cap P_6^g$. By inspection these have Levi subgroups D_5T_1, D_4T_2 and A_4T_2. In the first case $P_1 \cap P_6^g = L_1 = D_5T_1$, and the central torus T_1 of this centralizes X, a contradiction. In the second case, D_4 has 3 trivial composition factors on V_{27} (see [23, 2.3]), whereas X has only 2 such. Finally, in the third case, we see from Table 8.7 of [23] that A_4 has three composition factors on V_{27}, each of which are natural 5-dimensional modules or their duals. This implies that X has a composition factor appearing with multiplicity at least 3, which is not the case.

Thus $M \not\leq P_1$. Similarly, $M \not\leq P_6$.

Finally, if $M = F_4$ then by the maximality of X, M is not σ-stable. So X lies in $M \cap M^\sigma$, a subgroup of F_4 of dimension at least 26. From our knowledge of the maximal connected subgroups of F_4 $(p = 5)$ (see [31]), we see that $M \cap M^\sigma$ must lie in a parabolic or reductive subgroup of maximal rank in F_4. Maximal rank reductive subgroups have nontrivial centralizer, which is not possible. Hence, X is contained in a maximal parabolic P of

F_4. The composition factors X on $V_{26} = V_{F_4}(\lambda_4)$ are as above with one less fixed point. Let F be the fixed point space in V_{26} of the unipotent radical of P. Then the Levi factor has irreducible, dual actions on F and on $V_{26}/F^{\perp}$. But this is not consistent with the composition factors of X. This completes the proof for case (1).

The argument for case (2) is similar. Here the T-label is 222222, and as a linear combination of fundamental roots we have $\lambda_1 = \frac{1}{3}(435642)$, from which we see that the T-weights on V_{27} are $16, 14, 12, 10, 8^2, 6^2, 4^2, 2^2, 0^3$ and their negatives. Hence $V_{27} \downarrow X = 16/12/8/0^2$, and again it follows that X fixes a 1-space in V_{27}. Now we obtain a contradiction as in case (1) above.

Finally, in case (3), [31, p.65] tells us that $T < A_2A_5$, projecting to a regular torus in each factor. Letting $\hat{G}$ be the simply connected group E_7, we work with the 56-dimensional $\hat{G}$-module $V_{56} = V(\lambda_7)$. By [23, 2.3], $V_{56} \downarrow A_2A_5 = \lambda_1 \otimes \lambda_1/\lambda_2 \otimes \lambda_5/0 \otimes \lambda_3$, from which we calculate that the T-weights on V_{56} are $9, 7^3, 5^6, 3^9, 1^9$ and their negatives. It follows that $V_{56} \downarrow \hat{X} = 9/7^2/5^3/3^6/1^2$ (where $\hat{X}$ is the preimage of X in $\hat{G}$). Since the module is self dual we conclude that $\hat{X}$ stabilizes a 2-space corresponding to a module of high weight 1. If this space is uniquely determined, then by Lemma 2.2.3 it is ω-invariant and its stabilizer is σ-invariant. However any 2-space stabilizer has dimension at least $133 - 55 - 54 = 24$, so this is a contradiction. So assume there are two submodules of high weight 1.

We then have $V_{56} \downarrow \hat{X} = 9 \oplus 1^2 \oplus 7^2 \oplus (5^3/3^6)$. Letting u be a non-identity unipotent element of $\hat{X}$, we find that u has Jordan block decomposition $J_5^2 + J_4^2 + J_2^4$ on the first three terms. Hence from [18] we see that u is in one of the following G-classes: $A_3A_2A_1$, $(A_3A_1)'$,$(A_3A_1)''$, $2A_2A_1$. Now consider the possible Jordan form on the summand $5^3/3^6$. By [32, 2.4] we can write this space as a direct sum of submodules of the form $5, 3, W(5), W(5)^*$ or $T(5)$, where $T(5)$ denotes the tilting module of high weight 5. The Jordan forms of u on these modules are, respectively, J_2, J_4, $J_5 + J_1$, $J_5 + J_1, J_5^2$ (see the proof of Lemma 3.3.7). From [18] we see that only $A_3A_2A_1$ and $(A_3A_1)''$ remain as possibilities. Moreover, in the former case the Jordan decomposition forces $V_{56} \downarrow X = 9 \oplus 1^2 \oplus 7^2 \oplus 3^4 \oplus 5 \oplus T(5)$. So here there is a unique irreducible submodule of high weight 5, which yields a contradiction as in the previous paragraph. Hence u has type $(A_3A_1)''$.

Now consider the action on $L = L(G)$. From the above and [18] we see that $L \downarrow u = J_5^{17} + J_3^9 + J_1^{21}$. By Lemma 2.2.10 we have $C_L(X) = 0$. Using this it is easy to argue that the block with composition factors $10^3/8^6/0^3$ must be a tilting module. Hence $L \downarrow X = T(10)^3 \oplus 4^4 \oplus (6^5/2^{11})$. The last summand is a direct sum of submodules of the form $T(6), W(6), W(6)^*, 6$ or 2, and it is clearly impossible to get a sufficient number of J_1 blocks. This completes the proof. ∎

We now move on to the case where $G = E_8$, $p = 5$, which requires a great deal more effort. The reason is that $p = 5$ is not a good prime for G, and so $A = C_L(L(X))$ can be non-zero. However, we do know by Lemma 2.3.4 that if $A \neq 0$ then $A \leq L(D)$, where $D = A_4A_4$, a subsystem subgroup of G; in particular, the number of T-weights on L which are multiples of $2p = 10$ is equal to $\dim D = 48$. Using the Weight Compare Program together with this condition and Lemmas 3.2.3-3.2.8, we obtain the following.

Lemma 3.4.2 *The possibilities for $L \downarrow X$ are as in the table below:*

Case	$L \downarrow X$	T-*labelling*
(1)	$18^2/16/14^3/12^4/10^5/8^6/6^8/4/2^8/0^3$	00020020
(2)	$28/26/22^3/20/18^3/16^4/14^3/12/10^4/8^3/6/4/2^2/0$	20020202
(3)	$22/18^2/16^4/14^3/12^3/10^5/8^6/6^2/4/2^6/0^3$	00020022
(4)	$38/34/30/28/26/22^3/20/18^2/16^4/14^2/12/10^3/8^2/6/2^2$	22202022
(5)	$42/38/34/28/26^2/22^2/20/18^2/16^4/14^2/10^4/8^2/2^2/0$	22022022
(6)	$34/28/26^2/22^3/20/18^3/16^4/14^2/12/10^4/8^3/2^2/0$	20020222
(7)	$46/42/38/34/30/28/26/22^2/20/18^2/16^2/14^2/10^3/8^2/2^2$	22202222

In each case in the table, we shall need to establish that $A \neq 0$. For this we require the structure of various Weyl modules:

Lemma 3.4.3 *For $p = 5$ the following are co-socle series for the indicated Weyl modules:*

$W(8) = 8|0$, $W(10) = 10|8$, $W(18) = 18|10$, $W(20) = 20|18$,

$W(28) = 28|18|20$, $W(30) = 30|(28 + 10)|18$,

$W(38) = 38|(30 + 8)|10$, $W(40) = 40|(38 + 0)|8$.

Proof The first line follows from Lemma 3.2.5. For the other cases, we first find the composition factors of the given Weyl modules using the Sum Formula. The nontrivial extensions between these are given by Lemma 2.1.6. The indicated series follow from this together with Lemma 2.1.5, the universality of Weyl modules.

∎

Lemma 3.4.4 *In each of cases* (1) – (7) *in Lemma* 3.4.2*, we have* $A \neq 0$ *and* $A \leq L(D)$ *with* $D = A_4A_4$*. Also,the multiplicities of the* T*-weights which are divisible by* 10*, and the* T*-labelling of* $D = A_4A_4$ *(up to graph automorphisms) are as in the table below.*

Case	T-weights divisible by p	T-labelling of A_4A_4
(1)	$0^{24}, 10^{12}$	0 0 10 0, 0 0 10 0
(2)	$0^{16}, 10^{12}, 20^4$	10 0 10 0, 10 0 10 0
(3)	$0^{22}, 10^{12}, 20^1$	10 0 0 10, 0 0 10 0
(4)	$0^{12}, 10^{10}, 20^6, 30^2$	10 0 10 10, 10 0 10 10
(5)	$0^{12}, 10^{10}, 20^5, 30^2, 40^1$	10 0 10 0, 10 10 10 10
(6)	$0^{14}, 10^{11}, 20^5, 30^1$	10 0 10 10, 10 0 10 0
(7)	$0^{10}, 10^9, 20^6, 30^3, 40^1$	10 0 10 10, 10 10 10 10

Proof The multiplicities and labellings of $D = A_4A_4$ are routinely calculated from $L \downarrow X$. Less obvious is the fact that $A \neq 0$ in each case, which we now establish.

For cases (1) and (3) of Lemma 3.4.2, it follows from Lemma 3.2.6(ii) that $A \neq 0$.

Consider cases (2) and (6). Suppose $A = 0$. Working in $L_X(0)$, generating with a vector of weight 28, then one of weight 20, and then one of weight 18, we see that $L \downarrow X$ has a singular subspace $Z \cong 18$, and $Z^\perp/Z$ has a non-degenerate subspace $M \cong 28 + 20 + 18$, and $M^\perp$ (perp taken in $Z^\perp/Z$) has composition factors $10^4/8^3/0$. It is easy to see that $M^\perp$ has a submodule 10^2, and hence L has a submodule 10. Hence in fact $A \neq 0$, as required.

Now consider (4) and (7). Working in $L_X(0)$, let v be a vector of weight 38, and $Y = \langle Xv \rangle$. Then by Lemma 3.4.3, either $A \neq 0$ or $Y \cong 38|8^a$ with $a \leq 1$. Suppose the latter, and let $S = 8^a$ be the radical of Y. In $S^\perp \cap L_X(0)/S$, let $Z/S = (Y/S)^\perp$. Then $Z/S = 30/28/20/18^2/10^3/8^{2-2a}$, and using Lemma 3.2.4 we see that A has a composition factor 30.

Finally, consider (5). Define Y, S, Z as in the previous paragraph, so $Z/S = 28/20/18^2/10^4/8^2/0$. Considering cyclic submodules generated by weight vectors in the usual way, we see that there is a submodule 10^2, whence A has a submodule 10. ∎

At this point we study the subalgebra A in detail. Recall that R is the subalgebra of A generated by its nilpotent elements.

Lemma 3.4.5 *The T-weights on $L(D)$ are at most* 40.

Proof This follows from the previous lemma. ∎

Lemma 3.4.6 *In the T-labelling of $D = A_4A_4$, it is not possible that one of the A_4's has all labels either* 0 *or* 10 *with at least one* 0.

Proof Assume false. First suppose R has a T-weight vector e of weight strictly greater than 10. Then e is nilpotent and there is a unique expression $e = e_1 + e_2$, with each e_i nilpotent in one of the sl_5 factors. Each e_i is a T-weight vector of the corresponding factor, so Lemma 3.4.5 implies that $e_i^3 = 0$ for each i. Hence, Lemma 2.3.8 implies that $N_D(R)$ contains a unipotent element, contrary to Lemma 3.2.7.

Now suppose that the largest weight of R is 10, so that $R \cong (2^{(5)})^k$ for some k. If there exists a T-weight vector $e = e_1 + e_2 \in R$ of weight 10 such that both projections e_i satisfy $e_i^4 = 0$, then we can apply Lemma 2.3.10 to each factor to obtain a unipotent element $\exp(e_1)\exp(e_2) \in N_D(R)$ (see (*) in the proof of Lemma 2.3.8), again contradicting Lemma 3.2.7. Therefore we may assume the condition on the projections fails, which forces one of the e_i to be a regular nilpotent element in $L(A_4)$. Say e_2 is regular.

In view of Lemma 3.4.4 this forces all labels of the corresponding A_4 to be 10, so by hypothesis the first factor has labels 10 and 0. The centralizer (modulo center) of e_2 in the second factor sl_5 has dimension 4 and is spanned by the powers of e_2. So the weights of these vectors are $10, 20, 30, 40$, with each weight space of dimension 1.

We may assume that σ normalizes T, so σ normalizes D and ω (as in Lemma 2.2.3) normalizes $L(D)$. Since T has non-isomorphic centralizers in the A_4 factors it follows that σ stabilizes each factor and ω stabilizes the corresponding Lie algebras.

We claim the projections of R (see the discussion preceding 2.3.6 for the definition) to the two $L(A_4)$ factors are both faithful. For suppose otherwise and let J be a minimal ideal in the kernel of one of the projections. If J is X-invariant, then taking the sum of the images of J under powers of ω we obtain a subspace that is stable under X as well as ω. On the other hand the centralizer of this subspace contains the other factor, contradicting Lemma 2.2.10(iii). Thus Lemma 2.3.2 implies the existence of an abelian X-invariant ideal in R, which, as above, must project nontrivially to both factors.

Now e_2 is nilpotent so centralizes an element in the projection of this ideal to the second factor. However, by earlier remarks on the centralizer of e_2, this forces e_2 to lie in the projection and, as the ideal is abelian, e_2 must actually span the projection. Write $N_X(T) = T\langle s\rangle$, where s sends each T-weight to its negative. So s normalizes D. As $N_G(D)/D \cong Z_4$, it follows that s normalizes each A_4 factor. But this forces the projection of the ideal to contain a weight vector of weight -10, a contradiction. This establishes the claim.

The weight space of $R \cong (2^{(5)})^k$ for weight 10 is abelian, so the claim, together with earlier information on the centralizer of e_2, implies that $k = 1$. The centralizer information and Lemma 2.3.3 together imply that R is simple. Hence, $R \cong sl_2$.

Now consider the projection to the first A_4 factor. Here there is a 0 label, so e_1 cannot be regular. Therefore, working in sl_5, Lemma 2.3.10 shows that we can exponentiate all scalar multiples of e_1, obtaining a 1-dimensional unipotent group, U_1. Similarly, we get U_1^s, using multiples of e_1^s.

Let $\hat{M} = \langle U_1, U_1^s\rangle$, a subgroup of SL_5. Then $\hat{M}$ is connected, and Lemma 2.3.10 shows that $\hat{M}$ normalizes the preimage, say F, of the projection of R. The action must be faithful, as otherwise the kernel would centralize R contradicting Lemma 2.3.1. Hence $\hat{M}$ induces a subgroup of PSL_2.

As $e \in R, e_1 \in F$ and so $L(U_1) \leq F$ (see the argument in [33, 2.5]) and similarly $L(U_1^s) \leq F$. But then $L(\hat{M}) \geq F$, so $\hat{M}$ has type A_1 and [1] implies that $\hat{M}$ is completely reducible on the natural module for SL_5. If $\hat{M}$ is reducible, then it is centralized by a torus of the SL_5 factor, and this torus would then centralize R, a contradiction. On the other hand, if $\hat{M}$ is irreducible, then it contains a regular unipotent element and so the projection of R contains a regular nilpotent element, which we have already seen to be false. This completes the proof of the lemma. ∎

It is now immediate to establish the main result in this subsection:

Lemma 3.4.7 *There is no maximal S-invariant A_1 in $G = E_8$ when $p = 5$.*

Proof By Lemma 3.4.4, we have $A = C_L(L(X)) \neq 0$, and $A \leq L(D)$ where $D = A_4A_4$. The T-labelling of D is given in Lemma 3.4.4, and in each case we have a contradiction by Lemma 3.4.6. ∎

This completes the proof of Theorem 3.1 for $p = 5$.

3.5 The case $p = 3$

In this section we prove the main theorem assuming $p = 3$. Let $X = A_1$ be a maximal S-invariant subgroup of G as in the hypothesis. The proof proceeds along the lines of the previous section, but is necessarily much more involved at a number of points, since $p = 3$ is a bad prime for all exceptional groups.

The first order of business is to settle the case $G = G_2$.

Lemma 3.5.1 *$G \neq G_2$.*

Proof Assume that $G = G_2$. Since $p = 3$, $L = L(G)$ has an ideal I generated by all root vectors for short roots. In particular I and L/I both have dimension 7. Lemma 2.2.2 shows that S does not contain special isogenies, so that the argument of Lemma 2.2.3 extends to yield an action of S on I and on L/I. By Lemma 2.2.6, T determines a labelling of the Dynkin diagram of G, and this labelling is $20, 02$ or 22.

If the T-labelling is 20 then checking T-weights on short root vectors, we find that $(L/I) \downarrow X = 6^2/0$. It follows that X has a unique fixed point on L/I, so the stabilizer of this in G is S-invariant. However, this stabilizer has dimension at least $14 - 6 = 8$, contradicting maximality of X. Similarly, if the labelling is 02, then we find that $I \downarrow X = 2^2/0$ and again we have a fixed point.

Finally, suppose the labelling is 22. Here we find that $I \downarrow X = 6/4$ and as I is a self-dual module, this must be a direct decomposition. But then $A = C_L(L(X)) \neq 0$. Hence by Lemma 2.3.4 we have $A \leq L(D)$ with $D = A_2$. However the number of T-weights on L divisible by $2p$ is only 4, so this is a contradiction. ∎

Now we prove a number of representation-theoretic lemmas which will be useful in restricting the possibilities for $L \downarrow X$.

The first lemma gives the co-socle series of all Weyl modules for A_1 of high weight up to 46 which correspond to possible irreducibles in $L_X(0)$.

Lemma 3.5.2 *For $p = 3$ and $r \leq 46$ with $V(r) \in L_X(0)$, the co-socle series of the Weyl module $W(r)$ is as given in the table below.*

r	co-socle series of $W(r)$
4	$4\|0$
6	$6\|4$
10	$10\|4\|6$
12	$12\|(10+0)\|4$
16	$16\|12\|0$
18	$18\|12\|16$
22	$22\|(18+10)\|12$
24	$24\|22\|10$
28	$28\|10\|22\|24$
30	$30\|(28+12)\|(18+10)\|22$
34	$34\|(30+16)\|12\|18$
36	$36\|(30+0)\|(34+12)\|16$
40	$40\|(36+28+4)\|(30+10+0)\|12$
42	$42\|(40+6)\|(28+4)\|10$
46	$46\|40\|(42+4)\|6$

Proof This follows from the Lemma 2.1.6, together with the universal property of Weyl modules. The structure of $W(30)$ is a little more complicated than other cases and here we also use the Sum Formula.

■

Lemma 3.5.3 (i) *Either* $n_0 = 0$ *or* $n_0 < n_4 + n_{12} + n_{36}$.

(ii) *If the highest* $L_X(0)$*-weight is* 10 *or less, then* $n_4 \geq 2n_0$.

(iii) *If the highest* $L_X(0)$*-weight is* 16 *or less, then* $n_{16} + n_4 \geq n_0$.

Proof (i) The only irreducibles appearing in $L \downarrow X$ which extend the trivial module are 4, 12 and 36. So (i) follows as in the proof of Lemma 3.2.5(ii).

(ii) Let $Y_1 = \langle Xv : v \in L_X(0)_{10} \rangle$ (recall $L_X(0)_{10}$ denotes the T-weight 10 subspace of $L_X(0)$), so that Y_1 is a sum of images of $W(10)$. Then $Y_1 \leq L_X(0)$ and in $L_X(0)/Y_1$ let $Y_2/Y_1 = \langle Xv : v \text{ of weight } 6 \rangle$. Then Y_2/Y_1 is a sum of images of $W(6)$, and so from 3.5.2 we see that only 10, 6 and 4 occur as composition factors of Y_2. Moreover, $L_X(0)/Y_2 = 4^a/0^{n_0}$. Generating in $L_X(0)/Y_2$ with vectors of weight 4, we see that since L has no trivial quotient (see Lemma 2.2.10(iv)), $L_X(0)/Y_2 \cong (4|0)^{n_0} + 4^{a-n_0}$. Hence, since L has no trivial submodule (again by Lemma 2.2.10(iv)), Y_2 has at least n_0 composition factors 4. Therefore $n_4 \geq 2n_0$, as required.

(iii) If $v \in L_X(0)$ is a vector of T-weight 16, then $\langle Xv \rangle$ is an image of $W(16)$, which by Lemma 3.5.2 is of the form $16|12|0$. As $C_L(X) = 0$, $Y = \langle Xv : v \in L_X(0)_{16} \rangle = 16^{n_{16}}/12^a$, where $a \le n_{16}$. Now work in $L_X(0)/Y$. Generating with a weight 12 vector gives an image of $W(12) = 12|(10+0)|4$. Say $Z/Y = \langle Xv : v \in (L_X(0)/Y)_{12} \rangle$ has b composition factors 0 and c composition factors 4. Since L has no trivial submodule, we have $a + c \ge b$.

In L/Z, the only composition factors present are $10, 6, 4, 0$, and only 4 extends 0. Say the multiplicities of 0,4 are d, e respectively. As there is no trivial quotient (otherwise L would have a fixed point), $e \ge d$. We now have

$$n_0 = b + d, \ n_4 = c + e, \ n_{16} \ge a, \ a + c \ge b, \ e \ge d.$$

Therefore $n_4 \ge c + d = n_0 - b + c \ge n_0 - (a + c) + c = n_0 - a \ge n_0 - n_{16}$, as required. ∎

The next result is a variation of Lemma 3.5.3 in a couple of special cases.

Lemma 3.5.4 (i) *If the highest $L_X(0)$-weight is* 16 *or less, then either $n_{10} \ge 2n_{12}$, or there is a composition factor* 12 *in A.*

(ii) *If the highest $L_X(0)$-weight is* 22 *or less, then either $n_{16} \ge 2n_{18}$, or there is a composition factor* 18 *in A.*

Proof (i) If v is a maximal vector of weight 16, then $\langle Xv \rangle$ is an image of $W(16) = 16|12|0$. So, assuming there is no submodule 12, it follows that $\langle Xv \rangle \cong 16$. Thus by Lemma 3.2.2, $Y = \langle Xv : v \in L_{16} \rangle$ is a non-degenerate subspace, and applying Lemma 3.2.4 to the space $Y^\perp$ we obtain $n_{10} \ge 2n_{12}$.

(ii) Assume there is no composition factor 18 in A. Let v be a maximal vector in $L_X(0)$ of weight 22. Then $\langle Xv \rangle$ is an image of $W(22) = 22|(18+10)|12$. By assumption the composition factor 18 does not appear, as otherwise there exists a submodule 18 or $18|12$ and Lemma 3.2.4 implies that 18 occurs as a composition factor of A. Therefore $\langle Xv \rangle \cong 22, 22|10$, or $22|10|12$ for each weight vector v of weight 22. Let $Y = \langle Xv : v \in L_X(0)_{22} \rangle$, and let S be the radical of Y, a singular (see 2.1.6) subspace with all composition factors of weight 10 or 12. Now apply Lemma 3.2.4 to $S^\perp/S$ to see that $n_{16} \ge 2n_{18}$ or $C_{S^\perp/S}(L(X))$ has 18 as a composition factor. In the latter case, taking preimages and applying the argument of Lemma 3.2.4 we see that A also has a composition factor of weight 18. ∎

Lemma 3.5.5 *Assume the highest $L_X(0)$-weight is less than* 40. *If $n_4 + n_{12} + n_{16} + n_{36} < n_6 + n_0$, then $A \neq 0$.*

Proof Assume $A = 0$. Let r_1 be the highest weight in $L_X(0)$. Define $Y_1 = \langle Xv : v \in L_X(0)_{r_1} \rangle$. Repeat in the space $L_X(0)/Y_1$, generating a space Y_2/Y_1 with vectors of maximal weight in $L_X(0)/Y_1$. Continue until we have generated a space, Y_k/Y_{k-1} say, with high weight vectors of weight 6.

The only values of $r < 40$ for which the Weyl module $W(r)$ has a composition factor 0 are $r = 4, 12, 16, 36$. Hence Y_k has at most $n_{36} + n_{16} + n_{12}$ composition factors 0. Also, as $A = 0$ by assumption, Y_k has no submodule 6. As the only module in the required range which extends 6 is 4, it follows that Y_k has at least n_6 composition factors 4. Hence $L_X(0)/Y_k$ has at most $n_4 - n_6$ composition factors 4, and has at least $n_0 - n_{12} - n_{16} - n_{36}$ composition factors 0. Since $L_X(0)$ has no trivial quotient, it follows that $n_4 - n_6 \geq n_0 - n_{12} - n_{16} - n_{36}$, giving the conclusion. ■

Lemma 3.5.6 *Assume the highest $L_X(0)$-weight is less than* 40. *Then A has a submodule which is a direct sum of at least $n_6 - \frac{1}{2}n_4$ copies of* 6. *In particular, if $n_6 - \frac{1}{2}n_4 > 0$, then $A \neq 0$.*

Proof Begin as in the previous lemma. Let $r_1 < 40$ be the highest weight in $L_X(0)$. Define $S_1 = \langle Xv : v \in L_X(0)_{r_1} \rangle$ and let Y_1 be the radical of S_1. Then Y_1 is a singular space and, taking perpendicular spaces in $L_X(0)$, $Y_1^{\perp}/Y_1 = (S_1/Y_1) \perp W$. Now similarly generate a submodule of W using the highest weight vectors of W. Say the radical of this submodule is Y_2/Y_1. Then Y_2 is a singular space. Continue in this way until we have generated by weight vectors of weight 16 (if any such exist). At this point we have a singular space, Y_k.

Observe from Lemma 3.5.2 that for $40 > r \geq 16$, the Weyl module $W(r)$ has no composition factors 4 or 6, and hence neither does Y_k.

We continue the process further, paying more attention to the structures obtained. In $Y_k^{\perp}/Y_k$, generate with weight 12 vectors (if any exist) in a suitable non-degenerate summand. This gives us a singular space Y_{k+1}/Y_k with composition factors among those of the maximal submodule of $W(12)$, namely 10, 4, 0; say 4 appears with multiplicity a.

In $Y_{k+1}^{\perp}/Y_{k+1}$, generate a submodule with vectors of weight 10. Since $W(10) = 10|4|6$, the weight 10 vectors generate a submodule of form $(10|4|6)^b + (10|4)^c + 10^d$. We then get the next singular subspace Y_{k+2} such that $Y_{k+2}/Y_{k+1} = (4|6)^b + 4^c$.

Finally, we pass to $Y_{k+2}^{\perp}/Y_{k+2}$ where the highest weight is at most 6. The submodule P/Y_{k+2} generated by vectors of weight 6 has shape $(6|4)^e + 6^f$ and we get a singular space $Y_{k+3}/Y_{k+2} \cong 4^e$.

Now the module 6 does not extend the indecomposable $4|6$ (generate with weight 6 vectors in a putative such extension). Hence P/Y_{k+1} contains a submodule 6^{b+f-c} (interpreted as the 6^b if $f \leq c$), and consequently P/Y_k contains $6^{b+f-c-a}$. Since Y_k has no composition factor 4, it has no composition factor extending 6, and hence L contains a submodule $6^{b+f-c-a}$.

The singular space Y_{k+3} has composition factors $4, 6$ occurring with multiplicities $a+b+c+e, b$ respectively. We deduce that

$$n_4 \geq 2(a+b+c+e),\ n_6 = 2b+e+f.$$

Hence $n_6 - \frac{1}{2}n_4 \leq b+f-a-c$, giving the conclusion of the lemma. ∎

Lemma 3.5.7 *If $G = F_4$ or E_6 then the possibilities for $L \downarrow X$ are as in the table below. In each case $A \neq 0$, and $A \leq L(D)$ with $D = A_2^2$ ($G = F_4$) or A_2^3 ($G = E_6$). The T-labellings of G and of D are as in the table.*

G	Case	$L \downarrow X$	T-labelling of G	T-labelling of D
F_4	(1)	$10^2/8/6^3/4^4/2^3/0$	0202	$6\,0, 6\,0$
	(2)	$14/10^2/6^3/4^4/2^2/0$	2202	$6\,0, 6\,6$
	(3)	$16/14/10/6^3/4^2/2^2/0$	2022	$6\,6, 6\,0$
E_6	(1)	$10^2/8^2/6^4/4^7/2^3/0^3$	200202	$6\,0, 6\,0, 6\,0$
	(2)	$14/10^3/8/6^4/4^6/2^2/0^3$	220202	$6\,6, 6\,0, 6\,0$
	(3)	$16/14/12/10^2/8/6^3/4^4/2^2/0^2$	222022	$6\,6, 6\,6, 6\,0$
	(4)	$22/18/16/14/12^2/10^2/8/6/4/2^2/0$	222222	$6\,6, 6\,6, 6\,12$

Proof By Lemma 2.3.4, if $A \neq 0$, then $A \leq L(D)$ with $D = A_2^2$ or A_2^3 for $G = F_4$ or E_6, respectively. Thus the number of T-weights which are multiples of 6 is equal to 16 or 24, respectively. Using the Weight Compare Program, together with this condition and Lemmas 3.5.3 - 3.5.6, we see that the possibilities for $L \downarrow X$ are as in the table. Moreover, Lemma 3.5.6 shows that in all cases A has a composition factor of high weight 6. Finally, the T-labellings of D are easily calculated. ∎

Lemma 3.5.8 *If $G = E_7$ then the possibilities for $L \downarrow X$ are as in the table below. In each case $A \neq 0$, $A \leq L(D)$ with $D = A_2A_5$, and the T-labellings of G and of D are as in the table.*

Case	$L \downarrow X$	T-labelling of G	T-labelling of D
(1)	$16/14/12^3/10^4/8^3/6^3/4^8/2^2/0^7$	2002020	$6\,6, 6\,0\,6\,0\,0$
		2002020	$6\,0, 0\,6\,0\,6\,0$
(2)	$10^3/8^3/6^8/4^{12}/2^6/0^4$	0002002	$6\,0, 0\,0\,6\,0\,0$
(3)	$14/12/10^5/8^2/6^7/4^{10}/2^5/0^3$	2002002	$6\,0, 0\,6\,0\,0\,6$
(4)	$18/16^2/14/12^3/10^4/8^3/6^2/4^5/2^2/0^7$	2202002	$6\,0, 6\,6\,0\,0\,6$
(5)	$12/10^4/8^4/6^5/4^{11}/2^3/0^9$	2000202	$6\,6, 0\,0\,0\,6\,0$
		2000202	$6\,0, 6\,0\,0\,0\,6$
(6)	$16/14^2/12/10^4/8/6^7/4^8/2^4/0^3$	2200202	$6\,6, 0\,6\,0\,0\,6$
(7)	$14^2/12/10^4/8^2/6^5/4^{11}/2/0^9$	0020202	$6\,0, 6\,0\,6\,0\,0$
(8)	$22/18^2/16^2/14^2/12^3/10^3/8/6^3/4^3/2^4/0$	2220202	$6\,6, 0\,6\,0\,6\,6$
(9)	$16^2/14/12^3/10^3/8^3/6^3/4^6/2^2/0^7$	0220022	$6\,6, 6\,0\,6\,0\,0$
		0220022	$6\,0, 0\,6\,0\,6\,0$
(10)	$20/16^2/14^2/12^3/10^3/8/6^3/4^6/2/0^7$	2220022	$6\,6, 6\,6\,0\,0\,6$
(11)	$18/16^2/14^2/12^2/10^3/8/6^5/4^6/2^4/0^2$	2002022	$6\,0, 6\,0\,6\,0\,6$
(12)	$22/20/18/16/14^2/12^4/10^4/8/6/4^5/2/0^6$	2202022	$6\,6, 6\,0\,6\,6\,0$
(13)	$28/24/22^2/20/18/16/14^2/12^3/10^3/8/4^2/2/0^4$	2222022	$6\,12, 6\,6\,0\,6\,6$
(14)	$22^2/18^2/16/14^2/12^3/10^4/8/6^3/4^2/2^4/0$	0220222	$6\,6, 0\,6\,0\,6\,6$
(15)	$26/22/18^2/16^2/14/12^3/10^3/6^3/4^3/2^2/0$	2220222	$6\,6, 6\,0\,6\,6\,6$
(16)	$28/26/22/18^3/16^2/14/12^3/10^3/6^2/4^2/2^2/0$	2202222	$6\,12, 6\,0\,6\,6\,6$

Proof By Lemma 2.3.4, if $A \neq 0$ then $A \leq L(D)$ with $D = A_2A_5$, and so the number of T-weights divisible by 6 is equal to $\dim D = 43$. Using the Weight Compare Program, together with this condition and Lemmas 3.5.3 - 3.5.6, and 3.2.5 we see that the possibilities for $L \downarrow X$ are as in the table. The T-labellings of $D = A_2A_5$ are easily calculated from the weights, and we see that these must be as indicated. It remains to show that $A \neq 0$.

In cases (2),(3),(6),(8),(11),(14),(15) and (16), Lemma 3.5.6 shows that $A \neq 0$. For cases (5) and (7) Lemma 3.5.5 gives the same conclusion. And in cases (1) and (12), Lemma 3.5.4 gives $A \neq 0$.

In cases (9) and (10), $L_X(0) = 16^2/12^3/10^3/6^3/4^6/0^7$. If v is a vector of weight 16, then $\langle Xv \rangle$ is an image of $W(16)$. Assume $A = 0$. Then Lemma 3.5.2 implies that $\langle Xv \rangle$ is irreducible, so that $L_X(0) = 16^2 \perp W$ for some (non-degenerate) subspace W. Applying Lemma 3.2.4 to W we conclude that $A \neq 0$, which we are assuming false.

Now consider case (4) where $L_X(0) = 18/16^2/12^3/10^4/6^2/4^5/0^7$. Assume $A = 0$. We will show that there is a fixed point. Let v be a weight vector of weight 18. Then Lemma 3.5.2 implies that $\langle Xv \rangle \cong W(18)$. Let Y be the maximal module so that Y is singular with composition factors $16, 12$.

We can write $Y^\perp/Y = 18 \perp J$ where J has highest weight 12. Generate by a maximal vector to obtain a subspace $E = 12/0^a/4^b/10^c$, with $a, b, c \leq 1$. First assume $a = 1$ and $b = 0$. Then the preimage of E has a submodule 16 with quotient $12 \oplus (12/0/10^c)$. As 0 does not extend 10 or 16 we conclude that $L \downarrow X$ has a fixed point, a contradiction.

Let N be the preimage of the radical of E, so that $E = 16/12/0^a/4^b/10^c$ and E is singular. We have $E^\perp/E = 18 \perp 12 \perp F$, where F has all weights at most 10 and the irreducibles 4 and 0 occur with multiplicities $5-2b$, $7-2a$, respectively. The remaining cases are $a = b = 0; a = 0$ and $b = 1$; and $a = b = 1$. For these cases we find that E has a submodule $0^2, 0^4, 0^2$, respectively. Taking preimages of this submodule we obtain a submodule of L with composition factors $0^2/10^c/12/16$; $0^4/4/10^c/12/16$; $0^3/4/10^c/12/16$, respectively. In each case the submodule has a fixed point.

Finally, consider case (13) where we show that L has a trivial submodule. We have $L_X(0) = 28/24/22^2/18/16/12^3/10^3/4^2/0^4$. Let v_1 be a vector of weight 28, and generate $Y_1 = \langle Xv_1\rangle$. This is an image of $W(28)$, say $Y_1 = 28|10^a|22^b$ $(a, b \leq 1)$ with radical $Z_1 = 10^a|22^b$. Now work in the space $Z_1^\perp/Z_1$. After splitting off Y_1/Z_1 we generate with a vector v_2 of weight 24 to get $\langle Xv_2\rangle/Z_1 = Y_2 = 24|22^c|10^d$, with radical $Z_2 = 22^c|10^d$. Likewise, in $Z_2^\perp/Z_2$, generate with a vector of weight 22 to get $Y_3 = 22^{2-2b-2c}|10^e|12^f$, with radical $Z_3 = 10^e|12^f$; and in $Z_3^\perp/Z_3$ generate with a vector of weight 18 to get $Y_4 = 18|12^g$ with radical $Z_4 = 12^g$. Finally, in $Z_4^\perp/Z_4$, generate with a suitable weight vector of weight 16 to get $Y_5 = 16|12^h|0^k$. Taking preimages of all the Z_i subspaces we obtain a singular space S for which

$$S^\perp/S = 28 \perp 24 \perp 22^{(2-2b-2c)} \perp 18 \perp 16 \perp M,$$

where

$$M = 12^{3-2(f+g+h)}/10^{3-2(a+d+e)}/4^2/0^{4-2k}.$$

If $k > 0$ then by Lemma 3.5.2 we have $h = 1$, and hence $f = g = 0$. But then S has a trivial submodule. So assume $k = 0$.

Suppose $f + g + h = 1$. Then $M = 12/10^{3-2((a+d+e)}/4^2/0^4$. As M is self-dual, first generating with a weight vector of weight 12 and arguing as above we see that 0^2 occurs as a submodule of M. (Indeed, as 12 occurs with multiplicity 1, only composition factors of high weight 4 can block trivial factors). Taking the preimage of 0^2 over S we conclude that $L_X(0)$ contains a trivial submodule. Finally suppose $f + g + h = 0$. Here we see that $L_X(0)$ has a trivial submodule provided M does. By way of contradiction assume $M = 12^3/10^{3-2(a+d+e)}/4^2/0^4$ does not have a trivial submodule. Let J be the submodule of M generated by cyclic submodules with generator a weight vector of weight 12. Then $J = 12^3/10^x/4^y/0^z$. Since we are assuming M has no fixed point we have $1 \geq y \geq z$. But then we see that M/J must have a trivial quotient and hence a fixed point as required. ∎

Lemma 3.5.9 *If $G = E_8$ then the possibilities for $L \downarrow X$ are as in Tables 1 and 2 below. In each case $A \neq 0$, and $A \leq L(D)$, with $D = A_2E_6$ for the cases in Table 1 and $D = A_8$ for those in Table 2. In all cases with $D = A_2E_6$, at least one of the T-labels of the A_2 factor is nonzero.*

Table 1 : $D = A_2E_6$

Case	$L \downarrow X$	T-labelling
(1)	$22^3/18^3/16^3/14^3/12^7/10^6/8^3/6^6/4^3/2^6/0^4$	02200200
(2)	$16^2/14^3/12^3/10^7/8^3/6^{13}/4^{14}/2^6/0^8$	22000020
(3)	$14^2/12^3/10^8/8^4/6^{13}/4^{19}/2^7/0^8$	00200020
(4)	$18^2/16^3/14^3/12^6/10^6/8^3/6^8/4^{12}/2^6/0^5$	00020020
(5)	$30^2/28/26/24/22^3/20/18^3/16^3/14^3/12^4/10^5/6^5/4^3/2^3/0^2$	20020220
(6)	$12/10^6/8^6/6^{15}/4^{21}/2^9/0^{10}$	00200002
(7)	$20/18^3/16^3/14^3/12^6/10^8/8^2/6^7/4^{10}/2^6/0^5$	20020002
(8)	$28/26^2/22/20/18^4/16^4/14/12^7/10^6/6^5/4^5/2^2/0^4$	02202002
(9)	$32/30^2/28/22^4/20/18^6/16^3/14^3/12^7/10^5/8/6/2^5/0^2$	22202002
(10)	$22/20/18^3/16^3/14^4/12^6/10^6/8/6^7/4^9/2^5/0^5$	22000202
(11)	$26/24/22^2/18^4/16^4/14^2/12^6/10^5/8^2/6^4/4^4/2^4/0^2$	02200202
(12)	$36/34^2/30^2/26^2/22/18^6/16^5/14/12^6/10^2/8/6/4/2^2/0^2$	20220202
(13)	$24/22/20/18^4/16^4/14^4/12^4/10^5/8/6^6/4^7/2^5/0^4$	20002022
(14)	$30/28/26/24/22^3/20/18^4/16^2/14^3/12^5/10^6/6^4/4^5/2^3/0^2$	22002022
(15)	$32/28/26/24/22^3/18^6/16^3/14^3/12^5/10^4/6^3/4^4/2^2/0^3$	20202022
(16)	$34/30^2/28/26/24/22^3/20/18^4/16^2/14^3/12^5/10^5/$ $6^3/4^3/2^3/0$	02202022
(17)	$28/24/22^4/20/18^5/16/14^4/12^5/10^7/8/6^4/4^4/2^5/0^3$	22000222
(18)	$48/46/42/40/38/34/32/30/28/26/22^2/18^4/16^2/14^2/$ $12^3/10^2/6^2/4^2/2^2/0$	22220222
(19)	$38/32/30^2/28^3/24/22^4/20/18^5/16/14^3/12^5/10^7/$ $6^2/4^2/2^4/0$	22002222
(20)	$42/40/34/32/30^2/28/26/22^2/20/18^4/16^2/14^2/12^5/$ $10^5/6^2/4^2/2^3/0$	02202222
(21)	$54^2/52/48/44/36^2/34^2/30^2/26/22/18^3/16^2/14/$ $12^3/10^2/6/4/2^2/0$	22222220

Table 2 : $D = A_8$

Case	$L \downarrow X$	T-labelling
(1)	$10^4/8^6/6^{14}/4^{24}/2^{10}/0^{10}$	00002000
(2)	$22^2/20/18^3/16^2/14^4/12^6/10^8/8/6^5/4^8/2^6/0^4$	20020020
(3)	$28^2/26/24/22^2/20/18^5/16^3/14^3/12^4/10^7/6^3/4^4/2^4/0^3$	22020020
(4)	$34^2/30^2/26/24/22^2/20/18^4/16^4/14^3/12^4/10^3/6^2/4/2^4/0$	22202020
(5)	$16/14^3/12^3/10^8/8^3/6^{11}/4^{17}/2^7/0^7$	00020002
(6)	$14/12^2/10^9/8^5/6^{12}/4^{19}/2^9/0^8$	00002002
(7)	$18/16^3/14^3/12^4/10^7/8^3/6^9/4^{12}/2^7/0^6$	20002002
(8)	$24/22^3/18^3/16^3/14^4/12^4/10^6/8^2/6^5/4^4/2^6/0^3$	22002002
(9)	$20/18/16^3/14^5/12^3/10^6/6^{10}/4^{13}/2^5/0^7$	00200202
(10)	$30/28^2/26/22^3/20/18^4/16^3/14^3/12^6/10^5/6^3/4^3/2^4/0^3$	22200202
(11)	$28/26/24/22^3/18^4/16^3/14^3/12^5/10^5/8/6^3/4^4/2^4/0^2$	20020202
(12)	$42/40^2/30^2/28^4/26/22^3/20/18^4/14^3/12^4/10^5/6/4^2/2^4/0$	22220202
(13)	$40/38/32/30^2/28^2/26/24/22^3/18^4/16^2/14^2/12^4/10^5/$ $6^2/4^2/2^3/0$	22202202
(14)	$22/18^3/16^5/14^3/12^6/10^4/8^3/6^5/4^6/2^7/0^4$	00020022
(15)	$26/22^2/20/18^3/16^3/14^3/12^6/10^6/6^5/4^7/2^4/0^4$	20020022
(16)	$38/34/32/30/28/26/24/22^3/18^4/16^3/14^2/12^5/10^4/$ $6^2/4^2/2^3/0$	22202022
(17)	$32/26^2/22^3/18^4/16^3/14/12^6/10^6/6^4/4^4/2^2/0^4$	02200222
(18)	$34/30/28/26^2/24/22^2/18^4/16^4/14/12^5/10^4/8/6^2/4^2/2^3/0$	20020222
(19)	$46/42/40/38/34/32/30/28/26/24/22^2/18^3/16^2/14^2/$ $12^3/10^3/6^2/4^2/2^3$	22202222

Proof By Lemma 2.3.4, if $A \neq 0$ then $A \leq L(D)$ with $D = A_2E_6$, A_8 or A_2^4, and so the number of T-weights divisible by 6 is equal to $\dim D = 86, 80$ or 32. Using the Weight Compare Program, together with this condition, Lemmas 3.5.3 - 3.5.6, and the fact that there must be a composition factor isomorphic to $L(X)$, we find that the possibilities for $L \downarrow X$ are as in Table 1 when $D = A_2E_6$ and as in Table 2 when $D = A_8$, and there are no possibilities when $D = A_2^4$. The Weight Compare Program gives the multiplicities of all weights and checking those that are a multiple of 6 we see that in Table 1 the A_2 factor of D must have a nonzero label.

It remains to show that $A \neq 0$ in each case. For Table 1, this follows from Lemma 3.5.6 in all cases except (18), (20) and (21). And in these cases, Lemma 3.2.3 gives $A \neq 0$.

Now consider Table 2. For all cases except (12),(13) and (19), we have $A \neq 0$ by Lemma 3.5.6. In case (13), the only composition factors present which extend 30 are 28 and 12. Here we consider $\langle Xv \rangle$ for v a weight vector of weight 40. If the radical E of this has a composition factor 30 then clearly $A \neq 0$ from the structure of $W(40)$ given in Lemma 3.5.2. So suppose E has no composition factor 30. Taking perps in $L_X(0)$ we have $E^\perp/E = 40 \perp W$, where W has high weight 30. Generate with weight vectors of W of weight 30 and take preimages to obtain a submodule J of L such that J has a singular submodule S such that $J/S = 30^2$. As S is singular, 28 can occur as a composition factor with multiplicity at most 1. Hence the argument of Lemma 3.2.3 shows that $A \neq 0$.

In case (19), let v be a maximal vector of weight 46. Then $\langle Xv \rangle$ is an image of $W(46)$. Since 40 appears with multiplicity only 1, it follows that $\langle Xv \rangle = Y$ is a non-degenerate submodule isomorphic to 46. Now apply Lemma 3.2.4 to $Y^\perp$ to see that $L \downarrow X$ has a composition factor 42, whence $A \neq 0$ in this case.

Finally, consider case (12). Here

$$L_X(0) = 42/40^2/30^2/28^4/22^3/18^4/12^4/10^5/6/4^2/0.$$

Suppose $A = 0$. Consider $Y_1 = \langle Xv \rangle$ for v a vector of weight 42. and let Z_1 be the image of the maximal submodule. Lemma 3.2.3 implies that 40 occurs as a composition factor of Z_1, so that $Z_1 = 40/28^a/10^b/4^c$. Next work in $Z_1^\perp/Z_1 = 4 + W_1$. Generate by weight vectors of weight 30 in W_1 to get a module of shape $30^2/28^d/12^e/18^f/22^g/10^h$. The module has a singular submodule with quotient 30^2 and we take the preimage of this submodule to get a singular submodule S of shape $40/28^{a+d}/22^g/18^f/12^e/10^{b+h}/4^c$. We repeat this procedure two more times generating by vectors of weight 22 and then weight 18. In this way we are able to construct modules of shapes $22^{3-2g}/18^i/10^j/12^k$ and $18^{4-2f-2i}/12^x$ which occur as appropriate sections of L.

From the constructions of the previous paragraph we can find a submodule N of $L_X(0)$ such that $N = 40/28^{a+d}/22^g/18^{4-i-f}/\cdots$. Note that $g \leq 1$ and that $4 - i - f \geq 2$. Choose a submodule $E \leq F \leq N$ with $E/F = 22^g$, taking $E = F = 0$ in case $g = 0$.

Now $W(18) = 18|12|16$ and 16 does not occur as a composition factor of $L_X(0)$. Also, neither 18 nor 12 extends either 40 or 28. So if 18 occurs as a composition factor of E, then there is a submodule of E with highest weight 18 and Lemma 3.2.3 implies that 18 occurs as a composition factor of A. So we may assume that 18 does not occur as a composition factor of E.

Now use the above information on extensions to conclude that there is a submodule $M/F < N/F$ with shape $18^{4-i-f}/12^r$ and this submodule is the sum of cyclic modules each with generator of high weight 18. Now 22 does not extend 12 and $\mathrm{Ext}^1_X(22, 18)$ has dimension 1. It follows that M has a submodule for which 18 occurs as a composition factor, while 22 does not. But as in the previous paragraph this implies 18 occurs as a composition factor of A. ∎

We have established in the previous lemmas that $A = C_L(L(X)) \neq 0$ in all cases, and $A \leq L(D)$. At this point we study the algebra A in detail. In the following we will consider certain projections of R, where we recall that R is the subalgebra of A generated by all nilpotent elements. We refer the reader to the discussion preceding Lemma 2.3.6 for definitions.

Lemma 3.5.10 *Assume that D has a factor $K \cong SL_3$. Then K can be chosen so that R projects faithfully to this factor and one of the following holds:*

(i) $R \cong 2^{(r)}$ *(isomorphism of X-modules) for r a power of 3, and the T-labelling of the A_2 factor has equal labels $2r$.*

(ii) $R \cong 2^{(r)} \oplus (1^{(r)} \otimes 1^{(s)})$ *or* $2^{(s)} \oplus (1^{(r)} \otimes 1^{(s)})$ *for $r < s$ nontrivial powers of 3. The T-labels of the A_2 factor are $2r, s-r$ or $s+r, s-r$, respectively.*

Proof We first claim that K can be chosen to be invariant under $N_S(T)$ and that R projects faithfully to $L(K)/Z(L(K))$, a simple algebra of dimension 7. This is immediate from Lemma 2.3.6 together with Lemmas 3.5.7, 3.5.8, and 3.5.9, with the possible exception of the first E_6 case where the labels of the A_2 factors are equal. In the exceptional case note that since $X = A_1$, we have $N_X(T)/T = Z_2$. An element in this group which inverts T must lie in $N_G(D) - D$ so interchanges two of the A_2 factors while fixing the third. Hence $N_S(T)$ leaves invariant one of the factors. The proof of Lemma 2.3.6 shows that the projection of R to such a factor is faithful.

Assume first that R has dimension greater than 3. As R has no trivial X-submodules, the existence of a trivial composition factor of R also implies the existence of a composition factor $1^{(r)} \otimes 1^{(3r)}$ for r a power of 3. Let the T-labels of $K = A_2$ be x, y, each a multiple of 6.

Now $\dim(R) \leq 7$ and R has no trivial submodules. It follows that $R \cong 2^{(r)} \oplus 2^{(s)}$, $1^{(r)} \otimes 1^{(s)}$, $0|1^{(r)} \otimes 1^{(s)}$, or $2^{(r)} \oplus (1^{(s)} \otimes 1^{(t)})$, where r, s, t are powers of 3. Notice that if there exists a trivial quotient, then the corresponding maximal submodule is an ideal of R, as can be seen by taking commutators.

Suppose $R \cong 1^{(r)} \otimes 1^{(s)}$ or $0|1^{(r)} \otimes 1^{(s)}$. Comparing T-weights we see that the projection of R to $L(K)$ contains a pair of root vectors for opposite roots. The commutator of these vectors is a toral element. Now R has no weight 0 vectors unless $R \cong 0|1^{(r)} \otimes 1^{(s)}$. But here R' is contained in the maximal submodule. This is a contradiction as the toral element is a commutator in R.

Next assume $R \cong 2^{(r)} \oplus 2^{(s)}$. As R has a 2-dimensional weight space for weight 0, we see that one of the labels x, y must be 0. It follows that $r = s$ and the labels are $2r$ and 0. Let E be the 2-dimensional subalgebra of R spanned by T-weight vectors for positive weights. Comparing weights we see that E projects to the Lie algebra of the unipotent radical of a maximal parabolic of K. Similarly, the span F of the negative weight vectors projects to the Lie algebra of the opposite parabolic. Therefore, the projections of E, F generate the 7-dimensional algebra $L(K)/Z(L(K))$, a contradiction.

Now suppose $R \cong 2^{(r)} \oplus (1^{(s)} \otimes 1^{(t)})$, with $s < t$. Notice that the projection to $L(K)$ has weight vectors for three positive weights. It follows that these must project onto the nilpotent radical of a Borel subalgebra. In particular, two of the weight vectors commutate to yield a third. It follows that $r = s$ or $r = t$. Comparing weights we have (ii).

