Finite subgroups of simple algebraic groups with irreducible centralizers

Martin W. Liebeck and Adam R. Thomas

December 15, 2016

Abstract

We determine all finite subgroups of simple algebraic groups that have irreducible centralizers – that is, centralizers whose connected component does not lie in a parabolic subgroup.

1 Introduction

Let G be a simple algebraic group over an algebraically closed field. Following Serre [15], a subgroup of G is said to be G-*irreducible* (or just *irreducible* if the context is clear) if it is not contained in a proper parabolic subgroup of G. Such subgroups necessarily have finite centralizer in G (see [13, 2.1]). In this paper we address the question of which finite subgroups can arise as such a centralizer. It turns out (see Corollary 4 below) that they form a very restricted collection of soluble groups, together with the alternating and symmetric groups Alt_5 and Sym_5.

The question is rather straightforward in the case where G is a classical group (see Proposition 3 below). Our main result covers the case where G is of exceptional type.

Theorem 1 *Let G be a simple adjoint algebraic group of exceptional type in characteristic $p \geq 0$, and suppose F is a finite subgroup of G such that $C_G(F)^0$ is G-irreducible. Then $|F|$ is not divisible by p, and F, $C_G(F)^0$ are as in Tables $7-12$ in Section 5 at the end of the paper (one G-class of subgroups for each line of the tables).*

Remarks (1) The notation for the subgroups F and $C_G(F)^0$ is described at the end of this section; the notation for elements of F is defined in Proposition 2.2.

(2) The theorem covers adjoint types of simple algebraic groups. For other types, the possible finite subgroups F are just preimages of those in the conclusion.

(3) We also cover the case where $G = \mathrm{Aut}\, E_6 = E_6.2$ (see Table 10 and Section 3.3).

(4) A complete determination of all G-irreducible connected subgroups is carried out in [16].

Every finite subgroup F in Theorem 1 is contained in a maximal such finite subgroup. The list of maximal finite subgroups with irreducible centralizers is recorded in the next result.

Corollary 2 *Let G be a simple adjoint algebraic group of exceptional type, and suppose F is a finite subgroup of G which is maximal subject to the condition that $C_G(F)^0$ is G-irreducible. Then F, $C_G(F)^0$ are as in Table* 1.

For the classical groups we prove the following.

Proposition 3 *Let G be a classical simple algebraic group in characteristic $p \geq 0$ with natural module V, and suppose F is a finite subgroup of G such that $C_G(F)^0$ is G-irreducible. Then $p \neq 2$ and F is an elementary abelian 2-group. Moreover, $G \neq SL_n$ and the following hold.*

(i) *If $G = Sp_{2n}$, then*

$$C_G(F)^0 = \prod_i Sp_{2n_i} = \prod_i Sp(W_i),$$

where $\sum n_i = n$ and W_i are the distinct weight spaces of F on V.

(ii) *If $G = SO_n$, then*

$$C_G(F)^0 = \prod_i SO_{n_i} = \prod_i SO(W_i),$$

where $n_i \geq 3$ for all i, $\sum n_i = n$ or $n-1$, and W_i are weight spaces of F.

Remark In Section 4 we prove a version of this result covering finite subgroups of $\mathrm{Aut}\,G$ for G of classical type.

Corollary 4 *Let G be a simple adjoint algebraic group, and let D be a proper connected G-irreducible subgroup. Then the finite group $C_G(D)$ is either elementary abelian or isomorphic to a subgroup of one of the following groups:*

$$2_-^{1+4},\ G_{12},\ Sym_4 \times 2,\ SL_2(3),\ 3^2.Dih_8,\ Sym_5.$$

Notation Throughout the paper we use the following notation for various finite groups:

Z_n, or just n	cyclic group of order n
p^s (p prime)	elementary abelian group of order p^s
Alt_n, Sym_n	alternating and symmetric groups
Dih_{2n}	dihedral group of order $2n$
$4 \circ Dih_8$	order 16 central product with centre Z_4
2_-^{1+4}	extra-special group of order 32 of minus type
$Frob_{20}$	Frobenius group of order 20
G_{12}	dicyclic group $\langle x, y \mid x^6 = 1, x^y = x^{-1}, y^2 = x^3 \rangle$ of order 12

In the tables in Section 5, and also in the text, we shall sometimes use $\bar{A}_1$ to denote a subgroup A_1 of a simple algebraic group G that is generated by long root subgroups; we use the notation $\bar{A}_2$ similarly. Also, B_r denotes a natural subgroup of type SO_{2r+1} in a group of type D_n.

We use the following notation when describing modules for a semisimple algebraic group G. We let $L(G)$ denote the Lie algebra of G. If λ is a dominant weight, then

Table 1: Maximal finite subgroups F with irreducible centralizer

G	F	p	$C_G(F)^0$
E_8	2^4	$p \neq 2$	A_1^8
	Q_8	$p \neq 2$	B_2^3
	2_-^{1+4}	$p \neq 2$	B_1^5
	Dih_6	$p \neq 2,3$	B_4
	G_{12}	$p \neq 2,3$	$\bar{A}_1 A_1 A_3$
	$Sym_4 \times 2$	$p \neq 2,3$	$\bar{A}_1 A_1 A_1$
	$SL_2(3)$	$p \neq 2,3$	$\bar{A}_1 A_2$
	$3^2.Dih_8$	$p \neq 2,3$	A_1^2
	Sym_5	$p \neq 2,3,5$	A_1
	Q_8	$p = 3$	$\bar{A}_1 D_4$
	$Dih_8 \times 2$	$p = 3$	$\bar{A}_1^2 B_1^2 B_2$
	3	$p = 2$	A_8
	3^2	$p = 2$	A_2^4
	5	$p = 2$	A_4^2
	$Frob_{20}$	$p = 3$	B_2
E_7	2^3	$p \neq 2$	A_1^7
	Q_8	$p \neq 2$	$\bar{A}_1 B_1^4$
	Dih_6	$p \neq 2,3$	$A_1 A_3$
	Alt_4	$p \neq 2,3$	A_2
	Sym_4	$p \neq 2,3$	$\bar{A}_1 A_1$
	2^2	$p = 3$	D_4
	Dih_8	$p = 3$	$\bar{A}_1 B_1^2 B_2$
	3	$p = 2$	$A_2 A_5$
E_6	2	$p \neq 2$	$A_1 A_5$
	Dih_6	$p \neq 2,3$	$A_1 A_1$
	3	$p = 2$	A_2^3
F_4	2^3	$p \neq 2$	A_1^4
	Q_8	$p \neq 2$	B_1^3
	Sym_4	$p \neq 2,3$	A_1
	3	$p = 2$	$A_2 A_2$
	Dih_8	$p = 3$	$B_1 B_2$
G_2	Dih_6	$p \neq 2,3$	A_1
	2	$p = 3$	$A_1 A_1$
	3	$p = 2$	A_2

Table 2: Centralizers of graph automorphisms in simple algebraic groups

G	order of t	$C_G(t)^0$
A_{2n}	2	$B_n\,(p \neq 2)$
A_{2n-1}	2	C_n
		$D_n\,(p \neq 2)$
D_n	2	B_{n-1}
		$B_kB_{n-k-1}\,(1 \leq k \leq n-2,\, p \neq 2)$
D_4	3	G_2
		$A_2\,(p \neq 3)$
E_6	2	F_4
		$C_4\,(p \neq 2)$

$V_G(\lambda)$ (or simply λ) denotes the rational irreducible G-module with high weight λ. When G is simple the fundamental dominant weights λ_i are ordered with respect to the labelling of the Dynkin diagrams as in [3, p. 250]. If $V_1, \ldots, V_k$ are G-modules, then $V_1/\ldots/V_k$ denotes a module having the same composition factors as $V_1 + \cdots + V_k$. Finally, when H is a subgroup of G and V is a G-module we use $V \downarrow H$ for the restriction of V to H.

2 Preliminaries

In this section we collect some preliminary results required in the proof of Theorem 1.

Proposition 2.1 *Let G be a simple adjoint algebraic group in characteristic p. Suppose $t \in \mathrm{Aut}\, G \setminus G$ is such that t has prime order and $C_G(t)^0$ is G-irreducible. Then t, $C_G(t)^0$ are given in Table 2.*

If $G = D_4$ and t has order 3 with $C_G(t) = A_2$, there is an involutory graph automorphism of G that inverts t and acts as a graph automorphism on $C_G(t)$.

Proof. The first part follows from [8, Tables 4.3.3, 4.7.1] for $p \neq 2$, from [1, §8] for $G = D_n$, $p = 2$, and from [1, 19.9] for $G = A_n, E_6$, $p = 2$. The last part follows from [9, 2.3.3]. ■

Proposition 2.2 *Let G be a simple adjoint algebraic group of exceptional type in characteristic p, and let $x \in G$ be a nonidentity element such that $C_G(x)^0$ is G-irreducible. Then x and $C_G(x)$ are as in Table 3; we label x according to its order, which is not divisible by p.*

Proof. First observe that if p divides the order of x then $C_G(x)^0$ is G-reducible by [2, 2.5]. Hence x is a semisimple element and $C_G(x)^0$ is a semisimple subgroup of maximal rank. It follows from [14, 4.5] that this implies the order of x is equal to one of the coefficients in the expression for the highest root in the root system of G; these are at most 6 for $G = E_8$, and at most 4 for the other types. The classes and centralizers of elements of these orders can be found in [5, 3.1, 4.1] and [6, 3.1]. ■

We shall need a similar result for the group $\mathrm{Aut}\, E_6 = E_6.2$.

Table 3: Elements of exceptional groups with irreducible centralizers

G	x	$C_G(x)$
E_8	$2A$	A_1E_7
	$2B$	D_8
	$3A$	A_8
	$3B$	A_2E_6
	$4A$	A_1A_7
	$4B$	A_3D_5
	$5A$	A_4^2
	$6A$	$A_1A_2A_5$
E_7	$2A$	A_1D_6
	$2B$	$A_7.2$
	$3A$	A_2A_5
	$4A$	$A_1A_3^2.2$
E_6	$2A$	A_1A_5
	$3A$	$A_2^3.3$
F_4	$2A$	B_4
	$2B$	A_1C_3
	$3A$	A_2A_2
	$4A$	A_1A_3
G_2	$2A$	A_1A_1
	$3A$	A_2

Table 4: Elements with irreducible centralizers in $G = \mathrm{Aut}\,E_6$

x	$C_{G'}(x)$
$2B$	F_4
$2C$	$C_4\,(p \neq 2)$
$4A$	$A_1A_3\,(p \neq 2)$
$6A$	$A_2A_2\,(p \neq 3)$

Proposition 2.3 *Let $G = \mathrm{Aut}\,E_6$ and let $x \in G \setminus G'$ be such that $C_G(x)^0$ is G'-irreducible. Then x and $C_{G'}(x)$ are as in Table 4.*

Proof. If x is an involution then the result follows from Proposition 2.1. Suppose now that $1 \neq x^2 \in G'$. Then x^2 has order 2 or 3 and $C_{G'}(x^2) = A_1A_5$ or $A_2^3.3$, respectively, by Proposition 2.2. In the former case x acts as a graph automorphism on the A_5 factor and $C_{G'}(x) = A_1C_3$ or A_1A_3 by [8, Table 4.3.1]. Here A_1C_3 is not possible since this lies in a subgroup F_4 and hence centralizes an involution in $G \setminus G'$.

In the case where $C_{G'}(x^2) = A_2^3.3$, the element x^2 is of order 3 in $C_{G'}(x^3)$. By Proposition 2.2, the latter group must be F_4 and $C_{F_4}(x^2) = A_2A_2$. ■

We also require information on normalizers of certain maximal rank subgroups. The following proposition can be deduced from [4, Tables 7–11] and direct calculation in the Weyl groups of exceptional algebraic groups; many of the results can be found

Table 5: Normalizers of maximal rank subgroups of G

G	M	$N_G(M)/M$
E_8	A_8	2
	A_2E_6	2
	A_1A_7	2
	A_4^2	4
	D_4^2	$Sym_3 \times 2$
	$A_1^4D_4$	Sym_4
	A_2^4	$GL_2(3)$
	A_1^8	$AGL_3(2)$
E_7	A_2A_5	2
	A_1^7	$GL_3(2)$
E_6	A_2^3	$Sym_3 \times 2$
F_4	A_2A_2	2
	D_4	Sym_3
G_2	A_2	2

in [12, Chapter 11].

Proposition 2.4 *Let G be a simple algebraic group of exceptional type. Then Table 5 gives the groups $N_G(M)/M$ (or $N_{\mathrm{Aut}G}(M)/M$ for G of type E_6) for the given maximal rank subgroups M of G.*

Next we have a result about the Spin group $Spin_n$ in characteristic $p \neq 2$. Recall that the centre of $Spin_n$ is 2^2 if n is divisible by 4 and is Z_2 if n is odd. In the former case the quotients of $Spin_n$ by the three central subgroups of order two are SO_n and the two half-spin groups $HSpin_n$.

Proposition 2.5 *Let G be $HSpin_n$ (where $4|n$) or $Spin_n$ (n odd), in characteristic $p \neq 2$. Let $\langle t \rangle = Z(G)$ (so that $G/\langle t \rangle = PSO_n$) and suppose F is a finite 2-subgroup of G containing t such that $C_G(F)^0$ is G-irreducible. Then the preimage of $F/\langle t \rangle$ in SO_n is elementary abelian. Moreover, an element $e \in F$ has order 2 if and only if its preimage in SO_n has -1-eigenspace of dimension divisible by 4.*

Proof. If the preimage contains an element e of order greater than 2, then $C_{SO_n}(e)$ has a nontrivial normal torus, and hence $C_G(F)^0$ cannot be irreducible. The assertion in the last sentence is well known. ■

In the following statement, by a *pure* subgroup of G we mean a subgroup all of whose nonidentity elements are G-conjugate.

Proposition 2.6 *Let $G = E_8$ in characteristic p.*

(i) *If $p \neq 2$ then G has two conjugacy classes of subgroups $E \cong 2^2$ such that $C_G(E)^0$ is G-irreducible, and one class of pure subgroups $E \cong 2^3$; these are as* *follows:*

E	*elements*	$C_G(E)^0$
2^2	$2B^3$	D_4^2
	$2A^2, 2B$	$A_1^2 D_6$
2^3	$2B^7$	A_1^8

Further, G has no pure subgroup 2^4.

(ii) *If $p \neq 3$ then G has one class of subgroups $E \cong 3^2$ such that $C_G(E)^0$ is G-irreducible. For this class, $C_G(E) = A_2^4$.*

Proof. Part (i) follows from [5, 3.7, 3.8]. For (ii), let $E = \langle x, y \rangle < G$ with $E \cong 3^2$ and $C_G(E)^0$ irreducible. Then $C_G(x) \neq A_8$, so Proposition 2.2 implies that $C_G(x) = A_2E_6$, and also that $C_{A_2E_6}(y)^0 = A_2^4$, as required. ■

The next result is taken from [13, Lemma 2.2].

Proposition 2.7 *Suppose G is a classical simple algebraic group in characteristic $p \neq 2$, with natural module V. Let X be a semisimple connected subgroup of G. If X is G-irreducible then either*

(i) *$G = A_n$ and X is irreducible on V, or*

(ii) *$G = B_n, C_n$ or D_n and $V \downarrow X = V_1 \perp \ldots \perp V_k$ with the V_i all nondegenerate, irreducible and inequivalent as X-modules.*

3 Proof of Theorem 1

3.1 The case $G = E_8$

We now embark on the proof of Theorem 1 for the case $G = E_8$. Let F be a finite subgroup of G such that $C_G(F)^0$ is G-irreducible. Then $C_G(F)^0$ is semisimple (see [13, 2.1]). Moreover $C_G(E)^0$ is irreducible for all nontrivial subgroups E of F. Also F is a $\{2,3,5\}$-group by Proposition 2.2.

Lemma 3.1 *If F is an elementary abelian 2-group, then F is as in Table 7.*

Proof. Suppose $F \cong 2^k$. If $k \leq 2$, or if $k = 3$ and F is pure, the conclusion follows from Proposition 2.6(i).