It remains to argue that if $R \cong 2^{(r)}$, then the labels are both $2r$, as in (i). Assume false. Then it is easy to see that weight vectors for nonzero weights project to root elements of $L(K)$. Suppose R is abelian. Let $e \in R$ be a weight vector for the positive weight $2r$. There is a unique expression $e = e_1 + e_2$, where e_1 is a root element in $L(K)$ and e_2 a nilpotent element in the product of the remaining factors of D. Viewing e_1 as an element of sl_3 it is straightforward to check that there does not exist $y \in sl_3$ for which $[e_1 y]$ is a nonzero element of the center. It follows that e_1 commutes with the preimage of the projection of R, hence $1 + e_1 \in K$ is a unipotent element of K centralizing R, a contradiction. Therefore, Corollary 2.3.3 implies that R must be simple. Then the corresponding root elements must be opposite and generate sl_2. But then the projection of R is centralized by a torus of A_2, a contradiction. ∎

At this point we can handle all cases for which D has a factor A_2.

Lemma 3.5.11 *No case in Lemma* 3.5.7, 3.5.8 *or Table* 1 *of* 3.5.9 *can occur.*

Proof For $G = F_4$ or E_6, the labellings of $D = A_2^2$ or A_2^3 are given in Lemma 3.5.7. Lemma 3.5.6 implies that in each case R has a composition factor $2^{(3)}$. Consider a factor $E = A_2$ of D, where the projection of R is faithful (see Lemma 3.5.10). If the T-labelling of E is $6\,0$, then this contradicts Lemma 3.5.10. If the labelling is $6\,12$, then Lemma 3.5.10 implies that $R \cong 2^{(9)} \oplus (1^{(3)} \otimes 1^{(9)})$, whereas R has a composition factor $2^{(3)}$. Finally, suppose the labelling is $6\,6$. By 3.5.10 we have $R \cong 2^{(3)}$ or $2^{(3)} \oplus (1^{(3)} \otimes 1^{(9)})$. In the latter case, a vector $e \in R$ of weight 12 has projection squaring to 0 in all factors of D, which yields a contradiction by Lemma 2.3.9 (together with 3.2.7). So assume $R \cong 2^{(3)}$. If R is abelian then projections to E of weight vectors for nonzero weights must be root vectors (otherwise an element of weight 6 would be a regular nilpotent element and cannot centralize the projection of elements of weight 0 or -6) and we now obtain a contradiction as at the end of the proof of Lemma 3.5.10. Otherwise, R is simple and projects faithfully to all simple factors of D (since $C_G(R) = 1$). Lemma 3.5.7 shows that some such factor has T-labels $6\,0$ or $6\,12$, so this gives a contradiction as above.

Now consider $G = E_7$, with possibilities given in Lemma 3.5.8. In cases (2),(3),(6),(8),(11),(14) and (15), Lemma 3.5.6 implies that A contains 6^2, which gives a contradiction by Lemma 3.5.10.

In cases (4) and (7) the labelling on the A_2 factor of D is $6\,0$, which is impossible by Lemma 3.5.10.

In cases (1),(9) and (10) A has 12 as a composition factor by Lemma 3.5.4(i). Then a vector $e \in R$ of weight 12 has projection squaring to 0 in both factors A_2 and A_5, so 2.3.9 and 3.2.7 give a contradiction. In case (12), Lemma 3.5.4(ii) shows that 18 is a composition factor of A and we get the same contradiction.

Now consider case (5). Here we must have the first case where the T-labelling of A_2A_5 is 6 6, 0 0 0 6 0. Hence Lemma 3.5.10 implies that $R \cong 6$ or $6 \oplus 12$. In the latter case we obtain a contradiction using 2.3.9 as above, so assume $R \cong 6$. As in the last paragraph of the proof of Lemma 3.5.10, R is simple and projects faithfully to $L(A_5)/Z(L(A_5))$, with image $R_1 = \langle e_1, h_1, f_1 \rangle$, say, where e_1, f_1 have T-weights 6, -6, respectively. In the group A_5 there is a Levi subgroup $A_3A_1T_1$ acting on the 8-dimensional space of weight 6 vectors in $L(A_5)$. This Levi subgroup has a subgroup of dimension at least 11 centralizing e_1; likewise, this centralizer has a subgroup of dimension at least 3 centralizing f_1, hence centralizing R_1. It follows that $C_G(R)$ has positive dimension, a contradiction by Lemma 2.2.10.

This leaves cases (13) and (16). In the former the proof of 3.5.8 showed that L has a fixed point, an immediate contradiction. In case (16) the A_2

factor is labelled $6\,12$, so 3.5.10 implies that $R \cong 18+12$. But Lemma 3.5.6 shows that R has 6 as a composition factor, a contradiction.

Finally, consider $G = E_8$, with $D = A_2E_6$. The possibilities are given in Table 1 of Lemma 3.5.9.

In cases (1)-(8),(10),(11),(13),(14),(16) and (17), Lemma 3.5.6 shows that A contains at least two composition factors of high weight 6, which contradicts Lemma 3.5.10.

In cases (18), (20) and (21), Lemma 3.2.3 implies that 48, 42 or 54^2 is a composition factor of A, which is impossible by Lemma 3.5.10. In cases (9) and (19), Lemma 3.2.3 shows that 30 is a composition factor of A; by Lemma 3.5.10, this means that the labelling of the A_2 factor of D is $6, 24$. Consider the labelling of the E_6 factor. The non-negative weights in $L(D)$ are respectively $0^{16}, 6^{12}, 12^{10}, 18^7, 24^3, 30^3$ and $0^{14}, 6^{11}, 12^9, 18^8, 24^4, 30^3, 36^1$. In neither case can we obtain a compatible labelling of the E_6 factor.

In the last two cases (12) and (15), the non-negative T-weights in $L(D)$ are $0^{14}, 6^{11}, 12^9, 18^9, 24^4, 30^2, 36$ and $0^{16}, 6^{13}, 12^{10}, 18^8, 24^3, 30$, and using this we check that the labelling of the A_2 factor must be $6\,12$. Hence $R \cong 18+12$ by Lemma 3.5.10. However, 3.5.6 shows that R has a composition factor 6, which is a contradiction. ■

It remains to exclude the cases in Table 2 of Lemma 3.5.9 - that is, when $G = E_8$ and $D = A_8$. Thus for the rest of this section we assume that $D = A_8$. Let C denote the sum of all X-invariant subspaces of L of type $2^{(3)}$. Then $C \leq R$ and C is ω-invariant, where ω is the semilinear map on L as given in Lemma 2.2.2.

Lemma 3.5.12 *C is a subalgebra of $A = C_L(L(X))$ unless A contains a two-step indecomposable submodule with socle* 12 *and quotient* 0*.*

Proof Suppose W and V are X-submodules of C isomorphic to $2^{(3)}$. We then get a map $W \times V \to [V, W] \leq A$ given by Lie commutation. Correspondingly, there exists a map $W \otimes V \to [V, W]$. On the other hand, we have $2 \otimes 2 \cong T(4) \oplus 2$, where $T(4)$ denotes the tilting module of high weight 4; $T(4)$ is uniserial of the form $0|4|0$. Twisting by a field morphism we obtain

$$2^{(3)} \otimes 2^{(3)} \cong (0|12|0) \oplus 2^{(3)}.$$

The conclusion follows as otherwise A would contain a trivial submodule. ■

Lemma 3.5.13 *Assume that $D = A_8$ and the largest T-weight in $L(D)$ is strictly less than* 24. *Then $R \cong 2^{(3)}$.*

Proof First suppose there is a weight vector of R with weight at least 12. The hypothesis implies that this vector corresponds to a nilpotent element of sl_9 with square 0. Hence Lemma 2.3.9 yields a contradiction. Since R has no fixed points under the action of X, we conclude that $R \cong (2^{(3)})^k$ for some $k \geq 1$. Then Lemma 2.3.7 gives the result. ■

We can now compete the proof of Theorem 3.1 for $p = 3$.

Lemma 3.5.14 *No case in Table* 2 *of Lemma 3.5.9 can occur.*

Proof In cases (1),(5),(6),(7) and (9), Lemma 3.5.6 implies that 6 occurs as a composition factor of A with multiplicity at least 2. However, the highest T-weight in $L(D)$ is less than 24, so Lemma 3.5.13 yields a contradiction. In cases (2),(14),(15) and (17), Lemma 3.5.4(ii) implies that A contains 18. Let $e \in A$ be a vector of weight 18. As the highest T-weight in $L(D)$ is less than 36, we have $e^2 = 0$. Now Lemma 2.3.9 gives a contradiction.

Now consider case (3). We claim that A has a composition factor 18 or 24 in this case. Suppose false. Let $Y = \langle Xv : v \in L_X(0)_{28} \rangle$ and let S be the radical of Y. Then $S^{\perp}/S = 28 \perp W$ where W has high weight 24. Next generate by a weight vector in W of weight 24 obtaining an image, say J, of $W(24)$. Let N be the radical of J. Then 22 must appear as a composition factor of the preimage of N, as otherwise 24 would occur as a composition factor of A, which we are assuming false. Now N is a singular space and $N^{\perp}/N = 28 \perp 24 \perp R$ where R has largest weight 18 and the composition factor 18 occurs with multiplicity 5. Generating by weight vectors for weight 18 and taking preimages over N we obtain a submodule E having a singular submodule F such that $E/F = 18^5$ and where F highest weight 22. The only irreducibles appearing in L that extend 18 are 22 and 12 and these occur with combined multiplicity 6, hence their combined multiplicity in F at most 3. It follows that 18 occurs as a submodule of E. If $e \in A$ is a vector of weight 18, then $e^2 = 0$ (as there are no vectors of weight 36), so Lemma 2.3.9 gives a contradiction.

In case (4), after first working in the usual way with two cyclic modules generated by weight vectors of weight 34 we obtain a submodule with highest composition factor 30. Then 3.2.4 implies that there is a composition factor 30 in A, and so a vector $e \in A$ of weight 30 satisfies $e^2 = 0$, giving a contradiction by 2.3.9 again.

Cases (8),(10), and (11) are based on the following fact:

$(*)$ Suppose L has a submodule $M = 22^a/18^b/16^c\cdots$, where $a \leq 1$ and $c < b - a$. Then 18 occurs as a composition factor of A.

To establish $(*)$ we first choose submodules $E \leq F \leq M$ with $F/E = 22^a$ and choose F of largest possible dimension. We claim that 18 occcurs as a composition factor of M/F with multiplicity at most 1. Indeed weight vectors of weight 18 in M/F generate images of $W(18) = 18|12|16$. Moreover, 22 does not extend 12 or 16 and $\mathrm{Ext}^1_X(22, 18)$ has dimension 1. So if the claim is false we can replace F by a larger submodule of M, a contradiction. So the claim holds and $E = 18^d/16^e/\cdots$ with $d \geq b - a$ and $e \leq c$. At this point we generate cyclic submodules of E with weight vectors of weight 18. But now our hypothesis and Lemma 3.2.4 imply that 18 occurs as a composition factor of A. This proves $(*)$.

Now consider case (8). We claim that A has a composition factor 24 or 18 here. For suppose not. Then generating with a vector of weight 24 gives a cyclic submodule $24|22|10^a$ $(a \leq 1)$. Factoring out the radical R of this and generating with a vector of weight 22 gives $22/18^b/10^c/12^d$ with $b \leq 1$. If $b = 1$, we work in the preimage of this, and generate with vectors of weight 22 to obtain a submodule $22|(18 + 10^x)|12^y$; now Lemma 3.2.4 shows that A has a composition factor 18, contrary to assumption. Hence $b = 0$.

Generate with vectors of weight 18 to see that $L \downarrow X$ has a submodule M of the form $18^3/S$ where S is singular of shape $22/16^f/12^{d+e}/\cdots$. As $f \leq 1$ the hypotheses of $(*)$ hold which yields the claim. Letting $e \in A$ be a vector of weight 18 or 24, we have $e^2 = 0$, giving a contradiction by 2.3.9.

We next consider case (10). Here we argue that A has a composition factor 18 or 30. Suppose not. Generate with vectors of weight 30, then 22, then 18. The weight 30 vector yields a submodule $30/28/22^a/18^b$ and then the 22 vectors contribute a section $22^{3-2a}/18^c/12^d\cdots$. Here $a \leq 1$ and $b+c \leq 2$ as there exists a singular subspace where 18 occurs with multiplicity $b + c$. If $b = c = 1$ then there is a submodule $M = 22^a/18^2/12^d/\cdots$ and the claim is immediate from $(*)$. So assume $b + c \leq 1$. We then generate by 18 vectors to get a section of shape $18^{4-2b-2c}/16^d/12^e$. Taking appropriate preimages we construct a submodule $M = 22^a/18^{4-b-c}/16^d\cdots$. All composition factors 16 occur within a singular subspace of M so that $d \leq 1 < 4 - a - b - c$. Once again we can apply $(*)$ to get the claim. At this point we proceed as above, using nilpotent elements of weight 18 or 30 to get a contradiction.

Essentially the same argument settles case (11) where we first claim that either 24 or 18 occurs as a composition factor of A. We then complete the argument in the usual way using 2.3.9.

Next consider case (13). Let v be a vector of weight 40. If $\langle Xv \rangle$ has a composition factor 30, then 3.5.2 and 3.2.4 show that it has a submodule lying in A with composition factor 30. Otherwise, factoring out the radical of $\langle Xv \rangle$ and generating with vectors of weight 30, we see that L has a submodule $30^2/28^a/W$ where $a \leq 1$ and where W has all composition factors less than 28. It follows from 3.2.4 that A has a composition factor 30. Now 2.3.9 gives a contradiction in the usual fashion. Cases (16) and (18) are entirely similar but easier - in each case we obtain a composition factor 30 in A. And in case (19), the same argument shows that there is a composition factor 42 in A.

It remains to deal with case (12). The proof of Lemma 3.5.9 shows that A has a composition factor of high weight $42, 30$, or 18. In either of the first two cases we can take a nilpotent argument of weight 42 or 30 and obtain a contradiction as in previous cases. So assume 18 is the largest T-weight appearing in A. Suppose the weight space of A for weight 18 has dimension at least 2. Viewing $L(D)$ as an image of sl_9 we see that vectors in this weight space square to elements of weight 36, where the corresponding weight space has dimension only 1. Taking linear combinations of two inidependent weight vectors of weight 18 we can find a weight vector with square 0 and once again we obtain a contradiction. So we now assume that 18 occurs a composition factor of A with multiplicity 1.

The cyclic submodule of A, say Y, generated by a weight vector of weight 18 is either 18 or $18|12$, and we will consider cases accordingly.

The T-labelling of $D = A_8$ is 6606606(12). Working out the 1-dimensional torus T of X viewed as a torus in SL_9 we have

$$\begin{aligned} T(c) = \quad & h_1(c^{56/3})h_2(c^{94/3})h_3(c^{114/3})h_4(c^{134/3})h_5(c^{136/3})h_6(c^{120/3})\times \\ & h_7(c^{104/3})h_8(c^{70/3}). \end{aligned}$$

This torus has (non-integral) weights on the 9-space, V, as follows:

$$56/3, 38/3, 20/3, 20/3, 2/3, -16/3, -16/3, -34/3, -70/3.$$

First suppose $Y = 18$. By the above Y is uniquely determined and hence S-invariant. Taking preimages in sl_9 we obtain a subspace $\hat{Y} < sl_9$ with a basis of T-weight vectors of weights $18, 0, 0, -18$. Weight consideratons show that $\hat{Y}$ preserves a decomposition $V = V_2 \oplus V_7$ where V_2 is the 2-space spanned by basis vectors corresponding to weights $56/3$ and $2/3$, while V_7 is spanned by the remaining basis vectors. But then there is an involution in SL_9 inducing -1 on V_2 and 1 on V_7 which centralizes $\hat{Y}$. This contradicts Lemma 2.2.10(iii).

Now suppose $Y = 18|12$ and let I be the socle. Here too Y and hence I are uniquely determined, hence S-invariant. Then Y has dimension 7 with a basis of weight vectors for weights $18, 12, 6, 0 - 6, -12, -18$ while I has dimension 4 with weights $12, 6, -6, -12$. Let $y \in Y$ be a weight vector of weight 18, the highest weight of A. Then weight considerations imply that $[yI] \leq I$ so that $y \in N_L(I)$, an X-invariant subspace. It follows that $\langle Xy \rangle = Y \leq N_L(I)$. In particular, I is a subalgebra. Say I has basis $\{a, b, c, d\}$ where these are weight vectors of weights $12, 6, -6, -12$ respectively. As there is no weight vector of weight 0 in I we must have $[bc] = 0$. Moreover, weight considerations imply that $\langle b, c \rangle$ is an (abelian) ideal of I. As I is irreducible under the action of X, an application of Lemma 2.3.2 (with I the subalgebra and J a minimal ideal contained in $\langle b, c \rangle$) shows that I is abelian.

We claim that the preimage, $\hat{I}$, of I in sl_9 stabilizes a proper subspace of V. Let $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ be the (uniquely determined) nilpotent elements of sl_9 in the preimages of a, b, c, d, respectively and let z generate the center of sl_9. Consider $\hat{B}$, the subalgebra generated by $\hat{a}$ and $\hat{d}$. First suppose that $\hat{B}$ is abelian. Then $\hat{B}$ consists of nilpotent elements so that $C = C_V(\hat{B}) \neq 0$ and C is invariant under $\hat{I}$. Now suppose that $\hat{B}$ is non-abelian, so that $z \in \hat{B}$. Then $\hat{B}$ has a basis of weight vectors for weights $12, 0, -12$ and weight considerations show that $V = V_4 \oplus V_5$ where V_4 is the 4-space spanned by the basis vectors for weights $38/3, 2/3, -34/3, -70/3$ and V_5 is spanned by the other weight vectors. But then $\hat{B}$ induces a subalgebra of sl_4 on V_4, whereas z induces a nonzero scalar, which cannot have trace 0. This is a contradiction and the claim is established.

It follows from the claim that I is contained in a maximal parabolic subalgebra of $L(D)$ corresponding to the stabilizer of a proper subspace of V. Then I induces linear transformations on the nilpotent radical of this parabolic, acting as an abelian algebra of nilpotent matrices. Hence I centralizes an element n of this nilpotent radical. Then $n^2 = 0$ and $1 + n$ is a unipotent element of SL_9 centralizing $\hat{I}$. Hence $C_D(I) > 1$, contradicting Lemma 2.2.10. ∎

This completes the proof of Theorem 3.1 in all characteristics.

4 Maximal subgroups of type A_2

In this section we prove Theorem 1 in the case where the subgroup X is of type A_2. Recall that G is an exceptional adjoint algebraic group, and G_1 is a group satisfying $G \leq G_1 \leq \mathrm{Aut}(G)$. We consider only the small characteristic cases required by Proposition 2.2.1.

Theorem 4.1 *Suppose that $X = A_2$ is maximal among proper closed connected $N_{G_1}(X)$-invariant subgroups of G. Assume further that*

(i) $C_G(X) = 1$, *and*

(ii) $p \leq 5$ *if* $G = E_7, E_8$; $p \leq 3$ *if* $G = E_6, F_4$; *and* $G \neq G_2$.

Then $G = E_7, p = 5$, and G contains a single conjugacy class of maximal subgroups A_2; these satisfy

$$L(E_7) \downarrow A_2 = V_{A_2}(11) \oplus V_{A_2}(44).$$

Suppose X, p are as in the hypothesis of the theorem, with $X = A_2$. Write $S = N_{G_1}(X)$. Then Lemma 2.2.10 shows that $C_S(X) = 1$, whence $S = X\langle\sigma, \tau\rangle$, where σ is either trivial or a Frobenius morphism of G, and τ induces either a trivial or a graph automorphism of X. By Lemma 2.2.2, σ is not an exceptional isogeny of F_4 or G_2 in case $p = 2, 3$, respectively.

Recall that $\Sigma(G), \Pi(G)$ denote the root system and a fundamental system of G. Recall also that T is the 1-dimensional torus in X defined in Definition 2.2.4 and T determines a labelling of $\Pi(G)$ by 0's and 2's (see Lemma 2.2.6). Let $\Sigma(X)$ be the root system of X, and let $\Pi(X) = \{\alpha, \beta\}$ be a fundamental system. Denote by U_γ ($\gamma \in \Sigma(X)$) the corresponding root subgroup of X, and by e_γ the corresponding root vector in $L(X)$.

By Lemma 2.2.10(v), X is of adjoint type, so that each composition factor of $L(G) \downarrow X$ has the form ab with $a \equiv b \bmod 3$, where ab denotes the irreducible X-module $V_X(a\lambda_1 + b\lambda_2)$. Set $L = L(G)'$, and define n_{ab} to be the multiplicity of ab as a composition factor of $L \downarrow X$.

The rest of this section is divided into three subsections, according as $p = 2, 3$ or 5.

4.1 The case $p = 5$

Assume $p = 5$, so that $G = E_7$ or E_8. As usual we can use the Weight Compare Programme to obtain a list of possible composition factors of $L \downarrow X$

corresponding to each possible T-labelling of the Dynkin diagram of G with 0's and 1's.

We make a few observations regarding these T-labellings. The labellings limit the possible T-weights of composition factors of $L \downarrow X$. For example if $G = E_8$, then the largest potential T-weight is 58, as this is the weight afforded by e_δ if δ is the root of highest height in $\Sigma(G)$ and the labelling is 22222222. However, in fact, the largest T-weight that can occur is 36, which is established using (2.6) and (2.7) of [31]. Such information is helpful in reducing the number of composition factors that need to be considered.

The output of the Weight Compare Program shows that the only composition factors which can appear in $L \downarrow X$ are 00, 11, 30, 03, 22, 41, 14, 60, 06, 33, 52, 25 and 44. From [23, 1.9 and 1.14], we see that of these, only 33 extends the trivial module, and $\dim \mathrm{Ext}^1_X(V(33), K) = 1$. As $C_L(X) = 0$ by Lemma 2.2.10(iii), and $L = L(G)$ is self-dual, it follows that either $n_{00} = 0$ or $n_{00} < n_{33}$, where n_λ denotes the number of composition factors of $L \downarrow X$ of high weight λ. Inspection of the list provided by the Weight Compare Program now reduces the number of possibilities for $L \downarrow X$ to three:

Lemma 4.1.1 *One of the following holds:*

(i) $G = E_7$, *the T-labelling is* 0002000, *and* $L \downarrow X = 22^3/30/03/11^7$

(ii) $G = E_7$, *the T-labelling is* 2002020, *and* $L \downarrow X = 44/11$

(iii) $G = E_8$, *the T-labelling is* 00020000, *and* $L \downarrow X = 33/60^2/06^2/22^7/11^2$.

We handle these three cases separately. The E_8 case is easy:

Lemma 4.1.2 *Case* (iii) *of Lemma* 4.1.1 *does not occur.*

Proof Suppose 4.1.1(iii) holds. For $c \in K^*$ let $T_1(c)$ be the image of the matrix $\mathrm{diag}(c, c, c^{-2})$ in $X = PSL_3(K)$, and let $T_1 < X$ be the 1-dimensional torus $\{T_1(c) : c \in K^*\}$. We calculate $\dim C_L(T_1)$ by finding $\dim C_V(T_1)$ for each composition factor V of $L \downarrow X$. This is easily done using the following information on tensor products (see [23, 2.14]):

$$60 = 10 \otimes 10^{(5)},\ \ 10 \otimes 01 = 11/00,\ \ 20 \otimes 02 = 22/11^2/00,$$
$$30 \otimes 03 = 33/22/11^2/00^2.$$

We find:

$ab:$	60	11	22	33
$\dim C_{V(ab)}(T_1):$	0	4	5	15

It follows that $\dim C_L(T_1) = 58$. Hence $C_G(T_1)$ is a Levi subgroup of $G = E_8$ of dimension 58. However, a simple check shows that there is no such Levi subgroup. ∎

Lemma 4.1.3 *There is a unique conjugacy class of maximal subgroups $X = A_2$ in $G = E_7$ ($p = 5$) with $L(E_7) \downarrow X = 44/11$, as in Lemma 4.1.1(ii).*

Proof This is proved for $p \geq 7$ in [31, 5.8], and we follow that proof closely, indicating certain special considerations required for $p = 5$. First, assuming the existence of a maximal subgroup $X = A_2$ as in Lemma 4.1.1(ii), we prove the uniqueness of $L(X)$; and finally, we show the existence of such a subgroup X.

The uniqueness part of the argument is exactly as in [31, p.82-89], where it is argued that if α, β are fundamental roots for a maximal X as in Lemma 4.1.1(ii), then after suitable conjugations, $L(X)$ is generated by root elements $e_\alpha, f_\alpha, e_\beta, f_\beta$ as given in [31, p.89].

For the existence argument, the strategy is likewise as in [31, 5.8]. Let $e_\alpha, f_\alpha, e_\beta, f_\beta$ be as in [31, p.89], and let Y be their Lie algebra span in $L(G)$. The aim is to define suitable fundamental SL_2 subgroups J_α, J_β of G and show that $\langle J_\alpha, J_\beta \rangle$ leaves Y invariant.

The argument in [31, p.89,90] shows that Y is a Lie algebra of type A_2 having basis $\{e_\alpha, e_\beta, e_{\alpha+\beta}, h_\alpha, h_\beta, f_\alpha{\cdot}f_\beta, f_{\alpha+\beta}\}$. Moreover we can choose SL_2 subgroups J_α, J_β such that $L(J_\gamma) = \langle e_\gamma, f_\gamma \rangle$ for $\gamma \in \{\alpha, \beta\}$. Let t_γ be the central involution in J_γ. It is shown that $C_G(t_\gamma) = A_1D_6$. Also we construct $J_\gamma < A_1A_4 < C_G(t_\gamma)$, with projections corresponding to the representations of high weights 1,4 on the natural modules for the factors A_1, A_4. Moreover,

$$L \downarrow A_1D_6 = L(A_1D_6) \oplus (1 \otimes \lambda_5), \text{ and } V_{D_6}(\lambda_5) \downarrow A_4 = \lambda_1 \oplus \lambda_2 \oplus \lambda_3 \oplus \lambda_4 \oplus 0^2.$$

It follows that

$$L \downarrow J_\gamma = M \oplus (1 \otimes (0^2 \oplus 4^2 \oplus (\wedge^2 4)^2)),$$

where all composition factors in M have even weights. By Lemma 2.1.7, for SL_2 we have $1 \otimes 4 = T(5)$, the indecomposable tilting module of the form $3|5|3$; moreover $\wedge^2 4$, hence also $1 \otimes \wedge^2 4$, are tilting modules, from which we see that

$$1 \otimes \wedge^2 4 = T(5) \oplus T(7),$$

where $T(7) = 1|7|1$ (see Lemma 2.1.7). Thus

$$L \downarrow J_\gamma = M \oplus 1^2 \oplus T(5)^4 \oplus T(7)^2.$$

Hence the homogeneous component of $L \downarrow J_\gamma$ corresponding to the irreducible of high weight 1 is of the form 1^4, and the same is true of $L \downarrow L(J_\gamma)$. So any subspace of this homogeneous component which is fixed by $L(J_\gamma)$ is also fixed by J_γ. In particular, if $\gamma = \alpha$, then J_α leaves invariant $\langle e_\beta, e_{\alpha+\beta}\rangle$ and $\langle f_\beta, f_{\alpha+\beta}\rangle$. These spaces generate Y as a Lie algebra, so J_α leaves Y invariant. Similarly, so does J_β.

Set $X = \langle J_\alpha, J_\beta\rangle$. We now argue as in [31, p.90-91] that $X = A_2$, $S = L(X)$, and $L(G) \downarrow X = 44 \oplus 11$.

Finally, observe that X is maximal among closed connected subgroups of G, since if $X \leq Z < G$ with Z connected, then X fixes $L(Z)$; the restriction $L(G) \downarrow X = 44 \oplus 11$ clearly forces $L(Z) = L(X)$, whence $X = Z$. ■

It remains to handle case (i) of Lemma 4.1.1. This takes a great deal more effort than the previous cases.

For $p > 5$ the corresponding case is addressed in [31, 5.7]. However, there is an error in the proof of [31, 5.7], so we present a new argument that covers the case $p > 5$ as well as $p = 5$.

Proposition 4.1.4 *Let $X = A_2, G = E_7$, and $p \geq 5$. Assume that either*

(i) *$p > 5$ and $L \downarrow X = 22^3/03/30/11^4$, or*

(ii) *$p = 5$ and $L \downarrow X = 22^3/03/30/11^7$.*

Then X is contained in a subsystem subgroup A_7 in G, and $C_G(X) \neq 1$.

The proof begins along the lines of [31, 5.7]. We present these details for completeness.

Let T_X be a maximal torus of X containing T. One checks that the T_X-weight spaces in the irreducible modules $11, 30, 03, 22$ for weight 00 have respective dimensions $2, 1, 1, 3$ if $p > 5$ and $2, 1, 1, 1$ if $p = 5$. Consequently, $C_G(T_X)$ has dimension 19. We have $C_G(T_X) \leq C_G(T)$ and from the labelling of T we see that $C_G(T) = TA_1A_2A_3$. As $C_G(T_X)$ is a maximal rank subsystem subgroup of $C_G(T)$, we conclude that $C_G(T_X) = T_4A_3, T_3A_2A_2$, or $T_2A_2A_1^3$.

Let $V = V_{\hat{G}}(\lambda_7)$, a 56-dimensional irreducible module for $\hat{G}$, the simply connected cover of G. Identifying X with its connected preimage in $\hat{G}$, we can consider X acting on V. Note that $\lambda_7 = \frac{1}{2}(2346543)$ when expressed in terms of fundamental roots. Subtracting roots and using the fact that the T-labelling is 0002000, we find all T-weights on V, from which we determine that $V \downarrow X = 11^2 \oplus 30^2 \oplus 03^2$.

Now $C_{\hat{G}}(T_X)$ acts on each weight space of T_X on V and from the previous paragraph we see that weight spaces for nonzero weights have dimension 2 or 6, whereas the 0-weight space has dimension 8. If $C_{\hat{G}}(T_X) = T_3A_2A_2$, we see from [23, 2.3] that $V \downarrow A_2A_2$ has a 9-dimensional direct summand which is the tensor product of 3-dimensional modules for the A_2 factors. This is not consistent with the above information on weight spaces. Suppose $C_G(T_X) = T_4A_3$. Here we use [23, p.106] to see that $V \downarrow A_3 = 100^4 \oplus 001^4 \oplus 010^2 \oplus 000^{12}$. Now A_3 must act trivially on each weight space of dimension 2 and there are six of these. This accounts for all fixed points of A_3. However, there are 6 weight spaces with dimension 6, so A_3 must have fixed points on some of these spaces. This is inconsistent with the above expression for $V \downarrow A_3$.

It follows from the above paragraph that we must have $C_G(T_X) = T_2A_2A_1^3 = T_XA_2A_1^3$. In particular, viewing $T_X < C_G(T) = TA_1A_2A_3$, we have $T_X < TA_3$.

Let D be the subgroup generated by the root groups $U_{\pm\beta}$ for β in the subsystem generated by the roots

$$0001000,\ 0000100,\ 0111000,\ 1000000,\ 0011110,\ 0000001,\ 0101110.$$

Then $D = A_7$, and the given roots form a fundamental system for D. Let Y be a subgroup A_2 of D with embedding given by the adjoint representation. Our ultimate aim is to show that X is G-conjugate to Y, which will establish the proposition.

Let $\bar{T}$ be a 1-dimensional torus in an SO_3 subgroup of $Y$and let T_Y be a maximal torus of Y containing $\bar{T}$. A check of the $\bar{T}$-weights on the usual module for D (actually we must use a covering group of D) shows that $\bar{T}$ determines the labelling 2020202 of the Dynkin diagram of D.

By [23, 2.1] we see that $L \downarrow D = L(D) \oplus V_D(\lambda_4)$. Using the $\bar{T}$-labels of D we determine all weights on $L(G)$ and find that these are precisely the same as those of T. Thus by Lemma 2.2.8, $\bar{T}$ determines the same labelled Dynkin diagram as T, and we may conjugate X by an element of G to conclude $T = \bar{T}$. We also note that [23, p. 102] shows that Y has precisely the same composition factors on L, including multiplicities, as does X.

Now $T_X, T_Y < C_G(T) = TA_1A_2A_3$ and by the above we in fact have $T_X, T_Y < TA_3$. Hence each of T_X, T_Y has centralizer in TA_3 isomorphic to $T_2A_1A_1$. Thus conjugating by an element of A_3 we may assume $T_X = T_Y$.

Let $\Pi(X) = \{\alpha, \beta\}$, and let T_α, T_β be corresponding 1-dimensional tori in X (so in matrix form T_α, T_β consist of matrices $T_\alpha(c) = \mathrm{diag}(c, c^{-1}, 1)$, $T_\beta(c) = \mathrm{diag}(1, c, c^{-1})$). Then T consists of matrices $T(c) = T_\alpha(c^2)T_\beta(c^2)$.

We then have $T_X = T_\alpha T_\beta$. Similarly, setting $\Pi(Y) = \{\gamma, \delta\}$ we have tori $T_X = T_Y = T_\gamma T_\delta$. A direct calculation using the action of X, Y on V shows that each of the tori $T_\alpha, T_\beta, T_\gamma, T_\delta$ have weight decomposition: $(\pm 1)^{12}, (\pm 2)^6, (\pm 3)^4, 0^{12}$. Writing $T_\gamma(c) = T_\alpha(c^r)T_\beta(c^s)$ and using the known action of T_α, T_β on V, we conclude that $T_\gamma \in \{T_\alpha, T_\beta, T_{\alpha+\beta}\}$. Similarly, for T_δ. Now $N_G(T_X)$ induces S_3 on $\{T_\alpha, T_\beta, T_{\alpha+\beta}\}$, so conjugating, if necessary, we may now assume that $T_\alpha = T_\gamma$ and $T_\beta = T_\delta$. Indeed, replacing $\Pi(Y)$ by $-\Pi(Y)$, if necessary, we may assume that $T_\alpha(c) = T_\gamma(c)$ and $T_\beta(c) = T_\delta(c)$ for all $0 \neq c \in K$.

Define a further 1-dimensional torus $R < X$ to consist of the matrices $R(c) = T(c)T_\alpha(c^{-1}) = \mathrm{diag}(c, c, c^{-2})$. Then $R = C_X(J_\alpha)$, where J_α is the fundamental SL_2 in X corresponding to α. This torus plays a similar role in Y.

We will need the labellings of the Dynkin diagram of G afforded by R and T_α. For this we work with the embedding of Y in D. This embedding is given via the adjoint representation of Y where we take as basis

$$\{e_{\gamma+\delta}, e_\gamma, -e_\delta, -h_\gamma, h_\delta, e_{-\delta}, e_{-\gamma}, -e_{-\gamma-\delta}\}.$$

In this basis R has weights $3, 0, 3, 0, 0, -3, 0, -3$ and $T_\alpha = T_\gamma$ has weights $1, 2, -1, 0, 0, 1, -2, -1$ from which we determine the corresponding labellings of the Dynkin diagram of D. Now D has semisimple rank 7 and it is an easy matter to use use these labellings to determine the precise labellings of the Dynkin diagram of G. We find that R, T_α determine labellings as follows

$$R : 0003(-3)3(-3)$$

$$T_\alpha : 000(-1)3(-3)3.$$

From this we find that $C = C_G(R)' = A_1D_5$, with $\Pi(C) = \{\alpha_5 + \alpha_6\} \cup \{\alpha_1, \alpha_3, \alpha_4+\alpha_5, \alpha_2, \alpha_6+\alpha_7\}$ (where $\Pi(G) = \{\alpha_1, \ldots, \alpha_7\}$). Of course $J_\alpha < C$ and from the above labelling of T_α we see that T_α centralizes the A_1 factor of C.

Using the T_α-labelling of the Dynkin diagram of G we see that T_α determines a labelling of the D_5 Dynkin diagram where all labels are 0 except for a 2 over the triality node. It follows that $J_\alpha < D_5$ and J_α acts as $2 \oplus 2 \oplus 2 \oplus 0$ on the natural 10-dimensional D_5-module. In particular, $C_{D_5}(J_\alpha) = F$ is of type A_1 (one of the factors in an $SO_3 \otimes SO_3$ subgroup).

The above analysis also applies to $J_\gamma < C_G(R)$. Now J_γ and J_α share the torus T_α, so conjugating within D_5 by an element centralizing T_α we may assume that $J_\alpha = J_\gamma$. Notice that the conjugation also centralizes R and hence T_X. A consideration of weights shows that $U_\alpha = U_\gamma$ and $U_{-\alpha} = U_{-\gamma}$ So we may assume the corresponding root vectors are equal; that is $e_\alpha = e_\gamma$ and $f_\alpha = f_\gamma$.

We will require a precise expression for e_α. We take the basis of $L(Y)$ given earlier. Choose signs so that $[e_\gamma e_\delta] = e_{\gamma+\delta}$. This determines the embedding of $L(Y)$ into sl_8. Next we choose an isogeny of SL_8 to D for which the differential sends the usual generating set of elementary matrix units above and below the main diagonal to the corresponding elements e_μ and f_μ, respectively, where μ is among the positive roots defining D.

In this way we get expressions for e_α in terms of the usual basis for $L(D)$. We use signs for commutators among the root vectors of $L(G)$ as given in the E_7 table of [13, p.416]. With this convention there are differences in signs between Lie brackets of the usual generators of sl_8 and those given in [13] for the base of D. Taking this into account we obtain

$$e_\alpha = e_\gamma = -e_{0001100} - 2e_{0111100} + e_{1111100} + e_{1011111} - e_{0101111}.$$

At this point we proceed in a series of lemmas. We summarize notation as follows. As above $R < X$ is the 1-dimensional torus with $C_X(R) = RJ_\alpha$ and $C_G(R) = RD_5A_1$. This last group is a Levi factor of a parabolic subgroup P with unipotent radical Q, where $L(Q)$ is the sum of all weight spaces of R for positive weights. Further, $J_\alpha < D_5$ and $C_{D_5}(J_\alpha) = F$, where on the usual orthogonal module for D_5, $J_\alpha F$ acts as the sum of a trivial module and $2 \otimes 2$.

Let L_3 denote the R-weight space of L for weight 3. Note that $L(Q) = L_3 \oplus L(Q')$. We have $e_\beta, e_\delta \in L_3$. In the next few lemmas we analyse the action of $RJ_\alpha FA_1$ on L_3, ultimately showing that e_β and e_δ must be conjugate under RFA_1 (see Lemma 4.1.10).

Lemma 4.1.5 $L_3 \downarrow J_\alpha FA_1 = (3 \otimes 1 \otimes 1) \oplus (1 \otimes 3 \otimes 1)$.

Proof To see this first note that $L_3 \downarrow D_5A_1$ is a spin module for D_5 tensored with a natural module for the A_1. Restricting the spin module to $J_\alpha F$ and using [23, 2.13] we have the assertion. ∎

Let Y denote the summand $1 \otimes 3 \otimes 1$ given in the last lemma. Also, let $\hat{Y}$ denote the sum of the T_α-weight spaces of L_3 for weights $1, -1$.

Lemma 4.1.6 (i) $Y < \hat{Y}$, $\dim \hat{Y} = 24$ *and* $\dim Y = 16$.

(ii) *There is an F-invariant decomposition $Y = Y^+ \oplus Y^-$ such that $Y^+ = [e_\alpha, Y^-]$.*

(iii) *There is an F-invariant decomposition $\hat{Y} = \hat{Y}^+ \oplus \hat{Y}^-$ such that $\hat{Y}^+ = [e_\alpha, \hat{Y}^-]$.*

Proof Part (i) is clear from Lemma 4.1.5. Then (ii) and (iii) follow by decomposing Y and $\hat{Y}$ with respect to T_α-weight spaces corresponding to weights $1, -1$, respectively. ■

Lemma 4.1.7 (i) *For $v \in \hat{Y}^-$, there is an expression*

$$\begin{aligned} v = \; & a_1 e_{0001000} + a_2 e_{0101000} + a_3 e_{0011000} + a_4 e_{0111000} + \\ & a_5 e_{1011000} + a_6 e_{1111000} + a_7 e_{0001110} + a_8 e_{0101110} + \\ & a_9 e_{0011110} + a_{10} e_{0111110} + a_{11} e_{1011110} + a_{12} e_{1111110}. \end{aligned}$$

(ii) *If also $v \in Y$, then*

$$a_1 = -a_4, \;\; a_7 = -a_{10}, \;\; a_2 = a_3 + 2a_5, \;\; a_8 = a_9 + 2a_{11}.$$

Proof The first expression is obtained simply by listing all root vectors which afford R-weight 3 and T_α-weight -1 and then writing v as a linear combination of these root vectors.

For (ii) we take $v \in Y$ and use the relation $[[e_\alpha v]e_\alpha] = 0$. Calculation gives

$$\begin{aligned} [e_\alpha v] = \; & (a_4 + 2a_1)e_{0112100} + (a_6 - a_1)e_{1112100} + (2a_7 + a_{10})e_{0112210} \\ & +(a_{12} - a_7)e_{1112210} - (a_3 + 2a_5)e_{1122100} - (a_9 + 2a_{11})e_{1122210} \\ & +(a_2 - a_5)e_{1112111} + a_4 e_{1122111} + (a_8 - a_{11})e_{1112221} \\ & +a_{10} e_{1122221} - a_3 e_{0112111} - a_9 e_{0112221}. \end{aligned}$$

At this point a further calculation yields

$$\begin{aligned} 0 = [[e_\alpha v]e_\alpha] = \; & 2(a_4 + a_1)e_{1123211} - 2(a_7 + a_{10})e_{1123321} \\ & +2(a_3 - a_2 + 2a_5)e_{1223211} + 2(a_8 - a_9 - 2a_{11})e_{1223321}, \end{aligned}$$

which gives the assertion. ■

Lemma 4.1.8 *Define $\phi : \hat{Y}^- \to L(Q)'$ by $\phi(v) = [v, [e_\alpha v]]$. Then*

(i) $\phi(\hat{Y}^-) = \langle e_{1123210}, e_{1223210}, e_{1123221}, e_{1223221} \rangle$.

(ii) *$\phi(\hat{Y}^-)$ is an (FA_1)-invariant subspace of $L(Q)'$ on which A_1 acts trivially and F acts as $2 \oplus 0$.*

(iii) *$\phi(Y^-)$ is 1-dimensional, affording a trivial module for FA_1.*

Proof (i) This involves a direct calculation using the E_7-structure constants presented in [13, p.416]. We begin with v as in Lemma 4.1.7(i). We then compute $[e_\alpha v]$, obtaining the expression as in the proof of 4.1.7, and then $[[e_\alpha v]v]$. We find that $\phi(v)$ is a linear combination of the indicated vectors in (i), with coefficients as follows:

$$\begin{aligned}
e_{1123210}: \quad & -a_1(a_9+2a_{11})-a_3(a_{12}-a_7)+a_5(2a_7+a_{10}) \\
& +a_7(a_3+2a_5)+a_9(a_6-a_1)-a_{11}(a_4+2a_1) \\[1ex]
e_{1223210}: \quad & -a_2(a_9+2a_{11})-a_4(a_{12}-a_7)+a_6(2a_7+a_{10}) \\
& +a_8(a_3+2a_5)+a_{10}(a_6-a_1)-a_{12}(a_4+2a_1) \\[1ex]
e_{1123221}: \quad & a_1a_{10}-a_3(a_8-a_{11})-a_5a_9-a_7a_4+a_9(a_2-a_5)+a_{11}a_3 \\[1ex]
e_{1223221}: \quad & a_2a_{10}-a_4(a_8-a_{11})-a_6a_9-a_8a_4+a_{10}(a_2-a_5)+a_{12}a_3.
\end{aligned}$$

By choosing appropriate $a_i, a_j \neq 0$ and taking all others equal to 0 we easily check that $\phi(\hat{Y}^-)$ is the 4-space indicated in (i).

For (ii) first note that that $L(Q)'$ affords the natural module for the D_5 factor of $C_G(R) = RD_5A_1$ and the trivial module for A_1. Hence F acts on $L(Q)'$ as the sum of three adjoint modules and a trivial module. Consequently, any F-invariant 4-space of $L(Q)'$ satisfies the conclusion of (ii). So (ii) will follow if we can show that $\phi(\hat{Y}^-)$ is FA_1-invariant. If $x \in FA_1$, then we know that $xe_\alpha = e_\alpha$. So it is immediate from the definition of ϕ and the fact that FA_1 preserves the Lie bracket on L, that $\phi(xv) = \phi(v)$ for $v \in \hat{Y}^-$.

Finally, consider $\phi(Y^-)$. Since Y^- is F-invariant, the above argument shows that its image under ϕ is also F-invariant. For $v \in Y^-$ the conditions in Lemma 4.1.7(ii) hold. Using these relations one checks that the above coefficients of $e_{1223210}$ and $e_{1123221}$ are both 0. So from (ii) it follows that $\phi(Y^-)$ is either a 1-space or 0. To complete the proof we note that setting $a_2 = a_3 = a_{12} = 1$ and all other $a_i = 0$, the conditions of 4.1.7(ii) are satisfied and $\phi(v) = -e_{1123210} + e_{1223221}$. Hence (iii) holds. ■

Lemma 4.1.9 *Regarding $\phi(Y^-) = K^+$, the map $v \to \phi(v)$ is an FA_1-invariant quadratic form on Y^-.*

Proof In the penultimate paragraph of the proof of Lemma 4.1.8, we verified that $\phi(xv) = \phi(v)$ for $x \in FA_1, v \in Y^-$. So it will suffice to show that ϕ is a quadratic form. For $y \in Y^-$ set $y^* = [e_\alpha y]$. Then for $v, w \in Y^-$

and $a, b \in K$ we have

$$\begin{aligned}\phi(av+bw) = \; & [(av+bw)(av+bw)^*] = [(av+bw)(av^*+bw^*)] \\ & = a^2[vv^*] + b^2[bb^*] + ab[vw^*] + ab[wv^*] \\ & = a^2\phi(v) + b^2\phi(w) + ab(v,w),\end{aligned}$$

where $(v,w) = [vw^*] + [wv^*]$. Notice that this last expression is symmetric in v, w and is also bilinear. So this establishes the lemma. ■

We can now establish a key lemma.

Lemma 4.1.10 *e_β and e_δ are conjugate under the action of RFA_1.*

Proof We consider the action of RFA_1 on the space Y^-. Observe that $e_\beta, e_\delta \in Y^-$. We have just seen that FA_1 preserves the quadratic form ϕ on this space and we know that R induces scalars. We next observe that working within the root systems of our A_2 subgroups X and Y we certainly have $\phi(e_\delta) = \phi(e_\beta) = 0$. Hence, e_δ, e_β are singular vectors with respect to this quadratic form.

Now R does not preserve the form, but it does preserve the variety of singular vectors, hence RFA_1 acts on this variety. Let $v = e_\beta$ (resp. e_δ), and suppose that $C = C_{RFA_1}(v)$ has positive dimension. Then v centralizes $L(C)$. Also, $J_\alpha R$ centralizes C and $\langle L(J_\alpha R), v\rangle = L(P_X)$ (resp. $L(P_Y)$), the Lie algebra of a maximal parabolic of X (resp. Y). So $L(P_X)$ (resp. $L(P_Y)$) has fixed points on L. On the other hand, $L \downarrow X = 22^3/03/30/11^4$ (or $22^3/03/30/11^7$ if $p=5$), so that no composition factor has such a fixed point. Similarly for $L(P_Y)$. This is a contradiction, showing that $C^0 = 1$.

The variety of singular vectors in Y^- has dimension 7. It is also an irreducible variety as SO_8 acts transitively. It follows from the above that RFA_1 has an open dense orbit and both e_β, e_δ lie in this orbit, so this establishes the lemma. ■

Lemma 4.1.11 *There is a proper subspace of L left invariant by both X and Y.*

Proof We will consider weight spaces in L for $L(T_X) = L(T_Y)$. We have seen that weights are the same for T_X and T_Y, so there is no ambiguity in this.

By Lemma 4.1.10 we can conjugate by an element of RFA_1 to assume that $e_\beta = e_\delta$. Hence we may assume that $L(P_X) = L(P_Y)$, where as in the last lemma these are the Lie algebras of maximal parabolic subgroups of X, Y respectively.

Of particular interest is the weight space for weight -03, the low weight in an irreducible module of high weight 30. The only composition factors of $L \downarrow X$ containing this weight are 30 and 22. Hence the -03 weight space, say E, has dimension 4.

For the moment we work with X. If $p > 5$ then L has an X-submodule, say $L_0(X)$, of the form $22 \oplus 22 \oplus 22 \oplus 30$ (see [23, 1.9]). And if $p = 5$ then $W(22) = 22/11$ and we see that there is a submodule, which we again call $L_0(X)$, having a homogeneous submodule U of type 11 satisfying $L_0(X)/U = 22 \oplus 22 \oplus 22 \oplus 30$.

In either case $E \subseteq L_0(X)$, and we can choose a basis v_1, v_3, v_3, v_4 of E such that for $p > 5$ each v_i belongs to one of the summands of $L_0(X)$, and for $p = 5$ this is true in $L_0(X)/U$.

Now take v to be any nonzero linear combination of the v_i's and consider the subspace $S_v(X) = \langle L(P_X)v \rangle$. Clearly $S_v(X)$ is invariant under $L(P_X)$ and lies in $L_0(X)$. Consider the projections of $S_v(X)$ to the direct summands in $L_0(X)$ or $L_0(X)/U$. As $L(P_X)$ contains a Borel subalgebra containing $L(T_X)$, each nonzero projection contains an invariant 1-space stabilized by this Borel subalgebra. Such 1-spaces are uniquely determined in the projection and afford the high weight of the summand, either 22 or 30.

It follows that there is a uniquely determined 1-space, say $\langle v \rangle < E$, with the property that for all $v' \in \langle v \rangle$, $S_{v'}(X)$ contains no weight vector of weight 22. This conclusion holds whether or not $p = 5$. Since $L(P_X) = L(P_Y)$, we are led to the same subspace $\langle v \rangle$ whether we are working with X or Y.

Note that both $L \downarrow X$ and $L \downarrow Y$ have a unique direct summand of type 30. These summands are also direct summands of $L_0(X)$ and $L_0(Y)$, respectively, and each contains the weight vector v. From the representation theory of $L(X)$ and $L(Y)$ we see that $S_v(X)$ and $S_v(Y)$ must equal the irreducible summand 30. However, $S_v(X) = S_v(Y)$, so this establishes the lemma. ■

At this point we can establish Proposition 4.1.4. Indeed, the proof of Lemma 4.1.11 shows that the subspace constructed is S-invariant. Hence Lemma 2.2.10(iii), implies that $X = Y$. Moreover, Y was chosen to lie in a subsystem group of type A_7. This subsystem group has nontrivial center, so the proof is complete.

This completes the proof of Theorem 4.1 for $p = 5$.

4.2 The case $p = 3$

Suppose that $p = 3$. Recall that $L = L(G)'$, which is equal to $L(G)$ except when $G = E_6$, in which case L has dimension 77 (see Lemma 2.1.1).

Let $X = A_2$ be maximal S-invariant in G with $p = 3$. Recall that by Lemma 2.2.10(v), X is of adjoint type so that all composition factors of $L \downarrow X$ have high weight ab with $a \equiv b \bmod 3$. Let n_λ denote the multiplicity of $V_X(\lambda)$ as a composition factor of $L \downarrow X$.

Let $\Pi(X) = \{\alpha, \beta\}$ be a fundamental root system for X, and for $\gamma \in \Sigma(X)^+$, the positive roots in the root system for X, let e_γ be the corresponding root vector in $L(X)$ and $f_\gamma = e_{-\gamma}$.

Define $I = L(X)'$. As in the proof of Lemma 2.1.1, we have $\dim I = 7$, and as an X-module I affords the irreducible $V_X(11)$. Recall our definition from Section 2.2 that $A = C_L(I)$. As before we let T_1 denote the 1-dimensional torus consisting of the images of the diagonal matrices $T_1(c) = \mathrm{diag}(c, c, c^{-2})$.

We begin with a lemma giving the composition factors of various Weyl modules for A_2 (in characteristic 3).