Now assume that $k = 3$ and F is not pure. By considering the 2^2 subgroups of F, all of which must be as in (i) of Proposition 2.6, we see that one of these, say $\langle e_1, e_2 \rangle$, is $2B$-pure, so that $C_G(e_1, e_2)^0 = D_4^2$. We have $C_G(e_1)/\langle e_1 \rangle \cong PSO_{16}$, and consider the preimage of $F/\langle e_1 \rangle$ in SO_{16}. This preimage is elementary abelian by Proposition 2.5, so can be diagonalized, and we can take $e_2 = (-1^8, 1^8)$. Let e_3 be a further element of F that is in class $2A$. Then the -1-eigenspace of e_3 has dimension 4 or 12, and so the fact that $C_G(F)^0$ is G-irreducible means that we can take $e_3 = (-1^4, 1^4, 1^8)$, so that $C_{SO_{16}}(F)^0 = SO_4SO_4SO_8$, and so $C_G(E)^0 = A_1^4D_4$ as in Table 7.

Next suppose $k \geq 4$. Then F is not pure by Proposition 2.6(i), so F contains a subgroup $\langle e_1, e_2, e_3 \rangle \cong 2^3$ as in the previous paragraph. Arguing as above, we can take a further element e_4 of F to be $(1^8, -1^4, 1^4)$, so that $C_G(e_1, \ldots, e_4)^0 = A_1^8$. There is no possible further diagonal involution in F such that $C_G(F)^0$ is irreducible, so $k = 4$. ■

In view of the previous lemma, we assume from now on that F is not an elementary abelian 2-group. Hence if F is a 2-group, it has exponent 4 by Proposition 2.2.

Lemma 3.2 *Suppose F is a 2-group, and has no element in the class $4B$. Then one of the following holds:*

(i) *$F \cong Z_4$, generated by an element in class $4A$, and $C_G(F)^0 = A_1A_7$;*

(ii) *$F \cong Q_8$ with elements $2A, 4A^6$, and $C_G(F)^0 = A_1D_4$.*

In both cases F is as in Table 7.

Proof. Let $e \in F$ have order 4. By hypothesis, e is in class $4A$, so $C_G(e) = A_1A_7$. There is nothing to prove if $F \cong Z_4$, so assume $|F| > 4$ and pick $f \in N_F(\langle e \rangle) \backslash \langle e \rangle$. If $f \in A_1A_7$ then $C_G(e, f)$ has a normal torus, so $e^f = e^{-1}$ and f induces an involutory graph automorphism on A_7 (see Proposition 2.4). Hence $C_{A_7}(f)^0 = C_4$ or D_4 by Proposition 2.1. The subgroup C_4 lies in a Levi subgroup E_6 of $C_G(A_1) = E_7$ (see the proof of [7, 2.15]), so the irreducibility of $C_G(F)^0$ implies that $C_{A_7}(f)^0 = D_4$, hence $C_G(e, f)^0 = A_1D_4$. Also $\langle e, f \rangle \cong Q_8$, as shown in the proof of [7, 2.15]. Finally, if $N_F(\langle e, f \rangle) > \langle e, f \rangle$ then some element of order 4 in $\langle e, f \rangle$ has centralizer in F of order greater than 4, which we have seen to be impossible above. Hence $F = \langle e, f \rangle$. ■

Lemma 3.3 *Suppose F is a 2-group and has an element e in the class $4B$. If $C_F(e) \neq \langle e \rangle$, then F is as in Table 7 (one of the entries 4×2, $Dih_8 \times 2$, $4 \circ Dih_8$, $Q_8 \times 2$, 2_-^{1+4}).*

Proof. Assume that $C_F(e) \neq \langle e \rangle$. Then $C_F(e)$ contains a group of order 8 in which e is central, and hence there is an involution $e_1 \in C_F(e) \backslash \langle e \rangle$. Diagonalizing the preimage of $C_F(e^2)/\langle e^2 \rangle$ in SO_{16} (as in the proof of Lemma 3.1), we can take

$$e = (-1^6, 1^{10}),\ e_1 = (1^6, -1^4, 1^6),$$

so that $\langle e, e_1 \rangle \cong Z_4 \times Z_2$ and $C_G(e, e_1)^0 = A_1^2A_3^2$, as in the 4×2 entry in Table 7.

Write $E_0 = \langle e, e_1 \rangle$. If there exists $f \in (F \cap A_1^2A_3^2) \backslash E_0$, then $C_G(e, e_1, f)$ has a nontrivial normal torus, which is a contradiction. Hence $F \cap C_G(E_0)^0 = E_0$.

Suppose $C_F(e) \neq E_0$. If there is no involution in $C_F(e) \backslash E_0$, then $C_F(e)$ contains a subgroup $Z_4 \times Z_4$, which is impossible. So let e_2 be an involution in $C_F(e) \backslash E_0$. As e_2 does not centralize e_1, we can take

$$e_2 = (1^6, -1, 1^3, -1^3, 1^3).$$

Then $C_G(e, e_1, e_2)^0 = A_3B_1^3$ (where each B_1 corresponds to a natural subgroup SO_3 in SO_{16}), and $\langle e, e_1, e_2 \rangle = \langle e \rangle \circ \langle e_1, e_2 \rangle \cong 4 \circ Dih_8$, as in Table 7. If $F \neq E_1 := \langle e, e_1, e_2 \rangle$, then there is an element $e_3 \in N_F(E_1) \backslash E_1$, and adjusting by an element of E_1 we can take

$$e_3 = (-1^3, 1^3, -1, 1^9).$$

Then $C_G(E_1, e_3)^0 = B_1^5$ and $\langle E_1, e_3 \rangle \cong 2_-^{1+4}$, as in Table 7. Finally, there are no possible further elements of F such that $C_G(F)^0$ is irreducible.

Now suppose $C_F(e) = E_0$ and $F \neq E_0$. Pick $f \in N_F(E_0) \backslash E_0$. Then f centralizes e^2, so we can diagonalize in the usual way; adjusting by an element of E_0 and using the fact that $C_G(E_0, f)$ has no nontrivial normal torus, we can take $f \in \{e_4, e_5, e_6, e_7\}$, where

$$\begin{array}{ll} e_4 = (-1^3, 1^3, 1^4, -1, 1^5), & e_5 = (-1^3, 1^3, 1^4, -1^3, 1^3), \\ e_6 = (-1^3, 1^3, -1, 1^3, 1^6), & e_7 = (-1, 1^5, 1^4, -1^3, 1^3). \end{array}$$

Let $E_2 = \langle E_0, f \rangle$.

If $f = e_4$ then $C_G(E_2)^0 = B_1^2 \bar{A}_1^2 B_2$ and $E_2 = \langle e, e_4 \rangle \times \langle e_1 \rangle \cong Dih_8 \times Z_2$, as in Table 7. There are no possible further elements of F in this case.

If $f = e_5$ then $C_G(E_2)^0 = \bar{A}_1^2 B_1^4$ and $E_2 = \langle e, e_5 \rangle \times \langle e_1 \rangle \cong Q_8 \times Z_2$, as in Table 7. Any further element of F would centralize e^2, and hence would violate the fact that $C_F(e) = E_0$.

Finally, if $f = e_6$ or e_7, then E_2 is D_8-conjugate to $\langle e, e_1, e_2 \rangle$ or $\langle e, e_1, e_4 \rangle$, cases considered previously. ∎

Lemma 3.4 *If F is a 2-group, then it is as in Table 7.*

Proof. Suppose F is a 2-group. In view of the previous two lemmas, we can assume that F contains an element e in the class $4B$, and that $C_F(e) = \langle e \rangle$ – and indeed that F has no subgroup $Z_4 \times Z_2$. We can also assume that $F \neq \langle e \rangle$. Hence there exists $f \in F$ such that $e^f = e^{-1}$. As usual we can diagonalize $\langle e, f \rangle$, and hence take $e = (-1^6, 1^{10})$ and $f \in \{f_1, f_2, f_3, f_4, f_5\}$, where

$$\begin{array}{ll} f_1 = (-1^3, 1^3, -1, 1^9), & f_2 = (-1^3, 1^3, -1^3, 1^7), \\ f_3 = (-1^3, 1^3, -1^5, 1^5), & f_4 = (-1, 1^5, -1^3, 1^7), \\ f_5 = (-1, 1^5, -1^5, 1^5). & \end{array}$$

Moreover, the fact that F has no subgroup $Z_4 \times Z_2$ implies that $F = \langle e, f \rangle$.