Lemma 4.2.1 *For $X = A_2$, $p = 3$, the Weyl modules $W_X(ab)$ have the following composition factors.*

$$
\begin{aligned}
W_X(11) &= 11/00 \\
W_X(30) &= 30/11 \\
W_X(22) &= 22 \\
W_X(41) &= 41/30/03/11/00 \\
W_X(33) &= 33/41/14/30/03/11/00^2 \\
W_X(60) &= 60/41/00 \\
W_X(52) &= 52 \\
W_X(44) &= 44/60/06/33/41/14/30/03/11/00^2
\end{aligned}
$$

Proof The composition factors can be found using either the computer program described in [13] or the Jantzen Sum Formula. ■

More precise information will be required for the structure of $W_X(41)$ and $W_X(33)$.

Lemma 4.2.2 *Assume $X = A_2$, with $p = 3$.*

(i) $W_X(33)$ *has simple socle with high weight* 11.

(ii) $W_X(14)$ *and* $W_X(41)$ *have simple socle with high weight* 11.

(iii) $W_X(14)$ *and* $W_X(41)$ *both embed in* $W_X(33)$.

(iv) *The maximal submodule of* $W_X(33)$ *itself has a unique maximal submodule with simple quotient* 00.

Proof The previous lemma gives the composition factors of all these Weyl modules. We work within $W_X(33)$. If v is a maximal vector, then this Weyl module is spanned by images of vectors of the form $f_{\alpha+\beta}^c f_\beta^b f_\alpha^a v$.

Weight spaces in the Weyl module $W_X(33)$ have the same dimension as in the correspondng irreducible module in characteristic 0. In particular, the weight spaces for weights $03, 30, 11$ have dimensions $2, 2, 3$ respectively. It follows that bases for these weight spaces are as follows:

$03 : f_{\alpha+\beta} f_\alpha v, f_\beta f_\alpha^2 v$
$30 : f_\beta^2 f_\alpha v, f_{\alpha+\beta} f_\beta v$
$11 : f_{\alpha+\beta}^2 v, f_\beta^2 f_\alpha^2 v, f_{\alpha+\beta} f_\beta f_\alpha v.$

The module $Y = \langle X f_\alpha v\rangle$ is an image of the Weyl module of high weight 14. The irreducible module $V_X(14)$ has dominant weights $03, 30, 11$ appearing with respective multiplicities $1, 0, 1$. It follows from the above that Y must have composition factors of high weight $03, 30$ and 11. Since the T_X-weight space of Y for weight 30 is spanned by $w = f_\beta^2 f_\alpha v$, and since 30 is not subdominant to an other weight in the maximal submodule of Y, we see that $\langle Xw\rangle$ spans an image of $W_X(30)$. Also, since we know $f_\alpha w = f_\alpha f_\beta^2 f_\alpha v = f_\beta f_\alpha f_\beta f_\alpha v + f_{\alpha+\beta} f_\beta f_\alpha v = f_\beta^2 f_\alpha^2 v + f_\beta f_{\alpha+\beta} f_\alpha v + f_{\alpha+\beta} f_\beta f_\alpha \neq 0$, we conclude that $\langle Xw\rangle = W_X(30)$.

It follows from the above and symmetry (interchanging the roles of α and β or applying a graph automorphism to all considerations) that the socle of $W_X(33)$ cannot contain composition factors of high weights $41, 14, 30, 03$.

We also claim that 00 is not present in the socle. Let $T(11) = 00|11|00$ be the indecomposable tilting X-module of high weight 11. Then $T(11) \otimes 22$ is a tilting module of high weight 33. Since $W_X(33)$ is a subquotient of $T_X(33)$, we see from the universal property of Weyl modules that $W_X(33)$ occurs as a submodule of this tensor product. On the other hand, $\mathrm{Hom}_X(00, T(11) \otimes 22) \cong \mathrm{Hom}_X(00 \otimes T(11), 22) = \mathrm{Hom}_X(T(11), 22) = 0$. Hence, 00 does not occur as a submodule of $T(11) \otimes 22$ and hence not as a submodule of $W_X(33)$ either. This proves (i).

We now return to consideration of Y, aiming to prove (ii). We have seen that $14, 30, 03, 11$ all occur as composition factors of Y, which is an image of $W_X(14)$. From (i) we see that the socle of Y is 11. So it suffices to show that Y is isomorphic to $W_X(14)$, which will also establish (iii). Assume this is not the case. Then Y is isomorphic to the quotient of $W_X(14)$ by a trivial module. Let $Y' = \langle X f_\beta v\rangle$ denote the submodule generated by a vector of weight 41. Then Y' is the image of Y under the action of a graph automorphism of A_2, so that Y' is the quotient of $W_X(41)$ by a trivial module. Then $W_X(33)/(Y+Y') \cong 33/00^2$. However, any extension of 33 by 00 factors through a Frobenius morphism and hence $\dim \mathrm{Ext}_X(00,33) = \dim \mathrm{Ext}_X(00,11) = 1$. This implies that $W_X(33)$ has a quotient 00, a contradiction to the universal property of Weyl modules.

Finally, consider (iv). We have seen that the composition factors 30, 03, 11, 00 all occur within the submodule generated by a weight vector of weight 14 and similarly for 41. Now 00 occurs with multiplicity 2 in $W_X(33)$. If there is a quotient of type $33/14$, then there would also be one of type $33/41$. But then there would be a submodule with composition factors $30/03/11/00^2$, leading to a submodule 00 and contradicting (i). So there are no such images and (iv) follows. ■

Corollary 4.2.3 *We have the following co-socle series:*

$$\begin{aligned} W_X(41) &= 41|(30+03+00)|11 \\ W_X(33) &= 33|00|(41+14)|(30+03+00)|11. \end{aligned}$$

Proof The composition factors are given by 4.2.1, from which can be deduced those pairs of such composition factors between which there exists a nontrivial extension. The conclusion follows from this information together with the universal property of Weyl modules. ■

In the next lemma, $T(11)$ denotes the indecomposable tilting module for X of high weight 11. This is a uniserial module with series $00|11|00$.

Lemma 4.2.4 (i) *An X-module of shape $W(11)|11$ is isomorphic to $W(11)\oplus 11$.*

(ii) *A module of shape $T(11)|11$ is isomorphic to $T(11)\oplus 11$.*

(iii) *A module of shape $W(11)|14$ or $W(11)|41$ has a 00 submodule.*

(iv) *A module of shape $T(11)|14$ or $T(11)|41$ has a 00 submodule.*

Proof (i) Let v be a weight vector of weight 11 not contained in the given 11 submodule. Then the hypothesis implies that $\langle Xv\rangle$ is a cyclic module isomorphic to $W(11)$ which gives the conclusion.

(ii) Consider a module $B = T(11)|11$. Since $T(11)$ has a submodule $W(11)$ we see from (i) that B has a trivial submodule. Working modulo this trivial module and noting that $\dim \mathrm{Ext}_X(11, 00) = 1$ we see that B has a 11 quotient. But then the kernel is isomorphic to $T(11)$ and the required decomposition is the sum of this $T(11)$ and the given 11 submodule.

(iii) It will suffice to settle the case of $J = W(11)|14$. Consider the dual module, $J^* = 41|(00|11)$. Let $v \in J^*$ be a maximal vector of weight 41 and generate $C = \langle Xv\rangle$ to get a cyclic module which is an image of $W(41)$. By Lemma 4.2.2, 11 is the socle of $W(41)$, so it follows from Lemma 4.2.1 that the cyclic module C cannot contain the 11 submodule. Hence C must be irreducible, so $C = 41$ and J^* has a 00 quotient module. Taking duals we have the assertion.

(iv) Let J be the module in question and consider $J^* = 41|T(11)$. Let $v \in J^*$ be a weight vector of weight 41 and form the cyclic module $F = \langle Xv\rangle$, which is an image of the Weyl module $W(41)$. Since the socle of $W_X(41)$ is 11, we see that F cannot contain 11 as composition factor (otherwise 00 would occur as a submodule). Then $F = 41$ or $41|00$ and so J^*/F has quotient 00. Hence, so does J^* and taking duals we have the assertion. ∎

The next lemma is the A_2-analogue of Lemma 3.2.3.

Lemma 4.2.5 *Suppose that V is an X-invariant submodule of L for which the largest T-weight among composition factors is afforded by irreducibles of high weight $(ab)^{(p)}$ and $(ba)^{(p)}$ for some a, b, and only by these irreducibles. Then either $A = C_L(I)$ has $V_X(ab)^{(p)}$ as a composition factor, or $n_{pa-2,pb+1} > 0$, or $n_{pa+1,pb-2} > 0$.*

Proof Let $v \in L$ be a maximal vector of weight $(ab)^{(p)}$. Then e_α, e_β annihilate v. If also f_α, f_β annihilate v then $v \in A$, and so A has $V(ab)^{(p)}$ as a composition factor. Otherwise, either $w = f_\alpha v$ or $w = f_\beta v$ is nonzero and has weight $(ab)^{(p)} - \alpha$ or $(ab)^{(p)} - \beta$, respectively. However, this is not a weight of $V_X(ab)^{(p)}$ or $V_X(ba)^{(p)}$ and by our hypothesis on maximality of T-weights we see that this weight cannot occur within a composition factor of high weight other than that afforded by w. It follows w is fixed by U_X and so the universal property of Weyl modules implies that $L \downarrow X$ has a composition factor of high weight equal to that of w. The conclusion follows, as this weight is $pa-2, pb+1$ or $pa+1, pb-2$. ■

As usual we shall use the Weight Compare Program to determine the list of possible composition factors of $L \downarrow X$. For this the following two lemmas are useful.

Lemma 4.2.6 *If the highest T-weight on L is at most* 8, *then $n_{00} \leq 2n_{30}$ and $2n_{00} \leq n_{11} + n_{30}$.*

Proof By hypothesis the composition factors of $L \downarrow X$ are among 22, 30, 03, 11 and 00. By Lemma 4.2.1 the relevant Weyl modules are

$$W_X(22) = 22,\ W_X(30) = 30|11,\ W_X(11) = 11|00.$$

Thus the 22 composition factors form a non-degenerate subspace V_0, and $V = V_0^{\perp}$ has no 22 composition factors. Let $V_1 = \langle Xv : v \in V\rangle$, where v is a maximal vector of weight 30. Then as an X-module $V_1 = 30^{n_{30}}/11^b$ with $b \leq n_{30}$. Using Lemma 2.1.5(ii) and (iii) we see that V_1 is totally singular. Now work in $V_2 = (V \cap V_1^{\perp})/V_1$. As an X-module, $V_2 = 11^{n_{11}-2b}|00^{n_{00}}$. Generating in V_2 with maximal vectors of weight 11 gives a submodule $11^{n_{11}-2b}|00^c$ of V_2, where $c \leq n_{11} - 2b$, and in V_2 we have $(00^c)^{\perp}/00^c = 11^{n_{11}-2b} \oplus 00^{n_{00}-2c}$. Therefore V_2 has a submodule $00^{n_{00}-c}$. Since $C_L(X) = 0$ by Lemma 2.2.10(iv), this must be blocked by the submodule 11^b of V_1, and so $n_{00} - c \leq b$.

From the above we have established the following inequalities:

$$n_{00} - c \leq b,\ \ c \leq n_{11} - 2b,\ \ 2c \leq n_{00},\ \ b \leq n_{30}.$$

Hence

$$n_{11} + n_{30} \geq 3b + c \geq 3n_{00} - 2c \geq 2n_{00},$$

and

$$n_{30} \geq b \geq n_{00} - c \geq n_{00}/2,$$

as required. ■

Lemma 4.2.7 *Assume all composition factors ab of $L \downarrow X$ satisfy $a+b \leq 9$ and $ab \neq 17, 71$. Then either $n_{00} = 0$, or $n_{00} < n_{11} + n_{41} + n_{14} + n_{33}$.*

Proof We shall show that only irreducible X-modules ab, with $a \equiv b \bmod 3$ satisfying the hypothesis and for which $\mathrm{Ext}_X(ab, 00) \neq 0$ are 11, 41, 14 and

33, and for each of these $\mathrm{Ext}_X(ab, 00)$ is 1-dimensional. This will give the conclusion. Lemmas 4.2.1 and 4.2.2, together with the universal property of Weyl modules, cover all cases except $60, 06, 90, 09, 63, 36$ and 44. In each of these cases except 44, we see that the irreducible in question cannot extend the trivial module, since otherwise so would the corresponding restricted irreducible module $20, 02, 10, 01, 21, 12$, and this is not the case (see [23, 1.9]). Finally, for the 44 case we have

$$\mathrm{Ext}_X(44, 00) = \mathrm{Ext}_X(33 \otimes 11, 00) \cong \mathrm{Ext}_X(33, 11),$$

which is 0 by Lemma 4.2.2(i). ∎

Using the Weight Compare Program, together with the previous two lemmas and the fact that $n_{11} > 0$ (as $L(X) \leq L(G)$), we obtain the following list of possibilities for the composition factors of $L \downarrow X$. Denote by n_3 the number of T-weights on L which are divisible by $p = 3$.

Lemma 4.2.8 *The possibilities for $L \downarrow X$ are:*

G	*Case*	$L \downarrow X$	T-*labelling*	n_3
E_6	(1)	$33/41/14/30^2/03^2/11^2/00^2$	022020	30
	(2)	$22/30^2/03^2/11^5/00^3$	220002	30
	(3)	$41/14/30^2/03^2/11^3/00^2$	200202	24
	(4)	$44/33/30/03/11^2/00$	222022	24
	(5)	$90/09/44/33^2/11/00$	202222	30
E_7	(6)	$41/14/22^2/30^2/03^2/11^3/00^4$	0020020	43
	(7)	$44/60/06/33/41/14/30/03/11^2/00^3$	2002020	43
E_8	(8)	$41^2/14^2/22^2/30^6/03^6/11^{10}/00^4$	00002000	80
	(9)	$52/25/33/41/14/30^2/03^2/11^3/00^4$	00200020	86
	(10)	$63/36/71/17/44/33^2/41/14/30^2/03^2/$ $11^2/00^3$	00020020	86
	(11)	$90/09/71/17/44/60/06/33^4/41/14/$ $30^2/03^2/11^2/00$	00020020	86
	(12)	$63/36/44^3/33^2/30^2/03^2/11^6/00^3$	00020020	86
	(13)	$90/09/71/17/44/33^6/41/14/30/03/11^2/00^5$	00020020	86
	(14)	$90/09/44^3/60/06/33^4/30^2/03^2/11^6/00$	00020020	86
	(15)	$90/09/44^3/33^6/30/03/11^6/00^5$	00020020	86
	(16)	$33/41^3/14^3/30^7/03^7/11^9/00^{10}$	00200002	86
	(17)	$44/52/25/33/30/03/11^3/00^3$	00020002	80

Lemma 4.2.9 *Cases* (6) *and* (7) *of Lemma* 4.2.8 *do not occur.*

Proof Here $G = E_7$ and we will make use of $V = V_{\hat{G}}(\lambda_7)$, the restricted irreducible module of dimension 56. Let $\hat{X}$ denote the connected preimage of X in the simply connected group $\hat{G}$. Since $Z(\hat{G})$ has order 2, we have $\hat{X} \cong X$, an adjoint group. Note that expressed in terms of fundamental roots, $\lambda_7 = \frac{1}{2}(2346543)$.

In case (6) the T-labelling is 0020020, from which we find that the T-weights on V are $8^2/6^4/\ldots$. Recalling that all composition factors must be representations of the adjoint group, we see that $V \downarrow \hat{X} = 22^2/00^2$. Then Lemma 4.2.1 shows that $\hat{X}$ has fixed space on V of dimension 2 and Lemma 2.2.13 yields a contradiction.

Now consider case (7) where the labelling is 2002020. Here we check that V has T-weights $12^2/10^2/8^4/6^4/\ldots$. The action is adjoint so $V \downarrow \hat{X}$ must have composition factors $60, 06$ or 33^2. The former pair yields T-weights $12^2/6^2/\ldots$, while the latter yields T-weights $12^2/6^4/\ldots$. In either case we find that there must be two composition factors affording T-weight 10, so that $41, 14$ must occur. These composition factors contribute T-weights $10^2/8^4/6^2/\ldots$. We therefore conclude that $V \downarrow \hat{X} = 60/06/41/14/00^2$. Note that 60 does not extend the trivial module (such an extension factors through a Frobenius morphism, and 20 clearly does not extend the trivial module). Using the fact that V is self-dual we easily argue from Lemma 4.2.1 that $\hat{X}$ has fixed space of dimension 1 or 2 on V. Hence Lemma 2.2.13 again yields a contradiction. ∎

Lemma 4.2.10 *Case* (3) *of Lemma* 4.2.8 *does not occur.*

Proof Here $G = E_6$ and we will make use of the irreducible $\hat{G}$-module $V = V_{\hat{G}}(\lambda_1)$ of dimension 27. Let $\hat{X}$ denote the preimage of X in the simply connected group $\hat{G}$.

The T-labelling is 200202. As $\lambda_1 = \frac{1}{3}(234654)$, it follows that $\hat{T}$ has weights $8^1/6^2/4^4/\ldots$ on V. Hence, the highest weight of $\hat{X}$ on V is one of $22, 31, 13, 40, 04$. If the first case occurs, then $\hat{X}$ is irreducible on V and it follows from [41, Theorem 1] that $\hat{X}$ is contained in a uniquely determined subgroup G_2 of $\hat{G}$. This contradicts the maximality of X.

In the remaining cases $\hat{X}$ acts as a simply connected group on V, so that $Z = Z(L(\hat{X})) = Z(L(\hat{G}))$, inducing scalars on V. In particular, all composition factors afford faithful action of Z. In each case the composition factor for the largest weight affords $\hat{T}$-weights $8^1/6^1/4^1/\ldots$. So there must be another composition factor affording $\hat{T}$-weight 6. The only possibilities with Z acting nontrivially are 21 and 12. One of these must occur, with T-weights $6/4^2/\ldots$. This leaves a composition factor with largest T-weight 4 and this is not possible by dimension considerations, since $\dim V(31) = \dim(40) = 9, \dim V(21) = 15$. ∎

Lemma 4.2.11 *Case* (8) *of Lemma* 4.2.8 *does not occur.*

Proof Here we have $L \downarrow X = 41^2/14^2/22^2/30^6/03^6/11^{10}/00^4$. Again let $T_1 = \{\mathrm{diag}(c, c, c^{-2}) : c \in K^*\} < X$. It is possible to explicitly compute both the fixed points of T_1 on L and also the T-weights on this fixed point space. For purposes of this computation it is convenient to note that $20 \otimes 02 = 22/11/00^2$.

The result of the computation is that $\dim C_L(T_1) = 68$ and that T has non-negative weights $4^6, 2^{18}, 0^{20}$ on $C_L(T_1)$. Now $C_G(T_1)$ is a Levi factor, so the only possibility is $C_G(T_1) = D_6T_2$.

We now consider possible T-labellings of the Dynkin diagram of D_6 that are consistent with the weights indicated. We find that there is no possible labelling and so this is a contradiction. ∎

Lemma 4.2.12 *Cases* (2) *and* (5) *of Lemma* 4.2.8 *do not occur.*

Proof We shall establish that in each of these cases $A \neq 0$. Since $n_3 = 30$ in these cases, this leads to a contradiction by Lemma 2.3.4.

In case (5) we have $A \neq 0$ by Lemma 4.2.5.

Now consider case (2). Here we have $L \downarrow X = 22/30^2/03^2/11^5/00^3$. The Weyl module $W_X(22)$ is irreducible so we can write $L = 22 \perp J$, where J has the remaining composition factors.

Suppose that $A = 0$. Then if v is any vector of weight 30 we have $\langle Xv \rangle = 30/11$. It follows that there is a submodule $R = 30^2/11^2$ with socle 11^2. Then working entirely in J we have $R^{\perp}/R = 11/00^3$. There are two possibilities for this self-dual module: $11 \oplus 00^3$ or $T(11) \oplus 00$.

In the first case, the preimage of 00^3 over R must yield a submodule 00 of L, which is a contradiction. Suppose the second case occurs. Here the preimage, say F, of $T(11)$ has a quotient $T(11)|30$.

We claim that such a module must be a direct sum $T(11) \oplus 30$. For consider the dual of such a module. This has submodule $T(11)$ with quotient 03. Setting v to be a weight vector of weight $03, \langle Xv \rangle$ is an image of the

Weyl module $W_X(03) = 03/11$. But $T(11)$ is uniserial of shape $00|11|00$, so this forces $\langle Xv\rangle = V_X(03)$. Taking duals again, we have the claim.

Two applications of the claim show that F has a submodule of the form $T(11)|11^2$. Let v be a weight vector of weight 11 with $v \notin 11^2$. Then $\langle Xv\rangle \cong W_X(11)$, hence we again have a fixed point and a contradiction. ∎

Lemma 4.2.13 *Cases* (10) – (15) *of Lemma* 4.2.8 *do not occur.*

Proof We first claim that in each of these cases A has a composition factor which is one of $63, 36, 90, 09$ or 33. In cases (14) and (15) this is immediate from Lemma 4.2.5.

Next consider cases (10), (11) and (13). We prove the claim by the same argument for each of these, so we give the argument just for case (10). Let v, w be vectors of weights $63, 36$ respectively. If there is no submodule 63 or 36, then $\langle Xv, Xw\rangle = (63+36)|R$ where either R has 44 as a composition or both 71 and 17 occur as composition factors. Moreover R is totally singular by Lemma 2.1.5. However, this is impossible as each of $71, 17, 44$ appears in $L \downarrow X$ with multiplicity 1.

Now consider case (12), where establishing the claim is somewhat more complicated. First note that if $L \downarrow X$ contains a submodule where 33 is the highest weight, then there is a submodule 33 by Lemma 4.2.5, since neither 41 nor 14 occur as composition factors of L. So assume there is no such submodule.

Let R be the maximal submodule of $S = \langle Xv\rangle$, where v is a vector of weight 63. Another application of Lemma 4.2.5 shows that 44 occurs as a composition factor of R. Indeed, $Z = \langle Xf_\alpha v\rangle$ is a nontrivial image of $W_X(44)$. First assume that 33 is also a composition factor of R. The weight space of S for weight 33 is generated by $f_{\alpha+\beta}f_\alpha v, f_\beta f_\alpha^2 v$, both of which lie in Z. It follows that Z contains a submodule for which 33 is the highest weight, contradicting the previous paragraph.

Now assume that 33 does not occur as a composition factor of R. By 2.1.5, R is singular and hence so is Z. Consider $H = Z^\perp/Z = (63+36) \perp (44/33^2/\ldots)$. In the second factor consider submodules generated by a vector of weight 44. Each is an image of a Weyl module, and if any of these images have 33 as a composition factor, then generating by a suitable 44 weight vector in the preimage we find that L has a submodule where 33 is the highest weight. This yields a contradiction as above. So we may assume that all cyclic modules generated by 44 weight vectors of H yield submodules with no 33 composition factor. It follows that L has an image of form $44/33^2/\ldots$ Within this quotient generate by a weight vector of weight 44 and factor out the corresponding submodule. As 33 occurs with multiplicity 1 in $W_X(44)$, we conclude that L has a quotient for which the highest weight of a composition factor is 33. But L is self-dual, so there must also be a submodule with the same property, which is again a contradiction. This establishes the claim.

We have now established our claim that $63, 36, 90, 09$ or 33 appears as a composition factor of A.

By Lemma 2.3.4 we have $A \leq L(D)$, and since $\dim D = n_3 = 86$ in each of cases (10)-(15), we have $D = A_2E_6$. The T-labelling of the Dynkin diagram of G is 00020020 in each case, and the non-negative T-weights divisible by $p = 3$ are $0^{24}, 6^{20}, 12^9, 18^2$. It follows that the T-labelling of $D = A_2E_6$ is

$$6\,0,\ 0\,0\,0\,6\,0\,0$$

By Lemma 2.3.6, R, the subalgebra of A generated by nilpotent elements, projects faithfully to $L(A_2) \subseteq L(D)$. However, R has a vector of T-weight at least 12, whereas because of the labelling $6\,0$ on the A_2 factor of D, $L(A_2)$ has no T-weight vectors of weight more than 6. This is a contradiction, completing the proof. ■

Lemma 4.2.14 *Cases* (4) *and* (17) *of Lemma* 4.2.8 *do not occur.*

Proof Suppose first that A has a composition factor 33 in either of these cases. By Lemma 2.3.4, $A \subseteq L(D)$ where $D = A_2^3$ or A_8 (in case (4) or (17) respectively). By our supposition we can choose an element $e \in A$ of T-weight 12. From the labelled diagram we see that the largest T-weight of $L(D)$ is 12, so the square of the projection of e in each factor $L(A_k)$ of $L(D)$ is zero. We can then apply Lemma 2.3.9 to elements of $\langle e \rangle$ and obtain a 1-dimensional unipotent subgroup of D which stabilizes each $\mathrm{ad}(e)$-invariant subspace of $L(D)'$. In particular, the subalgebra R of A generated by all nilpotent elements is invariant under this unipotent group. But now Lemma 2.3.5 gives a contradiction.

So assume now that A has no composition factor 33.

In case (4) we have $L(G) \downarrow X = 44/33/30/03/11^2/00$. Let $C = \langle Xv \rangle$, where v is a weight vector of weight 44. Then C is an image of $W(44)$, and we let B be the image of the maximal submodule of $W(44)$. Lemma 2.1.5 shows that B is a singular subspace. Since it appears with multiplicity 1 in $L \downarrow X$, 33 does not appear as a composition factor of B and so 33 is a composition factor of $B^{\perp}/B$. In this quotient, the factor 44 is non-degenerate, so there is a module $B' > B$ such that 33 is the highest weight of B'. Applying Lemma 4.2.5 to B' we conclude that 41 or 14 must occur as a composition factor of B', a contradiction.

In case (17), $L(G) \downarrow X = 44/52/25/33/30/03/11^3/00^3$ and the argument is identical to that of the previous paragraph after an initial reduction. By Lemma 4.2.1, neither 52 nor 25 occurs within the Weyl module of high weight 44. Hence 44 does not extend either 52 or 25. Since the Weyl modules $W(52)$ and $W(25)$ are irreducible we have $L \downarrow X = (52 \oplus 25) \perp E$ for some nondegenerate module E. Now apply the previous argument to E to obtain a contradiction. ∎

Lemma 4.2.15 *Case* (9) *of Lemma* 4.2.8 *does not occur.*

Proof Suppose first that A has 33 as a composition factor. By Lemmas 2.3.4 and 4.2.8 we have $A \subseteq L(D)$ where $D = A_2E_6$. The non-negative T-weights divisible by p are $0^{32}, 6^{22}, 12^5$, from which we see that the T-labelling of D can be taken as 60, 006000. However, by Lemma 2.3.6 the projection of R to $L(A_2)$ is faithful, which is impossible as R has a vector of T-weight 12.

Thus we may assume that A does not have 33 as a composition factor. We have $L \downarrow X = 52/25/33/41/14/30^2/03^2/11^3/00^4$. Since the Weyl modules for high weights 52 and 25 are irreducible we can write $L = (52+25) \perp M$ for some module M. In the following we work entirely within M.

Let $m \in M$ have weight 33 so that $S = \langle Xm \rangle$ is an image of $W_X(33)$. Let R be the image of the maximal submodule. By Lemma 4.2.5 we see that either 14 or 41 is a composition factor of R. But as R is a singular subspace only one can occur and we will assume this to be 41.

By Lemma 4.2.2, 11 is the socle of $W_X(33)$, so if 11 occurs in S, then $S \cong W_X(33)$ contradicting the fact that 14 does not occur as a composition factor. Hence 11 does not occur. Since 14 does not occur as a composition factor of S, it follows from Lemmas 4.2.2 and 4.2.3 that $S = 33/00/41$.

Now R is a singular subspace and we set $J = R^{\perp}/R$ (the perp within M). Then J is a non-degenerate space of the form $33/30^2/03^2/11^3/00^2$, and the 33 factor splits off as a non-degenerate subspace. Consider the submodule generated by all weight vectors of weights 30 and 03 in the perpendicular space to 33. This space has the form $(30^2 + 03^2)/11^e$, and Lemma 2.1.5 shows that the subspace $E = 11^e$ is singular, so that $e \leq 1$. Then working in J, we have $E^{\perp}/E = 33 \perp (30^2 + 03^2) \perp (11^{3-2e}/00^2)$.

If the preimage over E of the last summand has a 00 submodule, then taking preimages over R we get a 00 submodule in L, a contradiction. Hence there is no such 00 submodule. In particular, 00^2 cannot be a summand of $E^{\perp}/E$. It follows that either $T(11) = 00/11/00$ occurs as a summand or both indecomposables $11/00$ and $00/11$ occur as summands. But both $T(11)/11^e$ and $(11/00)/11^3$ yield 00 submodules, as is easily seen by generating by suitable vectors of weight 11. This completes the proof. ∎

We are left with two remaining cases of Lemma 4.2.8: (1) and (16). These require substantially more argument than the other cases.

Lemma 4.2.16 *Case* (1) *of Lemma* 4.2.8 *does not occur.*

Proof Here we have $L \downarrow X = 33/41/14/30^3/03^2/11^2/00^2$. Note that Lemma 2.3.4 and the fact that $\dim D = 30$ together imply that $A = 0$. Let T_1 denote the 1-dimensional torus of X for which $T_1(c) = \mathrm{diag}(c, c, c^{-2})$. Then $C_X(T_1) = T_1 J_\alpha$, where $J_\alpha = \langle U_\alpha, U_{-\alpha} \rangle$. It is straightforward to compute the T_1-weights on each of the composition factors of $L \downarrow X$, and we find that T_1 has weights $9^4, 6^9, 3^{16}, 0^{19}, -3^{16}, -6^9, -9^4$ on L. (Recall that L has codimension 1 in $L(G)$.) It follows that up to symmetry under a graph automorphism, the T_1-labelling of G is 300030. Hence $C_G(T_1) = A_1 A_3 T_2$. One can also determine the action of T on $C_L(T_1)$ and we find that T determines the labelling $2, 222$ of the $A_1 A_3$. Working in X we have $T(c) = T_1(c) T_\alpha(c)$, so this is also the labelling afforded by T_α. It follows that the projection of J_α to the A_3 factor has composition factors $3/1$ on the 4-dimensional module $V_{A_3}(\lambda_1)$.

Let L_3 be the T_1-weight space of L for weight 3. There exists a parabolic subgroup P of G such that $C_G(T_1)$ is a Levi factor of P and $L_3 \cong L(Q)/L(Q)'$, where $Q = R_u(P)$. We consider the action of $A_1 A_3$ on L_3, where we let $\Pi(A_3) = \{\alpha_3, \alpha_4, \alpha_2\}$. From [3] we see that $L_3 \downarrow A_1 A_3 = (1 \otimes 010) \oplus (0 \otimes 001)$.

Conjugating by the reflection s_α, if necessary, which interchanges L_3 with L_{-3}, we can assume that J_α acts as either $3 \oplus 1$ or is indecomposable of shape $1|3$ on the summand 100 of L_3. We handle these possibilities separately in two subcases.

Subcase 1. The projection of J_α induces $3 \oplus 1$ on $V_{A_3}(\lambda_1)$.

Here we find that $L_3 \downarrow J_\alpha = (1 \otimes \wedge^2(3 \oplus 1)) \oplus (3 \oplus 1)$. Now $\wedge^2(3 \oplus 1) = 0^2 \oplus (3 \otimes 1)$, so $1 \otimes \wedge^2(3 \oplus 1) = 1^2 \oplus 5 \oplus 3$. It follows that the socle of $L_3 \downarrow J_\alpha$ has 1 appearing with multiplicity 3.

We consider the projection of U_α to A_3. Regarding A_3 as SL_4 and using a basis of the usual module corresponding to a basis of T_α-weight vectors of weights $3, 1, -1, -3$ we find that

$$U_\alpha(c) = U_4(c)U_{234}(c^3)U_6(c),$$

where here we recall that the A_3 has fundamental roots $\alpha_3, \alpha_4, \alpha_2$ and $U_{ij\ldots}(c)$ denotes the root element $U_{\alpha_i+\alpha_j+\ldots}(c)$. In particular,

$$\langle e_\alpha \rangle = L(U_\alpha) = \langle e_4 + e_6 \rangle$$

(recall $e_i = e_{\alpha_i}$), and so e_α is a nilpotent element of type A_1A_1 in $L(G)$.

Let $Y = \langle e_\beta, e_{\alpha+\beta} \rangle$, the Lie algebra of the unipotent radical of the standard parabolic subgroup of X corresponding to T_1J_α. Then Y is contained in L_3 and is a submodule of high weight 1 with $e_{\alpha+\beta}$ a maximal vector.

A maximal vector for A_1A_3 on the $1 \otimes 010$ summand of L_3 is given by e_{011211} and this affords T_α-weight 5. It then follows from the T_α-labelling of A_1A_3 and our expression for U_α that the root vectors $\{e_{010111}, e_{001111}\}$ form a basis for the T_α-weight 1 fixed points of U_α on this summand of L_3. In addition, e_{101100} is a weight 1 fixed vector in the other summand. Hence there is an expression

$$e_{\alpha+\beta} = ae_{010111} + be_{101100} + ce_{001111}.$$

The first two roots generate a subsystem of type A_2 and the third root is orthogonal to this subsystem. If $ab \neq 0$, then $ae_{010111}+be_{101100}$ is a nilpotent element of type A_2 and we have a contradiction, since $e_{\alpha+\beta}$ is conjugate to e_α, of type A_1A_1. Hence $ab = 0$ and then $c \neq 0$.

Next carry out the same considerations for L_{-3}, which affords the dual module to $L_3 \downarrow A_1A_3$. Here $e_{-5}(= e_{-\alpha_5})$ affords a maximal vector for the A_1A_3-submodule $1 \otimes 010$ and it follows that $\{e_{-001110}, e_{-010110}\}$ is a basis for the T_α-weight 1 fixed vectors of U_α in this submodule of L_{-3}. Thus we have an equation

$$e_{-\beta} = xe_{-101000} + ye_{-010110} + ze_{-001110}.$$

Reasoning as above we have $xy = 0$ and $z \neq 0$.

Working in $L(X)$ we have

$$[e_{\alpha+\beta}, e_{-\beta}] = e_\alpha \in \langle e_4 + e_6 \rangle.$$

Consideration of the above expressions for $e_{\alpha+\beta}, e_{-\beta}$ shows that an e_4 contribution to the commutator can only occur from $[e_{101100}, e_{-101000}]$. Hence $b, x \neq 0$. So at this point we have

$$e_{\alpha+\beta} = be_{101100} + ce_{001111}$$

$$e_{-\beta} = xe_{-101000} + ze_{-001110}.$$

From the precise embedding $J_\alpha < A_1A_3$ we find that $s_\alpha = s_3^{s_4s_2}s_4s_6$ (where s_α denotes the fundamental reflection in the Weyl group $W(X)$ corresponding to α, and $s_i = s_{\alpha_i} \in W(G)$).

Applying this to the above equations yields expressions

$$e_{-\alpha-\beta} = \pm(e_{-\beta})^{s_\alpha} = \pm xe_{-101100} \pm ze_{-001111}$$

$$e_\beta = \pm(e_{\alpha+\beta})^{s_\alpha} = \pm be_{101000} \pm ce_{001110}.$$

Now $L(X)'$ is generated as a Lie algebra by $e_\beta, e_{-\beta}, e_{\alpha+\beta}, e_{-\alpha-\beta}$. From the above expressions for these elements we see that they can all be generated by 4 pairs of opposite root vectors in $L(G)$. It follows that $L(X)'$ is centralized by a 2-dimensional torus in G, contradicting Lemma 2.2.10.

Subcase 2. The projection of J_α induces the indecomposable module $1|3$ on $V_{A_3}(\lambda_1)$.

Here we proceed as above although the contradiction is easier. One can realize the indecomposable representation $1|3$ as the space of homogeneous polynomials of degree 3 in the basis for the usual module for J_α. In this way we can find explicit matrices for J_α (acting from the left) and obtain

$$U_\alpha(c) = U_4(2c)U_3(c)U_{34}(c^2)U_{234}(c^3)U_6(c).$$

Hence $e_\alpha \in \langle e_3 + 2e_4 + e_6 \rangle$, from which we see that e_α is a nilpotent element of type A_2A_1.

Now consider L_{-3} with maximal vector e_{-5}. We then find that the vectors $e_{-101000}, e_{-001110}, e_{-010110}, e_{-000111}$ form a basis for the T_α-weight 1 space in L_{-3}. Looking for fixed points under the action of U_α we find that $e_{-\beta}$ is a multiple of $e_{0101100}$. Hence, $e_{-\beta}$ is of type A_1, contradicting the fact this it must be conjugate to e_α. ∎

The remaining case in this subsection is case (16), which is by far the most troublesome and takes up the next 12 pages.

Proposition 4.2.17 *Case* (16) *of Lemma* 4.2.8 *does not occur.*

Assume we are in case (16), so $A_2 = X < E_8, p = 3$ and

$$L(G) \downarrow X = 33/41^3/14^3/30^7/03^7/11^9/00^{10}.$$

We maintain the notation of previous cases. Let X have fundamental roots α, β and regard X as the image of SL_3 under the adjoint map.

As before let $T_1 < X$ be the torus for which $T_1(c)$ is the image of the diagonal matrix $\mathrm{diag}(c, c, c^{-2})$, so that $C_X(T_1) = T_1 J_\alpha$.

The proof proceeds in a series of lemmas.

Lemma 4.2.18 (i) $C_G(T_1) = T_1 A_7$.

(ii) T_α *determines the labelling* 2002002 *of the Dynkin diagram of* A_7.

(iii) $J_\alpha < A_7$ *and has composition factors* $3/1^3$ *on the usual* 8*-dimensional module* $V = V_{A_7}(\lambda_1)$.

(iv) *We have*

$$V \downarrow J_\alpha = 3 \oplus 1^3,\ (3|1) \oplus 1^2,\ (1|3) \oplus 1^2 \ \text{or}\ T(3) \oplus 1,$$

where $(3|1), (1|3)$ *are indecomposable modules, and* $T(3) = 1|3|1$ *is the indecomposable tilting module of high weight* 3.

Proof We can determine the fixed points of T_1 on each composition factor of $L \downarrow X$, and also the weights of T_α on these fixed point spaces, where $T_\alpha(c)$ is the image of $\mathrm{diag}(c, c^{-1}, 1)$. We find that $C_L(T_1)$ has dimension 64, and so $C_G(T_1)$ is a Levi subgroup of this dimension. Part (i) follows.

From the action of T_α on $C_L(T_1)$ we see that the T_α-labelling of the A_7 must be 2002002, giving (ii). Hence, $T_\alpha(c)$ acts on $V = V_{A_7}(\lambda_1)$ as $\mathrm{diag}(c^3, c, c, c, c^{-1}, c^{-1}, c^{-1}, c^{-3})$, and (iii) follows. Since $\mathrm{Ext}_{A_1}(3, 1)$ is 1-dimensional, (iv) is immediate from (iii).

■

Lemma 4.2.19 $A = C_L(I)$ *does not have* 33 *as an* X*-composition factor.*

Proof Suppose false. Since $n_3 = 86$ in this case, Lemma 2.3.4 gives $A \leq L(D)$ with $D = A_2E_6$. We check that the T-labelling of D is 60, 06000. Then Lemma 2.3.6 shows that R, the subalgebra of A generated by nilpotent elements, projects faithfully to the A_2 factor. But by assumption, R has a vector of T-weight 12, whereas the T-labelling of the A_2 factor is 60. This is a contradiction. ■

In the rest of the proof we consider separately each of the possibilities in Lemma 4.2.18(iv).

Lemma 4.2.20 *It is not the case that* $V \downarrow J_\alpha = 3 \oplus 1^3$.

Proof Let v be a weight vector for weight 33 and $S = \langle Xv \rangle$, an image of the Weyl module $W_X(33)$. Now $L \downarrow X$ has no submodule 33 by Lemma 4.2.19, so Lemma 4.2.5 shows that S has a composition factor 41 or 14. Suppose the latter holds, so that $e_{-\alpha}v$ spans the 14 weight space.

Consider L_9, the T_1-weight space of L for weight 9. By [3] we see that this weight space is irreducible under the action of A_7 and affords the module $V_{A_7}(\lambda_7)$, the dual of V. Now v affords T_1-weight 9 and one checks that the T_1-weight space of $V_X(33)$ for weight 9 has dimension 2 and affords the irreducible of high weight 3 for J_α. Similarly, the T_1-weight space of $V_X(14)$ for weight 9 has dimension 2 and affords the irreducible representation of J_α having high weight 1.

Since 33 and 14 are the only X-composition factors of S having nonzero T_1-weight 9 space, the full T_1-weight space of S for weight 9 has dimension 4, and by the above this weight space of S has shape 3/1 under the action of J_α. Since $e_{-\alpha}v \neq 0$, we conclude $L_9 \cong V^*$ has an indecomposable module for J_α of type $3|1$, contradicting our assumption that $V \downarrow J_\alpha$ is completely reducible. Hence 14 cannot occur as a composition factor of S. Then 41 must occur and the above analysis applies to J_β. But J_α and J_β are conjugate in X, so the complete reducibility hypothesis applies to this group as well and we have a contradiction. ■

Lemma 4.2.21 *It is not the case that* $V \downarrow J_\alpha = (3|1) \oplus 1^2$ *or* $(1|3) \oplus 1^2$ *(where* $(3|1), (1|3)$ *indicate indecomposables).*

Proof Suppose false and let v be a weight vector of weight 33, so that $S = \langle Xv \rangle$ is an image of the Weyl module $W_X(33)$. By Lemma 4.2.5 we can suppose that S has a composition factor 14 or 41. We assume the former, noting that the other case is entirely similar and just involves a change of notation. As in the previous lemma it follows that $L_9 \downarrow J_\alpha$ has the indecomposable module $3|1$ appearing. Indeed v is of T_α-weight 3 and $e_{-\alpha}v \neq 0$. Recall that L_9 affords V^*, so that this implies $V \downarrow J_\alpha = (1|3) \oplus 1^2$.

Suppose 41 also appears as a composition factor of S. Then $e_{-\beta}v \neq 0$. Now conjugate this by w_0, the long word in the Weyl group $W(X)$, to conclude that $e_\alpha w \neq 0$, where w is a weight vector of weight $-(33)$. It follows that $3|1$ occurs within $L_{-9} \cong V$. But this is a contradiction. Hence we take it that just 14 (and not 41) appears in S.

Since 41 does not appear as a composition factor of S it follows from Lemma 4.2.1 and 4.2.2 that $S = 33/00/14$. Let R be the maximal submodule of S, which by Lemma 2.1.5 is a singular subspace of L. We can write $R^\perp/R = 33 \perp J$ where J has 14 and 41 each appearing with multiplicity 2. Choose two weight vectors for weights in $\{14, 41\}$ (possibly both of the same weight), and generate cyclic X-modules containing these vectors in the usual way.

By 2.1.5 each is a singular space, and we form the sum of these cyclic modules, say H. Arguments with perpendicular spaces show that either H is itself singular (e.g. if both generating vectors were of the same weight) or $H/\mathrm{rad}(H) = 14 \perp 41$. In either case there is a submodule M of H such that M has all composition factors of H for weights $30, 03, 11, 00$. Indeed, M is just the sum of the images of the maximal submodules of the summands. Write $M = 00^x/11^y/30^j/03^k$. Note that $y \leq 2$.

Sublemma *We have $y = 2$.*

Proof The proof requires slightly different arguments depending on whether or not H is singular. By way of contradiction assume $y \leq 1$.

First assume that H is singular. Then the preimage of H over R, say $\bar{H}$, is singular with composition factors $14^l/41^{3-l}/30^j/03^k/00^{x+1}/11^y$. Note that $x \leq y$ as otherwise there would be a fixed point (the 00 composition factors are blocked only by 11 composition factors and the 14 submodule of the socle).

Write $\bar{H}^\perp/\bar{H} = 33 \perp C$. In C the largest dominant weights are 30 and 03 and as C is non-degenerate they occur with equal multiplicity. Consider a cyclic X-submodule generated by a 30 weight vector. This has the form 30 or 30/11, the latter being the Weyl module. If 30/11 occurs, Lemma 2.1.5 shows the 11 submodule is singular and all $30, 03$ weight vectors are orthogonal to this 11. Let E denote the sum of all 11 submodules obtained in this manner. Then $E = 11^e$ is a singular subspace and $E^\perp/E = (30^i + 03^i) \perp (11^{9-2y-2e}/00^{8-2x})$. Decompose the last summand as

$$T(11)^a + (11/00 + 00/11)^b + 00^c + 11^d.$$

The above decomposition gives:

(1) $a + 2b + d = 9 - 2y - 2e$

(2) $2a + 2b + c = 8 - 2x$.

From the structure of H and Lemma 4.2.4 we have

(3) $a + b \leq y - x$.

From (2) and (3) we have $c = 8 - 2x - 2(a + b) \geq 8 - 2x - 2(y - x)$. Adding this to (2) we have $2a + 2b + 2c \geq 8 - 2x + 8 - 2x - 2(y - x)$ and hence $a + b + c \geq 8 - x - y$.

We are now in position to establish the Sublemma in the case where H is singular. In $E^{\perp}/E$ the 00 submodule occurs with multiplicity $a + b + c$. The preimage of the sum of these trivials has 00 appearing with multiplicity $a + b + c + x + 1$. The only composition factors in $\bar{H}$ which can block a trivial are $11, 41, 14$ with combined multiplicity $e + y + 3$. Since there is no 00 submodule we must have $a + b + c + x + 1 \leq e + y + 3$. Combining with the above inequality yields $8 - x - y \leq e + y - x + 2$ and hence $6 \leq e + 2y$. But we are assuming $y \leq 1$, and from (1) we have $e + y \leq 4$. This is a contradiction.

Before proceeding with the next case we note a consequence of the previous case. If we form H so that the two generating vectors are of the same weight 14 or 41, then we are necessarily in the singular case. So it follows from the above case that each generator yields a submodule with 11 appearing as a composition factor.

Now consider the case where $H/M = H/\mathrm{rad}(H) = 41 \perp 14$. Let $\bar{M}$ be the preimage of M over R so that $\bar{M} = 00^{x+1}/11^{y}/30^{j}/03^{k}/14$. Note that by the previous paragraph we have $y = 1$.

Now consider $\bar{M}^{\perp}/\bar{M}$. We can split off non-degenerate spaces 33 and $41 + 14$ (from the image of H). In the orthogonal complement we have composition factors $14, 14$, each with multiplicity 1 and we generate corresponding cyclic modules. The sum of the images of the maximal submodules is a singular subspace, and we suppose this sum has composition factor 00 appearing with multiplicity r and 11 with multiplicity s. Note that $r, s \leq 2$.

Let J denote the preimage of this space over $\bar{M}$. Then J is a singular space with composition factors $14/30^{j'}/03^{k'}/11^{y+s}/00^{1+x+r}$. We claim that J has an image of type $00|14$. To see this start with $v \in R^{\perp} - R$ a weight vector for weight 14 or 41. Then $\langle Xv \rangle \cap R = 0$, as otherwise the intersection would be a trivial module. Factoring out the maximal submodule of $\langle Xv \rangle$

and repeating the procedure with other weight vectors of weights $14, 41$, we obtain the claim.

We now proceed as in the first case. In $J^\perp/J$ we can now split off the 33 factor and a non-degenerate space with composition factors $14^2, 41^2$. Let C be the orthogonal complement and take E, a, b, c, d, e as before. This time we get equations

(1′) $a + 2b + d = 9 - 2y - 2s - 2e$

(2′) $2a + 2b + c = 8 - 2x - 2r$.

From Lemma 4.2.4 and the remarks of the previous paragraph we get

(3′) $a + b \leq y + s - (x + r)$.

From (2′) and (3′) we have $c = 8 - 2x - 2r - 2(a + b) \geq 8 - 2x - 2r - 2(y + s - x - r) = 8 - 2y - 2s$. Adding this to (2′) we have $2a + 2b + 2c \geq 8 - 2x - 2r + 8 - 2y - 2s$, so that $a + b + c \geq 8 - x - y - r - s$. Now 00 occurs as a submodule of $E^\perp/E$ with multiplicity $a + b + c$ and the preimage over $\bar{M}$ of this fixed space has 00 appearing with multiplicity $a + b + c + 1 + r + x$. Also, the composition factors which extend 00 are 14 and 11 with combined multiplicity $1 + e + y + s$. Hence we must have $a + b + c + 1 + r + x \leq 1 + e + y + s$, so that $a + b + c \leq e + y + s - x - r$. Combining with the previous inequality we get $8 - x - y - r - s \leq e + y + s - x - r$, which reduces to $8 \leq e + 2y + 2s$. Now recall that $y = 1$. Also, (1′) shows that $e + y + s \leq 4$. Since we have already observed that $s \leq 2$, this is a contradiction, completing the proof of the Sublemma.

We are now in position to complete the proof of the Lemma 4.2.21. Work in $R^\perp/R$ where composition factors 14 and 41 each occur with multiplicity 2. Let $Y(14)$ denote the sum of the corresponding cyclic modules for vectors of weight 14 and similarly for $Y(41)$. The Sublemma and Lemmas 4.2.1 and 4.2.2 imply that $Y(14) = W_X(14) \oplus W_X(14)$ and $Y(41) = W_X(41) \oplus W_X(41)$. Note that each of $Y(14), Y(41)$ has socle 11^2.

We claim that these socles are disjoint. If the claim is false there is a vector v of weight 11 common to the socles of $Y(14)$ and $Y(41)$. We can choose weight vectors $v_{14} \in Y(14), v_{41} \in Y(41)$ of weights $14, 41$, respectively, such that $v \in \langle X v_{14} \rangle \cap \langle X v_{41} \rangle$. However, this contradicts the Sublemma. Indeed, taking H to be the image of $\langle X v_{14} \rangle + \langle X v_{41} \rangle$ modulo R we have a configuration with $y = 1$. This establishes the claim.

We have now established that the socle of $Y = Y(14) + Y(41)$ is 11^4

and so Y is the direct sum of four Weyl modules. Let P be the maximal submodule of one of the summands. Then all weight 14 and 41 vectors appear in $P^{\perp}$ and so all the other Weyl module summands are orthogonal to P. Repeating this with each of the summands we conclude that the sum of the maximal submodules is a singular subspace. Then Lemmas 4.2.1 and 4.2.2 show that this singular subspace has 30 and 03 each occuring with multiplicity 4. But L is self-dual, so this implies 30 and 03 each occur with multiplicity at least 8 in L. This is a contradiction. ■

By the previous two lemmas together with Lemma 4.2.18(iv), to complete the proof of Proposition 4.2.17 it remains to handle the configuration $V \downarrow J_\alpha = T(3) \oplus 1$. This is done in the next proposition.

Proposition 4.2.22 *It is not the case that* $V \downarrow J_\alpha = T(3) \oplus 1$.

The proof of this proposition takes a considerable amount of work, which is carried out in the following lemmas. These require many calculations within $L(G) = L(E_8)$, for which we need the table of E_8-structure constants $N(\gamma, \delta)$ $(\gamma, \delta \in \Sigma(G)^+)$ given in the Appendix, Section 11. This table was computed by the method described in [13]. For our calculations we often need to know $N(\gamma, \delta)$ where γ or δ is a negative root; this can be deduced using the relations $N(\gamma, -\delta) = N(\gamma - \delta, \delta)$ if $\gamma, \delta, \gamma - \delta \in \Sigma(G)^+$ and $N(\gamma, -\delta) = N(\delta - \gamma, \gamma)$ if $\gamma, \delta, \delta - \gamma \in \Sigma(G)^+$.

We shall study in detail the embedding of $L(X)$ in $L(G)$. For the purpose of proving Proposition 4.2.22 we shall be applying transformations from the left. We choose our root system of G so that $A_7 = C_G(T_1)'$ is the standard Levi subgroup generated by the root subgroups corresponding to fundamental roots $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$.