It is easily seen that the possibilities for F and $C_G(F)^0$ are as follows:

F	$C_G(F)^0$
$\langle e, f_1 \rangle \cong Dih_8$	$B_1^2 B_4$
$\langle e, f_2 \rangle \cong Q_8$	$B_1^3 B_3$
$\langle e, f_3 \rangle \cong Dih_8$	$B_1^2 B_2^2$
$\langle e, f_4 \rangle \cong Dih_8$	$B_1 B_2 B_3$
$\langle e, f_5 \rangle \cong Q_8$	B_2^3

All these possibilities are in Table 7. ∎

Lemma 3.5 *If F is a 3-group, then $F = 3$ or 3^2 is as in Table 7.*

Proof. Suppose F is a 3-group. It has exponent 3, by Proposition 2.2. If $F = 3$ or 3^2, it is as in Table 7 by Propositions 2.2 and 2.6(ii), so assume $|F| > 9$.

If F has an element e with $C_G(e) = A_8$, then there is an element $f \in C_F(e) \setminus \langle e \rangle$, and $C_{A_8}(f)$ is reducible in A_8, a contradiction. Hence all nonidentity elements of F

have centralizer A_2E_6 (see Proposition 2.2). In particular, they have trace 5 on the adjoint module $L(G)$ (see [5, 3.1]).

Let V be a normal subgroup of F with $V \cong 3^2$. Then $C_G(V) = A_2^4$ by Proposition 2.6(ii). If $f \in F \setminus V$, then f acts as a 3-cycle on the four A_2 factors, so $C_G(V,f)^0 = \bar{A}_2A_2$. On the other hand, since every nonidentity element has trace 5 on $L(G)$, we have

$$\dim C_{L(G)}(V,f) = \frac{1}{27}(248 + 26 \cdot 5) = 14.$$

This is a contradiction, showing that $|F| > 9$ is impossible. ■

From now on, we assume that F is not a 2-group or a 3-group. Let $J = \mathrm{Fit}(F)$, the Fitting subgroup of F.

Lemma 3.6 *Suppose J is a nontrivial 2-group. Then F is as in Table 7.*

Proof. By Lemma 3.4, J and $C_G(J)^0$ are as in Table 7. Also $C_F(J) \leq J$, so F contains an element x of order $r = 3$ or 5 acting nontrivially on J and as a graph automorphism of $C_G(J)^0$. By inspection of Table 7, the possibilities for J with these properties are as follows:

J	$C_G(J)^0$	r
2^2	D_4^2	3
2^3	$A_1^4D_4$	3
	A_1^8	3, 5
2^4	A_1^8	3, 5
Q_8	A_1D_4	3
	B_2^3	3
	$B_1^3B_3$	3
$4 \circ Dih_8$	$A_3B_1^3$	3
$Q_8 \times 2$	$A_1^2B_1^4$	3
2_-^{1+4}	B_1^5	3, 5

For the last five cases, where $C_G(J)^0 \rhd B_1^r$ (or B_2^r), $C_G(J,x)^0$ has a factor B_1 (or B_2) which is a diagonal subgroup of this, and so $C_G(J,x)^0$ is a reducible subgroup of D_8 in these cases, by Proposition 2.7. Also if $C_G(J)^0 = A_1^8$, then $N_G(A_1^8)/A_1^8 \cong AGL_3(2)$ by Proposition 2.4, so $r = 3$ and x acts as a product of two 3-cycles on the eight A_1 factors. We claim that again $C_G(J,x)^0$ is reducible. To see this, regard A_1^8 as a subgroup of D_8 corresponding to SO_4^4 in SO_{16}. Observe that $C_J(x)^0 = \bar{A}_1^2A_1^2$ where each of the last two A_1 factors is diagonal in $\bar{A}_1^3$. There are two possible actions of the subgroup $A_1^2 < \bar{A}_1^6 < D_6$ on the 12-dimensional natural module, namely $(1,1)^3$ or $(2,0)+(1,1)+(0,2)+(0,0)^2$. In both cases the subgroup A_1^2 is D_6-reducible by Proposition 2.7 and hence $C_J(x)^0$ is D_8-reducible.

This leaves the following possibilities remaining, all with $r = 3$:

J	$C_G(J)^0$
2^2	D_4^2
2^3	$A_1^4D_4$
Q_8	A_1D_4

Suppose $J = 2^2$, $C_G(J)^0 = D_4^2$. Then x induces a triality automorphism on both D_4 factors (see Proposition 2.4). So by Proposition 2.1, $C_G(J,x)^0 = G_2G_2,\ A_2A_2$

or A_2G_2. In the first case $G_2G_2 < D_7 < D_8$, so is reducible. Hence if $F = \langle J, x\rangle$ we have the possibilities

$$F = \langle J, x\rangle \cong Alt_4,\ C_G(F)^0 = A_2A_2 \text{ or } A_2G_2,$$

both in Table 7. Now assume $F \neq \langle J, x\rangle$. Then $F = \langle J, x, t\rangle \cong Sym_4$, where t is an involution inverting x (since $\mathrm{Fit}(F) \cong 2^2$). If $C_G(J, x)^0 = A_2A_2$ then t acts as a graph automorphism on each A_2 factor (see Proposition 2.1), and so $C_G(F)^0 = A_1A_1$; and if $C_G(J, x)^0 = A_2G_2$ then t acts as a graph autormorphism on the A_2 factor and centralizes the G_2 factor, so $C_G(F)^0 = A_1G_2$. Hence we have the possibilities

$$F = \langle J, x, t\rangle \cong Sym_4,\ C_G(F)^0 = A_1A_1 \text{ or } A_1G_2,$$

both in Table 7.

Next suppose $J = 2^3$, $C_G(J)^0 = A_1^4D_4$. Then x acts as a 3-cycle on the A_1 factors and as a triality on D_4, so $C_G(J, x)^0 = \bar{A}_1A_1G_2$ or $\bar{A}_1A_1A_2$ (where $\bar{A}_1$ denotes a fundamental SL_2 generated by a root group and its opposite). The first subgroup is reducible as it is contained in a subgroup D_7 of D_8. So if $F = \langle J, x\rangle$, we have

$$F = \langle J, x\rangle \cong 2 \times Alt_4,\ C_G(F)^0 = \bar{A}_1A_1A_2,$$

as in Table 7. Now assume $F \neq \langle J, x\rangle$, and let $\langle v\rangle = Z(\langle J, x\rangle)$. As F/J is isomorphic to a subgroup of $GL_3(2)$ with no nontrivial normal 2-subgroup, we have $F/J \cong Dih_6$ and $F = \langle J, x, t\rangle$ where $x^t = x^{-1}$ and $t^2 = 1$ or v. Such an element t centralizes both A_1 factors of $C_G(J, x)^0$, and induces a graph automorphism on the A_2 factor, so $C_G(F)^0 = \bar{A}_1A_1A_1$. This is contained in the above centralizer G_2A_2 of an Alt_4 subgroup, and $\bar{A}_1A_1$ centralizes an involution in G_2. Hence in fact $t^2 = 1$ and we have

$$F = \langle J, x, t\rangle \cong 2 \times Sym_4,\ C_G(F)^0 = \bar{A}_1A_1A_1,$$

as in Table 7.

Finally, suppose $J = Q_8$, $C_G(J)^0 = A_1D_4 < A_1A_7$. Then x induces triality on the D_4 factor (see [7, 2.15]), so $C_G(J, x)^0 = A_1G_2$ or A_1A_2. The first subgroup is reducible in A_1A_7, so if $F = \langle J, x\rangle$, we have

$$F = \langle J, x\rangle = Q_8.3 \cong SL_2(3),\ C_G(F)^0 = \bar{A}_1A_2,$$

as in Table 7. If $F \neq \langle J, x\rangle$ then F has an element t inducing a graph automorphism on the A_2 factor, so $C_G(J, x, t)^0 = A_1A_1$, which is reducible in A_1A_7. This completes the proof. ∎

Lemma 3.7 *If $|J|_3 = 3$, then F is as in Table 7.*

Proof. Assume $|J|_3 = 3$, and let $x \in J$ be of order 3.

Suppose first that $|F|_3 = 3$ also. As F has no element of order 15 we have $|F|_5 = 1$, so $F/\langle x\rangle$ is a 2-group. The case where $|F| = 3$ is in Table 7, so assume $|F| > 3$.

Suppose $C_G(x) = A_2E_6$. If t is an involution in $C_F(x)$, then $C_G(x, t) = A_2A_1A_5$; moreover $C_F(x)$ has no element of order 4 (as F has no element of order 12), and no subgroup $V \cong 2^2$ (as $C_G(x, V)$ would have a normal torus). Hence $C_F(x)$ has order 3 or 6, and so $|F|$ is 6 or 12.

If $|F| = 6$, then either $F \cong Z_6$, $C_G(F) = A_1A_2A_5$, or $F = \langle x, t\rangle \cong Dih_6$ with t inducing graph automorphisms on both factors of $C_G(x) = A_2E_6$ (see Proposition 2.4), in which case $C_G(F)^0 = A_1F_4$ or A_1C_4. All these possibilities are in Table 7.

If $|F| = 12$, then $F = \langle y, u\rangle$ where y has order 6, $y^u = y^{-1}$ and $u^2 = 1$ or y^3. Then u induces a graph automorphism on the A_2, A_5 factors of $C_G(y) = A_1A_2A_5$, so $C_G(F)^0 = \bar{A}_1A_1C_3$ or $\bar{A}_1A_1A_3$. The subgroup $\bar{A}_1A_1C_3$ is contained in A_1F_4 and in A_1C_4, while the subgroup $\bar{A}_1A_1A_3$ is contained in neither. Hence u is an involution in the first case, and has order 4 in the second. This gives the possibilities

$$F = \langle y, u\rangle \cong Dih_{12} \text{ or } G_{12},\ C_G(F)^0 = \bar{A}_1A_1C_3 \text{ or } \bar{A}_1A_1A_3 \text{ (resp.)},$$

both in Table 7.

Next suppose that $C_G(x) = A_8$. Then $C_F(x) = \langle x\rangle$, so $F = \langle x, t\rangle \cong Dih_6$ where t induces a graph automorphism on A_8, giving $C_G(F)^0 = B_4$. This completes the case where $|F|_3 = 3$.

Finally, suppose $|F|_3 = 3^2$, and let $\langle x, y\rangle$ be a Sylow 3-subgroup of F. Again, $|F|_5 = 1$. If $F = \langle x, y\rangle$ then $C_G(F)^0 = A_2^4$ by Proposition 2.6(ii), as in Table 7. Otherwise, as $C_F(J) \le J$ there must be a subgroup $V \cong 2^2$ of J such that y acts nontrivially on V. But then $C_G(x, V)^0 = C_{A_2E_6}(V)^0$ is reducible, a contradiction. ∎

Lemma 3.8 *If $|J|_3 = 3^2$, then F is as in Table 7.*

Proof. Let $V \cong 3^2$ be a Sylow 3-subgroup of J (also of F, by Lemma 3.5). By Propositions 2.6(ii) and 2.4, $C_G(V) = A_2^4$ and $N_G(A_2^4)/A_2^4 \cong GL_2(3)$. There is no involution in $C_F(V)$, so $J = \mathrm{Fit}(F) = V$ and F/J is a nontrivial 2-subgroup of $GL_2(3)$.

Suppose first that $|F/J| = 2$. There are two classes of involutions in $GL_2(3)$, with representatives $i = -I$ and $t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then i induces a graph automorphism on each A_2 factor of $C_G(V)$, so $C_G(V, i)^0 = A_1^4$; and t fixes two A_2 factors, inducing a graph automorphism on one of them, so $C_G(V, t)^0 = A_1\bar{A}_2A_2$. Both these groups $F = 3^2.2$ are in Table 7.

If $F/J \cong 2^2$, we can take $F = \langle V, i, t\rangle$ and so $C_G(F)^0 = A_1^2A_1$, as in Table 7.

Next suppose $F/J \cong Z_4$. There is one class of elements of order 4 in $GL_2(3)$, with representative $u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; this swaps two pairs of A_2 factors, and squares to i. Hence $C_G(F)^0 = C_G(V, u)^0 = A_1^2$, as in Table 7.

If $F/J \cong Dih_8$, we can take $F = \langle V, u, t\rangle$, and again $C_G(F)^0 = A_1^2$.

Now suppose $F/J \cong Q_8$. Then F/J acts transitively on the four A_2 factors, and contains i, so $C_G(F)^0 = A_1$, a diagonal subgroup of $A_1^4 < A_2^4$. We claim that $C_G(F)^0$ is reducible. To see this, observe that A_1^4 centralizes the involution i; this involution corresponds to w_0, the longest element of the Weyl group of G, and so $C_G(i) = D_8$. Now it is easy to check that $C_G(F)^0 = A_1$ is reducible in this D_8. Indeed, A_1^4 acts as $(1, 1, 1, 1)$ on the natural module for D_8 and hence a diagonal subgroup A_1 acts as $4 + 2^3 + 0^2$. Thus $F/J \cong Q_8$ is impossible, and we have now covered all possibilities for F/J. ∎

Lemma 3.9 *If $|J|_5 \ge 5$, then F is as in Table 7.*

Proof. Suppose $|J|_5 \geq 5$, and let $x \in J$ have order 5. As $C_G(x) = A_4^2$ by Proposition 2.2, there is no element of order 5 in $C_F(x) \setminus \langle x \rangle$, and so $\langle x \rangle$ is a Sylow 5-subgroup of F.

As F has no element of order 10 or 15, we have $C_F(x) = \langle x \rangle$, and $|F| = 10$ or 20. By Proposition 2.4, $N_G(A_4^2)/A_4^2 = \langle t \rangle \cong Z_4$, where t interchanges the two A_4 factors and t^2 induces a graph automorphism on both. Hence F is either Dih_{10} or $Frob_{20}$, and $C_G(F)^0 = B_2B_2$ or B_2, respectively, as in Table 7. ∎

Lemmas 3.6 – 3.9 cover all cases where the Fitting subgroup J is nontrivial.

Lemma 3.10 *Suppose $J = \mathrm{Fit}(F) = 1$. Then $F = Alt_5$ or Sym_5 is as in Table 7.*

Proof. In this case $S := \mathrm{soc}(F)$ is a direct product of nonabelian simple groups. As 5^2 does not divide $|F|$, in fact S is simple. Proposition 1.2 of [10] shows that $S \cong Alt_5$ or Alt_6.

Suppose $S \cong Alt_5$. Then S has subgroups $D \cong Dih_{10}$ and $A \cong Alt_4$, and by what we have already proved, these subgroups are in Table 7. Hence the involutions in S are in the class $2B$ (since those in A are in this class). If the elements of order 3 in S are in class $3A$ (with centralizer A_8), then from [5, 3.1] we see that the traces of the elements in S of orders $2, 3, 5$ on $L(G)$ are $-8, -4, -2$ respectively, and hence

$$\dim C_{L(G)}(S) = \frac{1}{60}(248 - 8 \cdot 15 - 4 \cdot 20 - 2 \cdot 24) = 0,$$

which is a contradiction. It follows that the elements of order 3 in S are in the class $3B$, with centralizer A_2E_6 and trace 5, so that

$$\dim C_{L(G)}(S) = \frac{1}{60}(248 - 8 \cdot 15 + 5 \cdot 20 - 2 \cdot 24) = 3.$$

Since $C_G(D)^0 = B_2B_2 < A_4A_4$, it follows that $C_G(S)^0 = A_1$, embedded diagonally and irreducibly in A_4A_4. Also $C_G(A_1) = Sym_5$ by [10, 1.5]. Hence $F = Alt_5$ or Sym_5 and $C_G(F)^0 = A_1$, as in Table 7.

Finally, suppose $S \cong Alt_6$ and choose a subgroup $T < S$ with $T \cong Alt_5$. By the above, $C_G(T)^0 = A_1$ and so $C_G(S)^0$ must also be A_1. But as observed before, $C_G(A_1) = Sym_5$, a contradiction. ∎

We have now established that F and $C_G(F)^0$ must be as in Table 7. To complete the proof of Theorem 1, we need to establish that all these examples exist. This is proved in the following lemma.

Lemma 3.11 *Let F and $C_G(F)^0$ be as in Table 7. Then $C_G(F)^0$ is G-irreducible.*

Proof. Any subgroup containing a G-irreducible subgroup is itself G-irreducible. Thus we need only consider the subgroups $C_G(F)^0$ for which F is maximal. These subgroups are given in Table 1; let X be such a subgroup $C_G(F)^0$.