Then root vectors in $L(A_7) = sl_8$ for positive roots are taken as upper triangular matrices. We begin with precise information on the embedding of $L(J_\alpha)$ in $L(A_7)$. As always, we write $e_{ij\ldots}$ for $e_{\alpha_i+\alpha_j+\ldots}$ and $U_{ij\ldots}(c)$ for $U_{\alpha_i+\alpha_j+\ldots}(c)$.

Lemma 4.2.23 *We may assume that*

(i) $e_\alpha = e_1 + e_{13} + e_5 + e_8 - e_{78} + e_{3456} + e_{456} - e_{4567}$.

(ii) $f_\alpha = f_1 + f_5 + f_8 - f_{13} + f_{78} + f_{456} - f_{3456} + f_{4567}$.

Proof We begin with a concrete realization of $V \downarrow J_\alpha$. Regard $J_\alpha \cong SL_2$ as matrices corresponding to a 2-dimensional vector space with basis $\{x, y\}$.

We then have the relations $e_\alpha x = 0, e_\alpha y = x, f_\alpha x = y, f_\alpha y = 0$. In addition, $T_\alpha(c)$ affords the diagonal matrix (c, c^{-1}) with respect to this basis.

Since tensor products of tilting modules are again tilting modules, $1 \otimes 2$ is a tilting module of high weight 3. Dimension considerations imply that this is the indecomposable tilting module $T(3)$. Hence we can write $V \downarrow J_\alpha = (1 \otimes 2) \oplus 1$.

Now $V_{J_\alpha}(2)$ can be realised as the space of homogeneous polynomials of degree 2 in two variables u, v, so has basis u^2, uv, v^2 (we change from x, y to u, v here to avoid notational confusion). Let $\{x', y'\}$ be the basis for the direct summand of high weight 1 in $V \downarrow J_\alpha$. We then take the following as basis for V:

$$x \otimes u^2,\ y \otimes u^2,\ x \otimes uv,\ x',\ y',\ y \otimes uv,\ x \otimes v^2,\ y \otimes v^2.$$

It is straightforward to work out the matrices of e_α and f_α relative to this basis, from which we obtain the expressions (i) and (ii). ∎

Now $T_1 J_\alpha$ is a Levi factor of two parabolic subgroups of X whose corresponding unipotent radicals have Lie algebras affording the natural module for J_α and have bases $\{e_{\alpha+\beta}, e_\beta\}$ and $\{e_{-\beta}, e_{-\alpha-\beta}\}$, respectively. In each case the first basis vector listed is a maximal vector.

From the known action of J_α on V one can see that J_α is centralized by a 1-dimensional unipotent subgroup of A_7. Indeed it is easily observed that elements in this centralizer are products of two commuting root elements. The next lemma gives this centralizer explicitly and is verified by a direct check within sl_8 and then translating to the A_7 root system.

Lemma 4.2.24 *For $c \in K$, set*

$$S(c) = U_{-3}(-1)U_3(c)U_{-3}(1)U_{-7}(-1)U_7(-c)U_{-7}(1).$$

Then $S(c)$ centralizes $\langle e_\alpha, f_\alpha \rangle = L(J_\alpha)$.

Ultimately we shall show that $S(c)$ centralizes the whole of $L(X)'$, contradicting Lemma 2.2.10(iii).

Lemma 4.2.25 *The T_1-weights on L are $9^8, 6^{28}, 3^{56}, 0^{64}, -3^{56}, -6^{28}, -9^8$. The weight of T_1 on a root vector e_δ of L is 3 times the coefficient of α_2 in δ.*

Proof As T_1 centralizes A_7 a quick check shows that the T_1-labelling of G is 03000000. The lemma follows. ■

Lemma 4.2.26 *Let L_3, L_{-3} denote the T_1-weight spaces of L corresponding to weights* $3, -3$, *respectively. Then*

(i) $\{e_{\alpha+\beta}, e_\beta\} < L_3$ *and* L_3 *affords* $\wedge^3 V^*$ *for* $A_7 = C_G(T_1)'$.

(ii) $\{e_{-\beta}, e_{-\alpha-\beta}\} < L_{-3}$ *and* L_{-3} *affords* $\wedge^3 V$ *for* A_7.

Proof Working within X we easily verify that $T_1(c)v = c^3 v$ for $v \in \{e_{\alpha+\beta}, e_\beta\}$ and $T_1(c)v = c^{-3}v$ for $v \in \{e_{-\beta}, e_{-\alpha-\beta}\}$, giving the containments. The previous lemma shows that L_3 and L_{-3} have bases consisting of all root vectors for roots having coefficient of α_2 equal to 1 and -1, respectively. Using [3] we see that these weight spaces affords irreducible modules for A_7 of high weights λ_4 and λ_3 respectively, giving (i) and (ii). ■

For future reference we note that $e_{11233321}$ and $e_{-01000000}$ afford maximal vectors for $L_3 \downarrow A_7$ and $L_{-3} \downarrow A_7$, respectively.

We want to locate $e_{\alpha+\beta}, e_{-\beta}$ within L_3, L_{-3}, respectively. These are each fixed by $\mathrm{ad}(e_\alpha)$ and are vectors having T_α-weight 1. We will obtain a basis for such weight vectors of L_3 and L_{-3}. As a first step, in the next lemma we present a basis for the T_α-weight 1 subspace in each module.

Lemma 4.2.27 *The T_α-weight spaces of L_{-3} and L_3 corresponding to weight* 1 *each have dimension* 15. *Bases for these subspaces are respectively given by root vectors $v_i = e_{-\delta}, 1 \le i \le 15$ and $w_i = e_\delta, 1 \le i \le 15$, where δ is as follows:*

v_1 : 11111000	v_9 : 11221110	w_1 : 11232210	w_9 : 11122110
v_2 : 11111100	v_{10} : 01011111	w_2 : 11232110	w_{10} : 01122221
v_3 : 11111110	v_{11} : 01111111	w_3 : 11232100	w_{11} : 01122211
v_4 : 11121000	v_{12} : 01121111	w_4 : 11222210	w_{12} : 01122111
v_5 : 11221000	v_{13} : 01122100	w_5 : 11222110	w_{13} : 11221111
v_6 : 11121100	v_{14} : 01122110	w_6 : 11222100	w_{14} : 11121111
v_7 : 11221100	v_{15} : 01122210	w_7 : 11122100	w_{15} : 11111111
v_8 : 11121110		w_8 : 11122210	

Proof The fact that the weight spaces have dimension 15 is a direct check from Lemma 4.2.26 and the fact that the T_α-weights on V and V^* are each $3, 1^3, -1^3, -3$. It is also immediate that T_α has high weight 5 on each module.

As remarked above $e_{-01000000}$ affords a maximal vector for L_{-3}, while $e_{11233321}$ is a maximal vector for L_3.

It follows from Lemma 4.2.25 that a basis for L_{-3} and L_3 is obtained by taking all root vectors for roots with coefficient of α_2 equal to $-1, 1$, respectively. These roots are obtained by starting from the maximal vector and subtracting certain fundamental roots from the A_7 root system. Since the high weight of T_α on each module is 5, we obtain the weight space for weight 1 by removing appropriate fundamental roots such that the total T_α-weight is 4. We have seen that T_α determines labelling 2002002 of the A_7 diagram. It is then readily checked that the root vectors indicated form bases for the weight 1 subspaces of T_α on L_{-3} and L_3. ■

Lemma 4.2.28 *We have $e_{-\beta} = ar_1 + br_2 + cr_3 + dr_4 + er_5 + fr_6$, where $a, b, c, d, e, f \in K$ and $r_1, r_2, r_3, r_4, r_5, r_6$ are as follows:*

$$\begin{aligned}
r_1 &= v_4 + v_5 \\
r_2 &= v_2 - v_3 - v_{10} - v_{11} \\
r_3 &= v_2 + v_3 - v_{10} - v_{15} \\
r_4 &= v_6 + v_7 - v_8 - v_9 \\
r_5 &= v_6 + v_8 - v_9 - v_{12} \\
r_6 &= v_1 - v_6 - v_9 + v_{13} - v_{14}.
\end{aligned}$$

Proof From the previous lemma we have $e_{-\beta} = \sum_1^{15} a_i v_i$. We also know that $[e_\alpha e_{-\beta}] = 0$ in $L(X)$, and e_α is given by Lemma 4.2.23. We calculate $[e_\alpha e_{-\beta}]$ using the E_8 structure constants in the Appendix, Section 11. Setting the result equal to 0 yields the conclusion. ■

A similar calculation yields

Lemma 4.2.29 *We have $e_{\alpha+\beta} = rz_1 + sz_2 + tz_3 + uz_4 + vz_5 + wz_6$, where $r, s, t, u, v, w \in K$ and $z_1, z_2, z_3, z_4, z_5, z_6$ are as follows:*

$$\begin{aligned}
z_1 &= w_2 + w_3 \\
z_2 &= w_4 + w_8 + w_{10} + w_{15} \\
z_3 &= w_4 - w_8 + w_{10} - w_{11} \\
z_4 &= w_6 + w_9 + w_{12} \\
z_5 &= w_5 + w_6 - w_7 - w_9 \\
z_6 &= w_1 + w_5 + w_7 + w_{13} - w_{14}.
\end{aligned}$$

We next obtain some information on the coefficients in the above expressions for $e_{-\beta}$ and $e_{\alpha+\beta}$.

Lemma 4.2.30 *The following conditions hold:*
(i) $wf = 1$
(ii) $s = c = 0$
(iii) $uf = we$.

Proof The previous lemmas give expressons for $e_{\alpha+\beta}$ and $e_{-\beta}$. On the other hand, working in $L(X)$ we have $[e_{\alpha+\beta}e_{-\beta}] = e_\alpha$, which is given in Lemma 4.2.23. Using the structure constants in the Appendix, Section 11, we calculate $[e_{\alpha+\beta}e_{-\beta}]$. Equating coefficients of the result to those of e_α, we obtain a number of equations.

First, equating the coefficients of e_1 and e_{13} in $[e_{\alpha+\beta}e_{-\beta}]$ with the corresponding terms in e_α gives the equations

$$-cs + ct - fu + fw + ew - bs = 1,$$

$$ct - fu + fw + ew - bs = 1.$$

Subtracting these yields $cs = 0$.

Next, equate coefficients of e_8 and obtain

$$-ct + fu - we + wf + sb - sc = 1.$$

Since $sc = 0$ we can add this to the e_{13} equation to conclude that $wf = 1$, which is (i).

Equating coefficients of the e_{678} and e_{134} terms we obtain $sf = 0$ and $cw = 0$, respectively. Since (i) implies $w, f \neq 0$, we have $s = 0$ and $c = 0$, giving (ii).

Finally, we use the e_{13} equation together with (i) and (ii) to conclude that $uf = we$. ■

Lemma 4.2.31 *We have* $e_\beta = rl_1 + tl_2 + ul_3 + vl_4 + wl_5$, *where*

$$\begin{aligned}
l_1 &= e_{11221000} - e_{11121000}\\
l_2 &= e_{11111110} + e_{01111111} - e_{01011111} + e_{11111100}\\
l_3 &= e_{01121111} + e_{11221100} + e_{11121110}\\
l_4 &= e_{11221100} - e_{11121110} + e_{11221110} - e_{11121100}\\
l_5 &= e_{01122110} + e_{01122100} - e_{11221110} - e_{11121100} + e_{11111000}.
\end{aligned}$$

Proof Working in $L(X)$ we have $e_\beta = [f_\alpha, e_{\alpha+\beta}]$. The lemma then follows from Lemmas 4.2.23 and 4.2.29 via direct computation using the fact that $s = 0$ (see 4.2.30(ii)). ■

Lemma 4.2.32 *We have $u = e = 0$.*

Proof The previous lemma provides an expression for e_β, and working within $L(X)$ we have $[e_\beta, e_{\alpha+\beta}] = 0$. A direct calculation of the coefficient of e_δ for $\delta = 22343210$ shows that $u^2 = 0$, hence $u = 0$. Then (i) and (iii) of Lemma 4.2.30 show that $e = 0$, as required. ■

At this point the expressions in Lemmas 4.2.28 and 4.2.29 read as follows:

$$e_{-\beta} = ar_1 + br_2 + dr_4 + fr_6$$

$$e_{\alpha+\beta} = rz_1 + tz_3 + vz_5 + wz_6$$

Lemma 4.2.33 *The coefficients a, b, d, f can each be expressed in terms of the coefficients r, t, v, w.*

Proof First note that Lemma 4.2.30(i) gives $f = 1/w$. For information on a, b, d we return to the equation $[e_{\alpha+\beta}, e_{-\beta}] = e_\alpha$. Equating the coefficients of e_{56} we obtain $tf - aw = 0$, so that $a = tf/w = t/w^2$. Next equate coefficients of e_{45} to obtain the equation $-rf - bw = 0$, whence $b = -rf/w = -r/w^2$. Finally, equating the coefficients of e_{4567} we obtain the equation

$$-ar - bt - fv + dw = -1.$$

Solving for d we have

$$d = w^{-1}(-1 + ar + bt + fv) = w^{-1}(-1 + rt/w^2 - rt/w^2 + v/w)$$

so that $d = (v - w)/w^2$, completing the proof. ■

At this point we focus attention on e_β. In view of Lemmas 4.2.31 and 4.2.32 we have

$$\begin{aligned} e_\beta = \ & r(e_{11221000} - e_{11121000}) + \\ & t(e_{11111110} + e_{01111111} - e_{01011111} + e_{11111100}) + \\ & v(e_{11221100} - e_{11121110} + e_{11221110} - e_{11121100}) + \\ & w(e_{01122110} + e_{01122100} - e_{11221110} - e_{11121100} + e_{11111000}). \end{aligned}$$

Lemma 4.2.34 *Let $\delta \in \Sigma(G)$ be a root for which e_δ is one of the root vectors appearing in the above expression for e_β.*

(i) *If $\delta+\alpha_3 \in \Sigma(G)$, then $U_{-3}(-1)U_3(c)U_{-3}(1)e_\delta = (1-c)e_\delta + c(e_{\delta+\alpha_3})$.*

(ii) *If $\delta-\alpha_3 \in \Sigma(G)$, then $U_{-3}(-1)U_3(c)U_{-3}(1)e_\delta = (1+c)e_\delta - c(e_{\delta-\alpha_3})$.*

(iii) *If $\delta+\alpha_7 \in \Sigma(G)$, then $U_{-7}(-1)U_7(-c)U_{-7}(1)e_\delta = (1+c)e_\delta + c(e_{\delta+\alpha_7})$.*

(iv) *If $\delta-\alpha_7 \in \Sigma(G)$, then $U_{-7}(-1)U_7(-c)U_{-7}(1)e_\delta = (1-c)e_\delta - c(e_{\delta-\alpha_7})$.*

Proof This is a straightforward calculation using the E_8 structure constants in the Appendix in Section 11. In particular, we need the observation that if δ is as in (i), (ii), (iii), or (iv), then $N(\alpha_3,\delta) = 1, N(-\alpha_3,\delta) = 1, N(\alpha_7,\delta) = -1, N(-\alpha_7,\delta) = -1$, respectively. ■

Lemma 4.2.35 *$S(c)$ fixes e_β for each $c \in K^*$.*

Proof By Lemma 4.2.24 we have $S(c) = S_3(c)S_7(c)$, where $S_3(c) = U_{-3}(-1)U_3(c)U_{-3}(1)$ and $S_7(c) = U_{-7}(-1)U_7(-c)U_{-7}(1)$. Note that the factors $S_3(c)$ and $S_7(c)$ commute. We apply $S(c)$ to the above expression for e_β. The following observations simplify the computation. Note that the roots appearing in the term with coefficient r are only affected by the $S_3(c)$ factor of $S(c)$ and the roots appearing in this term differ by α_3. Then Lemma 4.2.34 immediately shows that the difference of these root vectors is fixed by $S_3(c)$ so that this term in e_β is fixed.

Similar considerations apply to the difference of the second and third root vectors appearing in the term with coefficient t and also the first and fourth root vectors of this term (but now using $S_7(c)$). So the terms with coefficients r and t are both fixed.

The terms with coefficients v and w are a little more complicated. The last root vector appearing in the w term is fixed by both $S_3(c)$ and $S_7(c)$. The remaining ones appear, so we can focus on the remaining root vectors, which appear in both the v and w terms of e_β. The computation is more complicated as both factors of $S(c)$ affect the terms, but here too we apply Lemma 4.2.34 and find that the sum of the v and w terms is fixed by $S(c)$. ■

We can now complete the proof of Proposition 4.2.22. We have just seen that $S(c)$ fixes e_β. An application of Lemma 4.2.33 then shows that $S(c)$ fixes

$e_{-\beta}$ as well. Also, Lemma 4.2.24 shows that $S(c)$ fixes e_α and f_α. Hence, for each $c \in K^*$, $S(c)$ centralizes $L(X)'$. But then $L(X)'$ is centralized by a 1-dimensional unipotent group, contradicting Lemma 2.2.10(ii).

This proves Proposition 4.2.22, and completes our proof of Proposition 4.2.17.

The proof of Theorem 4.1 for $p = 3$ is now complete.

4.3 The case $p = 2$

Assume $p = 2$, and let X be a maximal S-invariant subgroup of G with $X = A_2$. As usual, X is adjoint and all composition factors of $L(G) \downarrow X$ are of the form ab with $a \equiv b \bmod 3$. Recall that G is of adjoint type and $L = L(G)'$. We begin with a preliminary lemma giving the structure of some Weyl modules and Ext groups.

Lemma 4.3.1 *Let $X = A_2$ and $p = 2$.*

(i) $\dim \mathrm{Ext}_X(ab, K) \le 1$, *and equality holds if and only if ab is a field twist of* 30 *or* 03.

(ii) $W_X(30) = 30|00$ *and* $W_X(22) = 22|(30 + 03)|00$ *(co-socle series).*

(iii) *We have*

$$\begin{aligned}
W(41) &= 41/22/03/30,\\
W(60) &= 60/41/03/00,\\
W(52) &= 52/60/14/41/22/30/03/00,\\
W(33) &= 33,\\
W(44) &= 44/52/25/60/06/14/41/22/30/03/00.
\end{aligned}$$

Proof Part (i) follows from [35]. The remaining parts can be obtained using the Sum Formula. ∎

Let n_{ab} denote the multiplicity of the composition factor ab in $L \downarrow X$. Since $C_L(X) = 0$ by Lemma 2.2.10(iv), Lemma 4.3.1 immediately gives

Lemma 4.3.2 *We have $n_{00} < 2(n_{30} + n_{60} + n_{12,0} + \ldots)$.*

Using this lemma and the Weight Compare Program we obtain the following lemma, where n_4 denotes the number of T-weights on $L(G)$ divisible by 4.

Lemma 4.3.3 *The possibilities for $L \downarrow X$ are as follows:*

G	*Case*	$L \downarrow X$	T*-labelling*	n_4
F_4	(1)	$41/14/22/30/03/11$	0202	24
E_6	(2)	$22^3/30^2/03^2/11^2/00^2$	002020	46
	(3)	$41/14/22^2/30^2/03^2/11$	200202	38
	(4)	$44/52/25/22/11$	222022	38
E_7	(5)	$22^3/30^4/03^4/11^4/00^4$	0002000	69
	(6)	$41/14/22^4/30^4/03^4/11/00^2$	0020020	69
	(7)	$44/52/25/60/06/41/14/22/30/03/11$	2000202	69
	(8)	$33/41/14/22^2/30/03/11^2$	2000202	69
E_8	(9)	$41^2/14^2/22^6/30^8/03^8/11^2/00^4$	00002000	120
	(10)	$71/17/44/60/06/41/14/22^2/30^2/03^2/11$	00200200	136
	(11)	$44^3/52^2/25^2/60^2/06^2/41/14/22^2/30^2/03^2/11/00^2$	00200200	136
	(12)	$33^2/41/14/22^5/30^2/03^2/11^3/00^2$	02000020	136
	(13)	$60/06/41^3/14^3/22^7/30^6/03^6/11/00^4$	02000020	136
	(14)	$52/25/33/60/06/41^2/14^2/22^2/30^3/03^3/00^6$	00200020	120
	(15)	$82/28/63/36/44^2/60^2/06^2/41/14/22/11$	20002020	136
	(16)	$82/28/90/09/44^2/52^2/25^2/60^2/06^2/41^2/14^2/$ $22/11$	20002020	136
	(17)	$33/41^2/14^2/22^4/30^6/03^6/00^8$	00200002	120
	(18)	$44/52^2/25^2/60/06/41^2/14^2/22^3/30^2/03^2/11^2/00^2$	00020002	120
	(19)	$55/90/09/52/25/60^2/06^2/41^3/14^3/30/03/00^4$	20020002	120
	(20)	$44/52/25/60^3/06^3/41^3/14^3/22/30^3/03^3/11/00^8$	20000202	136
	(21)	$41/14/22^8/30^8/03^8/11/00^{14}$	20000020	136
	(22)	$22^7/30^7/03^7/11^7/00^{10}$	00200000	136

As usual, let $A = C_L(L(X))$ and let R be the subalgebra of A generated by nilpotent elements.

Lemma 4.3.4 *Cases* (14), (17) *and* (19) *of Lemma* 4.3.3 *do not occur.*

Proof In these cases $L \downarrow X$ has no composition factor 11, contradicting the fact that $L(X)$ must appear. ∎

Lemma 4.3.5 *Case* (1) *of Lemma* 4.3.3 *does not occur.*

Proof Here $G = F_4$, $L(G) \downarrow X = 41/14/22/30/03/11$ and the T-labelling of G is 0202. The short root elements of $L(G)$ generate a 26-dimensional ideal M, and as G-modules we have $M \cong V_G(\lambda_4)$ and $L(G)/M \cong V_G(\lambda_1)$ (see Lemma 2.1.1). Write $N = L(G)/M$.

The highest short root of G is 1232, which affords T-weight 8. It follows that $M \downarrow X$ has 22 as a composition factor, and hence

$$M \downarrow X = 22/30/03, \quad N \downarrow X = 41/14/11.$$

Now N has a submodule $L(X) \cong 11$. Let α, β be fundamental roots for X, with corresponding elements $e_\alpha, e_\beta, f_\alpha, f_\beta \in L(X)$ in the usual notation. Then e_α, e_β afford T-weight 2. The weight 2 subspace W of N is 4-dimensional, corresponding to the 4 long roots 0100, 1100, 0120, 1120 of F_4. Moreover, W admits the action of the subgroup $A_1A_1 = \langle U_{\pm\alpha_1}, U_{\pm\alpha_3}\rangle$ acting as SO_4. This group has two orbits on 1-spaces in W and singular vectors are long root vectors in $L(G)$ (modulo M). The 2-space $\langle e_\alpha, e_\beta\rangle$ in W must contain a singular vector, hence a long root vector modulo M. Now Lemma 2.2.12 gives a contradiction since $L(X) + M$ is S-invariant and contains a long root element of G. ■

Lemma 4.3.6 *Cases* (2), (4), (11), (15) *and* (16) *of Lemma* 4.3.3 *do not occur.*

Proof We begin with cases (2), (4), and (11). We first argue that in these cases A has a submodule of high weight 22, 44, 44, respectively. By [35], the only relevant irreducibles which extend 22 are 30, 03, and those which extend 44 are 60,06. Hence the assertion is immediate in case (4). Cases (2) and (11) are similar and we illustrate the argument in the latter case. Let v be a weight vector for weight 44. Assuming that 44 does not occur in A we see that $\langle Xv\rangle$ is an image of $W_X(44)$, and by Lemma 4.2.5 either 60 or 06 occurs as a composition factor. Also Lemma 2.1.5 shows that the maximal submodule is singular. Now there is a 3-space of such weight vectors and applying the above reasoning to all v in this space and adding the maximal submodules we obtain a singular subspace containing composition factors $60^a, 06^b$ with $a + b \geq 3$. But then L has composition factors $60^{a+b}, 06^{a+b}$, a contradiction.

In case (2), $D = T_1D_5$, whereas Lemma 2.3.4 forces D to be semisimple. In cases (4) $G = E_6$ and $D = A_1A_5$, while in case (11) $G = E_8$ with $D = A_1E_7$. One finds that in both cases the A_1 has label 4, while the other factor has label 44044 and 0004000, respectively. From the above A contains a T-weight vector of weight 16 and the labelling implies that this is a root vector of the second factor and hence a root vector of L. This contradicts Lemma 2.2.12.

Now consider cases (15) and (16). The labelling is the same in both case and $D = A_1E_7$, where the first factor has label 4 and the second has label

0040040. If either 82 or 28 appears as a composition factor of A, then as above A contains a root element and we have the same contradiction as in the previous paragraph. So assume this does not occur.

We give the argument for case (15), the other case being entirely similar. Let v, v' be weight vectors of weight 82 and 28, respectively so that $J = \langle Xv \rangle$ and $J' = \langle Xv' \rangle$ are images of the corresponding Weyl modules. We are assuming that 82 and 28 are not composition factors of A, so from our information on composition factors and Lemma 4.2.5 we conclude that 63 and 36 must occur as composition factors of J and J', respectively. Let R be the maximal submodule of J. Then Lemma 2.1.5 implies that R is singular and 63 is a composition factor of R. So 36 is a composition factor of $L/R^{\perp}$. Also, a consideration of composition factors shows that $v' \in R^{\perp}$ so that $J' \leq R^{\perp}$ and hence 36 is a composition factor of $R^{\perp}$. But this is a contradiction as 36 has multiplicity 1 as an X-composition factor of L. ■

Lemma 4.3.7 *Cases* (8), (9), (10), (12), (13) *and* (18) *of Lemma* 4.3.3 *do not occur.*

Proof The torus T of $X = A_2$ consists of elements $T(c) = \mathrm{diag}(c^2, 1, c^{-2})$ with $c \in K^*$. Define another 1-dimensional torus $T_1 = \{T_1(c) : c \in K^*\}$ in X, where $T_1(c) = \mathrm{diag}(c, c, c^{-2})$. Observe that $T < C_X(T_1) = T_1A_1$. Note that $T(c) = T_1(c)T_\alpha(c)$ where T_α is the torus of A_1 with $T_\alpha(c) = \mathrm{diag}(c, c^{-1}, 1)$.

In each of the cases under consideration we shall calculate $C_G(T_1)$ and its T-labelling. For this we need to find the T-weights on $C_V(T_1)$ for each composition factor V of $L \downarrow X$. This is routine, and the conclusions are as follows:

V	$\dim C_V(T_1)$	T-weights on $C_V(T_1)$
11	4	$2, 0^2, -2$
30	2	$2, -2$
41	0	
33	16	$6, 4^2, 2^3, 0^4, -2^3, -4^2, 6$
52	4	$6, 2, -2, -6$
71	8	$6, 4^2, 2, -2, -4^2, -6$

In cases (8), (10), and (12) we find that $C_G(T_1)$ has dimension 36, 44, and 122, respectively. However, one checks that G has no Levi subgroup of this dimension, so this is impossible.

In case (9) we have $\dim C_G(T_1) = 68$. The only possibility is that $C_G(T_1) = D_6T_2$. The non-negative T-weights here are $4^6, 2^{18}, 0^{20}$. However an easy check shows that there is no possible T-labelling of D_6 which is compatible with these weights. Similarly, in case (13) we have $\dim C_G(T_1) = 64$ and the only possibility is that $C_G(T_1) = A_7T_1$. But here the non-negative T-weights are $4^9, 2^{13}, 0^{20}$ and again we see that there is no compatible labelling.

Finally, in case (18) we have $\dim C_G(T_1) = 54$. A check of Levi subgroups shows that the only possibility is $C_G(T_1) = A_2D_5T_1$, and the non-negative T-weights on this are $8, 6^4, 4^5, 2^{10}, 0^{14}$. The T-labelling of A_2D_5 must be 02, 02022. Consider $T < A_1T_1 < A_2D_5T_1$. Now T and T_α have the same action on the A_2 factor. Since T has labelling 02 on this factor, it follows that T_α, and hence also the A_1 subgroup of X containing it, project nontrivially to the A_2 factor. But this latter projection must be either a long root A_1 or an irreducible A_1. In either case $C_{A_2}(T_\alpha) = C_{A_2}(T) = T_2$. But this is not consistent with the labellings of T. ■

Lemma 4.3.8 *Cases* (5) *and* (6) *of Lemma* 4.3.3 *do not occur.*

Proof Consider first case (5). Here T has labelling 0002000. We shall calculate the composition factors of X on the 56-dimensional $\hat{G}$-module $V_{56} = V_{\hat{G}}(\lambda_7)$. We can identify X with its connected preimage in $\hat{G}$ and consider X as acting on V_{56}. Now

$$\lambda_7 = \frac{1}{2}(2346543)$$

from which we calculate that the non-negative T-weights on V_{56} are 6^4, 4^6, 2^{12}, 0^{12}. Hence $V_{56} \downarrow X$ has composition factors ab with $2a + 2b = 6$, and since $a \equiv b \bmod 3$ it must be the case that $ab = 30$ or 03. As V_{56} is self-dual it follows that X has composition factors $30^2/03^2$. These take care of T-weights $6^4, 4^4, 2^8, 0^4$, leaving $4^2, 2^4, 0^8$ to account for. The 4^2 can only be accounted for by X-composition factors 11^2, and so we conclude that

$$V_{56} \downarrow X = 30^2/03^2/11^2/00^4.$$

By [35], $\dim \mathrm{Ext}_{A_2}(30, K) = 1$ and 11 does not extend the trivial module, so it follows that X has a non-trivial fixed space on V_{56}. This contradicts Lemma 2.2.13.

Case (6) is similar. Here the T-labelling is 0020020 and the non-negative T-weights on V_{56} are $8^2, 6^4, 4^8, 2^8, 0^{12}$. Hence arguing as above we obtain

$$V_{56} \downarrow X = 22^2/30^2/03^2/00^4,$$

and so X has a non-trivial fixed space on V_{56} again. ■

Lemma 4.3.9 *Cases (21) and (22) of Lemma 4.3.3 do not occur.*

Proof In these cases we argue that X has a fixed point on L which will contradict Lemma 2.2.10.

We begin with Case (21). Assume there is no fixed point. Here $L \downarrow X = 41/14/22^8/30^8/03^8/11/00^{14}$. Generating by weight vectors of weight 41 and 14 and using Lemma 2.1.5 we obtain a submodule J_1 containing a singular submodule, R_1 such that $J_1/R_1 = 41 \oplus 14$ and $R_1 = 22^a/30^b/03^c$.

Then $J_1^{\perp}/J_1 = (41 \oplus 14) \perp M_1$, where M_1 has highest weight 22. Generating by weight vectors in M_1 of weight 22 we obtain a module having a singular submodule of shape $30^x/03^y/00^d$ and quotient 22^{8-2a}. If $d > 0$, then taking preimages of this space (and noting that 22 is the highest weight) we see from the structure of $W(22)$ that there is a fixed point, against our assumption. Hence $d = 0$. Take preimages of the singular space to obtain a singular space $R_2 = 22^a/30^{b+x}/03^{c+y}$.

Repeat the argument in $R_2^{\perp}/R_2 = (41 \oplus 14) \perp 22^{8-2a} \perp M_2$, where $M_2 = 30^{8-(b+c+x+y)}/03^{8-(b+c+x+y)}/11/00^{14}$. Here we generate by weight vectors of weight 30 and 03 and take preimages of the corresponding singular submodule of the form 00^r to obtain a singular space $R_3 = 22^a/30^{b+x}/03^{c+y}/00^r$. Then $R_3^{\perp}/R_3 = (41 \oplus 41) \perp 22^{8-2a} \perp (30^{8-(b+c+x+y)} \oplus (03^{8-(b+c+x+y)}) \perp 11 \perp 00^{14-2r}$.

Take preimages of the submodule 00^{14-2r} and obtain a submodule $J = 22^a/30^{b+x}/03^{c+y}/00^{14-r}$. We are assuming that this does not contain a trivial submodule, so $14 - r \leq a + b + c + x + y$. By construction we have $r \leq 2(8 - (b + c + x + y))$, so combined with the previous inequality we have $14 \leq a + 16 - (b + c + x + y)$. So $b + c + x + y \leq 2 + a \leq 4$. Hence, $r \geq 14 - (a + b + c + x + y) \geq 14 - 2 - 4 \geq 8$. But then R_3 contains a trivial submodule, a contradiction.

Finally, consider case (22). This is similar to the previous case, but much simpler. Assume there does not exist a fixed point. Then from the structure of $W(22)$ we see that generating by weight vectors of weight 22 we obtain a submodule J_1 having singular submodule $R_1 = 30^a/03^b$. Then $R_1^{\perp}/R_1 = 22^7 \perp M_1$, with $M_1 = 30^{7-(a+b)}/03^{7-(a+b)}/11^7/00^{10}$. Here $a + b \leq 7$. Next generate by weight vectors in M_1 of weights 30 and 03. Each must be irreducible, since otherwise a weight vector in the preimage would generate the corresponding Weyl module which has a fixed point. It follows that

$M_1 = (30^7 \oplus 03^7) \perp 11^7 \oplus 00^{10}$. But then the preimage of the fixed space of M_1 contains a fixed point. This is a contradiction. ■

The three remaining cases of Lemma 4.3.3 - cases (3), (7) and (20) - are much less straightforward than the previous cases.

Lemma 4.3.10 *Case* (3) *of Lemma* 4.3.3 *does not occur.*

Proof Here $G = E_6$ and $L(G) \downarrow X = 41/14/22^2/30^2/03^2/11$. We have $T = \{T(c) : c \in K^*\}$, where $T(c) = \mathrm{diag}(c^2, 1, c^{-2})$, and T has labelling 200202 in G.

Define $R(c) = \mathrm{diag}(c, c, c^{-2}) \in X$, and let R be the 1-dimensional torus in X consisting of all $R(c)$ for $c \in K^*$. The R-weights on the X-composition factors in $L(G)$ are easily calculated, and are as follows:

λ	R-weights on $V_X(\lambda)$
11	$3^2, 0^4, -3^2$
30	$3^4, 0^2, -3^2, -6$
41	$6^2, 3^4, -6, -9^2$

Hence the non-negative R-weights on $L(G)$ are $9^2, 6^9, 3^{18}, 0^{20}$. In particular, $C_G(R)$ has dimension 20. Since $C_G(R)$ is a Levi subgroup, a quick check shows that the only possibility is $C_G(R) = A_1A_2A_2R$.

Let α, β be fundamental roots for $X = A_2$, chosen so that $C_X(R)' = J_\alpha$, the fundamental SL_2 in X corresponding to the root α. Then J_α induces an irreducible of high weight 2 on $C_{V_X(30)}(R)$, and an indecomposable 0/2/0 on $C_{V_X(11)}(R)$.

Now $J_\alpha \leq C_G(R)' = A_1A_2A_2$. The non-negative T-weights on $L(C_G(R))$ are $4^2, 2^5, 0^6$, from which it follows that the embedding of J_α in $A_1A_2A_2$ is via representations with composition factors $1, 2/0, 2/0$.

Let U be the unipotent radical of a parabolic subgroup $UJ_\alpha R$ of X, chosen so that $U < Q$, where Q is the unipotent radical of a parabolic subgroup $QC_G(R)$ of G. Then $L(U)$ is a 2-dimensional subspace of the 18-dimensional space V_3 of vectors in $L(Q)$ of R-weight 3. Moreover, $C_G(R)' = A_1A_2A_2$ acts on V_3 as a tensor product $1 \otimes 10 \otimes 10$. Thus $L(U)$ is an irreducible J_α-submodule of high weight 1 in $V_3 \downarrow J_\alpha$.

We next assert that the homogeneous component of the socle of $V_3 \downarrow J_\alpha$ corresponding to the irreducible of high weight 1 is 1^2, the sum of two irreducible 1's. To see this first observe that as a J_α-module $1 \otimes (2/0) = 3+1$.

We also note that $1 \otimes 2 \otimes 2$ is a tensor product of tilting modules and has highest weight 5. Dimension considerations then show that this tensor product is $T(5) = 1|5|1$. Therefore,

$$\begin{aligned} 1 \otimes (2/0) \otimes (2/0) &= (3+1) \otimes (2/0) = (3 \otimes (2/0)) + (3+1) \\ &= (1 \otimes 2 \otimes (2/0)) + 3 + 1 = ((3+1) \otimes 2) + 3 + 1 \\ &= (1 \otimes 2 \otimes 2) + 3^2 + 1 = T(5) + 3^2 + 1 \\ &= (1|5|1) + 3^2 + 1. \end{aligned}$$

This proves the assertion.

We take $C_G(R)' = A_1A_2A_2$ to have (ordered) fundamental root system $\{\alpha_2\}, \{\alpha_1, \alpha_3\}, \{\alpha_5, \alpha_6\}$, respectively, (with the usual labelling of roots for $G = E_6$). The embedding of J_α in each factor A_2 is either completely reducible $2+0$, or indecomposable $2|0$ or indecomposable $0|2$. These representations are the action of J_α on the module $V_{A_2}(\lambda_1)$. Choose a 1-dimensional torus $T_\alpha < J_\alpha$ such that $T_\alpha(c) = T(c)R(c)^{-1}$ for $c \in K^*$, and let $U_\alpha < U$ be a T_α-root group in J_α. Then on the J_α-modules $2+0$, $2|0$, $0|2$ respectively, $U_\alpha(c)$ acts as the following matrices (relative to a basis of vectors of T_α-weights $2, 0-2$ in this order):

$$\begin{pmatrix} 1 & 0 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & c^2 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & c & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence we may take it that one of the following holds, denoting $U_{\alpha_i+\alpha_j+\dots}(c)$ by $U_{ij\dots}(c)$:

(a) $U_\alpha(c) = U_2(c)U_3(c)U_{13}(c^2)U_5(c)U_{56}(c^2)$ (embedding $1, 2|0, 0|2$)
(b) $U_\alpha(c) = U_2(c)U_3(c)U_{13}(c^2)U_6(c)U_{56}(c^2)$ (embedding $1, 2|0, 2|0$)
(c) $U_\alpha(c) = U_2(c)U_3(c)U_{13}(c^2)U_{56}(c^2)$ (embedding $1, 2|0, 2+0$)
(d) $U_\alpha(c) = U_2(c)U_{13}(c^2)U_{56}(c^2)$ (embedding $1, 2+0, 2+0$).

We shall work within $L(Q)$. First choose notation so that $L(Q)$ is spanned by root vectors e_r with $r \in \Sigma(G)^+$ having positive α_4-coefficient. Then V_3 is spanned by e_r with r having α_4-coefficient 1. The highest such root is 111111, so e_{111111} is a maximal vector for the action of the Levi subgroup $C_G(R)'$ on V_3. This root affords R-weight 4. Hence we calculate that the 1-weight space for T_α in V_3 is

$$V_{1,+} = \langle e_{111100}, e_{010111}, e_{011110}, e_{001111}, e_{101110} \rangle.$$

We also consider the opposite unipotent radical $L(Q)^-$, spanned by root vectors $e_{-r} = f_r$ with $r \in \Sigma(G)^+$ having α_4-coefficient 1. Here f_{000100} is a maximal vector, and the 1-weight space for T_α on V_{-3} is

$$V_{1,-} = \langle f_{001110}, f_{000111}, f_{101100}, f_{010110}, f_{011100} \rangle.$$

We now consider separately the possibilities (a)-(d) above for $U_\alpha(c)$.

Case (a) Here $L(U_\alpha) = \langle e_{010000} + e_{001000} + e_{000010} \rangle$. The roots involved, namely $\alpha_2, \alpha_3, \alpha_5$, are mutually orthogonal, so the generator $e_{010000} + e_{001000} + e_{000010}$ is a nilpotent element of $L(G)$ of type $3A_1$.

We work with $V_{1,+}$. Calculation shows that the fixed space of U_α on this space is $\langle x, y \rangle$, where

$$x = e_{011110}, \ \ y = e_{111100} + e_{010111} + e_{001111} + e_{101110}.$$

Hence $L(U_{\alpha+\beta}) = \langle cx + dy \rangle$ for some $c, d \in K$. The roots involved in x, y lie in a subsystem

$$A_1A_2A_2 = \langle 011110 \rangle \times \langle 111100, 001111 \rangle \times \langle 010111, 101110 \rangle.$$

Hence $cx + dy$ is a nilpotent element of type $A_1A_2A_2$ if $c, d \neq 0$, of type A_2A_2 if $c = 0$, and of type A_1 if $d = 0$. In any case it is not of type $3A_1$, which is a contradiction.

Case (b) Here $L(U_\alpha) = \langle e_{010000} + e_{001000} + e_{000001} \rangle$, again of type $3A_1$. The fixed space of U_α on $V_{1,+}$ is $\langle v, w \rangle$ where

$$v = e_{111100} + e_{010111} + e_{011110}, \ w = e_{011110} + e_{001111},$$

and hence $L(U_{\alpha+\beta}) = \langle av + bw \rangle$ for some $a, b \in K$. The roots involved in v, w lie in a subsystem

$$A_1A_1A_2 = \langle 010111 \rangle \times \langle 011110 \rangle \times \langle 111100, 001111 \rangle.$$

Hence we see that the fact that $av + bw$ must be of type $3A_1$ forces $b = 0$, so

$$L(U_{\alpha+\beta}) = \langle v \rangle.$$

Now consider $V_{1,-}$. The fixed space of U_α on $V_{1,-}$ is $\langle t, u \rangle$ where

$$t = f_{001110} + f_{010110}, \ u = f_{001110} + f_{000111} + f_{101100}.$$

It follows that $L(U_{-\beta}) = \langle a't + b'u \rangle$ for some $a', b' \in K$. Now $[t, v] = e_{010000} + e_{001000} + e_{000001}$ and $[u, v] = e_{010000}$. Hence the fact that $[L(U_{\alpha+\beta}), L(U_{-\beta})] =$

$L(U_\alpha)$ forces $b' = 0$. But then $L(U_{-\beta}) = \langle t\rangle$ which is not of type $3A_1$, a contradiction.

Case (c) Here $L(U_\alpha) = \langle e_{010000}+e_{001000}\rangle$, of type $2A_1$. The fixed space of U_α on $V_{1,+}$ is

$$\langle e_{011110},\ e_{111100}+e_{010111}+e_{001111}\rangle.$$

However, no vector in this 2-space can be of type $2A_1$.

Case (d) Here $L(U_\alpha) = \langle e_{010000}\rangle$. This contains a root element, giving a contradiction by Lemma 2.2.12(ii). ■

Lemma 4.3.11 *Case* (7) *of Lemma* 4.3.3 *does not occur.*

Proof Here $G = E_7$ and $L \downarrow X = 44/52/25/60/06/41/14/22/30/03/11$. The strategy is very similar to that of the proof of the previous lemma. Let R be the 1-dimensional torus of X as in that proof. The non-negative R-weights on $L(G)$ are $12^5, 9^{10}, 6^{15}, 3^{22}, 0^{29}$. We deduce that the R-labelling of $\Pi(G)$ is 0000303, and in particular $C_G(R)' = A_1A_4$. The T-weights on $L(C_G(R))$ are $8, 6^2, 4^3, 2^5, 0^7$ and so the T-labelling of A_1A_4 is 2, 2222. Letting J_α be as in the previous proof, we see from this that the embedding $J_\alpha < A_1A_4$ is via representations with composition factors 1, 4/2/0. Let $V = V_{A_4}(\lambda_1)$, so that $V \downarrow J_\alpha = 4/2/0$.

As before, letting U be the unipotent radical of a maximal parabolic subgroup $UJ_\alpha R$ of X, we have $L(U) \subset V_3$, the 3-weight space for R in $L(Q)$ (where Q is the unipotent radical of a parabolic $QC_G(R)$). Here V_3 is a 22-dimensional space with A_1A_4-action $(1\otimes\lambda_3)\oplus(1\otimes 0)$.

By Lemma 2.1.6, the J_α-modules with composition factors 4/2/0 are the following, together with their duals:

$$\text{(a) } 2|0|4,\ \ \text{(b) } (0|2)+4,\ \ \text{(c) } (0|4)+2,\ \ \text{(d) } 4+2+0.$$

In cases (c) and (d) the module is a Frobenius 2-twist of a rational module (in (c) it is $((0|2)+1)^{(2)}$, in (d) it is $(2+1+0)^{(2)}$). It follows that if $V\downarrow J_\alpha$ is as in (c) or (d) then $L(U_\alpha)$ is spanned by a root element of $L(G)$, contradicting Lemma 2.2.12(ii).

Hence we may assume that $V \downarrow J_\alpha$ is as in case (a) or (b). We calculate the action of $U_\alpha(c)$ on V in each of these cases. In case (a), as a J_α-module V is the space of homogeneous polynomials of degree 4 in two variables x, y, with $U_\alpha(c)$ sending $x \to x, y \to cx + y$, so relative to an ordered basis of R-weight vectors of weights $4, 2, 0, -2, -4$, $U_\alpha(c)$ acts as the matrix

$$\begin{pmatrix} 1 & c & c^2 & c^3 & c^4 \\ & 1 & 0 & c^2 & 0 \\ & & 1 & c & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$$

Ordering the roots of $C_G(R)' = A_4A_1$ as $\alpha_1, \alpha_3, \alpha_4, \alpha_2, \alpha_6$, and performing similar calculations for case (b), we obtain the following expressions for $U_\alpha(c)$ in cases (a),(b):

(a) $U_\alpha(c) = U_1(c)U_4(c)U_{13}(c^2)U_{34}(c^2)U_{1342}(c^4)U_6(c)$
(b) $U_\alpha(c) = U_3(c)U_{34}(c^2)U_{1342}(c^4)U_6(c)$.

Working first with positive root vectors for $L(Q)$, and adopting notation as in the previous proof, we see that maximal vectors in V_3 are $e_{1122110}$, affording R-weight 7, and $e_{0000011}$ affording R-weight 1. Hence

$$V_{1,+} = \langle e_{0111110}, e_{1011110}, e_{0000011}, e_{1111100}, e_{0112100} \rangle.$$

Similarly,

$$V_{1,-} = \langle f_{0111100}, f_{1011100}, f_{0000001}, f_{0101110}, f_{0011110} \rangle.$$

Consider first case (a). Here $L(U_\alpha) = \langle e_{1000000} + e_{0001000} + e_{0000010} \rangle$. The fixed space of U_α in $V_{1,-}$ is $\langle f_{0111100}, f_{0000001} \rangle$, which must contain $L(U_{-\beta})$; however it contains no vector of type $3A_1$, giving a contradiction.

Now consider (b). Here $L(U_\alpha) = \langle e_{0010000} + e_{0000010} \rangle$. The fixed space of U_α on $V_{1,-}$ is $\langle f_{1011100}, f_{0000001}, f_{0111100} + f_{0101110} \rangle$, and hence

$$L(U_{-\beta}) = \langle af_{1011100} + bf_{0000001} + c(f_{0111100} + f_{0101110}) \rangle.$$

The roots involved lie in a subsystem

$$A_1A_3 = \langle 0111100 \rangle \times \langle 1011100,\ 0101110,\ 0000001 \rangle.$$

Hence the fact that $L(U_{-\beta})$ is of type $2A_1$ forces either $c \neq 0, a = b = 0$ or $c = 0, a \neq 0, b \neq 0$.

Next, the fixed space of U_α on $V_{1,+}$ is $\langle e_{0111110}, e_{1011110}, e_{0000011} \rangle$ and contains $L(U_{\alpha+\beta})$. Now the fact that $[L(U_{\alpha+\beta}), L(U_{-\beta})] = L(U_\alpha)$ forces

$$L(U_{-\beta}) = \langle f_{0111100} + f_{0101110} \rangle, \quad L(U_{\alpha+\beta}) = \langle e_{0111110} + ge_{1011110} + he_{0000011} \rangle,$$

where $gh = 0$ (as $e_{\alpha+\beta}$ has type A_1A_1).

Now J_α contains a Weyl group element s_α inverting T_α. Since the labelling of T_α in A_1A_4 is 2, 2222, we see that $s_\alpha = w_0(A_1)w_0(A_4) = s_{\alpha_6}w_0(A_4)$. As $w_0(A_4)$ sends α_5 to 1122100, we see that

$$L(U_\beta) = L(U_{\alpha+\beta})^{s_\alpha} = \langle e_{0111100} + ge_{1011100} + he_{0000001}\rangle,$$

$$L(U_{-\alpha-\beta}) = L(U_{-\beta})^{s_\alpha} = \langle f_{0111110} + f_{0112100}\rangle.$$

Now $L(X)$ is generated by $L(U_{\pm\beta}), L(U_{\pm(\alpha+\beta)})$. Since either $g = 0$ or $h = 0$ only six roots elements appear in the expressions for the generators, and so, since G has rank 7, there is a 1-dimensional torus T_1 centralizing all of them, hence centralizing $L(X)$. This contradicts Lemma 2.2.10(ii). ∎

The final case (20) of Lemma 4.3.3 takes a great deal more effort than all the other cases. We state it as a proposition, and prove it in a series of lemmas.

Proposition 4.3.12 *Case* (20) *of Lemma* 4.3.3 *does not occur.*

Assume case (20) holds, so $G = E_8$ and

$$L \downarrow X = 44/52/25/60^3/06^3/41^3/14^3/22/30^3/03^3/11/00^8.$$

Fix notation for X with T_X-root subgroups $U_{\pm\alpha}, U_{\pm\beta}, U_{\pm(\alpha+\beta)}$. For each root δ we let e_δ denote a generator of $L(U_\delta)$. Recall that the torus T consists of matrices $T(c) = \mathrm{diag}(c^2, 1, c^{-2})$ for $0 \neq c \in K$. Similarly let $T_1(c) = \mathrm{diag}(c, c, c^{-2})$ and T_1 the corresponding 1-dimensional torus of X. As before we have $C_X(T_1) = T_1J_\alpha$ and $T(c) = T_1(c)T_\alpha(c)$.

The proof proceeds in a series of lemmas.

Lemma 4.3.13 (i) *$C = C_G(T_1)' = A_1A_6$ is a Levi subgroup of a parabolic subgroup of G conjugate to P_3.*

(ii) *T_1 determines the labelling* 00300000 *of the Dynkin diagram of G.*

(iii) *T_α determines the labelling* 3 *for the A_1 factor and* 220022 *for the A_6 factor of $C_G(T_1)$.*

Proof To find $C_G(T_1)$ we find the T_1-weights on all composition factors in $L \downarrow X$, and also the T_α weights and their multiplicities on $C = C_G(T_1)$. We find that $\dim C = 52$ and the non-negative T_α-weights on $L(C)$ are

$8, 6^2, 4^7, 2^9, 0^{14}$. As C is a Levi subgroup, the only possibility is $C' = A_1A_6$. This gives (i) and (ii). Moreover, the T_α-weights indicated are only consistent with labelling indicated in (iii).

■

At this point we choose a fundamental sytem for G such that P_3 has Levi subgroup C. Then the A_1 factor corresponds to α_1 while the A_6 factor has base $\{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$. In view of the ordering of fundamental roots we let ω_1 denote the fundamental dominant weight for the A_1 factor and ω_i for $i = 2, 4, 5, 6, 7, 8$ denote the fundamental dominant weights for the A_6 factor of C. Let $W = V_{A_1}(1)$ and $V = V_{A_6}(100000)$ be the natural modules for the factors. Using projections we can consider both V, W as modules for J_α.

Lemma 4.3.14 (i) *The T_1-weight spaces L_3, L_{-3} of $L(G)$ corresponding to weights* 3 *and* -3 *are spanned by all root vectors with α_3 coefficient* 1 *or* -1 *respectively.*

(ii) *L_3, L_{-3} afford irreducible modules for $C = A_1A_6$ isomorphic to $V_{A_1}(1) \otimes V_{A_6}(000001) = W \otimes \wedge^2 V^*$ and $V_{A_1}(1) \otimes V_{A_6}(010000) = W \otimes \wedge^2 V$.*

(iii) *$\langle e_\beta, e_{\alpha+\beta}\rangle$ is a subspace of L_3 affording a restricted irreducible J_α-module with maximal vector $e_{\alpha+\beta}$.*

(iv) *$\langle e_{-\beta}, e_{-(\alpha+\beta)}\rangle$ is a subspace of L_{-3} affording a restricted irreducible J_α-module with maximal vector $e_{-\beta}$.*

Proof Parts (i) and (ii) follows directly from [3]. For (iii) and (iv) just note that the subspaces indicated are the Lie algebras of the unipotent radicals of parabolics of X with Levi factor J_α and have the correct T_1-weights. ■

We next consider possible embeddings of J_α in A_1A_6.