Firstly, if X has maximal rank then X is clearly G-irreducible. For the subgroups not of maximal rank we use the fact that a subgroup with no trivial composition factors on $L(G)$ is necessarily G-irreducible (since the Lie algebra of the centre of a Levi subgroup gives a trivial composition factor). It thus remains to show X has no trivial composition factors on $L(E_8)$. We find the composition factors of X on $L(G)$ by restriction from a maximal rank overgroup Y, as given in the last column of Table 7. The restrictions $L(G) \downarrow Y$ are given in [12, Lemma 11.2, 11.3] for all of the maximal rank overgroups Y except for A_1A_7, $A_1^4D_4$ and A_2^4. The latter subgroups are contained in A_1E_7, D_4^2 and A_2E_6, respectively, and it is straightforward to compute their composition factors on $L(G)$.

We finish the proof with two examples of how to calculate the composition factors of $L(G) \downarrow X$ from those of a maximal rank overgroup Y. The others all follow similarly and in each case there are no trivial composition factors.

For the first example, let $X = B_2^3$ so $p \neq 2$ and X is contained in the maximal rank overgroup D_8. From [12, Lemma 11.2],

$$L(G) \downarrow D_8 = V(\lambda_2) + V(\lambda_7),$$

the sum of the exterior square of the natural module for D_8 and a spin module. To find the restriction of the spin module $V_{D_8}(\lambda_7)$ to X we consider the chain of subgroups $X < B_2D_5 < B_2B_5 < D_8$. By [12, Lemma 11.15(ii)], $V_{D_8}(\lambda_7) \downarrow B_2B_5 = 01 \otimes \lambda_5$. Also, $V_{B_5}(\lambda_5) \downarrow D_5 = \lambda_4 + \lambda_5$ and $V_{D_5}(\lambda_i) \downarrow B_2^2 = 01 \otimes 01$ for $i = 4, 5$. Therefore,

$$L(G) \downarrow X = \bigwedge^2 (10 \otimes 00 \otimes 00 + 00 \otimes 10 \otimes 00 + 00 \otimes 00 \otimes 10 + 0) + (01 \otimes 01 \otimes 01)^2$$

and this has no trivial composition factors.

For the second example, let $X = A_1D_4$. Here $p = 3$ and X is contained in a maximal rank subgroup A_1A_7. Then using the restriction $L(G) \downarrow A_1E_7$ given in [12, Lemma 11.2] we find

$$L(G) \downarrow A_1A_7 = 2 \otimes 0 + 1 \otimes \lambda_2 + 1 \otimes \lambda_6 + 0 \otimes (\lambda_1 + \lambda_7) + 0 \otimes \lambda_4.$$

It is sufficient to show there are no trivial composition factors for D_4 acting on $V_{A_7}(\lambda)$ for $\lambda = \lambda_1 + \lambda_7$ and λ_4. By weight considerations, the first module restricts to D_4 as $V(2\lambda_1) + V(\lambda_2)$ and the second as $V(2\lambda_3) + V(2\lambda_4)$. Hence $L(G) \downarrow X$ has no trivial composition factors. ■

This completes the proof of Theorem 1 for $G = E_8$.

3.2 The case $G = E_7$

In this section we prove Theorem 1 for $G = E_7$, of adjoint type. Let F be a finite subgroup of G such that $C_G(F)^0$ is G-irreducible. As before, $C_G(F)^0$ is semisimple and $C_G(E)^0$ is G-irreducible for all nontrivial subgroups E of F. Also F is a $\{2,3\}$-group by Proposition 2.2.

Lemma 3.12 *If F is an elementary abelian 2-group, then F is as in Table 8.*

Proof. We may suppose that $|F| > 2$. If F has an element e in the class $2B$, then any further element $f \in F \backslash \langle e \rangle$ must lie in $C_G(e) \backslash C_G(e)^0 = A_7.2 \backslash A_7$, and hence $F = \langle e, f \rangle \cong 2^2$; moreover $C_G(F)^0 = D_4$, as in the proof of Lemma 3.2. Hence F is as in Table 8.

So now suppose that F is $2A$-pure. Let $1 \neq e \in F$ and $e_1 \in F \backslash \langle e \rangle$. Then $C_G(e) = A_1D_6$, and diagonalizing in SO_{12} as in Lemma 3.1, we can take $e_1 =$

$(-1^4, 1^8)$. Hence $C_G(e,e_1)^0 = A_1^3 D_4$. If there is an element $e_2 \in E \backslash \langle e, e_1 \rangle$, then we can take $e_2 = (1^4, -1^4, 1^4)$, and so $C_G(e,e_1,e_2)^0 = A_1^7$. Both these possibilities are in Table 7, and there are no further possible elements in F. ■

Lemma 3.13 *If F is a 2-group containing an element of order 4, then F is as in Table 8.*

Proof. Let $e \in F$ of order 4. By Proposition 2.2 we have $C_G(e)^0 = A_1 A_3^2$. Suppose $F \neq \langle e \rangle$, so there exists $f \in F$ such that $e^f = e^{-1}$. Now $C_G(e^2) = A_1 D_6$, and diagonalizing in SO_{12} as in Lemma 3.1, we may take $e = (-1^6, 1^6)$ and $f \in \{f_1, f_2\}$, where

$$f_1 = (-1, 1^5, -1^3, 1^3), \quad f_2 = (-1^3, 1^3, -1^3, 1^3).$$

If $f = f_1$ then $C_G(e,f)^0 = \bar{A}_1 B_1^2 B_2$ and $\langle e, f \rangle \cong Dih_8$; and if $f = f_2$ then $C_G(e,f)^0 = \bar{A}_1 B_1^4$ and $\langle e, f \rangle \cong Q_8$. Both possibilities are in Table 8. Finally, there are no possible further elements of F, as can be seen by diagonalizing in the usual way. ■

In view of the previous two lemmas we assume from this point that F contains an element x of order 3. Let J be the Fitting subgroup of F. Note that F does not contain an element of order 6 by Proposition 2.2. Therefore J is a 2-group or a 3-group.

Lemma 3.14 *If J is a 3-group then F and $C_G(F)^0$ are as given in Table 8.*

Proof. Suppose $|J| = 3$. If $|F| = 3$ then by Proposition 2.2 we have $C_G(F) = A_2 A_5$. Otherwise $F \cong Dih_6$ and $C_G(F)^0 = A_1 C_3$ or $A_1 A_3$.

Finally, $|J| > 3$ is impossible because the centralizer of an element of order 3 in $A_2 A_5$ is not $A_2 A_5$-irreducible. ■

We may now assume that J is a 2-group. By Lemmas 3.12 and 3.13, J is as in Table 8 and the action of x shows that the only possibilities are $J \cong 2^2$, 2^3 or Q_8.

Lemma 3.15 *If $J \cong 2^2$ then F and $C_G(F)^0$ are as given in Table 8.*

Proof. Suppose $C_G(J)^0 = A_1^3 D_4$. By Proposition 2.4, $N_G(A_1^3 D_4)/A_1^3 D_4 \cong Sym_3$ acting simultaneously on both the A_1^3 and the D_4 factors. Therefore $C_G(J,x)^0 = A_1 A_2$ or $A_1 G_2$ with $\langle J, x \rangle \cong Alt_4$. The subgroup $A_1 G_2$ is $A_1 D_6$-reducible by Proposition 2.7, and therefore does not appear in Table 8. If $F \neq \langle J, x \rangle$ then we must have $F \cong Sym_4$ with $C_G(F)^0 = A_1 A_1$.

Now suppose $C_G(J)^0 = D_4 < A_7$. By [7, Lemma 2.15], we have $N_G(D_4)/(D_4 \times C_G(D_4)) \cong Sym_3$. Therefore $C_G(J,x)^0 = A_2$ or G_2. The subgroup G_2 is A_7-reducible and therefore does not appear in Table 8. If $F \neq \langle J, x \rangle$ then $F \cong Sym_4$ with $C_G(F)^0 = A_1$. ■

Lemma 3.16 *There are no possible subgroups F with $J \cong 2^3$ or Q_8.*

Proof. Suppose $J \cong 2^3$ so $C_G(J)^0 = A_1^7$. By Proposition 2.4 we have $N_G(A_1^7)/A_1^7 \cong GL_3(2)$. The element $x \in F$ therefore acts as a product of two disjoint 3-cycles on the seven A_1 factors. But the centralizer $C_G(J,x)^0$ is then A_1D_6-reducible by an argument in the first paragraph of the proof of Lemma 3.6.