Lemma 4.3.15 (i) *J_α has composition factors $4/2/0^3$ on V.*

(ii) *In its action on V, J_α has either an irreducible submodule or irreducible quotient of high weight* 2.

Proof First note that $L(A_6)$ has codimension 1 in $V \otimes V^*$ and that as an expression in terms of A_6-roots, $\lambda_1 = \frac{1}{7}(654321)$. Also irreducible modules for J_α are all self-dual, so the composition factors of J_α on V and V^* are the same. It is now easy to see from the labelling in Lemma 4.3.13(iii) that the composition factors of J_α on V are as indicated in (i).

For (ii), let $v \in L$ be a T_X-weight vector of weight 44. Then v is a maximal vector and hence $E = \langle Xv \rangle$ is an image of the corresponding Weyl module. Of course v is annihilated by e_α and e_β. Suppose $f_\alpha v = 0 = f_\beta v$. Then 44 appears as a composition factor of A. The non-negative T-weights on $L(D)$ are $16, 12^8, 8^{17}, 4^{25}, 0^{34}$. The highest weight is 16 and this is also the highest weight of 44 which occurs within A. As this weight appears with multiplicity 1 it is afforded by a root vector of D (corresponding to the high root) and hence A contains a root vector of $L(G)$, contradicting Lemma 2.2.12. Hence either $f_\alpha v \neq 0$ or $f_\beta v \neq 0$. It follows that either 52 or 25 occurs as a weight of E and thus there is a composition factor of E having this high weight.

Suppose that 52 appears as a composition factor of E. Writing $52 = 40 \otimes 10 \otimes 02$ one checks that E_{-12} (the T_1-weight space in E for weight -12) affords an irreducible module of high weight 2 for J_α. Also, 44 and 52 are the only composition factors of E with nonzero contribution to L_{-12}. On the other hand one checks that L_{-12} has trivial action of the A_1 factor of C and affords V for the A_6 factor. So in this case 2 occurs as a submodule of $V \downarrow J_\alpha$. If instead 25 appears as a composition factor of E, then we consider L_{12}, which affords V^*, and get a submodule 2 of this, hence a quotient 2 of $V \downarrow J_\alpha$. This establishes (iii). ■

Lemma 4.3.16 *We may assume that $V \downarrow J_\alpha$ is one of the following, where each bracketed term represents an indecomposable module:*

(i) $(4|0|2) \oplus 0^2$

(ii) $(0|4|0|2) \oplus 0$

(iii) $(0|4|0) \oplus (0|2)$

(iv) $(0|4) \oplus (0|2) \oplus 0$

(v) $(4|0) \oplus (0|2) \oplus 0$

(vi) $4 \oplus (0|2) \oplus 0^2$

(vii) $(0|(2 \oplus 4)) \oplus 0^2$.

Proof First note that conjugation by s_α inverts T_α and interchanges the roles of L_3 and L_{-3}. Hence the previous lemma shows that we may assume that there is a submodule 2 in $V \downarrow J_\alpha$. We next claim that this submodule is not a direct summand. For if there is such a direct summand, then the representation of J_α on V can be factored through a Frobenius map and hence its differential is zero, giving $L(J_\alpha) = L(A_1)$. But then $L(X)$ contains a root element, contradicting Lemma 2.2.12(ii).

With these observations the conclusion follows easily by factoring out the submodule 2, considering possibilities for this quotient, and then looking at possible preimages. ■

The Levi factor $J_\alpha T_1$ is contained in two maximal parabolic subgroups of X. On the two corresponding unipotent radicals J_α induces an irreducible module of high weight 1, with maximal vectors $e_{\alpha+\beta}$ and $e_{-\beta}$, and T_1 has weights 3 and -3, respectively. Hence the Lie algebras of the unipotent radicals are contained in L_3, L_{-3}, respectively.

We will determine these embeddings explicitly, so we will be interested in J_α-modules of high weight 1 in L_3, L_{-3}. By Lemma 4.3.14, L_3 and L_{-3} afford the modules $W \otimes \wedge^2 V^*$ and $W \otimes \wedge^2 V$ for $C = A_1 A_6$.

As a first step we list the weight vectors of weights $1 \otimes 0$ and $-1 \otimes 2$ in these modules.

Lemma 4.3.17 *In the following we list the roots for which the corresponding root vectors in L_3, L_{-3} have T_α-weight 1. The first set of roots given corresponds to vectors of weight $1 \otimes 0$, and the second to vectors of weight $-1 \otimes 2$.*

$$\begin{array}{ll} L_3: & \{11111110, 11121100, 11122100, 10111111, 11121000\}\ (wt.\ 1 \otimes 0) \\ & \{01122210, 01122110, 01121110, 01111111\}\ (wt.\ -1 \otimes 2) \\ L_{-3}: & \{-01121000, -01121100, -01111110, -00111111, -01122100\} \\ & \{-11110000, -11111000, -11111100, -10111110\} \end{array}$$

Proof Each of L_3, L_{-3} affords an irreducible module for C and the modules have bases of root vectors for roots with coefficient of α_3 equal to $1, -1$, respectively. It is then an easy matter to check that the root vectors corresponding to roots 11122221 and and -00100000 are maximal vectors for the Borel subroup corresponding to positive roots. Moreover, from the labelling we see that these vectors afford T_α-weight 7. Hence to get roots of T_α-weight 1 we must subtract roots from the root system of C with total weight 6. From the labelled diagram one sees that the indicated roots are the only ones possible. ■

At this point we are position to consider the various possibilities for $V \downarrow J_\alpha$, obtaining a contradiction in each case. For each of the possibilities indicated Lemma 4.3.16 we can determine the precise embedding of J_α in $C = A_1 A_6$. In each case we use a basis of V for which T_α has weights $4, 2, 0, 0, 0, -2, -4$.

Lemma 4.3.18 *It is not the case that* $V \downarrow J_\alpha = (4|0|2) \oplus 0^2$.

Proof Assume false. The indecomposable module $4|0|2$ is just the Weyl module for J_α of high weight 4, which can be realized as the dual of the space of homogeneous polynomials of degree 4 in two variables. From here it is easy to choose a suitable basis for V. As usual, writing $U_{ij\ldots}(c)$ for the root element $U_{\alpha_i+\alpha_j+\ldots}(c)$ of C and $e_{ij\ldots}$ for $E_{\alpha_i+\alpha_j+\ldots}$, we find that

$$U_\alpha(c) = U_1(c)U_4(c)U_8(c)U_{45}(c^2)U_{78}(c^2)U_{245678}(c^4)$$

so that

$$L(U_\alpha) = \langle e_\alpha \rangle = \langle e_1 + e_4 + e_8 \rangle.$$

In particular, e_α is a nilpotent element of $L(G)$ of type $A_1A_1A_1$.

Next we consider the action of $U_\alpha(c)$ on the basis for L_3 given in the conclusion of Lemma 4.3.17. Using the E_8 structure constants given in the Appendix, Section 11, it is straightforward to calculate that

$$C_{L_3}(U_\alpha) = \langle e_\delta, e_\gamma \rangle$$

where $\delta = 10111111$ and $\gamma = 11121000$. These are orthogonal roots and so $e_{\alpha+\beta}$, which must lie in this centralizer, is a nilpotent element of type A_1 or A_1A_1. This is a contradiction since e_α and $e_{\alpha+\beta}$ are conjugate under the action of X. ■

Lemma 4.3.19 *It is not the case that* $V \downarrow J_\alpha = (0|4|0|2) \oplus 0$.

Proof Assume false. It is easy to argue from extension theory that there is at most one nonsplit extension of the Weyl module $4|0|2$ by the trivial module. Next note that the irreducible module 3 is also the Weyl module $W(3)$, so that $1 \otimes 3$ is a tilting module, which necessarily is the indecomposable tilting module $T(4)$ of high weight 4. This is a uniserial module of shape $2|0|4|0|2$, so the desired module occurs as the unique maximal submodule of $T(4)$. Using the tensor product expression $T(4) = 1 \otimes 3$, we can easily obtain a matrix expression for $U_\alpha(c)$ and we find that

$$U_\alpha(c) = U_1(c)U_{24}(c^2)U_{45}(c)U_{245}(c^2)U_8(c)U_{4567}(c^2)U_{678}(c^2)U_{5678}(c^2).$$

Then

$$L(U_\alpha) = \langle e_\alpha \rangle = \langle e_{45} + e_8 + e_1 \rangle,$$

so that e_α is a nilpotent element of type $A_1A_1A_1$.

We next compute the fixed points of U_α on L_3 using the basis given in Lemma 4.3.17. From the precise action we see that this fixed space has dimension 2 and is spanned by the vectors $v = e_{11121100} + e_{11122100}$ and $w = e_{10111111} + e_{11121000}$. The four root vectors involved in these expressions lie in a subsystem of type A_1A_3 having base $\{e_{10111111}\}, \{e_{11121000}, e_6, e_5\}$, and any nonzero element of $\langle v, w\rangle$ is of type A_1 or A_1A_1. As $e_{\alpha+\beta}$ lies in this space and is of type $A_1A_1A_1$, we have a contradiction. ■

Lemma 4.3.20 *It is not the case that* $V \downarrow J_\alpha = (0|(2 \oplus 4)) \oplus 0^2$.

Proof Assume false, and proceed as in the previous cases. Note that the first summand of $V \downarrow J_\alpha$ is a submodule of codimension 1 in the module $(0|2)\oplus(0|4)$, which can be regarded as the space of homogenous polynomials of degree 2 in two variables, plus a Frobenius twist of this module. We can then obtain a matrix expression for U_α and find that

$$U_\alpha(c) = U_1(c)U_4(c)U_{24}(c^2)U_{4567}(c^2)U_{245678}(c^2).$$

Hence,

$$L(U_\alpha) = \langle e_\alpha\rangle = \langle e_1 + e_4\rangle,$$

a space generated by a nilpotent element of type A_1A_1. The same considerations show that $L(U_{-\alpha}) = \langle e_{-1} + e_{-567}\rangle$.

We next compute the fixed points of U_α on L_3 and L_{-3}. This is a straightforward computation using the structure constants in the Appendix, Section 11, and we find that

$$e_{\alpha+\beta} = ae_{11121100} + be_{11122100} + ce_{11121000} + d(e_{11111110} + e_{01121110}),$$

where the indicated root vectors span the fixed space of U_α on L_3. The 5 roots in the expression for $e_{\alpha+\beta}$ lie in a subsystem of type A_1A_4, where the A_1 has base $e_{11111110}$ and the A_4 has base $\{e_{11121000}, e_6, e_5, e_{01121110}\}$. Since $e_{\alpha+\beta}$ must have type A_1A_1 we can work within this subsystem, projecting to sl_5 and find that $d \neq 0$ but $a = b = c = 0$. Hence

$$e_{\alpha+\beta} = d(e_{11111110} + e_{01121110}).$$

Similarly,

$$e_{-\beta} = re_{-01111110} + se_{-00111111} + te_{-01122100}.$$

Now $[e_{\alpha+\beta}, e_{-\beta}] = e_\alpha$ which is a multiple of $e_1 + e_4$. It follows that $r \neq 0$. Since $e_{-\beta}$ has type A_1A_1 exactly one of s, t is nonzero. In either case

$e_{-\beta}$ is the sum of two root vectors. So is $e_\beta = (e_{\alpha+\beta})^{s_\alpha}$. But then $L(X) = \langle e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}\rangle$ where each generator is the sum of two root vectors and there is at least one opposite pair of roots represented, and these have the same centralizer in T_X. It follows that there is a 1-dimensional torus centralizing $L(X)$, contradicting Lemma 2.2.10(ii). ∎

Lemma 4.3.21 $V \downarrow J_\alpha \neq (0|4|0) \oplus (0|2)$.

Proof Here we can take the first summand to be a Frobenius twist of $1 \otimes 1$ (the tilting module of high weight 2) and the second summand as the space of homogeneous polynomials of degree 2 in two variables. We then see that

$$U_\alpha(c) = U_1(c)U_{678}(c^2)U_{4567}(c^2)U_{2456}(c^2)U_{245}(c^2)U_{78}(c^2)U_4(c),$$

so that

$$L(U_\alpha) = \langle e_\alpha\rangle = \langle e_1 + e_4\rangle.$$

Computing fixed points of U_α on L_{-3} we find that

$$\begin{aligned} e_{-\beta} = \quad & ae_{-01111110} + b(e_{-00111111} + e_{-01122100}) + \\ & + c(e_{-01121000} + e_{-01121100} + e_{-11111000} + e_{-11111100}). \end{aligned}$$

The roots involved in this expression lie in an A_1A_5 subsystem, where the A_1 factor has base 01111110 and the A_5 factor has base

01121000, 00000100, 00001000, 11110000, 00111111.

Computing the matrix in sl_6 that $e_{-\beta}$ projects to, and using the fact that $e_{-\beta}$ is of type A_1A_1, we conclude that $c = 0$, whence $a = 0$ and we have

$$e_{-\beta} = b(e_{-00111111} + e_{-01122100}).$$

Next consider fixed points of U_α on L_3 and find that

$$e_{\alpha+\beta} = r(e_{11121100} + e_{11121000}) + s(e_{11122100} + e_{10111111}) + t(e_{11111110} + e_{01121110}).$$

We get a contradiction from the relation $[e_{\alpha+\beta}, e_{-\beta}] = e_\alpha$, since the commutator cannot contain the e_4 component of e_α.

∎

Lemma 4.3.22 $V \downarrow J_\alpha \neq (0|4) \oplus (0|2) \oplus 0$.

Proof Assume the contrary. Here the situation requires a bit more effort. We proceed as in previous cases. The nontrivial summands can be regarded as the space of homogeneous polynomials of degree 2 in two variables, and a Frobenius twist of this module. Using this we can determine the matrix expressions of U_α in C and we obtain

$$U_\alpha(c) = U_1(c)U_4(c)U_{2456}(c^2)U_{4567}(c^2)U_{245678}(c^4),$$

so that

$$L(U_\alpha) = \langle e_\alpha \rangle = \langle e_1 + e_4 \rangle.$$

We next calculate the fixed points of U_α on L_{-3} and find

$$e_{-\beta} = ae_{-01111110} + be_{-00111111} + c(e_{-01121000} + e_{-11111000}).$$

The second and fourth roots span an A_2 subsystem and the others are orthogonal to these. It follows $e_{-\beta}$ is in an $A_1A_1A_2$. On the other hand this nilpotent element is conjugate to e_α hence must have type A_1A_1. It follows that either

$$e_{-\beta} = c(e_{-01121000} + e_{-11111000})$$

or

$$e_{-\beta} = ae_{-01111110} + be_{-00111111}.$$

Next we calculate the U_α-fixed points on L_3, and hence find

$$e_{\alpha+\beta} = re_{11121100} + se_{11122100} + te_{11121000} + w(e_{11111110} + e_{01121110}).$$

Here the roots involved lie in a subsystem of type A_1A_4 with base equal to $\{e_{11111110}\} \cup \{e_{111210000}, e_6, e_5, e_{01121110}\}$. Then considering a matrix expression for $e_{\alpha+\beta}$ and using the fact that this element has type A_1A_1, we find that $w \neq 0$, and this forces $r = s = t = 0$. Hence

$$e_{\alpha+\beta} = w(e_{11111110} + e_{01121110}).$$

From the commutator $[e_{\alpha+\beta}, e_{-\beta}] = e_\alpha$ and the known expression for e_α we see that the second expression for $e_{-\beta}$ above must hold, with $a = b$.

There is no contradiction at this point so we must proceed further in the analysis. The next step is to use a reflection $s_\alpha \in N_{J_\alpha}(T_\alpha)$. From the embedding $J_\alpha < C = A_1A_6$ we see that we can take

$$s_\alpha = s_1 s_2^{s_4s_5s_6s_7s_8} s_4^{s_5s_6s_7}.$$

We have $\langle e_{-\alpha-\beta}\rangle = \langle (e_{-\beta})^{s_\alpha}\rangle$ and $\langle e_\beta\rangle = \langle e_{\alpha+\beta}^{s_\alpha}\rangle$. So at this point we can write

$$\begin{aligned}
\langle e_{-\beta}\rangle &= \langle e_{-01111110} + e_{-00111111}\rangle,\\
\langle e_{\beta}\rangle &= \langle e_{01111110} + e_{11110000}\rangle,\\
\langle e_{\alpha+\beta}\rangle &= \langle e_{11111110} + e_{01121110}\rangle,\\
\langle e_{-\alpha-\beta}\rangle &= \langle e_{-11111110} + e_{-10111111}\rangle.
\end{aligned}$$

There are 8 root elements involved in these expressions, and two of these are opposites. Since we are working in E_8 it follows that there is a 1-dimensional torus in G centralizing all of $L(X)$, and this contradicts Lemma 2.2.10(ii). ■

Lemma 4.3.23 $V \downarrow J_\alpha \neq (4|0) \oplus (0|2) \oplus 0$.

Proof The argument here resembles that of the previous lemma. The nontrivial summands can be regarded as the space of homogeneous polynomials of degree 2 and the Frobenius twist of the dual of this module. Using this we can determine the matrix expressions of U_α in C and we have

$$U_\alpha(c) = U_1(c)U_4(c)U_{78}(c^2)U_{4567}(c^2)U_{245678}(c^4),$$

so that

$$L(U_\alpha) = \langle e_\alpha\rangle = \langle e_1 + e_4\rangle.$$

We calculate the U_α-fixed points on L_3 and deduce

$$e_{\alpha+\beta} = ae_{11121000} + be_{10111111} + c(e_{11111110} + e_{01121110}).$$

Here the roots involved lie in a subsystem of type $A_1A_1A_2$ with base equal to $\{e_{11121000}\}\cup\{e_{11111110}\}\cup\{e_{10111111}, e_{01121110}\}$. Then considering a matrix expression for $e_{\alpha+\beta}$ and using the fact that this element is of type A_1A_1, we find that either $c \neq 0$ and $a = b = 0$, or else $c = 0$ and $a, b \neq 0$. Hence either

$$e_{\alpha+\beta} = c(e_{11111110} + e_{01121110})$$

or

$$e_{\alpha+\beta} = ae_{11121000} + be_{10111111}.$$

We next calculate the fixed points of U_α on L_{-3} and find

$$\begin{aligned}
e_{-\beta} = \ & re_{-011111110} + se_{-01122100} + t(e_{-11111000} + e_{-011210000})\\
& +w(e_{-11111100} + e_{-01121100}).
\end{aligned}$$

The roots in this expression lie in an A_1A_4 subsystem with base

$$\{e_{-01111110}\} \cup \{e_{-01121000}, e_{-6}, e_{-5}, e_{-11110000}\}.$$

First assume $e_{\alpha+\beta} = c(e_{11111110}+e_{01121110})$. Since $[e_{\alpha+\beta}e_{-\beta}] = e_\alpha$, we see that $r \neq 0$. This forces $t = w = 0$ so that $e_{-\beta}$ is the sum of two root elements as is $e_\beta = (e_{\alpha+\beta})^{s_\alpha}$. Also, $e_{-\alpha}$ is the sum of two root elements and clearly one of these is e_{-1}. As in the last case we have $L(X) = \langle e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}\rangle$ and this is centralized by a 1-dimensional torus of G, contradicting Lemma 2.2.10(ii).

Now assume $e_{\alpha+\beta} = ae_{11121000} + be_{10111111}$. Here the commutator equation $[e_{\alpha+\beta}e_{-\beta}] = e_\alpha$ implies that $t \neq 0$, which forces $r = w = 0$. Hence, $e_{-\beta} = se_{-01122100} + t(e_{-11111000} + e_{-011210000})$. Once again we use the reflection $s_\alpha \in N_{J_\alpha}(T_\alpha)$. From the embedding $J_\alpha < C = A_1A_6$ we again find that

$$s_\alpha = s_1 s_2^{s_4s_5s_6s_7s_8} s_4^{s_5s_6s_7}.$$

Hence $e_\beta = (e_{\alpha+\beta})^{s_\alpha} = xe_{01121000}+ye_{00111111}$. It follows that the generators of $L(X) = \langle e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}\rangle$ involve 9 root vectors, but there are two pairs of opposite roots involved. Consequently there is again a 1-dimensional torus centralizing $L(X)$, contradicting Lemma 2.2.10(ii). ■

Lemma 4.3.24 $V \downarrow J_\alpha \neq 4 \oplus (0|2) \oplus 0^2$.

Proof As in other cases we note that the summand $0|2$ can be regarded as homogeneous polynomials of degree 2 and using this we obtain an expression for elements of U_α:

$$U_\alpha(c) = U_1(c)U_4(c)U_{4567}(c^2)U_{245678}(c^4),$$

so that

$$L(U_\alpha) = \langle e_\alpha\rangle = \langle e_1 + e_4\rangle$$

and e_α is a nilpotent element of type A_1A_1. We next calculate fixed points of U_α in L_3 to get

$$e_{\alpha+\beta} = a(e_{01121110}+e_{11111110})+be_{11121100}+ce_{11122100}+de_{10111111}+ee_{11121000}.$$

Observe that all roots in the above expression occur in a subsystem of type $A_1A_1A_4$ with base $\{e_{11111110}\}\cup\{e_{10111111}\}\cup\{e_{11121000}, e_6, e_5, e_{01121110}\}$. Now $e_{\alpha+\beta}$ is conjugate to e_α so projecting $e_{\alpha+\beta}$ to the A_4 factor and considering matrices we conclude that either $a \neq 0$ and $b = c = d = e = 0$, or else $a = 0, d \neq 0$, and at least one of b, c, e is nonzero.

Next we calculate fixed points on L_{-3}, and deduce that

$$\begin{aligned} e_{-\beta} = \ & r(e_{-11111000} + e_{-01121000}) + s(e_{-11111100} + e_{-01121100}) + \\ & te_{-01111110} + ue_{-00111111} + ve_{-01122100}. \end{aligned}$$

All roots in the above expression are contained in an A_1A_5 subsystem with base $\{e_{-01111110}\} \cup \{e_{-01121000}, e_{-6}, e_{-5}, e_{-11110000}, e_{-00111111}\}$ and projecting $e_{-\beta}$ to the A_5 factor we obtain restrictions on the coefficients by considering matrices and using the fact that $e_{-\beta}$ is of type A_1A_1.

First assume $u \neq 0$. This forces $r = s = 0$ and $vt = 0$. From the commutator $[e_{\alpha+\beta}, e_{-\beta}] = e_\alpha = e_1 + e_4$ we conclude that $a \neq 0$ and thus

$$\langle e_{\alpha+\beta} \rangle = \langle e_{01121110} + e_{11111110} \rangle$$

$$\langle e_{-\beta} \rangle = \langle e_{-01111110} + xe_{-00111111} \rangle,$$

where $x \neq 0$. At this stage we conjugate the above vectors by s_α and find that $L(X)$ is generated by four nilpotent elements (namely, $e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}$) with each expressed in terms of 2 root vectors of E_8 and an opposite pair occurring. As in previous cases this implies that there is a 1-dimensional torus centralizing $L(X)$, which is a contradiction.

We now assume $u = 0$. Assume that in addition $r \neq 0$. Then the matrix considerations force $s = t = 0$. Here the commutator $[e_{\alpha+\beta}, e_{-\beta}] = e_\alpha = e_1 + e_4$ leads to expressions

$$\langle e_{-\beta} \rangle = \langle e_{-01121000} + e_{-11111000} + xe_{-01122100} \rangle,$$

$$\langle e_{\alpha+\beta} \rangle = \langle ye_{10111111} + e_{11121000} + ze_{11121100} \rangle.$$

Conjugating by $U_{-56}(x)$ and then $U_6(z)$, both of which commute with U_α, we can assume that $x = 0 = z$. At this point we can proceed just as in the previous paragraph.

Hence we can now assume that $u = r = 0$. Suppose $s \neq 0$. Here the matrix expression implies $t = 0$. Then the commutator identity $[e_{\alpha+\beta}, e_{-\beta}] = e_\alpha = e_1 + e_4$ implies $b \neq 0$, but $a = c = 0$. So here we have

$$\langle e_{-\beta} \rangle = \langle e_{-01121100} + e_{-11111100} + xe_{-01122100} \rangle,$$

$$\langle e_{\alpha+\beta} \rangle = \langle e_{11121100} + ye_{11121000} + ze_{10111111} \rangle.$$

Conjugating by $U_{-5}(x)$ and $U_{-6}(y)$ we can omit the terms with coefficients x, y. At this point we conjugate get the usual contradiction.

The final case is where $r = u = s = 0$. Then $t, v \neq 0$. The commutator identity forces $a \neq 0$, so that

$$\langle e_{-\beta} \rangle = \langle e_{-01111110} + x e_{-01122100} \rangle,$$

$$\langle e_{\alpha+\beta} \rangle = \langle e_{11111110} + e_{01121110} \rangle.$$

Once again we conjugate by s_α and find that there exists a 1-dimensional torus centralizing $L(X)$, a final contradiction. ■

We have now excluded all the cases in Lemma 4.3.16, completing the proof of Proposition 4.3.12.

Theorem 4.1 is now proved for all primes p.

5 Maximal subgroups of type B_2

In this section we prove Theorem 1 in the case where the maximal subgroup X is of type B_2. Recall that G is an exceptional adjoint algebraic group, and G_1 is a group satisfying $G \leq G_1 \leq \mathrm{Aut}(G)$. We consider only the small characteristic cases required by Proposition 2.2.1.

Theorem 5.1 *Suppose that $X = B_2$ is maximal among proper closed connected $N_{G_1}(X)$-invariant subgroups of G. Assume further that*

(i) *$C_G(X) = 1$, and*

(ii) *$p \leq 5$ if $G = E_8$; $p \leq 3$ if $G = E_7, E_6$; $p = 2$ if $G = F_4$; and $G \neq G_2$.*

Then $G = E_8, p = 5$, and G contains a single conjugacy class of maximal subgroups B_2.

Suppose X, p are as in the hypothesis of the theorem, with $X = B_2$. Write $S = N_{G_1}(X)$. Lemmas 2.2.2, 2.2.10 and 2.2.11 imply that $C_S(X) = 1$ and that $S = X\langle\sigma\rangle$, where σ is a field morphism of G (possibly trivial), or a graph-field morphism of G, the latter only if $G = E_6$.

We shall prove the theorem in subsections, one for each value of $p = 2, 3, 5$. The case $p = 2$ is the most complicated and we save this for last.

Set notation as follows. Choose a root system of X with base $\Pi(X) = \{\alpha, \beta\}$, with α long and β short, and positive roots $\Sigma^+(X) = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta\}$. Let T_X be a maximal torus of X with corresponding root elements and root subgroups labelled by $\Sigma(X)$. For $\gamma \in \Sigma^+(X)$, let e_γ be the corresponding root vector in $L(X)$, and $f_\gamma = e_{-\gamma}$. As in Definition 2.2.4, T is a 1-dimensional torus of X such that each of α, β affords weight 2 of T. That is, T gives the labelling 22 of the Dynkin diagram of X. The T_X-weight ab affords T-weight $4a + 3b$.

5.1 The case $p = 5$

Assume $p = 5$. Then from the hypothesis of Theorem 5.1, we have $G = E_8$.

By Lemma 2.2.6, T determines a labelling of the Dynkin diagram of G by 0's and 2's. As usual, we can use the Weight Compare Program to determine the composition factors of $L(G) \downarrow X$ corresponding to each possible labelling.

The first step is to determine the possible extensions among those composition factors which occur, and this amounts to determining the structures of the relevant reducible Weyl modules.

Lemma 5.1.1 *Excluding the T-labelling* 22202022 *of* E_8*, the only reducible Weyl modules for which the corresponding irreducible module appears as a composition factor of* $L(G) \downarrow X$ *are as follows:*

(i) $W(20) = 20|00$

(ii) $W(22) = 22|20$

(iii) $W(06) = 06|22$

(iv) $W(32) = 32|22$.

We see from this lemma that excluding the exceptional labelling, the only composition factor that extends the trivial module is 20. Since X has no fixed points on $L(G)$ and since $L(G)$ is self dual, it follows that 20 must have greater multiplicity than 00 when the latter multiplicity is positive. This observation pares down the list of possibilities provided by the Weight Compare Program to the following, which also covers the exceptional labelling 22202022.

Lemma 5.1.2 $L(G) \downarrow X$ *is one of the following:*

(i) $22^2/30/12/20^2/02^2/00$

(ii) $32/06/22^2/02$

(iii) $2(10)/0(10)/52/16/02$

(iv) $56/1(10)/0(10)/52/16/06^2/02$.

We consider each of the above configurations separately.

Lemma 5.1.3 *Case* (i) *of Lemma* 5.1.2 *does not hold.*

Proof Suppose 5.1.2(i) holds. Let $V = V_X(10)$, the 5-dimensional orthogonal module, and let T_1 denote a 1-dimensional torus of X such that for $c \in K^*$, $T_1(c)$ induces $\mathrm{diag}(c, c^{-1}, 1, 1, 1)$ on V. So $C_X(T_1) = T_1A_1$, where A_1 induces SO_3.

Noting that 02 is the wedge-square of V, we calculate that $T_1(c)$ induces $(c^{(3)}, (c^{-1})^{(3)}, 1^{(4)})$ on 02 (where a superscript (n) indicates that this eigenvalue occurs with multiplicity n); also $S^2(V) = 20/00^2$, from which we see that $T_1(c)$ induces $(c^{(3)}, (c^{-1})^{(3)}, c^2, c^{-2}, 1^{(5)})$ on 20.

We next note that $10 \otimes 02 = 12/02/10$, $10 \otimes 20 = 30/12$ and $20 \otimes 02 = 22/12/20^2/00$ (this can be seen using the program of [13]). From this and the above action of T_1 on $10, 02$, and 20 we conclude that T_1 has fixed point spaces on $02, 20, 12, 30, 22$ of dimensions $4, 5, 11, 10, 16$, respectively. It follows that $C_{L(G)}(T_1)$ has dimension 72, hence so has $C_G(T_1)$. This centralizer must be a Levi subgroup of G. However, there is no Levi subgroup of this dimension. ∎

Lemma 5.1.4 *Case* (iii) *of Lemma* 5.1.2 *does not hold.*

Proof In this case $L(G) \downarrow X$ is multiplicity free. In view of the fact that $L(G)$ is self-dual, it follows that $L(G) \downarrow X$ is completely reducible. Therefore, $A = C_{L(G)}(L(X))$ is an irreducible module for X of high weight $0(10)$. By Lemma 2.3.4 we have $A \leq L(D)$, where $D = A_4A_4$. The non-negative T-weights on $L(G)$ which are multiples of 5 and their multiplicities are as follows: $0^{12}, 10^{10}, 20^6, 30^2$. Now $T < A_4A_4$ and it is easy to check that up to possible graph automorphisms, the only possible labelling for each A_4 is $(10)(10)0(10)$.

Now A contains a nonzero weight vector v of T-weight 30, and consequently 2.3.5 and 2.3.8 yield a contradiction.

∎

Lemma 5.1.5 *Case* (iv) *of Lemma* 5.1.2 *does not hold.*

Proof We first claim that there does not exist an indecomposable $L(X)$-module with socle 01 and quotient 00. By way of contradiction assume V is such a module with socle W. Let T_1 be a 1-dimensional torus in X such that for $c \in K^*$, $T_1(c)$ acts diagonally as (c, c, c^{-1}, c^{-1}) on the 4-dimensional symplectic module 01, and let $L(T_1) = \langle h \rangle$. Then $C_X(h) = T_1J$, where $J = SL_2$ is a fundamental SL_2 corresponding to a short root. Also, $V = W \oplus C_V(h)$. Then J leaves $C_V(h)$ invariant, acting trivially. Let T_1' be a 1-dimensional torus of J. From the action on 01 we see that T_1' is a conjugate of T_1, so that $\langle h' \rangle = L(T_1')$ is a conjugate of $\langle h \rangle$. Set $C_X(h') = T_1'J'$. It now follows that $C_V(h) = C_V(h')$, so this space is invariant under $\langle J, J' \rangle = X$. This is a contradiction and establishes the claim.

Now assume 5.1.2(iv) holds. We next claim that there is an irreducible submodule $0(10)$ in $L \downarrow X$. All composition factors of $L \downarrow X$ appear with multiplicity 1, with the exception of 06, and $L(G)$ is self-dual. So if the claim fails to hold, then there must be a singular submodule $W \cong 06$. Then $W^{\perp}/W$ is multiplicity free, so has a submodule $0(10)$. Repeated application of the first claim implies that under the action of $L(X)$ the preimage splits as $W \oplus Z$, with Z the fixed point space of $L(X)$. Hence Z is X-invariant and affords $0(10)$. At this point the proof is completed as in Lemma 5.1.4. ■

It remains to handle case (ii) of Lemma 5.1.2. In this case maximal subgroups arise:

Lemma 5.1.6 *There is a unique conjugacy class of maximal B_2's in $G = E_8$ such that for X in the class, $L(G) \downarrow X = 32/06/22^2/02$.*

Proof For $p \geq 7$, (6.7) of [31] shows the existence and conjugacy of a unique class of maximal B_2 in E_8. A careful check shows that in the case $p = 5$ many of the same arguments apply. In particular, assuming such a group X exists we find that its Lie algebra must be conjugate to $J = \langle e_\alpha, e_\beta, f_\alpha, f_\beta \rangle$ (Lie algebra span), with the vectors $e_\alpha, e_\beta, f_\alpha, f_\beta$ as indicated on p.111 of [31]. It is shown that J is a simple algebra of type B_2, with α, β long and short roots, respectively. In the following we argue that this Lie algebra is indeed the Lie algebra of a corresponding group $X = B_2$.

Just as on p.109 of [31] there is a subgroup $E_\alpha \cong SL_2$ contained in a subsystem subgroup A_1A_5 of G, with $L(E_\alpha) = \langle e_\alpha, f_\alpha \rangle$. In this embedding the usual 6-dimensional module V_6 for A_5 affords the module $3 \oplus 1$ for E_α. Now $L \downarrow A_1A_5$ can be decomposed explicitly into the sum of irreducible modules (view $A_1A_5 < A_1E_7$ and use 2.1 and 8.6 of [23]). Aside from one adjoint module for each factor, the remaining irreducibles have the form $M \otimes N$, where M is a natural or trivial module for the A_1 factor and $N = V_6, V_6^*, \wedge^2 V_6, \wedge^2 V_6^*, \wedge^3 V_6$ or a trivial module. Using this and standard results on tilting modules (see Lemma 2.1.7 and its preamble), we see that $L(G) \downarrow E_\alpha$ is a tilting module with highest weight 6.

Now $T(6) = 2|(1 \otimes 1^{(5)})|2$ and $T(5) = 3|1^{(5)}|3$. We claim that neither of these has an irreducible 2-dimensional $L(E_\alpha)$-submodule. This is clear in the second case since there is no 2-dimensional composition factor. In the first case there are such composition factors; however, if such a submodule existed then the sum of all its images under E_α would be E_α-invariant and homogeneous for $L(E_\alpha)$, contradicting the fact that $T(6)$ is indecomposable.

At this point we can argue that each 2-dimensional $L(E_\alpha)$-submodule of L is invariant under E_α. In particular, E_α stabilizes the subspaces $\langle e_\beta, e_{\alpha+\beta} \rangle$ and $\langle e_{-\beta}, e_{-\alpha-\beta} \rangle$. So E_α also stabilizes the Lie algebra span of these spaces, which is J.

Next we carry out a similar analysis for a group $E_\beta < A_4A_2 < D_5A_2$ with $L(E_\beta) = \langle e_\beta, f_\beta \rangle$. In this embedding E_β has irreducible and restricted action on the natural modules for both the A_4 and A_2 factors. Restricting L to A_4A_2 and then to E_β we find that $L \downarrow E_\beta$ is a tilting module with highest weight 8 and all weights even. Now $T(8) = 0|3 \otimes 1^{(5)}|0$ and $T(6) = 2|1 \otimes 1^{(5)}|2$. Then $T(8) \downarrow L(E_\beta)$ has no composition factor of dimension 3 and the above argument shows that the only 3-dimensional submodule of $T(6) \downarrow L(E_\beta)$ in the socle 2.

It follows from the above remarks that any 3-dimensional irreducible $L(E_\beta)$-submodule of L is also E_β-invariant. In particular, $\langle e_\alpha, e_{\alpha+\beta}, e_{\alpha+2\beta}\rangle$ and $\langle e_{-\alpha}, e_{-\alpha-\beta}, e_{-\alpha-2\beta}\rangle$ are both E_β-invariant. So E_β stabilizes the Lie algebra they generate, namely J.

Now set $Y = \langle E_\alpha, E_\beta\rangle$. Then Y induces an irreducible subgroup on J preserving the Lie algebra structure. From the adjoint action of J on L we have $C_{L(G)}(J) = 0$ and hence $C_Y(J)$ is a finite group. It follows that Y is of type B_2 and this completes the argument. ■

5.2 The case $p = 3$

Here $G = E_6, E_7$ or E_8. We proceed as in the case for $p = 5$. The torus T determines a labelling of the Dynkin diagram of G, which in turn determines all weights of T on $L(G)$ and (via the Weight Compare Program) the possible composition factors of $L(G) \downarrow X$. We first determine which of these composition factors has its corresponding Weyl module being reducible.

We will say that a composition factor is *acceptable* if it appears in $L(G) \downarrow X$ for some labelling in which the adjoint module of X also appears (which of course must be the case).

Lemma 5.2.1 *The following are the only reducible Weyl modules of T-weight at most* 20 *whose simple quotient is acceptable:*

(i) $W(12) = 12/02$

(ii) $W(30) = 30/12$

(iii) $W(04) = 04/20/10$

(iv) $W(40) = 40/04/20$

(v) $W(06) = 06|14|02$ *(uniserial)*

(vi) $W(16) = 16/24/04/10$

(vii) $W(14) = 14/30/12/02/00$

(viii) $W(32) = 32/14/12/30/02$

(ix) $W(50) = 50/04/10$

(x) $W(24) = 24/40/04/20/10$.

Proof The composition factors were obtained using a computer implementation of the Sum Formula. To complete the proof we must verify that the $W(06)$ is uniserial with the indicated series. If this is not the case then there must exist an indecomposable module of the form $I = 06|02$. Let $v \in I$ be a maximal vector. Then e_α, e_β, and f_α all annihilate v. Moreover, $f_\beta v = 0$, as well, since 14 does not occur as a T_X-weight of this module. Hence $v \in C_I(L(X))$, the latter being X-invariant. But then, $L(X)$ annihilates I, which is clearly false since it does not annihilate the socle.

■

Lemma 5.2.2 *The irreducible X-module* 60 *does not extend the trivial module.*

Proof Assume false and assume V is an indecomposable module with submodule $W = 00$ such that $V/W = 60$. Then $L(X)$ annihilates V/W. Let $v \in V$ have weight 60. If $\gamma \in \Sigma(X)$ and e_γ is a root vector of $L(X)$ then $e_\gamma v$ has weight $60+\gamma$. On the other hand, this must lie in the trivial module. As γ cannot afford the weight -60, we conclude that $v \in C_V(L(X))$, the latter being X-invariant. Hence $L(X)$ is trivial on V and so the representation $X \to SL(V)$ factors through a Frobenius morphism (see 1.2 of [23]). But this is impossible as $V_X(02)$ does not extend the trivial module (see [23, 1.10]). ■

Recall that $L = L(G)'$ and $A = C_L(L(X))$. By Lemma 2.1.1, $L = L(G)$ except when $G = E_6$, in which case L has codimension 1 in $L(G)$. Denote by n_{ab} the multiplicity of the composition factor ab in $L \downarrow X$.

Lemma 5.2.3 *One of the following holds for X, G and $L \downarrow X$.*

(i) $G = E_6$ *and* $L \downarrow X = 06/40/04^2/02$

(ii) $G = E_8$ *and* $L \downarrow X = 12^4/20^2/02^{10}/10^4$

(iii) $G = E_8$ *and* $L \downarrow X = 06/32/14^2/04/20/02^3$

(iv) $G = E_8$ *and* $L \downarrow X = 06/32/14^2/30/12/02^3$.

Proof As in previous cases we make use of the Weight Compare Program to list the possibiities for the composition factors of $L \downarrow X$. We can immediately rule out all cases where there does not exist an adjoint module. Also, by

Lemma 2.2.10 we can rule out any case where $L \downarrow X$ has a nonzero trivial submodule.

First assume that the highest T-weight on L is at most 20. By Lemma 5.2.1, the only irreducible in this range that can possibly extend the trivial module is 14, and 00 occurs with multiplicity 1 in $W(14)$. It follows that either $n_{00} = 0$ or $n_{00} < n_{14}$. Working through the possible configurations we see that under the assumption, one of the cases (i)-(iv) occurs.

Now suppose that there is a T-weight greater than 20. Here we find that $G = E_8$ and one of the following occurs:

(a) $L \downarrow X = 06^2/14^3/30/02^3/00$

(b) $L \downarrow X = 34/16/06/32/14/02$

(c) $L \downarrow X = 60/16^2/06/32^2/20/02$.

We must rule out these exceptional configurations. Suppose (a) holds. Here we can use the fact that $W(06) = 06|14|02$ is uniserial to see that there exists a 06 submodule, which must then occur as a submodule of A. The labelling here is 00020020 so the non-negative T weights which are a multiple of 6 are $0^{24}, 6^{20}, 12^9, 18^2$. Then $\dim(D) = 86$, so Lemma 2.3.4 implies $D = A_2E_6$. In view of the T-weights the labellings of these factors must be 66, 0000600 and Lemma 2.3.6 yields a contradiction.

Cases (b) and (c) both occur for the T-labelling 00020202 where the non-negative T-weights which are a multiple of 6 are $0^{20}, 6^{18}, 12^{11}, 18^6, 24^1$. It follows that $\dim(D) = 92$. If we show that $A \neq 0$, this will contradict Lemma 2.3.4.

So it remains to establish that $A \neq 0$. In case (b), $L \downarrow X$ is multiplicity-free. Now L and all composition factors of $L \downarrow X$ are self-dual. It follows that each simple module is non-degenerate under the form on L, and hence $L \downarrow X$ is completely reducible. Then 06 occurs as a submodule and hence $A \neq 0$. Essentially the same argument works in case (c). The restriction $L \downarrow X$ is not multiplicity-free, since both 16 and 32 occur with multiplicity 2; but neither of these extends 06 and we conclude that 06 occurs as a submodule, hence $A \neq 0$. ■

Lemma 5.2.4 *It is not the case that* (i),(ii) *or* (iii) *of Lemma* 5.2.3 *holds.*

Proof Suppose 5.2.3(i) occurs. From Lemma 5.2.1, we conclude that there is an irreducible X-submodule 06, and hence $A \neq 0$. The non-negative T-weights on $L(G)$ which are multiples of 6 are $0^{10}, 6^8, 12^4, 18^1$. Hence the group D has dimension 36, which contradicts 2.3.4.

Now suppose 5.2.3(iii) holds. Let Y be a fundamental SL_2 of X corresponding to a long root and let T_1 be a maximal torus of Y. Then T_1 has non-negative weights $1, 0^2$ on 01 and $1^2, 0$ on 10. We can then calculate the T_1-weights for the other composition factors of $L \downarrow X$. Indeed, 02 is the wedge-square of 10 and 20 has codimension 1 in the symmetric square of 10. Using this and the Steinberg tensor product theorem one can easily determine the T_1-weights of all composition factors other than 14. Here we must first determine the weights of 11 and this follows using the fact that $10 \otimes 01 = 11/01$.

From the above considerations we calculate the T_1-fixed points on each of the composition factors. They are each of dimension 4 with the exception of the composition factor 14, where the fixed space has dimension 8. It follows that $C_G(T_1)$ has dimension 44, which is a contradiction since there is no such Levi subgroup in E_8.

Finally, consider 5.2.3(ii). This is actually the most complicated case, but the detailed analysis is the same as that carried out in 6.6 of [31], where it is shown that X is contained in an A_4A_4 subsystem group. Consequently, X is centralized by an element of order 5 in the center of this subsystem group and this contradicts Lemma 2.2.10(ii). ∎

We complete this section with

Lemma 5.2.5 *It is not the case that* (iv) *of Lemma* 5.2.3 *holds.*

Proof Suppose 5.2.3(iv) holds. Assume first that 06 occurs as a composition factor of A. The non-negative T-weights which are multiples of 3 are $0^{24}, 6^{20}, 12^9, 18^2$. Hence by Lemma 2.3.4, $A \leq L(D)$ where D is a reductive maximal rank subgroup of dimension 86. It follows that $D = A_2E_6$. Next, we compute the T-labelling of D and find that the A_2 has labelling 60 (up to a graph automorphism), while the E_6 has labelling 000600.

There is a T-weight vector $v \in A$ of weight 18. From the labelling of A_2E_6 we see that $v = ce_\gamma + de_\mu$, where γ is the root of highest height in $\Sigma(E_6)$ and μ is the next highest root. Now J_{α_2}, the fundamental SL_2 corresponding to α_2, is transitive on nonzero vectors of this form. Hence v is a root vector, contradicting Lemma 2.2.12(ii).

We may now assume 06 is not a composition factor of A. Let $v \in L$ be a weight vector of weight 06, so that $\langle Xv \rangle$ is an image of the Weyl module $W(06) = 06/14/02$. Our assumption implies that $L(X)$ does not annihilate

v, but weight considerations imply that $e_\alpha v = e_\beta v = f_\alpha v = 0$. It follows that $f_\beta v \neq 0$.

For $c \in K^*$, set $S_\beta(c) = h_\alpha(c^2)h_\beta(c)$ and $S_\beta = \langle S_\beta(c) : c \in K^* \rangle$. One checks that $S_\beta = C_X(J_\beta)$, where J_β is the image of the fundamental SL_2 generated by the root subgroups corresponding to short roots β and $-\beta$. Calculating fixed points of S_β on the various composition factors we find that $C_G(S_\beta)$ is a Levi factor of dimension 54, and hence $C_G(S_\beta) = S_\beta D_5 A_2$.

We next study the embedding of J_β in D_5A_2. Let T_β be a maximal torus of J_β. For each composition factor of $L \downarrow L(X)$, we can compute the action of J_β on the fixed points of S_β. We conclude that T_β has non-negative weights $8^1, 6^4, 4^6, 2^{10}, 0^{12}$ on the fixed space of S_β. It follows that T_β has labels 02022 on the D_5 factor and 22 on the A_2 factor. Hence, the projection of J_β to the A_2 factor corresponds to a subgroup acting irreducibly on the natural module with high weight 2, while the projection to the D_5 factor corresponds to a subgroup acting on the natural module with composition factors $(1 \otimes 1^{(3)})^2/0^2$. It follows that the projection of J_β leaves invariant a nested singular 1-space and singular 5-space, where the quotient affords $1 \otimes 1^{(3)}$. From the extension theory we see that there is also a 4-dimensional subspace affording $1 \otimes 1^{(3)}$ and there are two possibilities depending on whether this subspace is singular or non-degenerate. In the former case the projection is contained in a subsystem subgroup A_4 of D_5 in such a way that if V denotes the usual 5-dimensional module, then $V = 1 \otimes 1^{(3)}/0$. In the latter case the projection acts on the orthogonal module as a $T(4) \perp 4$, and hence is contained in a subgroup of type $SO_6 \cdot SO_4$.

Let $P = P_6$ denote the standard maximal parabolic subgroup of G with Levi factor $T_1D_5A_2$. Conjugating X, if necessary, we may assume this Levi factor is $C_G(S_\beta)$. We will label fundamental roots of D_5 and A_2 by the corresponding fundamental roots of E_8.

Case 1 First assume the projection of J_β to the D_5 factor of $C_G(S_\beta)$ is contained in an A_4 subsystem subgroup of the D_5. There is an element $s \in N_X(S_\beta)$ which inverts S_β. Then s induces a graph automorphism on D_5 interchanging the two classes of A_4 subsystem subgroups. Conjugating by an element of $D_5\langle s \rangle$, if necesssary, we may assume that the projection of J_β to D_5 is contained in the A_4 subgroup with base $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5\} \subset \Pi(E_8)$. This conjugation changes the T_β-labelling of D_5 and we will return to this later. The weights of $L \downarrow X$ are all integral combinations of roots in $\Sigma(X)$, so these weights afford even weights of S_β. In particular, α_6 must afford either 2 or -2. Replacing the base $\{\alpha, \beta\}$ of $\Sigma(X)$ by $\{-\alpha, -\beta\}$, if necessary, we can suppose the former holds.

Let Q denote the unipotent radical of P. We can write $L(Q) = W_1 \oplus W_2 \oplus W_3 \oplus W_4$, where W_i is the linear span of all root vectors of $L(G)$ for which the coefficient of α_6 is precisely i. It follows from [3] that W_3 affords the irreducible representation, $V_{D_5}(\lambda_5)$ (a maximal vector is given by $e_{23465321}$). So W_3 is a spin module for D_5, and setting $V = V_{A_4}(\lambda_5)$ we have $W_3 \downarrow A_4 = V \oplus \wedge^2 V^* \oplus E$, where E is a trivial module.

Now $v \in W_3$ and v affords the largest T_β-weight on W_3, namely 6, so that $\langle J_\beta v\rangle$ is an image of the corresponding Weyl module, $W(6) = 2^{(3)}|1 \otimes 1^{(3)}$. We have already seen that $f_\beta v \neq 0$, so $\langle J_\beta v\rangle \cong W(6)$. From this information we check that the projection of J_β must act on V as $W(4)^* = 0|1 \otimes 1^{(3)}$. Indeed, weight 6 for T_β occurs only within the factor $\wedge^2 V^*$ and for other embeddings the restriction of $\wedge^2 V^*$ contains $\wedge^2(1 \otimes 1^{(3)}) = 2^{(3)} \oplus 2$ as a submodule.

Let Q^- denote the opposite unipotent radical. Then $L(Q^-) = W_1^- \oplus W_2^- \oplus W_3^- \oplus W_4^-$. Here we study W_1^- which affords the irreducible representation $V_{D_5}(\lambda_5) \otimes V_{A_2}(\lambda_7)$. Note that f_α is a weight vector for S_β of weight -2, so that $f_\alpha \in W_1^-$. We aim to locate f_α, using the facts that f_α is fixed by U_β and has T_β-weight 2.

From the above we have the precise embedding of the projection of J_β in a Levi A_4 of D_5. Namely $V = V_{A_4}(\lambda_5)$ restricts as $W(4)^*$, and hence $V_{A_4}(\lambda_1)$ restricts as $W(4)$. Take a basis of $V_{A_4}(\lambda_1)$ consisting of T_β-weight vectors for descending weights with base $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$ and such that the labelling of T_β is 2222. Then

$$T_\beta(c) = h_{\alpha_1}(c^4)h_{\alpha_3}(c^6)h_{\alpha_4}(c^6)h_{\alpha_5}(c^4)h_{\alpha_7}(c^2)h_{\alpha_8}(c^2)$$

and

$$U_\beta(c) = U_1(c)U_5(c)U_4(2c)U_{134}(c^3)U_{345}(c^3)U_{45}(c^2)U_8(2c)U_7(c)U_{78}(c^2).$$

The first observation from these expressions is that $e_\beta = e_1+e_5+2e_4+2e_8+e_7$ and hence e_β is of type $A_2A_2A_1$. It also follows that T_β determines the labelling $2(-6)222(-6)22$ of the E_8 diagram. Then a direct check shows that the T_β-weight space of W_1^- for weight 2 has dimension 9 with basis

$$f_{00000111}, f_{00001110}, f_{00011100}, f_{01111111}, f_{11111110},$$

$$f_{11121100}, f_{01121110}, f_{01122100}, f_{12232100}.$$

At this point we can compute the fixed points of U_β on the above weight space and find that the fixed space has dimension 3 and there is an expression

$$\begin{aligned} f_\alpha = \quad & af_{12232100} + b(f_{00000111} + f_{00001110} + f_{00011100}) + \\ & c(f_{01111111} + f_{0112111} + f_{01122100}). \end{aligned}$$

Hence f_α lies in the Lie algebra of the subsystem subgroup of type $A_1A_1A_1A_3$ with base $\{f_{12232100}\} \cup \{f_{01121110}\} \cup \{f_{00001110}\} \cup \{f_{00000111}, f_{01111000}, f_{00011100}\}$. Using this we can easily identify the class of f_α, depending on the coefficients a, b, c.

We obtain additional information about f_α as follows. For $c \in K^*$ let $S_\alpha = h_\alpha(c)h_\beta(c)$ and set $S_\alpha = \langle S_\alpha(c) : c \in K^* \rangle$. Then $C_X(S_\alpha) = J_\alpha$, the fundamental SL_2 corresponding to the long root α. From the decomposition $L \downarrow X$ we find that $C_L(S_\alpha) \downarrow J_\alpha = 6/(1 \otimes 1^{(3)})^4/2^5/0^6$. We find $C_G(S_\alpha) = T_2A_1A_5$ and where T_α yields labellings 2 and 20202 of the Dynking diagrams of A_1 and A_5, respectively. Then J_α acts on the usual module for A_5 with composition factors $1^{(3)}/1^2$. Hence the precise action is one of $1^{(3)} \oplus 1^2, T(3), W(3) \oplus 1$, or $W(3)^* \oplus 1$. Taking into account the A_1 factor as well, we find that f_α has type $A_1^3, A_2^2A_1, A_2A_1^2$, or $A_2A_1^2$, respectively.

Combining this with the above, we have the following possibilities:

i) $b \neq 0 \neq c, a = 0$ and $f_\alpha = A_2A_1^2$

ii) $b \neq 0, a = c = 0$ and $f_\alpha = A_1^3$

iii) $b = 0 = a, c \neq 0$ and $f_\alpha = A_1^3$.

All we require from this information is that $a = 0$ in each case. One can find expressions for elements of $U_{-\beta}$ as was done earlier for U_β. Using this we find that

$$f_\beta = f_1 + 2f_3 + f_5 + 2f_7 + f_8.$$

Now $[f_\beta f_\alpha] = \pm f_{\alpha+\beta}$ and $f_{\alpha+\beta}$ is an X-conjugate of e_β, so must be of type $A_2A_2A_1$. On the other hand, from the above expressions for f_β and f_α we find that

$$[f_\beta f_\alpha] = cf_{11111110} + cf_{11121110} + cf_{11122100} - bf_{00111100}$$

$$-cf_{01122110} + bf_{00011110} - cf_{01121111}.$$

Conjugating this expression by $U_4(-c)$ we can delete the second term without affecting other terms. Hence

$$n = cf_{11111110} + cf_{11122100} - bf_{00111100} - cf_{01122110} + bf_{00011110} - cf_{01121111}$$

is of type $A_2A_2A_1$. If $b = 0$, this element lies in the Lie algebra of the subsystem group of type A_1A_3 with base $\{f_{01122110}\} \cup \{f_{11111110}, f_{01121111}, f_{11122100}\}$, whereas this Lie algebra contains no nilpotent element of type $A_2A_2A_1$. So assume $b \neq 0$. Now conjugate n by $U_{2345}(c/b)$ to obtain

$$n' = cf_{11111110} + cf_{11122100} - bf_{00111100} + bf_{00011110} - cf_{01121111},$$

another nilpotent element of type $A_2A_2A_1$. However this element lies in the Lie algebra of a the subsystem subgroup of type A_1A_4 with base $\{f_{00011110}\} \cup \{f_{00111100}, f_{11111110}, f_{01121111}, f_{11122100}\}$. Here also, it follows from the classification of nilpotent elements of A_4 that this subalgebra cannot contain a nilpotent element of type $A_2A_2A_1$. This is a contradiction and completes the argument for this case.

Case 2 The remaining case is the case where the projection of J_β to the D_5 factor of $C_G(S_\beta)$ is contained in a subsystem subgroup $J = D_3A_1A_1 < D_5$ acting as $SO_6 \cdot SO_4$ on the orthogonal module. In this case the projection of J_β acts on the orthogonal module as $T(4) \perp 4$. Conjugating if necessary we may take the subsystem group to have base $\{\alpha_5, \alpha_4, \alpha_2\}$, $\{\alpha_1, \delta\}$, where δ is the high root of $\Sigma(D_5)$. Consider the action of $J = D_3A_1A_1$ on the subspace W_3 of $L(Q)$. Here we again have W_3 irreducible under the action of D_5, affording $V_{D_5}(\lambda_5)$. Then $W_3 \downarrow J = (V_{D_3}(\lambda_5) \otimes E) \oplus (V_{D_3}(\lambda_2) \otimes F)$, where E, F are 2-dimensional restricted usual modules for the A_1 factors with base δ, α_1, respectively.

Using the fact that the projection of J_β to the D_3 factor acts as $T(4)$ on the orthogonal module, we see that this projection is indecomposable with composition factors $3/1$ on the 4-dimensional D_3-modules. There is an element of D_5 acting on J, inducing a graph automorphism of D_3, while interchanging the factors of the A_1A_1. Conjugating by this graph automorphism, if necessary, we may assume that the projection of J_β acts as $3|1$ on $V_{D_3}(\lambda_5)$. We recall from the above, that $\langle J_\beta v\rangle \cong W(6) = 2^{(3)}|1 \otimes 1^{(3)}$. It follows that there must be a twist in the projection to the A_1 factor with base α_1 and no twist on the other factor. We then have $(V_{D_3}(\lambda_5) \otimes E) \downarrow J_\beta = (3|1) \otimes 3$.

At this point we have the precise embedding of J_β in D_5A_2 from which we get the following expression for elements of T_β and U_β:

$$T_\beta(c) = h_1(c)h_\delta(c^3)h_5(c^3)h_2(c)h_7(c^2)h_8(c^2)$$

and

$$U_\beta(c) = U_1(c)U_\delta(c^3)U_5(c^3)U_2(2c)U_{245}(-c^2)U_{45}(c)U_8(2c)U_7(c)U_{78}(c^2).$$

We first observe that $e_\beta = e_1 + 2e_2 + e_{45} + 2e_8 + e_7$ so that e_β is a unipotent element of type $A_1A_2A_2$. Next, from the expression for elements of T_β we see that T_β determines the labelling $(-2)(-2)(-2)(4)(-6)(8)(-2)(-2)$ of the Dynkin diagram of G.

Now turn to Q^- and again consider W_1^- on which D_5A_2 acts irreducibly as $V_{D_5}(\lambda_5) \otimes V_{A_2}(\lambda_7)$. We again aim to determine f_α, a weight vector in W_1^-. This vector is fixed by U_β has T_β-weight 2, and $\langle J_\beta f_\alpha \rangle$ is the irreducible module of high weight 2. A direct check shows that the T_β-weight space of W_1^- for weight 2 has dimension 9 with basis

$$f_{00001100}, f_{00011111}, f_{01011110}, f_{01111100}, f_{00111110},$$

$$f_{10111100}, f_{11121110}, f_{01121111}, f_{11221100}.$$

We are now in position find the fixed points of U_β on the above weight space. A direct computation shows that this space has dimension 3 and that there is an expression

$$f_\alpha = a(f_{01111100} - f_{00111110}) + b(f_{01111100} + f_{10111100}) + c(f_{00001100} - f_{11221100}).$$

Next we find an expression for elements of $U_{-\beta}$, from which we see that

$$f_\beta = f_1 + 2f_2 + e_{24} + 2f_7 + f_8.$$

We then have

$$\begin{aligned} \pm f_{\alpha+\beta} = [f_\beta f_\alpha] = \ & 2bf_{01111110} - a(f_{11111100} + f_{01111111}) + \\ & (a-b)f_{10111110} + 2c(f_{00001110} + f_{11221110}). \end{aligned}$$

Now $f_{\alpha+\beta}$ is X-conjugate to e_β, so it must be of type $A_1A_2A_2$. This forces $a \neq 0$. Indeed, otherwise, $f_{\alpha+\beta}$ involves just 4 root vectors, hence is centralized by a 4-dimensional torus, whereas this is not the case for e_β. Conjugating by $U_8(2b/a)$ we can delete the first term of the expression without affecting the other terms. We can then view $f_{\alpha+\beta}$ as an element of the Lie algebra of the subsystem group of type A_1A_4 having base $\{11111100\} \cup \{00000111, 00001000, 01110000, 10111110\}$. Considering the projection of $f_{\alpha+\beta}$ to the A_4 factor and working within the matrix algebra sl_5, we check that $f_{\alpha+\beta}$ cannot have class $A_1A_2A_2$, a contradiction. ∎

5.3 The case $p = 2$

In this section we handle the case $X = B_2$ with $p = 2$. When $G = E_8$ this is more complicated than the previous cases because the Weight Compare

Program used in previous cases gives rise to hundreds of possible configurations for $L \downarrow X$. To deal with this situation we combine the techniques used for B_2 in odd characteristics with those used for A_1 with $p = 2$. Cases other than E_8 are relatively easy and will be settled first. Following this we develop machinery similar to what was used for A_1 and apply this to the E_8 configurations.

In view of Lemmas 2.2.2 and 2.2.11 we see that S is generated by X and a field or graph-field morphism of G, the latter possible only for $G = E_6$.

We begin with a lemma on extensions.

Lemma 5.3.1 ([34]) $\mathrm{Ext}^1_X(V_X(\lambda), K) \neq 0$ *only if* $\lambda = (2^i, 0)$ *for* $i \geq 0$ *or* $\lambda = (0, 2^j)$ *for* $j > 0$. *In the latter cases* $\mathrm{Ext}^1_X(V_X(\lambda), K)$ *is* 1*-dimensional.*

As in the A_1 case for $p = 2$, we will make use of a certain ideal in $L(X)$. Let $I = \langle e_\beta, f_\beta, e_{\alpha+\beta}, f_{\alpha+\beta}\rangle$ (Lie algebra span).

Lemma 5.3.2 (i) *I is an S-invariant abelian ideal of L, having basis e_β, f_β, $e_{\alpha+\beta}$, $f_{\alpha+\beta}$.*

(ii) $C_G(I) = 1$.

Proof (i) It can be checked from commutator relations that I is an ideal of $L(X)$. Also, as $p = 2$, the commutator relations imply $[e_{\pm\beta}, e_{\pm(\alpha+\beta)}] = 0$. Setting $[e_\beta, f_\beta] = t_\beta$, the fact that $p = 2$ implies that $t_\beta \in Z(L(X)) = 0$ (by Lemma 2.2.10(v)). Similarly for $t_\beta^{s_\alpha} = t_{\alpha+\beta}$. This gives (i).

Part (ii) follows from 2.2.10(iii). ■

For $\gamma \in \{\alpha, \beta\}$, set $T_\gamma = T_X \cap \langle U_\gamma, U_{-\gamma}\rangle$ and let $T_\gamma(c)$ be the element corresponding to the diagonal element (c, c^{-1}) of SL_2. Let t_γ, l_γ denote $T_\gamma(c)$ for c a cube root or fifth root of 1, respectively. The next lemma determines the actions of these tori on the basic modules $10, 01$. All other irreducibles are tensor products of twists of these, so the action of the tori and distinguished elements, in particular their fixed points, can be easily determined. These are recorded in the following lemmas.

Lemma 5.3.3 *There are bases for* $10, 01$ *such that* $T_\alpha(c)$ *and* $T_\beta(c)$ *have diagonal action as follows:*

on 10: $T_\alpha(c) \to (c, c, c^{-1}, c^{-1})$, $T_\beta(c) \to (c^2, c^{-2}, 1, 1)$

on 01: $T_\alpha(c) \to (c, c^{-1}, 1, 1)$, $T_\beta(c) \to (c, c, c^{-1}, c^{-1})$.

Lemma 5.3.4 *The dimensions of the fixed point spaces of $T_\alpha, T_\beta, t_\alpha, t_\beta, l_\alpha$ on certain irreducible X-modules are given below:*

$V =$	10	02	20	12	04	30	22	40	14	32	50	24
$\dim C_V(T_\alpha) =$	0	2	0	0	2	0	4	0	0	0	0	0
$\dim C_V(T_\beta) =$	2	0	2	4	0	4	0	2	0	12	4	4
$\dim C_V(t_\alpha) =$	0	2	0	4	2	8	4	0	4	24	8	4
$\dim C_V(t_\beta) =$	2	0	2	4	0	6	4	2	4	20	6	4
$\dim C_V(l_\alpha) =$	0	2	0	0	2	0	4	0	4	0	0	0

Lemma 5.3.5 *The number of trivial composition factors X on $L(G)$ is at least the rank of G.*

Proof Each nontrivial irreducible X-module is a tensor product of twists of the fundamental modules $10, 01$. The weights of 10 are $\{\pm\beta, \pm(\alpha+\beta)\}$ and the weights of 01 are $\{\pm\alpha, \pm(\alpha+2\beta)\}$. As α, β are independent weights of T_X it is clear that T_X cannot have fixed points on a nontrivial irreducible X-module. Since T_X acts trivially on $L(T_G)$, the result follows. ∎

The arguments for F_4, E_6, E_7 are easy and we settle these cases together in the next lemma.

Lemma 5.3.6 *G is not F_4, E_6 or E_7.*

Proof Suppose false. We begin by listing the labelled diagrams to consider. For E_6 we consider just one of each pair of labellings interchanged by a graph automorphism. The Weight Compare Program lists possible composition factors for each labelling. We have $L(X) = 10/02/00^2$, so as $L(X) \leq L(G)$ we certainly need these composition factors to occur within $L(G) \downarrow X$. Also by Lemma 5.3.5 we must have at least rank(G) trivial composition factors. Subject to these restrictions, the possible labellings and corresponding composition factors of $L(G) \downarrow X$ are as follows:

G	T-labelling	$L(G) \downarrow X$
F_4	2002	$20/02^5/10^4/00^{12}$
E_6	222200	$14/30^2/04/20^2/02/10^2/00^6$
	220002	$20/02^6/10^8/00^{18}$
	202002	$20^4/02^5/10^6/00^{18}$
	222002	$30/12^2/20/02^3/10^2/00^6$
	220202	$22/30/12/20^3/02/10^2/00^6$
E_7	2222000	$42/24/14^2/40/30^2/04^3/20/02/10/00^9$
	2000020	$20/02^8/10^{16}/00^{33}$
	2020020	$22/30/12^3/20^5/02^3/10^2/00^{13}$
	2220002	$40/22/30^3/04/12^2/20^2/02/10^2/00^9$
	2000022	$12/20^7/02^8/10^7/00^{29}$
	0020022	$30^2/12^4/02^4/10^3/00^9$ or $30/04/12^5/02^5/10/00^9$

We know from Lemma 2.2.10(iv) that X has no fixed points on L. Also note that $L = L(G)$ except for $G = E_7$ where L has codimension 1 in $L(G)$ (see 2.1.1). This implies that there are at most as many trivial composition factors on L as modules that extend the trivial module. We can do a bit better for $G = E_6$. Indeed, here $L(G)$ is a self-dual module, and this implies that there must exist strictly fewer trivial modules than modules that extend the trivial. Now Lemma 5.3.1 indicates precisely which composition factors can extend the trivial module and this quickly rules out each configuration in the table. ■

For the rest of this section we assume $G = E_8$. We will use a variation of arguments used in the case of A_1 with $p = 2$ (see Section 3.1).

Let T_G be a maximal torus of G and let δ be the root of highest height in $\Sigma(G)$. As in Lemma 2.2.7, T determines a parabolic subgroup P of G with Levi factor $L_P = C_G(T)$.

Let n denote the maximum possible T-weight of a vector in $L(G)$, and W_i the T-weight space of $L(G)$ for weight $n - i$.

The following key result is a variation of Proposition 3.1.3.

Proposition 5.3.7 *Suppose $L_P \cap E_7$ has at most two orbits on each of W_2, W_4 and W_6, with representatives being root elements and sums of two root elements for orthogonal roots. Then $C_G(I) \neq 1$.*

We begin the proof of the proposition with two lemmas. For $l \in L(X)$ and $v \in L(G)$ we will use the notation lv to denote $[lv]$.

Lemma 5.3.8 *Suppose $f_\beta f_\beta e_\delta = 0 = f_{\alpha+\beta} f_{\alpha+\beta} e_\delta$. Then $f_\beta f_{\alpha+\beta} e_\delta \in C_{L(G)}(I)$. If, in addition, $f_\beta e_\delta = 0$ (respectively $f_{\alpha+\beta} e_\delta = 0$), then $f_{\alpha+\beta} e_\delta \in C_{L(G)}(I)$ (respectively $f_\beta e_\delta \in C_{L(G)}(I)$).*

Proof The lemma is a consequence of the commutativity of I. For the first assertion, the hypothesis implies $f_\beta(f_\beta f_{\alpha+\beta} e_\delta) = f_{\alpha+\beta}(f_\beta f_\beta e_\delta) = 0$ and similarly $f_{\alpha+\beta}(f_\beta f_{\alpha+\beta} e_\delta) = f_\beta(f_{\alpha+\beta} f_{\alpha+\beta} e_\delta) = 0$. Also, $e_\beta(f_\beta f_{\alpha+\beta} e_\delta) = (f_\beta f_{\alpha+\beta})(e_\beta e_\delta) = 0$ and $e_{\alpha+\beta}(f_\beta f_{\alpha+\beta} e_\delta) = (f_\beta f_{\alpha+\beta})(e_{\alpha+\beta} e_\delta) = 0$, since e_δ is central in the Lie algebra of the maximal unipotent group of G corresponding to positive roots. This establishes the first assertion and the others follow in like manner. ∎

Lemma 5.3.9 *Suppose $f_\beta e_\delta = c_1 e_\gamma + c_2 e_\mu$, where either $c_2 = 0$ or $c_1 \neq 0 \neq c_2$ and γ and μ are orthogonal roots. Then $f_\beta e_\delta \in C_{L(G)}(\langle e_\beta, f_\beta \rangle)$.*

Proof This is just Lemma 3.1.4, except that we work there with T rather than a maximal torus of $\langle U_\beta, U_{-\beta} \rangle$. ∎

We now work towards the proof of Proposition 5.3.7. First observe that $f_\beta e_\delta$ has T-weight $n-2$ and hence is in W_2. So by hypothesis we can conjugate X by an element of $L_P \cap E_7$ so that the hypothesis of Lemma 5.3.9 holds. So we may suppose that we have this condition and hence Lemma 5.3.9 shows that $f_\beta f_\beta e_\delta = 0$. This conjugation fixes T and the corresponding labelled diagram, but may change T_G. Since the conjugation is from $L_P \cap E_7$ which centralizes e_δ, δ remains the highest root in the new system.

Now consider $f_{\alpha+\beta}$. This is an element of $L(G)$ of T-weight -4, so that $f_{\alpha+\beta} e_\delta \in W_4$. So by hypothesis we can conjugate by an element $l \in L_P \cap E_7$ so that $f_{\alpha+\beta} e_\delta$ is a root vector or the sum of two root vectors corresponding to orthogonal roots. Notice that l centralizes e_δ so by the first paragraph $f_\beta^l f_\beta^l e_\delta = 0$. At this point we replace X by X^l. Otherwise we maintain the previous notation.

Let $T_\beta = T_X \cap \langle U_\beta, U_{-\beta} \rangle$. Hence, T_β is the 1-dimensional torus of T_X centralizing $\langle e_{\alpha+\beta}, f_{\alpha+\beta} \rangle$. Then $C_X(T_\beta)' = J_{\alpha+\beta} = \langle U_{\alpha+\beta}, U_{-(\alpha+\beta)} \rangle$. We will work in $D = \langle U_\gamma : e_\gamma$ has T-weight a multiple of 4 $\rangle$.

We claim that $C_G(T_\beta) \leq D$. For suppose $U_\gamma \leq C_G(T_\beta)$. Since X is represented as an adjoint group on $L(G)$ we can write $\gamma \downarrow T_X = r\alpha + s\beta$

for integers r, s. Now $\alpha(T_\beta(c)) = c^{-2}$ and $\beta(T_\beta(c)) = c^2$. Consequently $\gamma(T_\beta(c)) = c^{-2r+2s}$. This must be trivial, so $r = s$ and $\gamma \downarrow T_X = r(\alpha + \beta)$. In particular, γ has T-weight $4r$. The claim follows and implies $J_{\alpha+\beta} < D$.

We now work through the proof of Lemma 3.1.4, using the fact that $f_{\alpha+\beta}e_\delta$ is a root vector or sum of two root vectors for orthogonal roots. In this argument expressions for $f_{\alpha+\beta}$ are all taken within $L(D)$. Similarly, towards the end of that proof there are arguments involving a certain class 2 unipotent group. We now take that group within D. The conclusion is that $f_{\alpha+\beta}f_{\alpha+\beta}e_\delta = 0$.

At this point Lemma 5.3.8 implies that one of $f_\beta f_{\alpha+\beta}e_\delta, f_{\alpha+\beta}e_\delta, f_\beta e_\delta$, or e_δ is a nonzero element of $C_{L(G)}(I)$. The hypothesis of Proposition 5.3.7 asserts that in each case the element is a root vector or sum of two root vectors corresponding to orthogonal roots. Therefore, the proof of Lemma 3.1.5 shows that $C_G(I) \neq 1$, completing the proof of the proposition. ■

In the A_1 case with $p = 2$ it was possible to produce general arguments verifying the analog of hypothesis of Proposition 5.3.7. However, in that case the hypothesis only concerned W_2. For the case at hand we require information on W_2, W_4, and W_6. For this we use the Weight Compare Program. This program allows us to reduce from all possible labelled diagrams to only those that yield a potential restriction of $L(G)$ to X. For the resulting labellings we either verify the hypothesis of the Proposition or apply other arguments.

Some of the labellings give rise to many possibilities for $L(G) \downarrow X$. In most situations it is possible to verify the hypothesis of Proposition 5.3.7 and thereby avoid further work.

We tabulate the possible labellings below into two groups. The first and largest group consists of those configurations where Proposition 5.3.7 applies. For each of these we give below the labelling, together with the 3-tuple $(\dim W_2, \dim W_4, \dim W_6)$:

20020000$(4,7,10)$	02200200$(3,3,6)$	20020200$(2,4,4)$	20220200$(2,2,2)$	02020020$(3,3,3)$
00002020$(2,10,10)$	02002020$(2,4,6)$	22000220$(1,4,5)$	02200220$(1,3,3)$	22200220$(1,3,2)$
22020002$(4,2,3)$	20002002$(3,6,7)$	02002002$(3,4,6)$	22002002$(3,3,4)$	00202002$(3,3,6)$
20202002$(3,3,3)$	02000202$(2,5,10)$	22000202$(2,4,5)$	02200202$(2,3,3)$	22200202$(2,3,2)$
20020202$(2,2,4)$	00220202$(2,2,2)$	22002202$(2,1,3)$	20202202$(2,1,3)$	22202202$(2,1,2)$
22000022$(1,5,6)$	00020022$(1,3,6)$	20020022$(1,3,4)$	22002022$(1,2,3)$	00202022$(1,2,3)$
20202022$(1,2,3)$	02202022$(1,2,2)$	22202022$(1,2,2)$	20200222$(1,1,4)$	20020222$(1,1,2)$
22020222$(1,1,2)$	00220222$(1,1,2)$	22220222$(1,1,2)$	20002222$(1,1,1)$	

For each of the above labellings one can use Proposition 5.3.7 to obtain a contradiction, and we will illustrate with some examples of how this is carried out. It is usually easy to see that there are at most two orbits with representatives given by root elements or the sum of two root elements. In the latter case we require for the hypothesis of the proposition that the corresponding roots are orthogonal. Say these roots are γ and μ. It will be clear from the labelling that these roots have coefficient of α_8 equal to 1. If their sum were a root, then this sum must be the highest root and in all cases it is clear from the situation that this is absurd. Suppose their difference is a root. Then this difference must be a root in $L_P \cap E_7$. Say $\mu - \gamma = \delta$, so that $\mu = \gamma + \delta$. It then follows that $\langle U_\delta, U_{-\delta}\rangle$ is transitive on the nonzero elements of $\langle e_\gamma, e_\mu\rangle$, so that nonzero elements in this 2-space are all root elements. So the orthogonality condition is not an issue.

We now illustrate the method in a couple of examples.

Consider the labelling 02002002(3,4,6). For notational purposes we set $L_{i,j,\ldots}$ to be the semisimple factor of L_P spanned by fundamental SL_2's corresponding to fundamental roots $\alpha_i, \alpha_j, \ldots$ With this notation we find that W_2 is a 3-space affording the usual module for $L_{6,7} = A_2$. The action on nonzero vectors is transitive so all elements are root vectors. Similarly W_4 affords a natural module for $L_{1,3,4}$ while W_6 affords the orthogonal 6-dimensional module for $L_{1,3,4}$. In the former case we have transitivity on nonzero veccctors, while in the latter case there are two orbits on 1-spaces and we obtain the hypotheses of Proposition 5.3.7.

Probably the most complicated configuration is the labelling 00002020(2,10,10). Here W_2 is no problem as this affords the usual module for L_6. But W_4 affords the wedge-square of the usual module for $L_{1,2,3,4}$ and W_6 affords the tensor product of usual modules for $L_6 \times L_{1,2,3,4}$. Indeed, $e_{01122221}$ is a maximal vector. In both these cases we find that there are two orbits on 1-spaces. For W_6 this follows using expressions for vectors in the tensor product together with transitive action of the factors. For W_4 we apply the results of [29] (working in an A_4-parabolic of a group of type D_5) to get the assertion and also the fact that representatives can be taken as root vectors and sums of two root vectors.

Using these techniques it is easy to deal with all the above labellings. We now consider the second group of labellings. These are tabulated below:

00200000	00020000	00002000	20002000	00200200
02000020	00200020	02200020	00020020	20002020
20202220	20000002	02000002	22000002	00200002
00020002	20000022			

A labelling determines the T-weights on $L(G)$, but each of these may yield several possibilities for the weights of T_X and hence several possibilities for composition factors of $L(G) \downarrow X$. The Weight Compare Program gives these explicitly although we will not reproduce these here.

Nearly all the configurations can be ruled out from the following observations. First, as noted before, $L(G) \downarrow L(X)$ must contain all composition factors of $L(X) = 10/02/00^2$. Also, by Lemma 5.3.5, X must have at least 8 trivial composition factors on $L(G)$. Next note that for $\gamma \in \{\alpha, \beta\}$, $C_G(T_\gamma)$ is a Levi factor of G, while $C_G(t_\gamma)$ and $C_G(l_\gamma)$ are reductive groups of maximal rank which contain $C_G(T_\gamma)$. The dimensions of these centralizers can be easily determined from Lemma 5.3.3 for a given set of composition factors of $L(G) \downarrow X$. After applying the above considerations we are left with the following cases to consider:

Table 1

Case	T-labelling	$L(G) \downarrow X$
(a)	00020000	$30^2/04^3/12^4/20^9/02^{12}/10^8/00^{24}$
(b)	00002000	$12^4/20^6/02^{16}/10^{16}/00^{32}$
(c)	00200020	$22^2/30^2/04^3/12^4/20^8/02^{10}/10^4/00^{20}$
(d)	00020020	$32^2/40/22^2/30/04^2/12/20^4/02^3/10^2/00^8$
(e)	00020002	$14/22^3/30^4/04^2/12/20^8/02^4/10^8/00^{16}$
(f)	00020002	$14/22^3/30^3/04^3/12^2/20^8/02^5/10^6/00^{16}$
(g)	00020002	$40/22^4/30^3/04^3/12/20^{10}/02^5/10^6/00^{20}$

Lemma 5.3.10 *X is not maximal in cases* (c), (d), (f) *and* (g) *of Table* 1.

Proof These cases are relatively straightforward in that they just require a slight extension of methods already used. In each case we calculate $\dim C_G(T_\beta)$ and $\dim C_G(t_\beta)$. This centralizer is a Levi factor in the first case and a reductive maximal rank subgroup in the second. The dimension of the centralizer is determined from Lemma 5.3.3 and we find that there is a unique possibility in each case. Clearly, $C_G(T_\beta) \leq C_G(t_\beta)$. Indeed, the smaller group is embedded as a Levi factor of the larger. However, the specific information on centralizers shows that this is impossible, yielding a contradiction. In the following table we present the information on centralizers which provides the contradiction.

Case	$\dim C_G(T_\beta)$	$\dim C_G(t_\beta)$	$C_G(T_\beta)$	$C_G(t_\beta)$
(c)	68	80	D_6T_2	A_8
(d)	54	80	$D_5A_2T_1$	E_6T_2, A_8
(f)	64	86	A_7T_1	A_2E_6
(g)	70	92	$D_6A_1T_1$	D_7T_1

Lemma 5.3.11 *Case* (a) *in Table* 1 *does not occur.*

Proof Here we have $L(G) \downarrow X = 30^2/04^3/12^4/20^9/02^{12}/10^8/00^{24}$. Let $v \in L(G)$ be a weight vector for weight 04. This weight is not subdominant to any other weight, so v is a maximal vector and hence $\langle Xv \rangle$ is an image of the Weyl module $W(04)$.

As v is a maximal vector, $e_\alpha v = e_\beta v = 0$. Also $f_\alpha v = 0$, since $04 - \alpha$ is not a weight of $L(G)$. Finally, $f_\beta v$ is a vector of weight 12. Now 12 does not occur in the irreducible of high weight 04. We conclude that either $f_\beta v = 0$, in which case $v \in C_{L(G)}(L(X))$, or $\langle Xv \rangle$ has a composition factor of high weight 12.

Suppose the former does not occur for any choice of v. Then letting v range over three independent vectors of weight 04, we conclude from Lemma 2.1.5 that the sum of the images of the maximal submodules is a singular subspace in which 12 appears with multiplicity 3. But then 12 must occur in $L(X)$ as a composition factor with multiplicity at least 6, a contradiction.

It follows that for some choice of v we have $v \in C_{L(G)}(L(X))$. Consider the T-weight space for T-weight 12. From the labelling we see that this is a 5-space which affords a natural module for the A_4 factor of $C_G(T)$. In particular, v is a root vector. But this contradicts Lemma 2.2.12. ∎

Lemma 5.3.12 *Case* (e) *of Table* 1 *does not occur.*

Proof In case (e), Lemma 5.3.3 implies that $\dim C_G(T_\alpha) = 40$, from which it follows that $C_G(T_\alpha)$ is either $A_5A_1T_2$ or $A_4A_3T_1$. All fixed points of T_α arise from composition factors with high weights among $00, 02, 04, 22$. We calculate the T-weights on the fixed points of each of these modules:

$$\begin{array}{ll} \text{on } C_{T_\alpha}(02): & T\text{-weights } 6, -6 \\ \text{on } C_{T_\alpha}(04): & T\text{-weights } 12, -12 \\ \text{on } C_{T_\alpha}(22): & T\text{-weights } 6, 6, -6, -6 \end{array}$$

From this information we can find all T-weights on $C_G(T_\alpha)$. The non-negative ones are as follows: $12^2, 6^{10}, 0^{16}$.

Now T determines a labelled Dynkin diagram on $C_G(T_\alpha)'$. Up to graph automorphisms, the only possible labellings consistent with the above information are: 60060, 6 for A_5A_1 and 6006, 606 for A_4A_3. In either case we find that $C_G(T_X) = C_G(T_\alpha, T) = A_2A_1T_5$.

Now let ω_G, ω_X denote representatives of the long words in the Weyl groups W_G, W_X, respectively. Since $T_X < T_G$, each of these elements inverts T_X, hence $\omega_G \cdot \omega_X$ centralizes T_X. As ω_G induces an outer automorphism of the A_2 factor of $C_G(T_X) = A_2A_1T_5$, this must also be true of ω_X (since $\omega_G \cdot \omega_X$ induces an inner autormorphism of A_2).

However, $N_X(T_X)$ acts on $C_G(T_X)$ and ω_X is in the derived group of W_X. As this Weyl group induces a group of automorphisms on the A_2 factor, this is a contradiction. ■

At this point we have settled all but case (b) in Table 1, which is more complicated.

Proposition 5.3.13 *Case* (b) *of Table* 1 *does not occur.*

The rest of this section consists of the proof of this proposition. We begin with the following lemmas.

Lemma 5.3.14 *The Weyl modules* $W_X(20)$ *and* $W_X(12)$ *are uniserial with composition series as follows:*

$W_X(20) = 20|00|02|00|10$

$W_X(12) = 12|02|00|20|00|02|00|10.$

Proof First consider $W_X(20)$ and let v be a maximal vector. A consideration of weights shows that the composition factors are as listed. The weight space for weight 02 is spanned by $f_\alpha v$ and this is a maximal vector within the maximal submodule. Hence $\langle X f_\alpha v\rangle$ is an image of the Weyl module $W_X(02)$ which is well-known to be uniserial of shape $02|00|10|00$. The weight space for weight 10 is 1-dimensional and spanned by $f_\alpha f_\beta v, f_\beta f_\alpha v$, and $f_{\alpha+\beta}v$. On the other hand $f_\alpha f_\beta v = f_\alpha 0 = 0$ and $f_\beta f_\alpha v = f_\alpha f_\beta v + f_{\alpha+\beta}v$. So it follows that the weight space is spanned by $f_\beta f_\alpha v$. Hence, this vector is contained in $\langle X f_\alpha v\rangle$, so we conclude that $\langle X f_\alpha v\rangle \cong W_X(02)$ or $W_X(02)/E$ with $E \cong 00$. On the other hand $V_X(20)$ extends the trivial module, so $W_X(20)$ has an indecomposable image of form $20|00$, and this forces the latter possibility. Finally, we note that by [34] there is no nontrivial extension of $V_X(20)$ by $V_X(02)$ and this forces $W_X(20)$ to be uniserial, as indicated.

Now consider $W_X(12)$ where we again use weights to check that the composition factors are as listed. We use a similar argument. Let v be a vector of weight 12, so that $f_\beta v$ spans the weight space of weight 20. Hence

$\langle X f_\beta v\rangle$ is an image of $W_X(20)$. Next, note that the weight space of $W_X(12)$ for weight 10 has dimension 3. It follows from the commutator relations that a basis for this weight space is $f_{\alpha+2\beta}v, f_{\alpha+\beta}f_\beta v, f_\beta^2 f_\alpha v$. In particular, this implies that $f_{\alpha+\beta}f_\beta v \neq 0$, so that 10 occurs as a weight in $\langle X f_\beta v\rangle$. It follows from the above paragraph that $\langle X f_\beta v\rangle \cong W_X(20)$.

Now consider $W_X(12)/\langle X f_\beta v\rangle$. This space has composition factors of high weights $12, 02, 00$. There is a unique simple quotient module, and by [34] there is no nontrivial extension of $V_X(12)$ by the trivial module. It follows that the quotient is uniserial of shape $12|02|00$. Finally, we note that by [34], $V_X(20)$ does not extend $V_X(02)$ or $V_X(12)$. It follows that $W_X(12)$ is uniserial of the indicated shape. ∎

Lemma 5.3.15 *Assume* (b) *in Table* 1 *holds. Then* $L(G)\downarrow X$ *contains a simple submodule of high weight* 02.

Proof Suppose false. Let v be a weight vector of T_X-weight 12. Then $\langle Xv\rangle$ is an image of $W_X(12)$ and we begin by determining the possibilities for this module. Suppose that for some v, $\langle Xv\rangle$ is irreducible. Letting I be the short ideal of $L(X)$ as before, we find that I acts trivially on $V_X(10)$ (indeed $V_X(10)\cong I$ as X-modules and I is abelian) and hence on $V_X(12) = V_X(10)\otimes V_X(02)$. As in the proof of Lemma 5.3.11, v is a root vector, so this contradicts Lemma 2.2.12. Hence $\langle Xv\rangle$ is not irreducible.

By Lemma 2.2.10, $\langle Xv\rangle$ contains no nonzero trivial submodule and by our supposition there is no irreducible submodule of high weight 02.

Consider the sum W of all modules of the form $\langle Xv\rangle$. These are each images of Weyl modules and by Lemma 2.1.5 the sum, say S, of the images of the maximal submodules is a singular subspace such that $W/S = (12)^4$. Since the multiplicity of $V_X(12)$ in $L\downarrow X$ is 4, while the multiplicity of $V_X(20)$ is 6, we conclude that $V_X(20)$ has multiplicity at most 3 in S. But then Lemma 5.3.14 implies that v can be chosen so that $\langle Xv\rangle$ is an image of the uniserial module of shape $12|02|00$. But we have seen that there are no trivial submodules, so the only possibility is that this image has the form $12|02$, establishing the lemma. ∎

Let $Y = A_1A_1 = J_\alpha J_{\alpha+2\beta} < X$ be the group generated by all long root subgroups of X with respect to T_X. We will determine the embedding of this subgroup in E_8 and then the fixed points of its Lie algebra. We note that $L(A_1A_1)$ acts trivially on the submodule 02 produced in Lemma 5.3.15. This will ultimately provide us with a contradiction.

Lemma 5.3.16 *There is a subsystem group $D_8 < E_8$ such that $Y < D_4D_4 < D_8$. The subgroup of SO_{16} corresponding to Y acts homogeneously on the natural module as the sum of 4 copies of $1 \otimes 1$.*

Proof We first note that $Z = Z(L(Y)) \neq 0$. Indeed, $Z = \langle h_\alpha \rangle$, so is generated by a semisimple element inducing the scalar 1 on the irreducible of high weight 10 and inducing zero on 02. All other irreducibles are tensor products of twists of these, so this determines the action on all irreducibles and it follows that $C_{L(G)}(Z)$ has dimension 120 and so $C_{L(G)}(Z) = D_8$.

Using Lemma 5.3.3 as in other cases we see that $C_G(T_\alpha)$ is a Levi factor of dimension 64 and hence $C_G(T_\alpha) = A_7T_\alpha$. Also, $C_G(T_\alpha) \leq C_G(h_\alpha) = D_8$. It follows that T_α has just two weights on the natural module $V = V(\lambda_1)$ for the preimage of D_8, both with multiplicity 8. So this shows that J_α is homogeneous on V, and similarly for $J_{\alpha+2\beta}$. Also h_α is acts as a nonzero scalar.

In the following we identify Y with its preimage in the cover of D_8, and then consider its action on the natural module V. It follows from the previous paragraph that Y acts on V as the direct sum of 4 irreducibles of the form $1 \otimes 1$. So $V \downarrow Y$ is the direct sum of 4 irreducibles, each of dimension 4. It is easy to argue that there are two pairs of irreducibles, each summing to a non-degenerate 8 space. Hence $Y < D_4D_4 < D_8$ and we have the assertion. ■

At this point we proceed with the proof of Proposition 5.3.13. We have $V_X(10) \downarrow Y = 1 \otimes 1$ and $V_X(01) \downarrow Y = (1 \otimes 0) \oplus (0 \otimes 1)$. All irreducible X-modules are tensor products of twists of $10, 01$, so we can determine all composition factors of Y on $L(G)$. The result is as follows:

$$L(G) \downarrow Y = (3\otimes 1)^4/(1\otimes 3)^4/(1\otimes 1)^{16}/(2\otimes 2)^6/(2\otimes 0)^{16}/(0\otimes 2)^{16}/(0\otimes 0)^{32}.$$

Now we also have $L(G) \downarrow D_8 = L(D_8) \oplus E$, where E is a spin module, and $L(D_8)$ can be realized as $\wedge^2 V$, where V is the usual orthogonal module $V(\lambda_1)$ for a cover of D_8. Therefore,

$$L(D_8) \downarrow Y = \wedge^2 \ ((1 \otimes 1) \oplus (1 \otimes 1) \oplus (1 \otimes 1) \oplus (1 \otimes 1)).$$

We next study $E \downarrow Y$. Arranging notation so that $V \downarrow D_4D_4 = V_{D_4}(\lambda_1) \oplus V_{D_4}(\lambda_1)$, it follows from 2.1 of [23] that

$$E \downarrow D_4D_4 = (V_{D_4}(\lambda_3) \otimes V_{D_4}(\lambda_3)) \ \oplus \ (V_{D_4}(\lambda_4) \otimes (V_{D_4}(\lambda_4)).$$

Now consider the projection, say Y_0, of Y to one of the D_4 factors. Then $V_{D_4}(\lambda_1) \downarrow Y_0 = (1 \otimes 1) \oplus (1 \otimes 1)$. This can be taken as the sum of two non-degenerate spaces with Y_0 diagonal in SO_4SO_4, or as the sum of two singular spaces with $Y_0 < A_3$. Conjugating by triality we can determine the possibilities for $V_{D_4}(\lambda_i) \downarrow Y_0$ for $i = 3, 4$. Let $0|2|0$ denote the indecomposable (tilting) module for one of the A_1 factors, obtained by tensoring two copies of the natural module. Also let $0|(2 \oplus 2)|0$ denote the wedge-square of the module $1 \otimes 1$ for Y_0, an indecomposable module. Then the possibilites for $V_{D_4}(\lambda_i) \downarrow Y_0$ are as follows:

(i) $(1 \otimes 1) \oplus (1 \otimes 1)$

(ii) $(0|2|0) \oplus (0|2|0)$ (one summand for each A_1 factor of Y_0)

(iii) $((0|(2 \oplus 2)|0) \oplus (0 \otimes 0)^2$ (arising from $Y_0 < A_3 = D_3$).

Now consider those possible restrictions of $L(G) \downarrow Y$ which are compatible with the known composition factors. We find that for $j = 3, 4$, $(V_{D_4}(\lambda_j) \otimes V_{D_4}(\lambda_j))$ is the tensor product of one factor of type (i) and one of type (ii) or (iii). In either case all composition factors of this summand have the form $1 \otimes 1, 1 \otimes 3$, or $3 \otimes 1$.

On the other hand, Lemma 5.3.15 implies that $V_X(02)$ occurs as an irreducible submodule, and $V_X(02) \downarrow Y = (2 \otimes 0) \oplus (0 \otimes 2)$. In view of the above considerations, this must occur within $L(D_8) \downarrow Y$, although from the earlier expression we see that this is impossible. Indeed, $L(D_8) \downarrow Y$ is a direct sum of modules of the form $\wedge^2(1 \otimes 1) = 0|(2 \oplus 2)|0$ and $(1 \otimes 1) \otimes (1 \otimes 1) = (0|2|0) \otimes (0|2|0)$. Restricting to one of the A_1 factors we see that in neither case does 2 occurs as a submodule. This is a contradiction, completing the proof of Proposition 5.3.13.

We have now completed the proof of Theorem 5.1.

6 Maximal subgroups of type G_2

In this section we prove Theorem 1 in the case where the subgroup X is of type G_2. As usual, we consider only the small characteristic cases required by Proposition 2.2.1.

Theorem 6.1 *Suppose that $X = G_2$ is a maximal proper closed connected $N_{G_1}(X)$-invariant subgroup of the exceptional group G, and assume further that*

(i) $C_G(X) = 1$, *and*

(ii) $p \leq 5$ *if* $G = E_8$; $p \leq 3$ *if* $G = E_7, E_6$; *and* $p = 2$ *if* $G = F_4$.

Then $G = E_6, p = 2$ *or* 3, *and X is unique up to* $\mathrm{Aut}\,G$*-conjugacy, with*

$$L(E_6) \downarrow X = 11/01^2/10^2/00, \;\; V_{27} \downarrow X = 20, \;\; \textit{if } p = 3$$

$$L(E_6) \downarrow X = 11 \oplus 01, \;\; V_{27} \downarrow X = 20/01/10/00 \;\; \textit{if } p = 2$$

where V_{27} is the 27-dimensional module $V_G(\lambda_1)$.

6.1 The case $p = 5$

Let X be as in the hypothesis of the theorem. Assume $p = 5$, so that $G = E_8$. In the usual way we use the Weight Compare Program to obtain a list of possible composition factors for $L(G) \downarrow X$. We find that the only irreducibles $V_X(\lambda)$ which can occur as composition factors are $\lambda = 30, 11, 20, 10, 01, 00$. In all cases the Weyl module $W_X(\lambda)$ is irreducible (see [13]), so $V_X(\lambda)$ does not extend the trivial X-module. Moreover, in all cases, $L(G) \downarrow X$ has at least one trivial composition factor. It follows that $C_{L(G)}(X) \neq 0$, contradicting Lemma 2.2.10(iv).

6.2 The case $p = 3$

Assume $p = 3$, so that $G = E_6, E_7$ or E_8. Recall that $L = L(G)'$, which is equal to $L(G)$ except when $G = E_6$, in which case L has codimension 1 in $L(G)$. As in other sections we use the notation n_{ab} to indicate the multiplicity of the irreducible module ab in $L \downarrow X$. Write $S = N_{G_1}(X)$.

The Weight Compare Program yields that the composition factors of $L \downarrow X$ are among $30, 11, 20, 10, 01, 00$. Of these, only 11 extends the trivial module, and $\dim(\mathrm{Ext}^1_X(11, 00)) = 1$ (by [37]).

Lemma 6.2.1 (i) *Either* $n_{00} = 0$ *or* $n_{00} < n_{11}$.

(ii) $n_{10} \geq 2$.

Proof (i) This follows from the paragraph preceding the lemma.

(ii) Observe first that $L(G_2)$ has a 7-dimensional ideal I generated by all e_β with β a short root; as a G_2-module $I \cong 10$.

Next we assert that $L(G_2)$ is indecomposable as a G_2-module, with composition factors 01/10: for if not, it is completely reducible for G_2, hence also for $L(G_2)$, and this implies that $[e_\alpha, e_\beta] = 0$ for α a long root and β a short root, which is not so. This proves the assertion.

Since $L(X) \subseteq L$ it follows from the previous paragraph that I is a singular subspace of L, and hence, as L is a self-dual X-module, it must have at least two composition factors isomorphic to 10. ∎

Of the list of possibilities for $L \downarrow X$ supplied by the Weight Compare Program, only one satisfies Lemma 6.2.1, whence we have the following.

Lemma 6.2.2 *We have* $G = E_6$ *and* $L \downarrow X = 11/01^2/10^2$, *with* T*-labelling* 222022.

In the situation of Lemma 6.2.2, consider the action of X on the 27-dimensional irreducible module $V_{27} = V_G(\lambda_1)$. Now as a linear combination of fundamental roots,

$$\lambda_1 = \frac{1}{3}(435642),$$

and hence T has highest weight 12 on V_{27}. Therefore $V_{27} \downarrow X$ has a composition factor ab, where $6a + 10b = 12$, hence $a = 2, b = 0$. The irreducible 20 has dimension 27 (see [13]), so we deduce that $V_{27} \downarrow X = 20$.

At this point, the existence and uniqueness of the subgroup X of $G = E_6$ $(p = 3)$ is provided by Testerman [41, Theorem 1(a)].

This completes the proof of Theorem 6.1 for $p = 3$.

6.3 The case $p = 2$

Assume now that $p = 2$. If $G = F_4$, then Lemma 2.2.2 implies that $S = N_{G_1}(X)$ does not contain special isogenies and hence Lemma 2.2.3 shows that we can regard S as acting on $L(G)$. Hence in any case, S acts on $L(G)$.

Use of the Weight Compare Program as usual gives us a list of possibilities for $L \downarrow X$, and the composition factors are among the following:

$$00, 10, 01, 20, 11, 30, 02, 21, 40, 12, 50, 04, 14, 80, 22. \qquad (*)$$

Lemma 6.3.1 (i) *Among the irreducibles in* $(*)$, *only* $10, 20, 40, 80$ *and* 21 *extend the trivial* X*-module, and for each of these the corresponding ext group* $\mathrm{Ext}^1_X(V(\lambda), K)$ *is* 1*-dimensional.*

(ii) *We have the following Weyl module structures:*

$$W_X(10) = 10|00,\ W_X(01) = 01,\ W_X(20) = 20|00|(10+01).$$

Proof All this follows from [36]. ∎

From part (i) of the previous lemma we immediately deduce

Lemma 6.3.2 *Either* $n_{00} = 0$ *or* $n_{00} < n_{10} + n_{20} + n_{40} + n_{80} + n_{21}$.

Combining this with the list already obtained from the Weight Compare Program gives the following. Let n_4 denote the number of T-weights on $L(G)$ which are divisible by 4.

Lemma 6.3.3 *The possibilities for* $L \downarrow X$ *are:*

G	*Case*	$L \downarrow X$	T*−labelling*	n_4
E_6	(1)	$11/01$	222022	38
E_7	(2)	$20^3/01^4/10^8/00^{10}$	0002020	69
	(3)	$11/20^2/01^3/10^2/00^2$	2002020	69
	(4)	$21/02/20/01^2$	2202022	69
E_8	(5)	$22/50/12/02/30/01$	22020022	136

Lemma 6.3.4 *Case* (2) *of Lemma* 6.3.3 *does not occur.*

Proof Here $L \downarrow X = 20^3/01^4/10^8/00^{10}$. By Lemmas 2.1.4 and 2.1.5, the maximal vectors of weight 20 generate an X-submodule M having a singular subspace Z, where $M/Z = 20^3$ and Z has $a \leq 3$ trivial composition factors and $b \leq 3$ composition factors 10. In $Z^\perp/Z$, generate with maximal vectors of weight 01, then 10, then 00. We find that $Z^\perp/Z = 20^3 \oplus 01^4 \oplus (10^{(8-2b)}/00^c) \oplus 00^{(10-2a-2c)}$. Taking the preimage of all trivial submodules we obtain a submodule $J = 10^b/00^{(10-a-c)}$. As $a+c \leq 5$ we have $10-a-c > b$ from which it follows that $L \downarrow X$ has a nonzero trivial submodule, contrary to Lemma 2.2.10(iv). ∎

Lemma 6.3.5 *Case* (3) *of Lemma* 6.3.3 *does not occur.*

Proof Here the T-labelling is 2002020. We shall consider the action of X on the 56-dimensional module $V_G(\lambda_7)$. Now

$$\lambda_7 = \frac{1}{2}(2346543)$$

from which we calculate that the non-negative T-weights on V_{56} are

$$12^2, 10^2, 8^4, 6^4, 4^6, 2^6, 0^8$$

The T-weights 12 arise from composition factors ab of $V_{56} \downarrow X$ with $6a + 10b = 12$, hence $a = 2, b = 0$. Thus $V_{56} \downarrow X$ has composition factors 20^2. Now T has non-negative weights $12, 8, 4$ on 20, so this leaves T-weights $10^2, 8^2, 6^4, 4^4, 2^6, 0^8$ to be accounted for by other composition factors. The T-weights 10^2 force composition factors 01^2, and as T has weights $10, 8, 6, 4, 2^2, 0^2$ on 01, this leaves T-weights $6^2, 4^2, 2^2, 0^4$. These lead to further composition factors $10^2/00^4$. We conclude that

$$V_{56} \downarrow X = 20^2/01^2/10^2/00^4.$$

Now V_{56} is self-dual and the only modules appearing which extend the trivial module are 20 and 10. It follows that $C_{V_{56}}(X) \neq 0$, which contradicts Lemma 2.2.13(ii). ■

Lemma 6.3.6 *Cases* (4) *and* (5) *of Lemma* 6.3.3 *do not occur.*

Proof We first claim that in cases (4) and (5), A contains a submodule 02 or 22, respectively. In case (5) 22 is the highest weight so if v is a weight vector of weight 22, then $\langle Xv \rangle$ is an image of the Weyl module $W(22)$. On the other hand, in this case $L \downarrow X$ is multiplicity-free and L is self-dual. So $\langle Xv \rangle$ must be irreducible. In case (4) the only composition factor present in $L \downarrow X$ which extends 02 is 21, and this occurs with multiplicity 1. So there is an X-submodule 02, establishing the claim.