Finally, if $J \cong Q_8$ and $C_G(J)^0 = A_1B_1^4$, then $C_G(J,x)^0 = A_1B_1B_1$ which is clearly A_1D_6-reducible. ■

The proof of Theorem 1 for $G = E_7$ is now complete, apart from showing that all the subgroups $C_G(F)^0$ in Table 8 are G-irreducible. This is proved in similar fashion to Lemma 3.11.

3.3 The case $G = \mathrm{Aut}\, E_6$

Let $G = \mathrm{Aut}\, E_6 = E_6.2$, and let F be a finite subgroup of G such that $C_{G'}(F)^0$ is G'-irreducible.

Lemma 3.17 *If F has an element x of order 4, then $F = \langle x\rangle$ and $C_{G'}(F)^0 = A_1A_3$.*

Proof. By Proposition 2.3, $C_G(x)^0 = A_1A_3$. By Proposition 2.7, A_1A_3 contains no proper A_1A_5-irreducible connected subgroups and therefore $F = \langle x\rangle$ as claimed. ■

We now assume that F has no element of order 4.

Lemma 3.18 *If F is an elementary abelian 2-group then it appears in Table 10.*

Proof. If $|F| = 2$ then F and $C_G(F)^0$ are as in Table 10 by Proposition 2.2. Now suppose $F = \langle t,u\rangle \cong 2^2$. Then $C_G(t)^0 = A_1A_5$, F_4 or C_4 and therefore $C_G(F)^0 = A_1A_3$, A_1C_3, B_4 or C_2^2. The A_1A_3 case is ruled out by Lemma 3.17. The B_4 and C_2^2 subgroups are both contained in D_5-parabolic subgroups. Therefore $C_G(F)^0 = A_1C_3$. Finally, if F has a further involution v then $C_G(F)^0 = A_1^2C_2$, which by Proposition 2.7 is A_1A_5-reducible. ■

We now let J be the Fitting subgroup of F. Since F is a $\{2,3\}$-group, J is nontrivial.

Lemma 3.19 *If J is not a 2-group or a 3-group, then F is as in Table 10.*

Proof. Under the assumptions of the lemma, J has an element x of order 6. Then $C_G(x)^0 = A_2A_2$ by Proposition 2.3. If $F \neq \langle x\rangle$ then there exists an element $t \in F \setminus J$ inverting x with $t^2 \in J$. Since F has no element of order 4 we have $t^2 = 1$ and $\langle J,t\rangle \cong Dih_{12}$. The element t induces a graph automorphism on both A_2 factors and so $C_G(J,t)^0 = A_1A_1$. Since the two factors are non-conjugate, there is no element swapping them and hence $F = \langle J,t\rangle$. ■

Lemma 3.20 *If J is a 2-group then $F = J$.*

Proof. The possibillities for the 2-group J are in Table 10, from which we see that no element of order 3 can act as a graph automorphism on $C_G(J)^0$. ■

Lemma 3.21 *If J is a 3-group then F is as in in Table 10.*

Proof. Suppose $J = \langle x\rangle \cong Z_3$. If $F = J$ then $C_G(F)^0 = A_2^3$ by Proposition 2.2. Otherwise, $F = \langle J, t\rangle \cong Dih_6$ where t is an involution in $N_G(A_2^3)/A_2^3 \cong 2 \times S_3$ by Proposition 2.4. This gives two possibilities for t. If t is the central involution then t induces a graph automorphism on each factor A_2 and $C_G(J,t)^0 = A_1^3$. If t is not central then $C_G(J,t)^0 = A_1A_1$.

Now suppose $|J| > 3$ and let $x, y \in J$ with $\langle x, y\rangle \cong 3^2$. Then $C_G(x) = A_2^3.3$ and y cyclically permutes the three A_2 factors, so $C_G(x,y)^0$ is a diagonal subgroup A_2. However, the elements in class $3A$ have trace -3 on $L(G)$ and so

$$\dim C_{L(G)}(S) = \frac{1}{9}(78 - 8 \cdot 3) = 6,$$

a contradiction. ■

Finally, we need to prove that all the subgroups $C_G(F)^0$ in Table 10 are G'-irreducible.

Lemma 3.22 *Let F and $C_G(F)^0$ be as in Table 9. Then $C_G(F)^0$ is G'-irreducible.*

Proof. This is proved in a similar fashion to Lemma 3.11 for most of the subgroups. Specifically, all of the subgroups have no trivial composition factors on $L(G)$ except for A_1A_5, $\bar{A}_1C_3$ and $\bar{A}_1A_3$ when $p = 3$, all of which have exactly one trivial composition factor. There are no Levi subgroups of G' containing a subgroup of type A_1A_5 or A_1C_3. Hence both are G'-irreducible.

Now consider $X = A_1A_3$. Assume X is G'-reducible and choose a minimal parabolic subgroup P containing X. By [11, Theorem 1], X is contained in a Levi subgroup L of P and by minimality X is L-irreducible. Hence $L = D_5T_1$ or $A_1A_3T_2$, where T_i denotes a central torus of rank i. The second possibility is ruled out since X has only one trivial composition factor on $L(G)$. So X is an irreducible subgroup of $L' = D_5$. The A_1 factor of X is generated by root groups of D_5 and so $C_{D_5}(A_1)^0 = A_1A_3$. Thus $C_{D_5}(X)^0$ contains a subgroup A_1, contradicting the L-irreducibility of X. Hence X is G'-irreducible, as required. ■

This completes the proof of Theorem 1 for $G = \mathrm{Aut}\, E_6$.

3.4 The case $G = F_4$

Let $G = F_4$, and let F be a finite subgroup of G such that $C_G(F)^0$ is G-irreducible.

Lemma 3.23 *If F is an elementary abelian 2-group, then F and $C_G(F)^0$ are given in Table 11.*

Proof. If $F \cong Z_2$ then $C_G(F) = B_4$ or A_1C_3 by Proposition 2.2. Now suppose $F = \langle t, u\rangle \cong 2^2$. Then $u \in C_G(t) = B_4$ or A_1C_3. Therefore $C_G(F)^0 = D_4$ or $A_1^2B_2$. Now suppose $|F| > 4$. A 2^2 subgroup of F must contain a $2A$ involution, say t, with centralizer B_4. Then $B_4/\langle t\rangle \cong SO_9$ and the image of F in SO_9 is elementary abelian by Proposition 2.5. Since $C_G(F)^0$ is G-irreducible, it follows that the image is $\langle u, v\rangle \cong 2^2$ with $u = (-1^8, 1)$ and $v = (-1^4, 1^5)$. Therefore $C_{SO_9}(F)^0 = SO_4SO_4$, and so $C_G(F)^0 = A_1^4$. ■

Lemma 3.24 *If F is a 2-group containing an element x of order 4, then F and $C_G(F)^0$ are given in Table 11.*

Proof. By Proposition 2.2, $C_G(x) = A_1A_3$. Now suppose $|F| > 4$. Since A_1A_3 is not contained in A_1C_3 it follows that $C_G(x^2) = B_4$. Therefore $F/\langle x^2\rangle$ is elementary abelian in SO_9. Diagonalizing as before we may assume $x = (1^3, -1^6)$. Since $C_G(F)^0$ is G-irreducible, it must be the case that $|F/\langle x^2\rangle| = 4$ and a further involution in F is either $u_1 = (-1^6, 1^3)$ or $u_2 = (1^5, -1^4)$. In the first case the order of xu_1 is 4 and hence $F \cong Q_8$ with $C_G(F)^0 = B_1^3$. In the second case the order of xu_2 is 2 and hence $F \cong Dih_8$ with $C_G(F)^0 = B_1B_2$. ∎

Now let J be the Fitting subgroup of F. Since F has no element of order 6 it follows that J is either a 2-group or a 3-group.