From Lemma 6.3.3, $n_4 = 69, 136$, respectively. Hence Lemma 2.3.4 gives $A \leq L(D)$ and we must have $D = A_1D_6$ or A_1E_7. Assume case (4) holds. Here the non-negative T-weights appearing in $L(D)$ are $0^{11}, 4^{10}, 8^8, 12^6, 16^3, 20^2$. It follows that the T-labelling of the A_1 factor is 8, while the the D_6 factor has labels 404044. Now A contains a weight vector of T-weight 20 and this must be in the subspace of $L(D_6)$ spanned by root vectors for the two highest roots. All elements in this space are root elements of $L(G)$, so this contradicts Lemma 2.2.12. Similarly, in Case (5) the non-negative T-weights are $0^{16}, 4^{14}, 8^{13}, 12^{11}, 16^8, 20^7, 24^3, 28^3, 32$. Here the labelling of the A_1 factor is 4 and the labelling of the E_7 factor is 4400404. With v as above, v has T-weight 32, so is a root vector of $L(E_7)$ and we obtain the same contradiction. ■

It remains to consider case (1) of Lemma 6.3.3.

Lemma 6.3.7 *The group $G = E_6$ $(p = 2)$ has exactly two conjugacy classes of maximal connected subgroups $X = G_2$ with $L(G) \downarrow X = 11 \oplus 01$. Writing $V_{27} = V_G(\lambda_1)$, $V_{27} \downarrow X$ is uniserial with co-socle series either* 01|20|00|10 *or the dual of this.*

Proof For the proof of existence, our starting point is the maximal subgroup $M = G_2(2)$ of $E_6(2)$ produced in [17]. Regard $E_6(2)$ as subgroup of G. In [17, Section 8] it is shown that $L(G) \downarrow M = 11 \oplus 01$ and $V_{27} \downarrow M$ is uniserial with co-socle series 01|10|00|10, and that M has a subgroup $N = L_3(2)$ such that $N < A_2^3 < G$, where the A_2^3 is a subsystem subgroup of G and N is diagonally embedded in A_2^3.

The restriction $V_{27} \downarrow A_2^3$ is given by [23, 2.3], from which we see that

$$V_{27} \downarrow N = (10 \otimes 10) \oplus (01 \otimes 01) \oplus (10 \otimes 01).$$

We take $N < A = A_2$, where A is diagonal in A_2^3 with the above action on V_{27}. As a module for A we see that $10 \otimes 10$ is the indecomposable tilting module of high weight 20, so this is uniserial of shape 01|20|01, and also indecomposable under the action of N. Likewise, $01 \otimes 01 = 10|02|10$, uniserial, and $10 \otimes 01 = 11 \oplus 00$. It follows that N fixes a unique 6-dimensional completely reducible subspace $W = 10 \oplus 01$ of V_{27}, which must therefore be the subspace 10 for $M = G_2(2)$.

Now define

$$X = \langle M, A \rangle.$$

Then X fixes the 6-space W, so $X < G$. We claim that $X = G_2$ and is maximal in G. At the outset we note that $X = X^0$. This follows from the facts that $A < X^0$ and that M is simple with $M \cap A \geq N$.

Now X contains $M = G_2(2)$ and $A = A_2$, and X acts on the 6-space W. It follows that X induces an irreducible subgroup of SL_6 on W. Hence, $L(X)$ has an X-invariant section of dimension at most 35. On the other hand, $L(G) \downarrow M = 11 \oplus 01$, forcing $L(X)$ to have dimension 14. Moreover, as M and hence X act irreducibly on $L(X)$, we conclude that X is simple of dimension 14 with a 6-dimensional representation. Hence, $X = G_2$. Moreover, the irreducibility of M on $L(G)/L(X)$ implies that X is maximal.

We have now demonstrated the existence of a maximal G_2 in G. We must have $L(G) \downarrow X = 11 \oplus 01$, since the previous lemmas have ruled out all other possibilities.

We next establish the uniqueness part of the statement of the lemma. Let $\tilde{X}$ be an arbitrary maximal G_2 in G satisfying $L(G) \downarrow \tilde{X} = 11 \oplus 01$. The analog of T then determines the same labelling of the Dynkin diagram of G, namely 222022. Using this we find that $V_{27} \downarrow \tilde{X}$ has composition factors 20/01/10/00. Maximality implies that there is no fixed point on this module or its dual. Replacing V_{27} by its dual, if necessary, we may assume that $\tilde{X}$ fixes a unique 6-dimensional subspace W of V_{27}, with W affording 10. There is a 3-element $t \in \tilde{X}$ with $C_{\tilde{X}}(t) = A_2$, and the only possibility for $C_G(t)$ is A_2^3; moreover, $C_{\tilde{X}}(t)$ must be a diagonal A_2 in A_2^3. As above we see that $C_{\tilde{X}}(t)$ fixes a unique 6-space in V_{27}, which must therefore be W, and $\tilde{X} = G_W$. Since $C_{\tilde{X}}(t) = A_2$ determines W, and this diagonal subgroup A_2 of A_2^3 is uniquely determined up to conjugacy in $\operatorname{Aut} G$, it follows that $\tilde{X}$ is also determined up to conjugacy in $\operatorname{Aut} G$.

In the last paragraph we saw that if X is a maximal G_2, then $V_{27} \downarrow X = 20/01/10/00$. Taking X to contain M we see that this restriction must be uniserial and it follows that $V_{27} \downarrow X = 01|20|00|10$ or its dual. This completes the proof. ∎

7 Maximal subgroups X with rank$(X) \geq 3$

In this section we complete the proof of Theorem 1 by handling the case where the subgroup X has Lie rank at least 3. In view of Proposition 2.2.1 it is sufficient to prove the following.

Theorem 7.1 *There is no maximal $N_{G_1}(X)$-invariant proper closed connected subgroup X of the exceptional group G such that $C_G(X) = 1$ and one of the following holds:*

(i) *$p = 2$, $X = B_3$ and $G = E_6, E_7$ or E_8;*

(ii) *$p = 2$, $X = A_3, C_3$ or B_4, and $G = E_8$.*

We proceed by way of contradiction, assuming such a group X exists. We will obtain a contradiction in each case. Write $S = N_{G_1}(X)$.

7.1 The case $X = B_3$

In this section we consider case (i) of Theorem 7.1, in which $X = B_3$, $p = 2$ and $G = E_6, E_7$ or E_8. In view of Lemma 2.2.10 we see that S is generated by X and a (possibly trivial) field or graph-field morphism of G, the latter possible only for $G = E_6$.

Set notation as follows. Choose a root system $\Sigma(X)$ of X with base $\Pi(X) = \{\alpha, \beta, \gamma\}$, where α and β are long roots and γ a short root. Let T_X be a maximal torus of X with corresponding root elements and root subgroups labelled by $\Sigma(X)$. For $\delta \in \Sigma^+(X)$, let $e_\delta \in L(X)$ be the corresponding root vector for T_X, and $f_\delta = e_{-\delta}$. Recall that T is a 1-dimensional torus of X defined in 2.2.4. Each of α, β, γ affords T-weight 2; that is, T gives the labelling 222 of the Dynkin diagram of X.

As in the B_2 case for $p = 2$ (see Section 5.3), we will make use of the ideal generated by root elements for short roots. This ideal is the Lie algebra span

$$I = \langle e_\gamma, f_\gamma, e_{\beta+\gamma}, f_{\beta+\gamma}, e_{\alpha+\beta+\gamma}, f_{\alpha+\beta+\gamma} \rangle.$$

Lemma 7.1.1 (i) *I is an S-invariant abelian ideal with basis the given generators.*

(ii) *I affords the irreducible module $V_X(100)$ for X.*

(iii) *$C_G(I) = 1$.*

Proof (i) Lemma 2.2.3 implies that I is S-invariant. It can be checked from the commutator relations that I is an ideal of $L(X)$. Also, the commutator relations imply $[e_{\pm\delta}, e_{\pm\mu}] = 0$ for δ, μ distinct elements in $\{\gamma, \beta+\gamma, \alpha+\beta+\gamma\}$. Set $[e_\gamma, f_\gamma] = t_\gamma$. Then $p = 2$ implies that $t_\gamma \in Z(L(X)) = 0$. Similarly for $t_\gamma^{s_\beta} = t_{\alpha+\beta}$ and $t_\gamma^{s_\beta s_\alpha} = t_{\alpha+\beta+\gamma}$. This gives (i).

Part (ii) is clear. For (iii) first note that $C_X(I) = 1$. As X is maximal among S-invariant connected subgroups of G, it follows that $C_G(I)$ is finite. But then X centralizes $C_G(I)$, whereas we know that $C_G(X) = 1$ by hypothesis. This forces $C_G(I) = 1$. ■

We shall require some information on Weyl modules for X.

Lemma 7.1.2 (i) $W(100) = 100|000$ *(uniserial).*

(ii) $W(010) = 010|100|000$ *(uniserial).*

(iii) $L(B_3) = (010 + 000)|100$ *(socle* 100*).*

(iv) $W(002) = 002/010/(100)^2/000$*, and* 002 *does not extend* 000.

(v) $W(200) = 200|(010 + 000)|100$.

(vi) $W(300)$ *does not have an image of the form* $300|000$.

(vii) $W(110)$ *is irreducible.*

(viii) $\mathrm{Ext}^1_X(102, 000)$ *has dimension* 1.

Proof Part (i) is clear since $W(100)$ has dimension 7. For (ii) we start with $L(\tilde{B}_3)$, where $\tilde{B}_3$ is the simply connected group. As above, there is a short ideal, $\tilde{I}$. A maximal torus $\tilde{T}_3$ is the direct sum of the 1-dimensional tori for each of the fundamental roots, and the corresponding fundamental A_1's are each SL_2. Consequently, here $\tilde{t}_\gamma = [\tilde{e}_\gamma, \tilde{f}_\gamma] \in \tilde{I}$ is nontrivial and generates $Z = Z(L(\tilde{B}_3))$. Moreover, $\tilde{t}_\gamma \in [\tilde{B}_3, \tilde{I}]$. Note that Z is the kernel of the differential of the map $\tilde{B}_3 \to B_3$. Commutators show that $[\tilde{B}_3, L(\tilde{B}_3)] > \tilde{I}$. It now follows that as a $\tilde{B}_3$-module, $L(\tilde{B}_3)$ is uniserial of form $010|100|000$. Since $L(\tilde{B}_3)$ is a cyclic high weight module of high weight 010 and dimension 21 it must be $W(010)$.

As indicated above, Z is the kernel of the map $L(\tilde{B}_3) \to L(B_3)$. Hence $L(B_3)$ has a submodule $010|100$ of codimension 1. It follows from (ii) that 010 does not extend the trivial module and $Z(L(B_3)) = 0$ (indeed, $B_3 \geq B_1^3$ and $Z(L(B_1^3)) = 0$ but contains a maximal toral subalgebra of $L(B_3)$). So (iii) holds.

(iv) The composition factors of $W(002)$ follow from either the Sum Formula or by using the computer program in [13]. Now 1.3 of [23] implies that

002 does not extend 000.

(v) There is an exceptional morphism $\tilde{B}_3 \to \tilde{C}_3$, and this factors through B_3. Now $L(\tilde{C}_3) = W_{C_3}(200) = 200|(010 + 000)$. As a module for B_3 the weights are the same, so that $W_{B_3}(200)$ has an image of the form $200|(010 + 000)$. From weight and dimension considerations we see that the kernel of this quotient is 100, proving (v).

(vi) Suppose a nonsplit extension $300|000$ exists, afforded by an indecomposable module V having trivial submodule W. We first claim that I annihilates V. It annihilates V/W since I annihilates all irreducibles with long support. Now consider $D = A_1^3$, the subsystem group corresponding to short roots. Then $100 \downarrow D$ is a direct sum of 3 irreducibles, one for each A_1 and these irreducibles have high weight 2. Now $300 \downarrow D$ is a sum of tensor products. For a given A_1 factor, it follows from Lemma 2.1.6 that $6 = 2 \otimes 4$ does not extend 0. The other modules to consider are of the form $2 \otimes 4$ for $A_1 \times A_1$. But $W_{A_1 \times A_1}(2 \otimes 4)$ is the tensor product of the corresponding Weyl modules, $2|0$ and $4|0|2$, which are both uniserial. Hence, $2 \otimes 4$ does not extend the trivial module. It follows that V splits under the action of D and this gives the claim.

Our supposition and the claim imply that V cannot split over W under the action of $L(D_3)$. However, $(V/W) \downarrow D_3 = 030$ (viewing $D_3 = A_3$) and the Sum Formula implies that $W_{A_3}(030) = 030/010/200/002$. Hence, $V \downarrow D_3 = 030 + W$. Hence the extension does indeed split under the action of D_3 and hence $L(D_3)$, a contradiction.

Part (vii) follows immediately from [13], and (viii) follows from [10]. ■

The following is immediate from 7.1.2(iii) and the fact that $L(X) < L(G)$ which is self-dual. Recall that $L = L(G)'$, of codimension 1 in $L(G)$ for $G = E_7$, and equal to $L(G)$ otherwise.

Lemma 7.1.3 (i) *If $G = E_6$ or E_8, then $L \downarrow X$ contains composition factors* $010, 100^2, 000$.

(ii) *If $G = E_7$, then $L \downarrow X$ contains composition factors* $010, 100^2$.

We will make use of a certain 1-dimensional torus $T_1 < X$. Define $T_1(c) = h_\alpha(c^2)h_\beta(c^2)h_\gamma(c)$ and $T_1 = \{T_1(c) : c \in K^*\}$. Then $T_1 = C_X(B_2)$, where $B_2 = \langle U_{\pm\beta}, U_{\pm\gamma}\rangle$. Let $t \in T_1$ be an element of order 3.

Lemma 7.1.4 (i) *$C_G(T_1)$ is a Levi factor of G.*

(ii) $C_G(T_1) \leq C_G(t)$.
(iii) *If* $G = E_8$, *then* $C_G(t) = A_8, A_2E_6, D_7T_1$ *or* E_7T_1.
(iv) *If* $G = E_8$, *then* $\dim C_{L(G)}(t) = 80, 86, 92$ *or* 134.

Proof Part (i) is standard and (ii) is obvious. Part (iii) is given in [14, 4.7.1], and (iv) follows from (iii) and the fact that $L(C_G(t)) = C_{L(G)}(t)$. ■

The next lemma gives the action of T_1 on fundamental modules. For this lemma we identify T_1 with its preimage in $\tilde{B}_3$, so that there is an action on 001.

Lemma 7.1.5 *There exist bases of the fundamental irreducible X-modules such that $T_1(c)$ has the following diagonal action:*
on $100 : (c^2, c^{-2}, 1, 1, 1, 1)$
on $010 : (c^2, c^2, c^2, c^2, c^{-2}, c^{-2}, c^{-2}, c^{-2}, 1^6)$.
on $001 : (c, c, c, c, c^{-1}, c^{-1}, c^{-1}, c^{-1})$.

Proof This is a straight forward computation, made easier by the fact that T_1 centralizes B_2. ■

Lemma 7.1.6 *The dimensions of the fixed point spaces of T_1 and t on certain irreducible modules are given below*

V	000	100	010	200	002	102	300	020	110
$\dim C_V(T_1)$	1	4	6	4	0	8	16	6	24
$\dim C_V(t)$	1	4	6	4	0	8	18	6	24

Proof With the exception of the module 110, this is immediate from the previous lemma combined with the Steinberg tensor product theorem. In the last case we use the program of [13] to show that $100 \otimes 010 = 110/002/100^2$, and now the result follows from 7.1.5. ■

At this point we begin considerations of the cases $G = E_6, E_7, E_8$. As usual, the 1-dimensional torus T defined in 2.2.4 determines a labelling of the Dynkin diagram of G by 0's and 2's, where a given label determines the weight of T on the root vector for the corresponding fundamental root. From here we get all weights of T on $L(G)$ and the Weight Compare Program then determines the possible composition factors of $L(G) \downarrow X$ which are consistent with this labelling. We now consider the possibilities.

Lemma 7.1.7 *If n_λ denotes the number of composition factors of high weight λ in $L \downarrow X$, then either $n_{000} = 0$ or $n_{100} + n_{200} + n_{102} > n_{000}$.*

Proof We know that L is self-dual, and $C_L(X) = 0$ by Lemma 2.2.10. Hence $L \downarrow X$ can have no nonzero trivial submodule or quotient. Among the high weights considered in Lemma 7.1.2 only $100, 200$ and 102 can extend the trivial module, and in these cases the Ext group has dimension 1. So this will yield the lemma provided there are no further high weights which can occur as composition factors of $L \downarrow X$. The Weight Compare Program shows that this is indeed the case with two exceptions. The exceptions occur only for $G = E_8$ with the labellings $22020002, 22000202$, and 22020022. The program gives various possibilities for the composition factors. In the first and third labellings, 100 does not occur and in the second labelling 010 does not occur. So these labellings are not consistent with Lemma 7.1.3. ■

Lemma 7.1.8 *Theorem* 7.1 *holds if $G = E_6$.*

Proof The only T-labellings of E_6 that are consistent with Lemmas 7.1.3 and 7.1.7 are $222202, 220222$, and 202022. For the first two of these cases we make use of $\tilde{E}_6$, the simply connected cover of $G = E_6$. Let $\pi : \tilde{E}_6 \to E_6$ be the natural surjection. Let $\tilde{X}$ be the derived group of the preimage of X and consider the restriction $\pi_X : \tilde{X} \to X$. We claim that this is an isomorphism. As $p = 2$, this is certainly the case at the level of groups, so it suffices to show (see 4.3.4 of [38]) that the differential is surjective. However, the kernel of $d\pi$ is trivial, so this is also the case for $d\pi_X$.

It follows from the claim that X acts on both of the 27-dimensional irreducible $\tilde{E}_6$-modules $V(\lambda_1)$ and $V(\lambda_6)$. The high weights of these modules can be represented as a rational combination of fundamental roots: $\lambda_1 = \frac{1}{3}(435642), \lambda_6 = \frac{1}{3}(234654)$. In both the first and second cases we find that the high weight restricted to T is non-integral. But this is impossible as the restriction of these modules to X has composition factors that are integral combinations of roots, hence integral upon restriction to T.

This leaves the third case. Here the labelling is 202022 from which it follows that $\dim(D) = 46$ and hence $D = D_5T_1$. This contradicts 2.3.4, provided we can show that $A \neq 0$. There are three possibilities for the composition factors of $L(G) \downarrow X$:

(a) $002^3/010^2/100^4/000^2$

(b) $002^2/200/010^2/100^4/000^4$

(c) $002/200^2/010^2/100^4/000^6$.

Notice that in each case either there are at least two composition factors of high weight λ for $\lambda = 002$ or 200. Choose independent weight vectors v, w of the corresponding weight. Neither 200 or 002 is subdominant to another dominant weight in L so these are maximal vectors and $\langle Xv \rangle$ and $\langle Xw \rangle$ are both images of $W_X(\lambda)$. The sum of the images of the maximal submodules is a singular space so it follows from the above that 010 can appear as a composition factor in this sum with multiplicity at most 1. Therefore we can rechoose v, if necessary, so that 010 does not appear as a composition factor of $\langle Xv \rangle$.

The choice of v implies that v is annihilated by $e_\alpha, e_\beta, f_\alpha, f_\beta, e_\gamma, f_\gamma$. It follows from the commutator relations that the subalgebra of $L(X)$ generated by these elements contains I, the short ideal of $L(X)$ (although they do not generate $L(X)'$, as can be seen by considering the image in $L(X)/I$). Hence $v \in C_L(I)$. Now v has T-weight 12, the largest weight in L, and it follows from the labelling that the corresponding weight space has dimension 3 and there is an A_2 subgroup acting transitively on nonzero vectors of this space. Therefore v is a root vector. In the proof of 3.1.5 it was shown that $C_L(v) = L(C_G(u))$ for u a root element of G. Hence $u \in C_G(I)$, contradicting 7.1.1(iii). ■

Lemma 7.1.9 *Theorem* 7.1 *holds if* $G = E_7$.

Proof We proceed as in the previous lemma. There are two T-labellings of E_7 which are consistent with Lemmas 7.1.3 and 7.1.7: $0002020, 2002020$.

Suppose the labelling is 0002020. Here we consider $\pi : \tilde{E}_7 \to E_7$ and let $\tilde{X}$ denote the derived group of the preimage. Consider $\pi_X : \tilde{X} \to X$. Let $\tilde{T}$ correspond to T. Then $\tilde{T}$ has the same weights on $L(\tilde{E}_7)$ as T has on $L(E_7)$. Also $\lambda_7 = \frac{1}{2}(2346543)$, so the non-negative weights of $\tilde{T}$ on $V(\lambda_7)$ are $10^2, 8^2, 6^6, \ldots$. It follows that $V(\lambda_7) \downarrow \tilde{X} = 010^2/100^a/001^b/000^c$, where $a+b = 4$. If $b \neq 0$, then $Z = Z(L(\tilde{X})) = Z(L(\tilde{E}_7))$. But then Z induces the group of scalars on $V(\lambda_7)$, whereas Z must be trivial on 010 (this appears within the adjoint representation of $\tilde{X}$, where Z induces the identity). Hence $b = 0$ and a dimension count shows that $c = 4$. Since $V(\lambda_7)$ is self-dual we conclude as in Lemma 7.1.7 that $\tilde{X}$ has a nonzero fixed point on $V(\lambda_7)$. This contradicts Lemma 2.2.13(ii).

Now suppose the labelling is 2002020. Here the Weight Compare Program yields two possibilities for the composition factors of $L(G) \downarrow X$, namely $110/200^2/010^3/100^2/000^3$ and $110/002/200/010^3/100^2/000$. In the first case

Lemma 7.1.6 shows that $\dim C_G(T_1) = 61$, whereas there is no Levi factor of G with this dimension.

Consider the second case. Here there is a unique trivial composition factor of $L(G) \downarrow X$, so this must occur as $L(G)/L(G)'$. Hence there is no trivial composition factor within $L(G)'$ and since this is the image of the differential under the projection $\tilde{E}_7 \to E_7$, we conclude that the preimage, $\tilde{X}$ of X is simply connected and $Z = Z(L(\tilde{X})) = Z(L(\tilde{E}_7))$. Then Z induces scalars on $V(\lambda_7)$, so all composition factors of $\tilde{X}$ on this module must be faithful modules for the simply connected group $\tilde{X}$. The preimage $\tilde{T}$ of T has the same labelling as T, and this implies that its non-negative weights on $V(\lambda_7)$ are $12^2, 10^2, 8^4, 6^4, \ldots$. The irreducible $\tilde{X}$-modules whose high weight affords T-weight 12 are 200, 002, and 101. Now 101 has dimension 48, so there can be at most one of these in $V(\lambda_7) \downarrow \tilde{X}$. Therefore, either 200 or 002 must occur as a composition factor. In either case Z is trivial on this factor, a contradiction. ∎

Lemma 7.1.10 *Theorem* 7.1 *holds if* $G = E_8$.

Proof We again consider the possible labelled Dynkin diagrams and corresponding composition factors of X on $L(G)$. There are just three labellings yielding composition factors consistent with Lemmas 7.1.3 and 7.1.7, namely $20002002, 02002002$, and 20000202. In the first case there is just one possibility for composition factors consistent with the lemmas and in the third case just three. However, the second case gives rise to many possibilities.

Most of the possibilities are settled with the aid of Lemmas 7.1.4 and 7.1.6. In the table below we list the possible composition factors of X on $L(G)$, and the corresponding dimensions of $C_G(T_1)$ and $C_G(t)$.

Applying Lemma 7.1.4(iv) we see that only cases $5, 9, 10, 15, 22, 27$ in the table are possible So it remains to settle these configurations. In cases $9, 10, 15, 22$ we have $\dim C_G(T_1) = 88$, 78, 90, 128, respectively. On the other hand $C_G(T_1)$ is a Levi factor of G and one easily checks that there do not exist Levi factors of any of these dimensions. This leaves cases 5 and 27.

Case	Comp. factors of $L(G) \downarrow X$	$\dim C_G(T_1)$	$\dim C_G(t)$
1	$300/110^2/002/200/010^4/100^2/000^2$	102	104
2	$020/102^3/002^7/200/010/100^2/000^2$	50	50
3	$020/300/102^2/002^7/200/010/100^4/000^2$	66	68
4	$020/102^3/002^6/200^2/010/100^2/000^4$	56	56
5	$020/300^2/102/002^7/200/010/100^6/000^2$	82	86
6	$020/300/102^2/002^6/200^2/010/100^4/000^4$	72	74
7	$020/102^3/002^5/200^3/010/100^2/000^6$	62	62
8	$020/300^3/002^7/200/010/100^8/000^2$	98	104
9	$020/300^2/102/002^6/200^2/010/100^6/000^4$	88	92
10	$020/300/102^2/002^5/200^3/010/100^4/000^6$	78	80
11	$020/102^3/002^4/200^4/010/100^2/000^8$	68	68
12	$020/300^3/002^6/200^2/010/100^8/000^4$	104	110
13	$020/300^2/102/002^5/200^3/010/100^6/000^6$	94	98
14	$020/300^3/002^5/200^3/010/100^8/000^6$	110	116
15	$020/300/102^2/002^3/200^5/010/100^4/000^{10}$	90	92
16	$020/300^2/102/002^4/200^4/010/100^6/000^8$	100	104
17	$020/300/102^2/002^4/200^4/010/100^4/000^8$	82	84
18	$020/300^3/002^4/200^4/010/100^8/000^8$	116	122
19	$020/300^2/102/002^3/200^5/010/100^6/000^{10}$	106	110
20	$020/300^3/002^3/200^5/010/100^8/000^{10}$	122	128
21	$020/300^2/102/002^2/200^6/010/100^6/000^{12}$	112	116
22	$020/300^3/002^2/200^6/010/100^8/000^{12}$	128	134
23	$020/300^3/002/200^7/010/100^8/000^{14}$	134	140
24	$110/002^5/200/010^7/100^6/000^4$	98	98
25	$110/002^4/200^2/010^7/100^6/000^6$	104	104
26	$110/002^3/200^3/010^7/100^6/000^8$	110	110
27	$110/002^6/010^7/100^6/000^2$	92	92

Assume case 5 holds, where $\dim C_G(T_1) = 82$ and $\dim C_G(t) = 86$. This does not give a contradiction, as we could have $C_G(T_1) = E_6A_1T_1$ and $C_G(t) = E_6A_2$. To settle this case we consider another torus. Let $T_1'(c) = h_\alpha(c)h_\beta(c^2)h_\gamma(c^{3/2})$ and let $T_1' = \{T_1'(c) : c \in K^*\}$. This torus is chosen so as to centralize the A_2 Levi factor $\langle U_{\pm\alpha}, U_{\pm\beta}\rangle$ of X. Let t' be an element of order 3 in T_1'.

Using the fact that T_1' centralizes A_2, one checks that $T_1'(c)$ has the following eigenvalues on the irreducible X-modules $100, 010, 002$:

$$\begin{array}{ll} \text{on } 100: & (c, c, c, c^{-1}, c^{-1}, c^{-1}) \\ \text{on } 010: & (c^2, c^2, c^2, c^{-2}, c^{-2}, c^{-2}, 1^8) \\ \text{on } 002: & (c^3, c, c, c, c^{-1}, c^{-1}, c^{-1}, c^{-3}) \end{array}$$

From this, together with the Steinberg tensor product theorem it follows that $\dim C_G(T_1') = 36$ and $\dim C_G(t') = 86$. Hence $C_G(t') = E_6A_2$. Also $C_G(T_1') = A_2A_1A_4T_1'$ or $A_1A_1D_4T_2$. Of course $C_G(T_1') < C_G(t')$ and this rules out the latter case since E_6A_2 contains no such subsystem. Hence $C_G(T_1') = A_2A_1A_4T_1'$.

As noted earlier, $C_X(T_1') = \bar{A}_2T_1'$, where $\bar{A}_2 = SL_3$. Hence $\bar{A}_2$ is contained in the A_2A_4 subsystem group of $C_G(T_1')$. First assume that $\bar{A}_2$ projects trivially to the A_4 factor. Then $\bar{A}_2$ is generated by root subgroups of G hence X is determined up to conjugacy by [22, 2.1]. In particular Table 3 of Section 4 of [22] shows that $C_G(X) = B_4$, a contradiction.

Hence $\bar{A}_2$ projects nontrivially to the A_4 factor. The only copy of A_2 in A_4 is a Levi A_2 which has nontrivial center not in the center of A_4. But this is impossible, as $Z(\bar{A}_2) = \langle t' \rangle$ and $t' \in C_G(A_4)$.

Now assume case 27 holds. Here we use a variation of the argument in the last two paragraphs of the proof of 7.1.8. First note that from the list of composition factors there is a 6-space, say L_{002}, of vectors of weight 002. If $0 \neq v \in L_{002}$, then $\langle Xv \rangle$ is an image of $W(002)$. The sum of the images of the maximal submodules is a singular subspace, so the composition factor 010 can occur in this sum with multiplicity at most 3. It follows that there is a 3-space, E, of L_{002} centralized by f_γ. Now $e_\alpha, f_\alpha, e_\beta, f_\beta, e_\gamma$ annihilate any weight vector of weight 002 and these together with f_γ generate a subalgebra containing I. Hence $E \leq C_L(I)$.

Now 002 has T-weight 12, and it follows from the labelling that the full space of vectors of T-weight 12 has dimension 8 and affords an orthogonal module for a D_4 subsystem group and this space is $\langle D_4e_\delta \rangle$, where $\delta = 24635321$. Hence E contains a singular vector in this subspace, which must then be a root vector. Consequently I centralizes a root vector of L so the argument of 3.1.5 implies that I is centralized by a root element of G, contradictiing 7.1.1(iii). This completes the proof of the lemma. ■

The proof of Theorem 7.1 is now complete for $X = B_3$.

7.2 The cases $X = C_3, B_4$

In this section we establish Theorem 7.1 when $X = C_3$ or B_4. Here we have $p = 2$ and $G = E_8$. As usual, by Lemma 2.2.10, S is generated by X and a (possibly trivial) field or graph-field morphism of G.

As in previous cases the 1-dimensional torus $T < X$ defined in 2.2.4 determines a labelling of the Dynkin diagram of G by 0's and 2's. In turn, this determines the weights of T on $L(G)$ and gives a finite number of possibilities for the composition factors of X on $L(G)$. We then make use of the Weight Compare Program to obtain the following lemma.

Lemma 7.2.1 *One of the following holds:*

(i) $X = C_3$ *and* $L(G) \downarrow X = 202/220^2/400^2/020^4/000^4$.

(ii) $X = B_4$ *and* $L(G) \downarrow X = 0010^2/2000/0100^4/1000^4/0000^8$.

We must settle the two cases in the above lemma. This is easy in the first case. Indeed, if $X = C_3$, then $L(G) \downarrow X$ does not contain the composition factor 010 which occurs within $L(X)$, so this is impossible.

So now assume $X = B_4$. Take a base for the root system of X to be $\Pi(X) = \{\beta_1, \beta_2, \beta_3, \beta_4\}$, with β_4 a short root.

We shall require information on certain Weyl modules for X.

Lemma 7.2.2 *Let $X = B_4$. The following Weyl modules for X have the indicated composition factors and are uniserial.*

(i) $W(1000) = 1000|0000$.

(ii) $W(0100) = 0100|0000|1000|0000$.

(iii) $W(0010) = 0010|0100|0000|1000|0000$.

(iv) $W(2000) = 2000|0000|0100|0000|1000$.

Proof In this proof take X to be simply connected and let I denote the ideal generated by short root elements. We first use either the program of [13] or the Sum Formula to see that the composition factors are as indicated. The main issue is verifying that module is uniserial with the indicated series.

Part (i) is clear as $W(1000)$ can be realized as the 9-dimensional orthogonal module. For (ii) consider $L(X)$. We first compute that $Z(L(X))$ is generated by h_{β_4}. Also I is the Lie algebra of 4 commuting copies of B_1, each simply connected, and we find that I affords the module $W(1000) = 1000|0000$, with the trivial submodule generated by h_{β_4}. Now $L(X)/I = 0100|0000$

and the trivial module can be obtained using the image of an element of $L(T_X)$. As $C_{L(T_X)}(I) = \langle h_{\beta_4} \rangle$ we conclude that $L(X)$ must be uniserial with the indicated composition series. However, $\dim L(X) = 36 = \dim W(0100)$ and $L(X)$ is a cyclic module with generator of weight 0100. It follows that $L(X) \cong W(0100)$ and (ii) follows.

(iii) Set $W = W(0010)$ and consider W/IW. There is a maximal rank subgroup $D_4 < X$, and $L(X) = I + L(D_4)$. View the quotient as a module for D_4. It is well known that irreducible modules for X whose high weight has short support remain irreducible upon restriction to D_4 (see [30, 4.1]).

In particular, 0010 restricts to D_4 as the irreducible module 0011, and $W_{D_4}(0011)$ is irreducible. It follows that this irreducible splits off in the restriction $(W/IW) \downarrow D_4$. But then W/IW splits as a module for $L(D_4)$, hence for $L(X)$, and hence for X. The only possibility is that W/IW is irreducible. Now consider IW. Let v we a maximal vector of W of high weight 0010.

We claim that $IW = \langle X f_{0011} v \rangle$. This follows from consideration of certain commutators. For example, $f_{1111} = [f_{1100} f_{0011}]$ so that $f_{1111} v = f_{1100} f_{0011} v - f_{0011} f_{1100} v = f_{1100} f_{0011} v$. In this way we see that $Iv = L(X) f_{0011} v$. On the other hand $IW = \langle IXv \rangle = \langle XIv \rangle = \langle X f_{0011} v \rangle$, as claimed (for the last equality note that $\langle X f_{0011} v \rangle$ is X-invariant, hence $L(X)$-invariant, hence I-invariant and so contains $I f_{0011} v$ and all images under X). It follows from the claim that IW is an image of the Weyl module with high weight $0010 - (\beta_3 + \beta_4) = 0100$. In view of the known composition factors of $W(0100)$, we see that $IW \cong W(0100)$, so (iii) follows from (ii).

In case (iv) we make use of the isogeny $B_4 \to C_4$, taking both groups to be simply connected. Arguing as we did above for $L(B_4)$ we find that $L(C_4) = W_{C_4}(2000) = 2000|0000|0100|0000$, a uniserial module. Viewing this as a module for B_4 (via the isogeny) we conclude that $W_{B_4}(2000)$ has a uniserial image of the same shape. The kernel of this map must be irreducible of high weight 1000.

Let $W = W_{B_4}(2000)$ and let v be a maximal vector. Then $w = f_{1000} v$ spans the weight space for weight 0100 which is the highest weight of the maximal submodule of W. Let $z \in W$ be a weight vector of weight 1000. Using the computer program of [13] we see that this weight space is 1-dimensional. We claim that $z \in \langle B_4 w \rangle$. Suppose that we have established this claim. It then follows that $\langle B_4 w \rangle$ is an image of the Weyl module containing 1000 as a submodule and having composition factors

0100/0000/1000. Then (ii) implies that $\langle B_4 w\rangle$ is isomorphic to a factor module $W_{B_4}(0100)/U$ with $U \cong 0000$. Hence $\langle B_4 w\rangle$ is uniserial, and this gives (iv). So it remains to establish the claim. For this we note that $1000 = 2000 - (\beta_1+\beta_2+\beta_3+\beta_4)$. It follows that z is a linear combination of terms of the form $f_{\delta_1} f_{\delta_2}\cdots f_{\delta_r} v$ where $\sum \delta_i = \beta_1+\beta_2+\beta_3+\beta_4$. As v has weight 2000, such a term is 0 unless δ_r involves β_1, in which case the commutator relations imply that $f_{\delta_r} v = f_{\delta_r-\beta_1} f_{\beta_1} v = f_{\delta_r-\beta_1} w$. Hence $z \in L(B_4)w \leq \langle B_4 w\rangle$, as required. ■

Lemma 7.2.3 *Case* (ii) *of Lemma* 7.2.1 *does not occur.*

Proof We begin by letting v be a T_X-weight vector of $L = L(G)$ of weight 2000. Since 2000 is not subdominant to any other weight in L, $E_1 = \langle Xv\rangle$ is an image of $W_X(2000)$. Let S_1 be the image of the maximal submodule, so S_1 is a singular subspace of L.

First assume $A \neq 0$ (where $A = C_L(L(X)'))$. Since there is no trivial submodule of $L \downarrow X$, we see that 2000 must be the highest weight of A. The Weight Compare Program gives all T-weights on L, from which we find using Lemma 2.3.4 that $A \leq L(D)$ with $D = D_8$, and T determines the labelling 00400400 of the D_8 diagram. Now 2000 affords T-weight 16, which is the largest T-weight of D. It is clear from the labelling of the D_8 diagram that the T-weight space for weight 16 has dimension 3 and is spanned by the root vectors corresponding to roots 12222211, 11222211, 01222211. There is a subgroup of D_8 acting as SL_3 on this weight space, so it follows that A contains a root vector of L, contradicting Lemma 2.2.12.

From now on we assume $A = 0$. It follows that $f_{\beta_i} v \neq 0$ for some i. As v has weight 2000 the only possibility is that $i = 1$, showing that 0100 is a weight of E_1. We conclude that 0100 appears as a composition factor of E_1. Using Lemma 7.2.2 and the fact that there do not exist trivial submodules, we see that there are just two possibilities: either $E_1 = W_{B_4}(2000) = 2000|0000|0100|0000|1000$ or $E_1 = 2000|0000|0100$. Consequently we write $E_1 = 2000|0000|0100|0000^x|1000^x$, where $x = 0$ or 1.

We argue as in earlier cases. Write $S_1^\perp/S_1 = 2000 \perp W_1$, where the highest weight of W_1 is 0010, which occurs with multiplicity 2. Generating with maximal vectors of W_1 having weight 0010 and using Lemma 7.2.2 we obtain a submodule of W_1 with composition factors $0010^2/0100^a/1000^b/0000^c$, having a singular submodule S_2/S_1 with quotient 0010^2.

We can now repeat the argument. Working in $S_2^\perp/S_2$ we split off a nondegenerate space $2000 \perp 0010^2$ and in the orthogonal complement generate by high weight vectors of weight 0100 to get a space E_3/S_2 having composition factors $0100^{2-2a}/1000^d/0000^e$ and having a singular subspace S_3/S_2 with composition factors $1000^d/0000^e$. We do this two more times, generating by high weight vectors first of weight 1000 and then 0000, obtaining sections E_4/S_3 and E_5/S_4. In the following we record the structure of the various sections:

$$\begin{aligned}
&E_2/S_1 = 0010^2/0100^a/1000^b/0000^c, \quad S_2/S_1 = 0100^a/1000^b/0000^c,\\
&E_3/S_2 = 0100^{2-2a}/1000^d/0000^e, \quad S_3/S_2 = 1000^d/0000^e,\\
&E_4/S_3 = 1000^{4-2b-2d-2x}/0000^f, \quad S_4/S_3 = 0000^f,\\
&E_5/S_4 = 0000^{6-2c-2e-2f-2x}.
\end{aligned}$$

Now $E_5 = 0100^{1+a}/1000^{b+d+x}/0000^{7-c-e-f-x}$. Since $L \downarrow X$ contains no trivial submodule we must have $7-c-e-f-x \leq (1+a)+(b+d+x)$ and hence $6 \leq a+b+c+d+e+f+2x$. In addition, $S_4 = 0100^{1+a}/1000^{b+d+x}/0000^{1+c+e+f+x}$ and S_4 is singular. This implies $1+a \leq 2$, $b+d+x \leq 2$, and $1+c+e+f+x \leq 4$. Hence, $a \leq 1$, $b+d+x \leq 2$, and $c+e+f+x \leq 3$. It follows that these must all be equalities. From $a=1$ we see that E_3/S_2 is trivial, forcing $d=e=0$. Hence $b+x=2, c+f+x=3$. The first of these forces E_4/S_3 to be trivial, so that $f=0$ and hence $c+x=3$.

There are now two cases depending on the value of x. First suppose $x=1$. Here $b=1$ and $c=2$ so that $E_2/S_1 = 0010^2/0100/1000/0000^2$. In view of Lemma 7.2.2 we must have $E_2/S_1 = W(0010) \oplus 0010$. But then, taking a vector $w \in E_2$ whose image generates the $W(0010)$ summand, we see that w is also a maximal vector in E_2 and must generate a submodule $W(0010)$ of E_2. This yields a trivial submodule of $L \downarrow X$, a contradiction.

Now assume $x=0$. This time we get $b=2$ and $c=3$. Then $E_2/S_1 = 0010^2/0100/1000^2/0000^3$. However, Lemma 7.2.2 implies that there is no such module generated by two weight vectors of weight 0010. This is a final contradiction. ∎

This establishes Theorem 7.1 for $X = C_3, B_4$.

7.3 The case $X = A_3$

The final case to consider is $X = A_3$, where we again have $p=2$ and $X = E_8$. As in previous sections we take $T < T_X$, a maximal torus of X. Take a base of the root system of X, say $\Pi(X) = \{\alpha, \beta, \gamma\}$.

Once again we use the Weight Compare Program, discarding any configurations where the composition factors of $L(X)$ do not occur among those of $L \downarrow X$. The possibilities are listed in the following lemma.

Lemma 7.3.1 *One of the following holds:*

(a) $L(G) \downarrow X = 020^7/101^{14}/000^{10}$.

(b) $L(G) \downarrow X = 202/400^2/004^2/210^3/012^3/020^7/101^2/000^4$.

(c) $L(G) \downarrow X = 202^3/400/004/210^3/012^3/020^3/101^2/000^8$.

(d) $L(G) \downarrow X = 210^2/012^2/020^6/101^8/000^4$.

(e) $L(G) \downarrow X = 202/210^3/012^3/020^4/101^4/000^{10}$.

To settle these cases we make use of a certain torus $T_1 < X$. For $0 \neq c \in K$, let $T_1(c) = h_\alpha(c)h_\beta(c^2)h_\gamma(c^3)$ and $T_1 = \{T_1(c) : c \in K^*\}$. A consideration of matrices shows that $C_X(T_1) = T_1A_2$.

Lemma 7.3.2 *The dimensions of the fixed point spaces of T_1 on certain irreducible X-modules are as follows.*

$V =$	101	020	400	210	202
$\dim C_V(T_1) =$	8	0	0	9	8

Proof The weights of T_1 on the irreducible usual module 100 are immediate from the definition. This immediately yields the weights of T_1 on the other irreducible modules 010 and 001. The irreducible module 101 has codimension 1 in the adjoint module, where the fixed point space has dimension 9 (the dimension of of $C_X(T_1) = T_1A_2$). Also $210 = 100^{(2)} \otimes 010$, from which we see that the fixed point space on this module has dimension 9. The remaining modules are twists of ones already considered. ∎

Lemma 7.3.3 *None of the cases* (a) - (e) *of Lemma* 7.3.1 *can occur.*

Proof We calculate the dimension of $C_G(T_1)$ in each case, using the information provided in Lemma 7.3.2 and noting that the fixed point space has the same dimension on a module and its dual. We find that these dimensions are $122, 82, 102, 104, 104$ in the respective cases (a)-(e). On the other hand $C_G(T_1)$ must be a Levi factor of G, and it is easy to check that the only possibility occurs in case (b) with $C_G(T_1) = E_6A_1T_1$. To settle this case we consider another 1-dimensional torus. Indeed consider T_1', a 1-dimensional torus in a fundamental A_1 of X. Here $T_1'(c)$ has eigenvalues

$(c, c^{-1}, 1, 1)$ on the natural module for SL_4 and using this we easily compute the fixed points on each irreducible in the decomposition given by 7.3.1(b). We find that $C_G(T_1')$ has dimension 62. However, there is no Levi subgroup of this dimension, so this is a contradiction. ■

At this point we have established Theorem 7.1, and hence the proof of Theorem 1 is complete.

8 Proofs of Corollaries 2 and 3

Proof of Corollary 2 To obtain the corollary from Theorem 1 we need only determine the maximal reductive subgroups of maximal rank in G, and to a large extent this is settled in [19]. Let M be a maximal closed subgroup of the exceptional algebraic group G such that M^0 is reductive of maximal rank. Note that if M^0 is a maximal torus of G, then $G \neq G_2$ or F_4 because of the containments $N_{G_2}(T_2) < N_{G_2}(A_2) = A_2.2$ and $N_{F_4}(T_4) < N_{F_4}(D_4) = D_4.S_3$.

Now assume that M^0 is not a maximal torus. Then the root system Δ of M^0 is a non-empty subsystem of $\Sigma(G)$. By maximality, M satisfies the conditions of Lemmas 2.1 and 2.2 of [19]. Tables A and B in [19, p.302] list all subsystems Δ which satisfy these conditions; then Lemmas 2.3 and 2.4 of [19] rule out various possibilities in Tables A,B. What remains is the list in Table 10.3, together with two more possibilities, namely $M^0 = D_5T_1 < E_6$ or $B_2B_2 < F_4$ ($p = 2$). In the former case $N_{E_6}(M_0)$ lies in a D_5-parabolic, and the latter possibility is ruled out by observing that $N_{F_4}(B_2B_2) = (B_2B_2).2 < N_{F_4}(B_4)$.

So at this point we have a list of possibilities for M^0, including the case where M^0 is a maximal torus of $G = E_6, E_7, E_8$. To obtain Corollary 2(i) we must determine for which cases $N_G(M^0)$ is maximal. If this normalizer is not maximal, then it is contained in either a proper parabolic subgroup or in the normalizer of another subsystem group from the list. But an easy check shows this does not occur. This establishes Corollary 2(i). Part (ii) follows by inspection, deleting those subgroups in Table 10.3 for which M^0 is non-maximal.

Remark If we wish to extend Corollary 2 to groups G_1 containing a graph morphism of G, then the following additions are needed to the lists of subgroups in Table 10.3:

G_2, $p = 3$: add T_2

F_4, $p = 2$: add T_4 and B_2B_2

E_6: add D_5T_1.

Proof of Corollary 3 Here H is a simple algebraic group and we are trying to show that there are only finitely many classes of maximal closed subgroups of positive dimension. This follows immediately from Theorem 1 if the simple algebraic group H is of exceptional type, so assume that H is of classical type. For the purpose of proving the result we may assume that H is a classical group acting faithfully on its natural module V, of dimension n over the algebraically closed field K. By [25, Theorem 1], if M is a maximal closed subgroup of positive dimension in H, then one of the following holds:

(i) M is one of the subgroups in the families $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ defined in [25];

(ii) M^0 is simple and acts irreducibly and tensor-indecomposably on V.

There are only finitely many conjugacy classes of subgroups in $\mathcal{C}_1 \cup \ldots \cup \mathcal{C}_4$. So consider subgroups M^0 as in (ii). First observe that $\mathrm{rank}(M^0) \leq \mathrm{rank}(H) < n$. Also, being tensor-indecomposable, V is a restricted module for M^0 (see 2.1.3). Write $V = V(\lambda)$, where λ is a restricted dominant weight for M^0; say $\lambda = \sum c_i\lambda_i$, where λ_i are fundamental dominant weights and c_i integers with $0 \leq c_i \leq p-1$. The restriction of V to the A_1 corresponding to the i^{th} fundamental root has a composition factor of high weight c_i, dimension c_i+1, and hence $c_i \leq n-1$ for all i. In particular, given n, there are only finitely many possibilities for the simple group M^0 (since $\mathrm{rank}(M^0) < n$), and for each possible M^0, only finitely many restricted n-dimensional irreducible KM^0-modules. Each such module gives rise to only a finite number of conjugacy classes of subgroups in H. This completes the proof.

9 Restrictions of small G-modules to maximal subgroups

In this section we address the issue of determining the precise actions of the maximal subgroups X, given in Table 1 of Theorem 1, on the adjoint modules $L = L(G)'$, and also on the minimal modules $V = V_{F_4}(\lambda_1), V_{E_6}(\lambda_1), V_{E_7}(\lambda_7)$ for $G = F_4, E_6, E_7$, of dimensions $26 - \delta_{p,3}, 27, 56$ respectively. The conclusions are recorded in Table 10.1 (for $L \downarrow X$) and in Table 10.2 (for $V \downarrow X$).

We begin with the analysis of $L \downarrow X$.

9.1 Proof of the assertions in Table 10.1

Let G be an exceptional algebraic group, and assume that $X < G$ is one of the maximal subgroups given in Table 1 of Theorem 1. In each case the composition factors of X on L are given either in [31, p.193] or in 4.1.3, 5.1.2(ii), or 6.1. We aim to decompose L into a direct sum of explicit indecomposable modules for X.

We begin with two lemmas which will be applied in several instances. The first is taken from [22, 1.6].

Lemma 9.1.1 (i) *Let $G_2 < D_4$ be the usual embedding and let V be a restricted irreducible 8-dimensional module for D_4. If $p \neq 2$, then $V \downarrow G_2 = 10 \oplus 00$, while if $p = 2$, then $V \downarrow G_2 = T(10) = 00|10|00$.*

(ii) *Let $F_4 < E_6$ be the usual embedding and let V be a restricted irreducible 27-dimensional module for E_6. If $p \neq 3$, then $V \downarrow F_4 = 0001 \oplus 0000$, while if $p = 3$, then $V \downarrow F_4 = T(0001) = 0000|0001|0000$.*

We require some notation before stating the next result. Assume X is a semisimple group and λ, γ, μ are dominant weights such that the tilting module $T_X(\lambda) = \mu|\lambda|\mu$ and $T_X(\gamma) = \mu|\gamma|\mu$, both uniserial. We use the notation $\Delta(\lambda;\gamma)$ to denote an indecomposable module of shape $\mu|(\lambda \oplus \gamma)|\mu$ with socle and cosocle both of type μ, and which is obtained as a section of $T(\lambda) \oplus T(\gamma)$, by taking a maximal submodule and then factoring out a diagonal submodule of the socle.

Lemma 9.1.2 *Let X be semisimple and let M be an indecomposable and self-dual X-module with composition factors $(\mu)^2/\lambda/\gamma$. Assume that $T_X(\lambda) = \mu|\lambda|\mu$ and $T_X(\gamma) = \mu|\gamma|\mu$. Then $M \cong \Delta(\lambda;\gamma)$ if either of the following conditions holds:*

(i) *each of the composition factors* λ, μ, γ *is a self-dual* X*-module;*

(ii) M *has socle and cosocle of type* μ.

Proof (i) First note that our hypotheses imply that $W_X(\lambda) = \lambda|\mu$ and $W_X(\gamma) = \gamma|\mu$. Let $v, w \in M$ be weight vectors for weights λ, γ, respectively. Then $\langle Xv \rangle \cong W(\lambda)$ and $\langle Xw \rangle \cong W(\gamma)$: for otherwise, there would be an irreducible submodule of high weight λ or γ and our assumptions would force this submodule to be non-degenerate, contradicting the fact that M is indecomposable.

The information on composition factors implies that $\langle Xv \rangle \cong W(\lambda)$ and $\langle Xw \rangle \cong W(\gamma)$ have the same socle, say S. Then $S \cong \mu$ is a singular submodule and $S^\perp/S \cong \lambda \oplus \gamma$. It is clear from the above that $S^\perp = \langle Xv \rangle + \langle Xw \rangle$ and there is a surjection $W(\lambda) \oplus W(\gamma) \to S^\perp$. Let Z be the kernel of this surjection.

The Weyl module structures imply that $\mathrm{Ext}^1_X(\lambda \oplus \gamma, \mu)$ has dimension 2 and $\mathrm{Ext}^1_X(\mu, \mu) = 0$. It follows that $\mathrm{Ext}^1_X(S^\perp, \mu)$ has dimension at most 2. On the other hand, consider $T(\lambda) \oplus T(\mu)$. Each of the tilting module summands is uniserial of length 3 with socle and simple quotient isomorphic to μ. Hence, $W(\lambda) \oplus W(\mu)$ is a submodule of $T(\lambda) \oplus T(\mu)$ with quotient module $\mu \oplus \mu$ and it follows that a 2-dimensional group of extensions of $S^\perp$ by μ can be realized as $(T(\lambda) \oplus T(\mu))/Z$.

It follows from the above paragraph that $M \cong E/Z$, where E is a maximal submodule of $(T(\lambda) \oplus T(\mu))/Z$ and we have designated such a self-dual indecomposable module as $\Delta(\lambda; \gamma)$.

Part (ii) is similar, but easier. We are assuming the socle is of type μ so starting with the second paragraph the above proof gives the assertion. ■

Lemma 9.1.3 *If* $X = A_1$*, then* $L \downarrow X$ *is as indicated in Table* 10.1.

Proof Assume that $X = A_1$. In each of the cases in Table 1 of Theorem 1, either $p = 0$ or p is a good prime with the highest X-weight on L at most $2p - 2$. Hence, X is a *good* A_1, in the sense of [32]. Therefore [32, Theorem 1.1(iii)] shows that $L \downarrow X$ is a tilting module. The precise decomposition of $L \downarrow X$ into indecomposables follows from knowledge of the weights. An example of how this is done is provided at the start of the next section. ■

Lemma 9.1.4 *If* X *is simple, then* $L \downarrow X$ *is as indicated in Table* 10.1.

Proof Assume that X is simple. By the last result we may assume X has rank at least 2. In some cases the result has already been established.

If $G = F_4$, then $X = G_2$ and $p = 7$. Here we see from [31, p.193] that $L \downarrow X = 11 \oplus 01$. Next suppose $G = E_7$. Here $X = A_2$, and by [31, p.193] and Theorem 4.1 we see that either $L \downarrow X = 44 \oplus 11$ or $p = 7$ and $L \downarrow X$ has composition factors $44/11^2$. In the latter case it follows from [27, Theorem 4], together with the fact that L is self-dual, that $L \downarrow X = T(44)$.

The cases $G = E_6, E_8$ require a little more work.

Case $G = E_6$.

First assume $X = F_4$ or $C_4\,(p \neq 2)$. By [31, p.193], we have $L \downarrow X = L(X)/W_X(0001)$. Except for $p = 2, 3$, all the relevant Weyl modules are irreducible and the assertion follows. When $p = 3, W_{F_4}(0001) = 0001|0000$. However, in this case, $L = L(G)'$ has co-dimension 1 in $L(G)$ so we still have $L \downarrow F_4 = 1000 \oplus 0001$.

Now assume $p = 2$ with $X = F_4$. Here $L \downarrow X$ has composition factors $1000/0001^2$ and, since $L(X) = W_X(1000) = 1000|0001$, we must have $L \downarrow X = T(1000)$.

If $X = A_2$, then $p \geq 5$ and it follows from [31, p.193] and the irreducibility of the relevant Weyl modules that $L \downarrow X = 41 \oplus 14 \oplus 11$, as required.

Now suppose $X = G_2$, where the only restriction is $p \neq 7$. By [31, p.193] and Theorem 6.1, either $p \neq 3$ and $L \downarrow X = 11 \oplus L(G_2)$ or $p = 3$ and $L \downarrow X = 11/01^2/10^2$. In the latter case X is determined up to conjugacy and [41, Proposition G.1] gives precise generators for X by giving the root groups for the fundamental roots and their negatives. Using this one immediately obtains expressions for e_α and e_β, root elements of $L(X)$ corresponding to fundamental short and long roots, respectively. From the commutator relations we compute $e_{2\alpha+\beta}$. The result is a linear combination of root vectors in $L(E_6)$ corresponding to positive roots.

Let δ be the positive root in $\Sigma(G)$ of highest height and set $v = f_\delta$. So v is a weight vector of weight -11 for X. From information already obtained we can compute $e_{\alpha+\beta}e_\alpha v$ and $e_{2\alpha+\beta}v$ and find that these are linearly independent weight vectors of weight -01. From [13] we see that the -01 weight space of $V_{G_2}(11)$ has dimension 1, whereas the dimension in the Weyl module is 2. It follows that $\langle Xv \rangle$ has a composition factor of high weight 01. We also know that $L(X) \cong W_{G_2}(01) = 01|10$. Since L is self-dual it follows that $L(X) < \langle Xv \rangle$ and that $L \downarrow X = 10|01|11|01|10$ and is uniserial.

Case $G = E_8$

Here the only case to consider is $X = B_2$ and $p \geq 5$. By [31, p.193],

$L \downarrow X = W(06)/W(32)/W(02)$. For $p > 5$ each of these Weyl modules is irreducible, so $L \downarrow X = 06 \oplus 32 \oplus L(B_2)$. The situation is more complicated for $p = 5$, where $W(06) = 06|22$ and $W(32) = 32|22$.

On p. 111 of [31] precise expressions are given for the elements f_α, f_β of $L(X)$. Moreover, as indicated on p.112 of [31], if δ is the high root of $\Sigma(G)$, then $v = e_\delta$ and $w = e_{\delta-\alpha_8}$ are maximal vectors affording T_X-weights 06 and 32 respectively. From the expression for f_β we check that $f_\beta^2 v \neq 0$, and this affords a weight vector of weight 22. As $p = 5$, this is not a weight in $V_X(06)$, so we conclude that $\langle Xv\rangle = W(06) = 06|22$.

Now consider $L(X)w$. From the expressions for f_α and f_β it is easily checked that $f_\alpha f_\beta w$ and $f_\beta f_\alpha w$ are linearly independent and both afford weight vectors of weight 22. On the other hand the weight space of $V_X(32)$ for this weight has dimension 1, so $\langle Xw\rangle$ must be the Weyl module $W_X(32) = 32|22$. The composition factors of $L \downarrow X$ are $06/32/22^2/02$ and L is self-dual. It follows that $L \downarrow X = M \perp 02$, where M has shape $22|(06\oplus 32)|22$. It now follows from Lemma 9.1.2, that $M \cong \Delta(06; 32)$ which gives the result here. ■

Lemma 9.1.5 *If X is not simple, then $L \downarrow X$ is as indicated Table* 10.1.

Proof We are assuming that X is not simple. In all but one case we can write $X = X_1X_2$, a product of two simple groups. The exception occurs for $G = E_8$ with $X = A_1G_2G_2$. With this one exception the composition factors appearing in $L \downarrow X$ are given in [31, p.193]. We first determine the precise action on L of certain of the simple factors. If $X_i = A_1$, then we see from the information provided and the prime restrictions given (if any), that each composition factor of $L \downarrow X_i$ has high weight at most $2p - 2$. Hence, this is a *good* A_1 in the sense of [32]. If p is a good prime for G, then it follows from [32, Theorem 1.1(iii)] that $L \downarrow X_i$ is a tilting module. In fact, we will show that this holds in all cases where $X_i = A_1$ and $p > 2$.

Case $A_1G_2 < F_4$

In this case $p \geq 3$ and if $p > 3$ then the high weights of all composition factors of $L \downarrow X$ correspond to irreducible Weyl modules. So here, $L \downarrow X$ is completely reducible as given in Table 10.1. So now assume $p = 3$. We first determine the action of the simple factors on L.

We have $G_2 < D_4 < F_4$ and $L \downarrow D_4 = L(D_4) \oplus V_1 \oplus V_2 \oplus V_3$, where the modules V_i are the restricted irreducible 8-dimensional modules. On each of these, G_2 acts as $10 \oplus 00$. Also, $L(D_4)$ can be identified as $\wedge^2 V_1$, a direct summand of $V_1 \otimes V_1$. Recall that the tensor product of tilting modules is a tilting module and that direct summands of tilting modules are again tilting modules. It follows that $L(D_4) \downarrow G_2$ is a tilting module and so $L \downarrow G_2 = (T(01) \oplus 10^4) \perp 00^3$.

Next we note that $A_1 = C_{F_4}(G_2) < C_{F_4}(A_2) = \hat{A}_2$, where $A_2, \hat{A}_2$ denote subsystem subgroups generated by root groups for long roots and short roots, respectively. The embedding $A_1 < \hat{A}_2$ corresponds to the fixed points under a graph automorphism (arising from the graph automorphism of E_6 which fixes F_4). From [31, 1.8] we have $L \downarrow A_2\hat{A}_2 = L(A_2\hat{A}_2) \oplus (10 \otimes 02) \oplus (01 \otimes 20)$. As L is self-dual and $\hat{A}_2$ is simply connected, we must have $T(11) = 10 \otimes 01$ as a direct summand of $L(A_2\hat{A}_2)$. Also the module 20 is a direct summand of $10 \otimes 10$, and 10 and 01 restrict to A_1 as 2. It follows that $L \downarrow A_1 = T(4)^7 \oplus 2 \oplus 0^7$.

We can now establish the required restriction. First note that the A_1 factor leaves invariant the fixed point space of G_2, so from information on composition factors we can write $L \downarrow X = M \perp (2 \otimes 00)$, where $M = 4 \otimes 10/0 \otimes 01/(0 \otimes 10)^2$. The information of the previous two paragraphs implies that M is indecomposable. Also $T(4 \otimes 10) = (0 \otimes 10)|(4 \otimes 10)|(0 \otimes 10)$ and $T(0 \otimes 01) = (0 \otimes 10)|(0 \otimes 01)|(0 \otimes 10)$. At this point Lemma 9.1.2 implies that $M \cong \Delta(4 \otimes 10; 0 \otimes 01)$, as required.

Case $A_2G_2 < E_6$

Here the composition factors of $L \downarrow X$ are the union of those of $W(11) \otimes W(10)$, $W(11) \otimes 00$ and $00 \otimes W(01)$. If $p > 3$, then all the relevant Weyl modules are irreducible and so $L \downarrow X$ is completely reducible as in Table 10.1. So it remains to consider the cases $p = 2, 3$.

We will use the following information about the factors of X. As above, the G_2 factor X_2 is embedded in a subsystem subgroup of type D_4 and so a subsystem subgroup $A_2 < G_2$ corresponding to long roots is also a subsystem subgroup of E_6. Hence X_1 is contained in the centralizer of this A_2, a subsystem group of type A_2A_2.

Assume $p = 2$. In this case we see from the composition factors, that $L \downarrow A_2 = 11^8 \perp 00^{14}$. It follows that $L \downarrow X = M \perp L(G_2)$, where M affords a self-dual representation of X which restricts to the A_2 factor as 11^8. Working within $Y = SL_{64}$ we see that $C_Y(X_1)' = SL_8$. It follows that X_2 acts on M as it does on 8 copies of the natural module for SL_8. However, $G_2 < D_4$ and $L \downarrow D_4$ contains copies of each of the 8-dimensional restricted modules. So Lemma 9.1.1 implies $M \downarrow X = 11 \otimes T(10)$, as required.

Now assume $p = 3$. Here we start with $L(E_6) \downarrow D_4T_2 = L(D_4) \oplus L(T_2) \oplus V_1^2 \oplus V_2^2 \oplus V_3^2$, where the V_i are the restricted irreducible 8-dimensional modules for D_4. Lemma 9.1.1 implies $V_i \downarrow G_2 = 10 \oplus 00$ and $L(D_4)$ can be realized as the wedge-square of any of the V_i. So from previously mentioned results on tilting modules it follows that $L(E_6) \downarrow G_2 = (10^7 \oplus T(01)) \perp 00^8$, and the last summand must be $L(X_2)$. As L has dimension one less than $L(E_6)$ we have $L \downarrow X = M \perp (11 \otimes 00)$, where $M = (11 \otimes 10)/(00 \otimes 01)/(00 \otimes 10)^2$. From the embedding $X_1 < A_2A_2$ we see that $L \downarrow X_1$ has a direct summand of the form $(10 \otimes 01)^6 = T(11)^6$. Using this, we see that M is indecomposable and Lemma 9.1.2 implies $M \cong \Delta(11 \otimes 10; 00 \otimes 01)$.

Case $A_1A_2 < E_8$.

Here $p \geq 5$. If $p > 5$, then the Weyl modules for all relevant composition factors are irreducible, so $L \downarrow X$ is completely reducible as indicated in Table 10.1.

So assume $p = 5$. The subgroup A_1A_2 is constructed in 3.13 of [31]. The A_2 factor, X_2, is a subgroup of a Levi A_7, such that $V \downarrow X_2 = 11$, where V is a natural module for A_7. Now $L \downarrow A_7$ is the sum of an adjoint module plus $V, \wedge^2 V, \wedge^3 V, V^*, \wedge^2(V^*), \wedge^3(V^*)$. As $p = 5$ each of these is an irreducible summand of the tensor product of at most 3 copies of V and V^*. Now V restricts to an (irreducible) tilting module for X_2 and tilting modules are closed under the operations of tensor products and direct sums. It follows that $L \downarrow X_2$ is a tilting module, so our information on composition factors implies $L \downarrow X_2 = T(22)^3 \oplus 30^5 \oplus 03^5 \oplus 11^5 \oplus 00^3$.

Let T_{X_2} denote a maximal torus of X_2. Using the information on composition factors we find that $C_L(T_{X_2})$ has dimension 38 and so $X_1 \leq C_L(T_{X_2}) = D_4A_2T_{X_2}$. Since the X_1-composition factors have weight at most 6, it follows that the projections of X_1 to D_4 and A_2 correspond to restricted completely reducible modules and from the action of D_4A_2 on L we see that $L \downarrow X_2$ is also a tilting module and hence $L \downarrow X_2 = T(6)^8 \oplus 4^{20} \oplus 2^{20} \oplus 0^8$.

We can now obtain the decomposition. First note that X_1 stabilizes the X_2 summands of form 30^5 and 03^5. Since we know that $4 \otimes 30$ and $4 \otimes 03$ occur as composition factors, we conclude that these both occur as summands of $L \downarrow X$. Similarly, each simple factor of X stabilizes the fixed points of the other. The sum of the modules so far described has shape $30^5 \oplus 03^5 \oplus L(A_1) \oplus L(A_2)$. The perpendicular space of this, say M, has composition factors $(6\otimes 11)/(2\otimes 22)/\,(2\otimes 11)^2$. Generating by weight vectors of weight $2\otimes 22$ and $6\otimes 11$ we get images of the Weyl modules $W_X(2\otimes 22) = 2\otimes W_{X_2}(22) = 2\otimes 22|2\otimes 11$ and $W_X(6\otimes 11) = W_{X_1}(6)\otimes 11 = 6\otimes 11|2\otimes 11$.

As $L \downarrow X_i$ is tilting for $i = 1, 2$ we conclude that M is indecomposable of shape $(2 \otimes 11)|((2 \otimes 22) \oplus (6 \otimes 11))|(2 \otimes 11)$. Hence, Lemma 9.1.2 implies $M = \Delta(2 \otimes 22; 6 \otimes 11)$, as required.

Case $G_2F_4 < E_8$.

If $p > 3$, then all composition factors correspond to irreducible Weyl modules, so $L \downarrow X$ is completely reducible as in Table 10.1. Now consider $p = 3$. We first consider $L \downarrow F_4$. Let $F_4 < E_6$, with E_6 a subsystem group, and let V denote the irreducible 27-dimensional E_6-module $V(\lambda_1)$. Then Lemma 9.1.1(ii) implies that $V \downarrow F_4 = T_{F_4}(0001)$. Now $L \downarrow E_6$ contains the sum of three copies of V plus three copies of V^*. Hence $L \downarrow F_4$ contains the sum of 6 copies of $T_{F_4}(0001)$. Also note that 0001 occurs with multiplicity 7 in $L \downarrow F_4$.

The G_2 factor X_1 arises from an embedding within a subsystem group of type D_4. From [31, 1.8] we have $L \downarrow D_4 = L(D_4) \oplus J \oplus C_L(D_4)$, where J is the direct sum of 24 restricted 8-dimensional modules. In particular, we see that $C_L(G_2)$ has dimension 52, and hence $L(F_4)$ is a nondegenerate summand of L. Now $L(D_4)$ can be realized as the wedge-square of an 8-dimensional module. Hence, $L \downarrow G_2$ is a tilting module so that $L \downarrow G_2 = (T(01) \oplus 10^{25}) \perp 00^{52}$.

We can now write $L \downarrow X = M \perp (00 \otimes 1000)$, where $M = (10 \otimes 0001)/(01 \otimes 0000)/(10 \otimes 0000)^2$. From information on $L \downarrow G_2$ and $L \downarrow F_4$ together with Lemma 9.1.2, it follows that $M \cong \Delta(10 \otimes 0001; 01 \otimes 0000)$, as required.

Now assume $p = 2$. As before consider $F_4 < E_6$. We have already established that $L(E_6) \downarrow F_4 = T(1000)$. This and 9.1.1 give $L \downarrow F_4 = (T(1000)) \oplus 0001^6) \perp 0000^{14}$. As G_2 centralizes F_4, the decomposition is stabilized by G_2 and we have $L \downarrow X = M \perp (01 \otimes 0000)$, where $M = (10 \otimes 0001)/(00 \otimes 1000)/(00 \otimes 0001)^2$. We have $G_2 < D_4$ and 9.1.1 shows that G_2 acts on each of the 8-dimensional restricted representations as $T(10) = 00|10|00$. So it follows from [31, 1.8] that $L \downarrow G_2$ contains $T(10)^{24}$ as a direct summand. This together with the information on the restriction to F_4 implies that M is indecomposable, so that we obtain the result from Lemma 9.1.2.

Case $A_1G_2G_2 < E_8$.

Here X lies in a subgroup F_4G_2 of G and $X \cap F_4 = A_1G_2$ is maximal in the F_4 factor. Moreover, $p > 2$ and the G_2 factors are conjugate. When

$p > 3$ we have the following restrictions:

$$L \downarrow G_2F_4 = L(G_2) \oplus L(F_4) \oplus (10 \otimes 0001)$$
$$L(F_4) \downarrow A_1G_2 = L(A_1) \oplus L(G_2) \oplus (4 \otimes 10)$$
$$V_{F_4}(\lambda_4) \downarrow A_1G_2 = (2 \otimes 10) \otimes (4 \otimes 00)$$

where the last restriction is obtained from [23, 2.5] using the embedding $A_1G_2 < A_2G_2 < E_6$. At this point we compute $L \downarrow A_1G_2G_2$ and obtain the result in Table 10.1.

Now assume $p = 3$. We will produce certain submodules of $L \downarrow X$. Consideration of the centralizer of one of the G_2 factors leads to $A_1G_2 < F_4$, which acts on $L(F_4)$ as described in the first case of this lemma. In particular there is a submodule $\Delta(4 \otimes 10; 0 \otimes 01)$. The other G_2 factor of X acts trivially on this submodule and it follows that $L \downarrow X$ contains $\Delta(4 \otimes 10; 0 \otimes 01) \otimes 00 = \Delta(4 \otimes 10 \otimes 00; 0 \otimes 01 \otimes 00)$ as a submodule. Now, $N_G(X)$ contains an involution which interchanges the G_2 factors. Hence, $\Delta(4 \otimes 00 \otimes 10; 0 \otimes 00 \otimes 01)$ also occurs as a submodule. Next note that $L \downarrow G_2F_4$ contains a summand $10 \otimes 0001$, and the restriction of 0001 to A_1G_2 contains $2 \otimes 10$ as a composition factor. Indeed, a check of Weyl modules shows that this occurs as a direct summand and hence $2 \otimes 10 \otimes 10$ occurs as a direct summand of $L \downarrow X$. At this point we have accounted for summands of total dimension 245 and G_2G_2 acts nontrivially on each composition factor. Thus $L(A_1)$ is an additional direct summand and the result follows.

Case $A_1F_4 < E_7$.

Here we see as in other cases that if $p > 3$, then all composition factors of $L \downarrow X$ have corresponding Weyl modules irreducible and hence the restriction is completely reducible as indicated in Table 10.1.

Assume $p = 3$. In this case $W_{F_4}(0001)$ is reducible. We have the embedding $F_4 < E_6T_1$ and $L \downarrow E_6T_1 = L(E_6T_1) \oplus V \oplus V^*$, where V restricts to E_6 as the irreducible 27-dimensional module $V(\lambda_1)$. By Lemma 9.1.1, $V \downarrow F_4 = T(0001) = 0000|0001|0000$. Also, $L \downarrow X_1 = 2^{27} \oplus 0^{52}$, and both summands must be invariant under X_2. The only possibility is $L \downarrow X = (2 \otimes T(0001)) \oplus (0 \otimes 1000)$.

Now assume $p = 2$. Here we have $L \downarrow F_4 = 0001^4/1000/0000^2$ and we have seen earlier that $L(E_6) \downarrow F_4 = T(1000)$. Since $F_4 < E_6 < E_7$ we have $L \downarrow F_4 = (T(1000) \oplus 0001^2) \perp 0000^2$. So from the known composition factors we have $L \downarrow X = M \perp (2 \otimes 0000)$, where $M = (2 \otimes 0001)/(0 \otimes 1000)/(0 \otimes$

$0001)^2$. To apply Lemma 9.1.2 we must verify that M is indecomposable and for this we discuss the action of X_1 on L. There is a subgroup $D_4 < F_4$ which is a subsystem subgroup of E_7. Then $X_1 < C_{E_7}(D_4) = (A_1)^3$. Using [31, 1.8] for $L(E_8) \downarrow D_4D_4$ and then restricting to $E_7 = C_{E_8}(A_1)$ we find that $L \downarrow (A_1)^3D_4$ contains the direct sum of three submodules, each of which is the tensor product of natural modules for two of the three A_1 factors with a restricted 8-dimensional module for D_4. It follows that $L \downarrow X_1$ contains a direct summand of the form $(T(2))^{24}$. This forces M to be indecomposable and Lemma 9.1.2 implies $M \cong \Delta(2 \otimes 0001; 0 \otimes 1000)$, as required.

Case $G_2C_3 < E_7$.

If $p > 3$ then the composition factors of $L \downarrow X$ correspond to irreducible Weyl modules, so the restriction is completely reducible as in Table 10.1. We have embeddings $G_2 < D_4, C_3 < A_5$, where in each case the larger group is a subsystem subgroup of E_7. Also, $D_4 < A_1^3D_4$ and $L \downarrow A_1^3D_4 = L(A_1^3D_4) \oplus V$, where V restricts to D_4 as the direct sum of 12 restricted 8-dimensional representations.

Suppose $p = 3$. Noting that $L(D_4)$ can be realized as the wedge square of a restricted 8-dimensional representation, we obtain from the above that $L \downarrow G_2 = (T(01) \oplus 10^{13}) \perp 00^{21}$ and C_3 fixes the summands. Then $L \downarrow X = M \perp (00 \otimes 200)$, where the second summand is $L(C_3)$. From the information on composition factors, $M = 10 \otimes 010/01 \otimes 000/(10 \otimes 000)^2$. Next we note that $L \downarrow A_5$ has a direct summand which is the sum of 3 copies of $\wedge^2 F$ and 3 copies of its dual, where F is the usual 6-dimensional module. Hence this summand restricts to C_3 as $T(010)^6$, where $T(010) = 000|010|000$. Another copy of $T(010)$ appears in $L(A_5)$. At this point it follows that M is indecomposable, so Lemma 9.1.2 implies $M \cong \Delta(10 \otimes 010; 01 \otimes 000)$.

Now assume $p = 2$. Here 9.1.1 implies that the restricted 8-dimensional representations of D_4 restrict to G_2 as $T(10) = 00|10|00$, so $L \downarrow G_2$ contains at least 12 copies of this tilting module. Also $W_{G_2}(01)$ is irreducible, so we can write $L \downarrow G_2 = M \perp L(G_2)$ where $M \downarrow G_2 = 10^{14}/00^{34}$. The decomposition is preserved by C_3, so in view of the known composition factors, we have $M \downarrow X = (10 \otimes 010)/(000 \otimes 200)/(00 \times 010)^2$. Since $L(C_3)$ is indecomposable, we have M indecomposable and Lemma 9.1.2 gives the result.

Case $A_1G_2 < E_7$.

Here $p > 2$. If $p \neq 3, 7$, then the composition factors involved all correspond to irreducible Weyl modules, so $L \downarrow X$ is completely reducible as indicated in Table 10.1.

Assume $p = 7$. The construction in 3.12 of [31] shows that $X_2 = G_2$ is contained in a Levi factor of type A_6. From the action of this Levi factor we see that $L \downarrow X_2$ is a tilting module and $L \downarrow X_2 = (T(20)^3) \oplus (10^5) \oplus 01$. Also, our information on composition factors implies $L \downarrow X_1 = 4^7 \oplus 2^{28} \oplus 0^{14}$, with each summand invariant under X_2 and affording a tilting module. The information on composition factors then implies that $L \downarrow X = (4 \otimes 10) \oplus (2 \otimes T(20)) \oplus (0 \otimes 01)$, as required.

Now assume $p = 3$. We have $L \downarrow G_2 = 20^3/10^6/01/00^3$ and here all composition factors correspond to irreducible Weyl modules with the exception of 01, which occurs within $L(G_2) = 01|10$, an indecomposable module. As L is self-dual we can write $L \downarrow G_2 = 20^3 \perp (T(01) \oplus 10^4) \perp 00^3$. Each summand is A_1-invariant and from knowledge of composition factors we have $L \downarrow X = M \perp (2 \otimes 20) \perp (2 \otimes 00)$, where $M = (4 \otimes 10)/(0 \otimes 01)/(0 \otimes 10)^2$.

To complete the argument in this case we claim that $L \downarrow A_1$ is a tilting module. This will imply M is indecomposable so that Lemma 9.1.2 applies to yield the result. Let T_{G_2} be a maximal torus of G_2. From the composition factors we find that $C_G(T_{G_2})$ has dimension 19 and so $X_1 < C_G(T_{G_2}) = A_2A_1^3T_{G_2}$. Now $C_G(X_1) = X_2 = G_2$, so X_1 must project nontrivially to each of the simple summands. On the other hand, in view of the composition factors, none of the projection factors involves a field twist. Now $L \downarrow A_2A_1^3$ is a direct sum of $L(A_2A_1^3T_{G_2})$ together with irreducibles each of which restricts to $A_2A_1^3$ as a tensor product of natural or dual modules for the factors. Finally, $L(A_2A_1^3T_{G_2})$ is non-degenerate and has $L(A_2T_{G_2})$ as a self-dual direct summand. Restricting to A_2 this must have shape $T(11) \oplus 00 = (10 \otimes 01) \oplus 00$, so restricting to $X_1 = A_1$ we have the claim.

Case $A_1A_1 < E_7$.

Here $p \geq 5$. If $p > 7$ then as in other cases we see that the action of X on L is completely reducible, as in Table 10.1.

Now suppose $p = 7$. From the construction of X given in 3.12 of [31] we see that $X_1 < A_1A_2A_3$ and $X_2 < A_4A_2$, where in each case the larger group is the semisimple part of a Levi factor and where each projection corresponds to an irreducible restricted representation of X_i. It follows from this, the well-known actions of the Levi factors on L, and results on tilting modules, that $L \downarrow X_i$ is a tilting module for each $i = 1, 2$.

The weights of maximal tori of X_1, X_2 are precisely those which occur for larger primes. So the corresponding Weyl modules for X_1 are all irreducible and we have $L \downarrow X_1 = 6^5 \oplus 4^{10} \oplus 2^{15} \oplus 0^3$. Each summand is invariant under the action of X_2. Considering the action of X_2 on each summand and using the information on composition factors and the fact that each restriction affords a tilting module for X_2, we have

$$L \downarrow X = (2 \otimes T(8)) \oplus (6 \otimes 4) \oplus (4 \otimes 6) \oplus (4 \otimes 2) \oplus (0 \otimes 2) \oplus (2 \otimes 0).$$

Now assume $p = 5$. Here we have $L \downarrow X_1 = T(6)^5 \oplus 4^{10} \oplus 2^{10} \oplus 0^3$, while $L \downarrow X_2 = T(8)^3 \oplus T(6)^5 \oplus 4^{10} \oplus 2$. We note that each factor leaves invariant the summand of form 4^{10} for the other factor with the restriction affording a tilting module. From information on composition factors we obtain non-degenerate summands $4 \otimes T(6)$ and $T(6) \otimes 4$. Also, X_2 leaves invariant the fixed space of X_1. So far we have $L \downarrow X = M \oplus (4 \otimes T(6)) \oplus (T(6) \otimes 4) \oplus (0 \otimes 2)$. Then $M \downarrow X_1 = 2^{10}$ and M affords a tilting module for X_2. It follows that $M \downarrow X = 2 \otimes T(8)$, as required. ∎

The completes the proofs of all the assertions in Table 10.1 concerning $L \downarrow X$.

We now turn our attention to Table 10.2.

9.2 Proof of the assertions in Table 10.2

Let $G = F_4, E_6$ or E_7 and let V be one of the G-modules $V_{F_4}(\lambda_1), V_{E_6}(\lambda_1)$ or $V_{E_7}(\lambda_7)$, of dimension $26 - \delta_{p,3}, 27$ or 56 respectively. We now analyse the precise actions on V of the maximal subgroups X in Table 1 of Theorem 1. The composition factors can be read off from [23, 2.5], together with Theorem 6.1. The information to be proved is recorded in Table 10.2.

Lemma 9.2.1 *If $G = F_4$ then $V \downarrow X$ is as in Table* 10.2.

Proof First consider $X = A_1\,(p \geq 13)$. Embedding F_4 in E_6, we have $L(E_6) \downarrow F_4 = L(F_4) \oplus V$. From [23, 2.4,2.5] we see that the highest weight of X on $L(E_6)$ is 22, which is less than $2p - 2$, and hence X is a good A_1 in E_6, in the sense of [32]. Therefore [32, Theorem 1.1(iii)] implies that $L(E_6) \downarrow X$ is a tilting module. As a direct summand, $V \downarrow X$ is therefore also a tilting module, as in Table 10.2.

If $X = G_2\,(p = 7)$ then $V \downarrow X$ is the irreducible module 20, by [41, Theorem 2].

Finally, consider $X = A_1G_2\,(p \geq 3)$. This lies in a maximal subgroup A_2G_2 of E_6, so from [23, 2.5] we see that the composition factors of $V \downarrow X$ are $2 \otimes 10/4 \otimes 00$. These do not extend each other, so $V \downarrow X$ is completely reducible as in Table 10.2. ∎

Lemma 9.2.2 *If* $G = E_6$ *then* $V \downarrow X$ *is as in Table* 10.2.

Proof Consider first $X = A_2\,(p \geq 5)$. Here, by [23, 2.5], $V \downarrow X$ has the same composition factors as $W(22)$, which is irreducible if $p > 5$ and has composition factors $22/11$ if $p = 5$. So it remains only to show that $V \downarrow X$ is not $22 \oplus 11$ when $p = 5$, and this is remarked in the proof of [41, Theorem (A.2)] (bottom of p.314).

Next let $X = G_2\,(p \neq 7)$. If $p > 2$ then by [41], $V \downarrow X$ is the irreducible 20, while if $p = 2$, Lemma 6.3.7 gives the desired conclusion.

If $X = C_4\,(p \neq 2)$ then $V \downarrow X$ is the irreducible 0100 (see [23, 2.5]), while if $X = F_4$ the conclusion follows from Lemma 9.1.1.

Finally, consider $X = A_2G_2$. Here $V \downarrow X$ has the same composition factors as $(10 \otimes W(10))/(W(02) \otimes 00)$. When $p > 2$ the relevant Weyl modules are irreducible, so $V \downarrow X$ is completely reducible as in Table 10.2. Now assume $p = 2$, so $V \downarrow X = (10 \otimes 10)/(02 \otimes 00)/(10 \otimes 00)^2$. The factor G_2 of X lies in a subsystem D_4, and $V \downarrow D_4 = \lambda_1 \oplus \lambda_3 \oplus \lambda_4 \oplus 0^3$. Hence using Lemma 9.1.1 we have

$$V \downarrow G_2 = T(10)^3 \oplus 00^3.$$

The factor A_2 of X lies in a subsystem A_2A_2, where $V \downarrow A_2A_2 = (01 \otimes 01) \oplus (10 \otimes 00)^3 \oplus (00 \otimes 10)^3$ (see [23, 2.3]). Hence

$$V \downarrow A_2 = T(02) \oplus 10^6.$$

Now the conclusion follows in the usual way from Lemma 9.1.2. ∎

Lemma 9.2.3 *If* $G = E_7$ *then* $V \downarrow X$ *is as in Table* 10.2.

Proof If X is in one of the two classes of maximal A_1's, we see that $V \downarrow X$ is a tilting module exactly as in the first paragraph of the proof of Lemma 9.2.1, noting that $L(E_8) \downarrow E_7$ has V as a direct summand.

Next consider $X = A_2\,(p \geq 5)$. Here the proof of [23, 2.5] shows that $V \downarrow X = W(60)/W(06)$ (recall this denotes a module having the same composition factors as $W(60) \oplus W(06)$). For $p > 5$ the Weyl module $W(60)$

is irreducible while for $p = 5$ we have $W(60) = 60|22$. So assume now that $p = 5$, $V \downarrow X = 60/06/22^2$. Let J be a fundamental SL_2 in X. As in the proof of Lemma 4.1.3, J lies in a subsystem subgroup A_1A_4 of G (lying in a subsystem A_1D_6), with projections corresponding to the irreducible representations 1,4. Using [23, 2.3,2.6], we see that

$$V \downarrow A_1A_4 = (1 \otimes (\lambda_1 \oplus \lambda_4 \oplus 0^2)) \oplus (0 \otimes (\lambda_1 \oplus \lambda_2 \oplus \lambda_3 \oplus \lambda_4 \oplus 0^2),$$

from which it follows that

$$V \downarrow J = T(5)^2 \oplus T(6)^2 \oplus 4^2 \oplus 1^2 \oplus 0^2.$$

In particular, as $T(6) = 2|6|2$, $V \downarrow J$ has no irreducible submodule of high weight 6. If $V \downarrow X$ has a submodule 60, this would restrict to J as $6 \oplus 5 \oplus 1 \oplus 0$, giving a submodule 6. Hence $V \downarrow X$ has no submodule 60 or 06, and so $V \downarrow X$ is indecomposable of shape $22|(60+06)|22$, as in Table 10.2.

Now let $X = A_1A_1$ $(p \geq 5)$. Here $V \downarrow X = (W(6)\otimes 3)/(4\otimes 1)/(2\otimes W(5))$. If $p > 5$, $V \downarrow X$ is completely reducible as in Table 10.2, so suppose $p = 5$; then $W(6) = 6/2, W(5) = 5/3$. Write A, B for the two factors A_1 of X. By [31, p.37], one of the factors, say A, lies in a subsystem A_2A_4 of G, with irreducible projections $2, 4$. Then using [23, 2.3] we find that

$$V \downarrow A = T(6)^4 \oplus 4^2 \oplus 2^2.$$

Likewise, $B < A_1A_2A_3$, which yields

$$V \downarrow B = T(5)^3 \oplus 3^4 \oplus 1^5.$$

Using also the structure of $V \downarrow A$, it now follows in the usual way from Lemma 9.1.2 that $V \downarrow AB = \Delta(6 \otimes 3; 2 \otimes 5) \oplus (4 \otimes 1)$, as in Table 10.2.

Now consider $X = A_1G_2$ $(p \geq 3)$. Here $V \downarrow X = (1\otimes W(01))/(W(3)\otimes 10)$ (see [23, 2.5]). For $p > 3$ the relevant Weyl modules are irreducible, so assume now that $p = 3$. Then $V \downarrow X = (3 \otimes 10)/(1 \otimes 01)/(1 \otimes 10)^2$. The G_2 factor of X is contained in a subsystem A_6 of E_7, and $V \downarrow A_6 = \lambda_1 \oplus \lambda_2 \oplus \lambda_5 \oplus \lambda_6$ by [23, 2.3]. Now $V_{A_6}(\lambda_1) \downarrow G_2 = 10$, and $V_{A_6}(\lambda_2) \downarrow G_2 = \wedge^2(10) = 10|01|10 = T(01)$. Hence

$$V \downarrow G_2 = T(01)^2 \oplus 10^2.$$

We now study the restriction of V to the A_1 factor of X. As in the proof of 9.1.5, this A_1 lies in a subsystem $A_1^3A_2$ with irreducible restricted projections, from which we calculate that $V \downarrow A_1 = T(3)^7 \oplus 1^7$. Combining this with the above decomposition of $V \downarrow G_2$, and using Lemma 9.1.2, we obtain the conclusion.

Next let $X = A_1F_4$, so $V \downarrow X = (1 \otimes W(\lambda_4))/(W(3) \otimes 0)$ (see [23, 2.5]). If $p > 3$ or $p = 2$ this is completely reducible as in Table 10.2. Now let $p = 3$. The factor F_4 of X lies in a subsystem E_6, and $V \downarrow E_6 = V_{E_6}(\lambda_1)^2 \oplus 0^2$, so Lemma 9.1.1 gives $V \downarrow F_4 = T(0001)^2 \oplus 0^2$. As for the A_1 factor of X, we argue as in the previous case that $V \downarrow A_1 = T(3) \oplus 1^{25}$. Now the conclusion follows from Lemma 9.1.2.

Finally, consider $X = G_2C_3$. Here $V \downarrow X = (W(10) \otimes 100)/(00 \otimes W(001))$. If $p > 2$ this is completely reducible as in Table 10.2, so assume $p = 2$. The G_2 factor of X lies in a subsystem D_4 of E_7, and $V \downarrow D_4 = \lambda_1^2 \oplus \lambda_3^2 \oplus \lambda_4^2 \oplus 0^8$, whence $V \downarrow G_2 = T(10)^6 \oplus 00^8$. The C_3 factor of X lies in a subsystem A_5 of E_7, from which we similarly see that $V \downarrow C_3 = 100^6 \oplus T(001)$ (note that the wedge-cube of the natural 6-dimensional A_5-module restricts to C_3 as the indecomposable $T(001) = 100|001|100$ - the indecomposability can easily be seen by restricting to the subgroup C_1C_2). Now the conclusion follows in the usual way from Lemma 9.1.2. ■

This completes the proof of all the information in Tables 10.1 and 10.2.

10 The tables for Theorem 1 and Corollary 2

This section contains Tables 10.1-10.4 referred to in the remarks following Theorem 1 and in Corollary 2. Before presenting the tables we make a few remarks concerning how to read off information from them.

Notation We remind the reader of the notation used in the tables. We identify a dominant weight λ with the irreducible module $V(\lambda)$.

The notation $T(\lambda;\mu;...)$ will be used only for $X = A_1$ and denotes a tilting module having the same composition factors as $W(\lambda) \oplus W(\mu) \oplus ....$ In situations to follow such tilting modules exist and we illustrate with an example.

Assume $X < G = E_7$ is the maximal A_1 corresponding to the label 2222222. Then a check using root heights shows that the maximal torus T has precisely the same weights as in the direct sum of Weyl modules

$$W(34) \oplus W(26) \oplus W(22) \oplus W(18) \oplus W(14) \oplus W(10) \oplus W(2).$$

So for $p > 31$ the restriction is just as above, but differs for smaller primes. For instance, consider $p = 23$. The highest weight is 34, so one summand is $T(34)$ which is uniserial of shape $10|34|10$. The highest weight not already accounted for is 26, so $T(26) = 18|26|18$ is also a summand. We continue in the way, but the remaining weights are all less than p, so the tilting modules are each irreducible. So in this case

$$T(34; 26; 22; 18; 14; 10; 2) = T(34) \oplus T(26) \oplus 22 \oplus 14 \oplus 2.$$

Finally, assume X is a semisimple group and λ, γ, μ are dominant weights such that $T(\lambda) = \mu|\lambda|\mu$ and $T(\gamma) = \mu|\gamma|\mu$. As in Section 9, we use the notation $\Delta(\lambda;\gamma)$ to denote an indecomposable module of shape $\mu|(\lambda \oplus \gamma)|\mu$ with socle and cosocle both of type μ, and which is obtained as a section of $T(\lambda) \oplus T(\gamma)$, by taking a maximal submodule and then factoring out a diagonal submodule of the socle.

Table 10.1: In this table we record, for each maximal subgroup X of G appearing in Table 1 of Theorem 1, the precise action of X on $L(G)'$, the index $t = |N_G(X) : X|$, and, in cases where X is simple, the labelled diagram determined by the torus T in X defined in Definition 2.2.4.

Proofs of the decompositions of $L(G)' \downarrow X$ are provided in Section 9.1.

Table 10.2: Let $V_{27} = V_{E_6}(\lambda_1)$, an irreducible 27-dimensional E_6-module, let $V_{56} = V_{E_7}(\lambda_7)$, an irreducible 56-dimensional E_7-module, and let $V_{26-\delta_{p,3}} = V_{F_4}(\lambda_1)$, an irreducible F_4-module of dimension $26-\delta_{p,3}$. In Table 10.2 we record the precise actions of X on $V = V_{26-\delta_{p,3}}$, V_{27} or V_{56} for each maximal subgroup X of F_4, E_6 or E_7 appearing in Table 1 of Theorem 1. Proofs are in Section 9.2.

Tables 10.3, 10.4: Table 10.3 lists the maximal subgroups M in exceptional groups with M^0 reductive of maximal rank; and Table 10.4 lists the maximal connected subgroups of maximal rank. In the tables, the symbols $\tilde{A}_1, \tilde{A}_2$ indicate that these subgroups correspond to subsystems having a base consisting of short roots. Proofs are in Section 8.

Table 10.1: actions of maximal subgroups of Table 1 on $L(G)'$

G	X	diagram	t	$L(G)' \downarrow X$
E_8	$A_1\ (p \geq 23)$	22202022	1	$T(38;34;28;26;22^2;18;16;14;10;6;2)$
	$A_1\ (p \geq 29)$	22202222	1	$T(46;38;34;28;26;22;18;14;10;2)$
	$A_1\ (p \geq 31)$	22222222	1	$T(58;46;38;34;26;22;14;2)$
	$B_2\ (p \geq 5)$	00020020	1	$06 \oplus 32 \oplus 02,\ p > 5$ $\Delta(06;32) \oplus 02,\ p = 5$
	A_1A_2 $(p \geq 5)$		2	$(6 \otimes 11) \oplus (2 \otimes 22) \oplus (4 \otimes 30) \oplus (4 \otimes 03) \oplus$ $(2 \otimes 00) \oplus (0 \otimes 11),\ p > 5$ $\Delta(2 \otimes 22; 6 \otimes 11) \oplus (4 \otimes 30) \oplus (4 \otimes 03) \oplus$ $(2 \otimes 00) \oplus (0 \otimes 11),\ p = 5$
	$A_1G_2G_2$ $(p \geq 3)$		2	$(2 \otimes 10 \otimes 10) \oplus (4 \otimes 10 \otimes 00) \oplus (4 \otimes 00 \otimes 10) \oplus$ $(2 \otimes 00 \otimes 00) \oplus (0 \otimes 01 \otimes 00) \oplus (0 \otimes 00 \otimes 01),\ p > 3$ $(2 \otimes 10 \otimes 10) \oplus \Delta(4 \otimes 10 \otimes 00; 0 \otimes 01 \otimes 00) \oplus$ $\Delta(4 \otimes 00 \otimes 10; 0 \otimes 00 \otimes 01) \oplus (2 \otimes 00 \otimes 00),\ p = 3$
	G_2F_4		1	$(10 \otimes 0001) \oplus (01 \otimes 0000) \oplus (00 \otimes 1000),\ p > 3$ $\Delta(10 \otimes 0001; 01 \otimes 0000) \oplus (00 \oplus 1000),\ p = 3$ $\Delta(10 \otimes 0001; 00 \otimes 1000) \oplus (01 \otimes 0000),\ p = 2$
E_7	$A_1\ (p \geq 17)$	2220222	1	$T(26;22;18;16;14;10^2;6;2)$
	$A_1\ (p \geq 19)$	2222222	1	$T(34;26;22;18;14;10;2)$
	$A_2\ (p \geq 5)$	2002020	2	$44 \oplus 11,\ p \neq 7$ $T(44),\ p = 7$
	A_1A_1 $(p \geq 5)$		1	$(2 \otimes 8) \oplus (4 \otimes 6) \oplus (6 \otimes 4) \oplus (2 \otimes 4) \oplus (4 \otimes 2) \oplus$ $(2 \otimes 0) \oplus (0 \otimes 2),\ p > 7$ $(2 \otimes T(8)) \oplus (4 \otimes 6) \oplus (6 \otimes 4) \oplus (4 \otimes 2) \oplus$ $(2 \otimes 0) \oplus (0 \otimes 2),\ p = 7$ $(2 \otimes T(8)) \oplus (4 \otimes T(6)) \oplus (T(6) \otimes 4) \oplus (0 \otimes 2),\ p = 5$
	A_1G_2 $(p \geq 3)$		1	$(4 \otimes 10) \oplus (2 \otimes 20) \oplus (2 \otimes 00) \oplus (0 \otimes 01),\ p > 3, p \neq 7$ $(4 \otimes 10) \oplus (2 \otimes T(20)) \oplus (0 \otimes 01),\ p = 7$ $\Delta(4 \otimes 10; 0 \otimes 01) \oplus (2 \otimes 20) \oplus (2 \otimes 00),\ p = 3$
	A_1F_4		1	$(2 \otimes 0001) \oplus (2 \otimes 0000) \oplus (0 \otimes 1000),\ p > 3$ $(2 \otimes T(0001)) \oplus (0 \otimes 1000),\ p = 3$ $\Delta(2 \otimes 0001; 0 \otimes 1000) \oplus (2 \otimes 0000),\ p = 2$
	G_2C_3		1	$(10 \otimes 010) \oplus (01 \otimes 000) \oplus (00 \otimes 200),\ p > 3$ $\Delta(10 \otimes 010; 01 \otimes 000) \oplus (00 \otimes 200),\ p = 3$ $\Delta(10 \otimes 010; 00 \otimes 200) \oplus (01 \otimes 000),\ p = 2$

Table 10.1, continued

G	X	diagram	t	$L(G)' \downarrow X$
E_6	$A_2\,(p \geq 5)$	200202	2	$41 \oplus 14 \oplus 11$
	$G_2\,(p \neq 7)$	222022	1	$11 \oplus 01,\ p \neq 3$ $10\vert 01\vert 11\vert 01\vert 10$ (uniserial), $p = 3$
	$C_4\,(p \neq 2)$	222022	1	$2000 \oplus 0001$
	F_4	222222	1	$0001 \oplus 1000,\ p > 2$ $T(1000),\ p = 2$
	A_2G_2		2	$(11 \otimes 10) \oplus (11 \otimes 00) \oplus (00 \otimes 01),\ p > 3$ $\Delta(11 \otimes 10; 00 \otimes 01) \oplus (11 \otimes 00),\ p = 3$ $(11 \otimes T(10)) \oplus (00 \otimes 01),\ p = 2$
F_4	$A_1\,(p \geq 13)$	2222	1	$T(22; 14; 10; 2)$
	$G_2\,(p = 7)$	2022	1	$11 \oplus 01$
	$A_1G_2\,(p \geq 3)$		1	$(4 \otimes 10) \oplus (2 \otimes 00) \oplus (0 \otimes 01),\ p > 3$ $\Delta(4 \otimes 10; 0 \otimes 01) \oplus (2 \otimes 00),\ p = 3$
G_2	$A_1\,(p \geq 7)$	22	1	$T(10; 2)$

Table 10.2: actions of maximal subgroups of F_4, E_6, E_7 on $V = V_{26-\delta_{p,3}}$, V_{27}, V_{56}

G	X	$V \downarrow X$
F_4	$A_1\,(p \geq 13)$	$T(16;8)$
	$G_2\,(p = 7)$	20
	$A_1G_2\,(p \geq 3)$	$(2 \otimes 10) \oplus (4 \otimes 00)$
E_6	$A_2\,(p \geq 5)$	$22,\ p > 5$ $W(22)$ or $W(22)^*,\ p = 5$ (2 classes in G)
	$G_2\,(p \neq 7)$	$20,\ p > 2$ $01\|20\|00\|10$ (uniserial) or dual, $p = 2$ (2 classes in G)
	$C_4\,(p \neq 2)$	0100
	F_4	$0001 \oplus 0000,\ p \neq 3$ $T(0001),\ p = 3$
	A_2G_2	$(10 \otimes 10) \oplus (02 \otimes 00),\ p > 2$ $\Delta(10 \otimes 10; 02 \otimes 00),\ p = 2$
E_7	$A_1\,(p \geq 17)$ $A_1\,(p \geq 19)$	$T(21;15;11;5)$ $T(27;17;9)$
	$A_2\,(p \geq 5)$	$60 \oplus 06,\ p > 5$ $22\|(60 \oplus 06)\|22,\ p = 5$
	$A_1A_1\,(p \geq 5)$	$(6 \otimes 3) \oplus (4 \otimes 1) \oplus (2 \otimes 5),\ p > 5$ $\Delta(6 \otimes 3; 2 \otimes 5) \oplus (4 \otimes 1),\ p = 5$
	$A_1G_2\,(p \geq 3)$	$(3 \otimes 10) \oplus (1 \otimes 01),\ p > 3$ $\Delta(3 \otimes 10; 1 \otimes 01),\ p = 3$
	A_1F_4	$(1 \otimes 0001) \oplus (3 \otimes 0000),\ p \neq 3$ $\Delta(1 \otimes 0001; 3 \otimes 0000),\ p = 3$
	G_2C_3	$(10 \otimes 100) \oplus (00 \otimes 001),\ p > 2$ $\Delta(10 \otimes 100; 00 \otimes 001),\ p = 2$

Table 10.3: maximal subgroups M with M^0 reductive of maximal rank

G	M^0	M/M^0
G_2	$A_1\tilde{A}_1,\ A_2,\ \tilde{A}_2\,(p=3)$	1, 2, 2
$F_4\,(p\neq 2)$	$B_4,\ D_4,\ A_1C_3,\ A_2\tilde{A}_2$	1, S_3, 1, 2
$F_4\,(p=2)$	$B_4,\ C_4,\ D_4,\ \tilde{D}_4,\ A_2\tilde{A}_2$	$1, 1, S_3, S_3, 2$
E_6	$A_1A_5,\ A_2^3,\ D_4T_2,\ T_6$	1, S_3, S_3, $W(E_6)$
E_7	$A_1D_6,\ A_7,\ A_2A_5,\ A_1^3D_4,$ $A_1^7,\ E_6T_1,\ T_7$	1, 2, 2, S_3, $L_3(2)$, 2, $W(E_7)$
E_8	$D_8,\ A_1E_7,\ A_8,\ A_2E_6,$ $A_4^2,\ D_4^2,\ A_2^4,\ A_1^8,\ T_8$	1, 1, 2, 2, 4, $S_3\times 2$, $GL_2(3)$, $AGL_3(2)$, $W(E_8)$

Table 10.4: maximal connected reductive subgroups M of maximal rank

G	M
G_2	$A_1\tilde{A}_1,\ A_2,\ \tilde{A}_2\,(p=3)$
$F_4\,(p\neq 2)$	$B_4,\ A_1C_3,\ A_2\tilde{A}_2$
$F_4\,(p=2)$	$B_4,\ C_4,\ A_2\tilde{A}_2$
E_6	$A_1A_5,\ A_2^3$
E_7	$A_1D_6,\ A_7,\ A_2A_5$
E_8	$D_8,\ A_1E_7,\ A_8,\ A_2E_6,\ A_4^2$

11 Appendix: E_8 structure constants

This section consists of a table of the structure constants $N(\alpha, \beta)$ for the E_8 Lie algebra, defined by the equation $e_\alpha e_\beta = e_\beta e_\alpha + N(\alpha, \beta) e_{\alpha+\beta}$ for positive roots α, β. This is computed by the method described in [13], where the corresponding tables for F_4, E_6 and E_7 can be found.

In the table, the first column lists the roots α in the form $c_1 \dots c_8$ (representing $\sum c_i \alpha_i$), and the top row lists the roots β in vertical form $(c_1 \dots c_8)^T$. The values taken by $N(\alpha, \beta)$ are 0,1 and -1, with -1 represented by the symbol A in the table.

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