Lemma 3.25 *If J is a 2-group then F and $C_G(F)^0$ are given in Table 11.*

Proof. By the previous two lemmas we may assume that F is not a 2-group, hence contains an element x of order 3. The only possibilities for J are 2^2, 2^3 or Q_8, with $C_G(J)^0 = D_4$, A_1^4 or B_1^3, respectively. The last two cases are ruled out since any proper diagonal connected subgroup of A_1^4 or B_1^3 such that each projection involves no nontrivial field automorphisms is not B_4-irreducible by Proposition 2.7.

Hence $J \cong 2^2$ and $C_G(J)^0 = D_4$. If $F = \langle J, x\rangle \cong Alt_4$, then $C_G(F)^0 = A_2$ or G_2; and G_2 is not possible since it is contained in a Levi subgroup of type B_3. And if $F \neq \langle J, x\rangle$ then $F \cong Sym_4$ and $C_G(F)^0 = A_1$. ∎

Lemma 3.26 *If J is a 3-group then F and $C_G(F)^0$ are given in Table 11.*

Proof. Let $J = \langle x\rangle$, so $C_G(x) = A_2A_2$. Proposition 2.4 gives $N_G(A_2A_2)/A_2A_2 = \langle t\rangle \cong Z_2$, where t acts as a graph automorphism on each factor. Therefore $F = \langle x, t\rangle \cong Dih_6$ with $C_G(F)^0 = A_1A_1$. ∎

As before, the fact that all the subgroups $C_G(F)^0$ in Table 11 are G-irreducible is proved in similar fashion to Lemma 3.11; in particular they all have no trivial composition factors on $L(G)$.

This completes the proof of Theorem 1 for $G = F_4$.

3.5 The case $G = G_2$

Lemma 3.27 *Let F be a finite subgroup of $G = G_2$ such that $C_G(F)^0$ is G-irreducible. Then F and $C_G(F)^0$ are as in Table 12.*

Proof. By Proposition 2.2, nonidentity elements of F have order 2 or 3. If F is a 2-group then F contains an involution t with $C_G(t) = A_1A_1$; the centralizer of an involution in A_1A_1 is reducible and therefore $F = \langle t\rangle$. Similarly, if F is a 3-group then $F = \langle u\rangle \cong 3$ and $C_G(F) = A_2$. The only remaining possibility is $F = \langle t, u\rangle \cong Dih_6$. Since $N_G(A_2)/A_2 \cong 2$, such an example exists and $C_G(F)^0 = A_1$. ∎

This completes the proof of Theorem 1.

4 Proof of Proposition 3

In this section we prove the following generalisation of Proposition 3.

Proposition 4.1 *Let G be a classical simple adjoint algebraic group in characteristic $p \geq 0$ with natural module V, and let $H = \mathrm{Aut}\, G$. Suppose F is a finite subgroup of H such that $C_G(F)^0$ is G-irreducible. Then F is an elementary abelian 2-group (or a group of order 3 or 6 in the case where $G = D_4$), and one of the following holds.*

(i) $G = PSL_n$, $F \cap G = 1$, $|F| = 2$ *and*

if n is even, then $C_G(F)^0 = PSp_n$ or $PSO_n\,(p \neq 2)$;
if n is odd, then $p \neq 2$ and $C_G(F)^0 = PSO_n$.

(ii) $G = H = PSp_{2n}$, $p \neq 2$, *and taking preimages in Sp_{2n},*

$$C_{Sp_{2n}}(F)^0 = \prod_i Sp_{2n_i} = \prod_i Sp(W_i),$$

where $\sum n_i = n$ and W_i are the distinct weight spaces of F on V.

(iii) $G = PSO_n\,(n \neq 8)$, $H = PO_n$, $p \neq 2$, *and taking preimages in O_n,*

$$C_{O_n}(F)^0 = \prod_i SO_{n_i} = \prod_i SO(W_i),$$

where $n_i \geq 3$ for all i, $\sum n_i = n$ or $n-1$, and W_i are weight spaces of F.

(iv) $G = PSO_{2n}\,(n \neq 4)$, $p = 2$, $F \cap G = 1$, $|F| = 2$ *and* $C_G(F)^0 = SO_{2n-1}$.

(v) $G = D_4 = PSO_8$, $H = D_4.Sym_3$, *and F, $C_G(F)^0$ are as in Table* 6.

Proof. First suppose $G = PSL_n$. If $F \cap G \neq 1$ then $C_G(F \cap G)^0$ is reducible, so $F \cap G = 1$. Hence $|F| = 2$ and now the conclusion in part (i) follows from Proposition 2.1. Similarly, if $G = D_n = PSO_{2n}$ with $n \neq 4$ and $p = 2$ (so that $H = G.2$), then $F \cap G = 1$, $|F| = 2$ and $C_G(F)^0 = B_{n-1}$ by Proposition 2.1, as in (iv).

Now suppose $G = PSp_{2n}$. Then $H = G$ and the centralizer in G of any element of order greater than 2 is reducible. Hence F is an elementary abelian 2-group and $p \neq 2$. The preimage $\hat{F}$ of F in Sp_n must also be elementary abelian, and if we let $W_i\,(1 \leq i \leq k)$ be the weight spaces of $\hat{F}$ on V, then $V = W_1 \perp \cdots \perp W_k$ and $C_{Sp_{2n}}(\hat{F}) = \prod Sp(W_i)$, as in conclusion (ii).

A similar proof applies when $G = PSO_n$ with $n \neq 8$ and $p \neq 2$, giving (iii).

It remains to handle $G = D_4 = PSO_8$. Here $H = G.Sym_3$. If $F \leq PO_8 = G.2$ then the above proof shows that $F = 2$ or 2^2 is as in Table 6. Now suppose 3 divides $|F|$, so that F contains an element x of order 3 inducing a triality automorphism on G. By Proposition 2.1, $C_G(x) = G_2$ or A_2, with $p \neq 3$ in the latter case.

If there is an element $y \in C_F(x) \setminus \langle x \rangle$, then $y \in G_2$ or A_2 has irreducible centralizer, which forces y to be an involution in G_2. So in this case $F = \langle x, y \rangle \cong Z_6$ and $C_G(F) = C_{G_2}(y) = \bar{A}_1 A_1$, as in Table 6. This subgroup has composition factors of dimensions 1, 3 and 4 on V, so is G-irreducible.

Table 6: $G = \mathrm{Aut}\, D_4$: finite subgroups F with irreducible centralizer

F	$F \cap G^0$	$C_G(F)^0$
2	2	$A_1^4\,(p \neq 2)$
	1	B_3
	1	$B_1B_2\,(p \neq 2)$
2^2	2	$A_1^2B_1\,(p \neq 2)$
3	1	G_2
	1	$A_2\,(p \neq 3)$
6	2	$\bar{A}_1A_1\,(p \neq 2)$
Dih_6	1	G_2
		$A_1\,(p \neq 2,3)$

We may now suppose that $C_F(x) = \langle x\rangle$ and $F \neq \langle x\rangle$. This implies that $F = \langle x, t\rangle \cong Dih_6$. If $C_G(x) = G_2$ then t must centralize G_2, so that $C_G(F) = G_2$. And if $C_G(x) = A_2$ then t induces a graph automorphism on A_2 (see Proposition 2.1), so $C_G(F) = A_1$ and $p \neq 2$, as in Table 6. This completes the proof. ∎

5 Tables of results

This section consists of the tables referred to in Theorem 1.

References

[1] M. Aschbacher and G.M. Seitz, Involutions in Chevalley groups over fields of even order, *Nagoya Math. J.* **63** (1976), 1–91.

[2] A. Borel and J. Tits, Éléments unipotents et sousgroupes paraboliques de groupes réductifs, *Invent. Math.* **12** (1971), 95–104.

[3] N. Bourbaki, *Groupes et Algèbres de Lie* (Chapters 4,5,6), Hermann, Paris, 1968.

[4] R.W. Carter, Conjugacy classes in the Weyl group, *Compositio Math.* **25** (1972), 1–59.

[5] A.M. Cohen and R.L. Griess, On finite simple subgroups of the complex Lie group of type E_8, *Proc. Symp. Pure Math.* **47** (1987), 367–405.

[6] A.M. Cohen and D.B. Wales, Finite subgroups of $E_6(\mathbb{C})$ and $F_4(\mathbb{C})$, *Proc. London Math. Soc.* **74** (1997), 105–150.

[7] A.M. Cohen, M.W. Liebeck, J. Saxl and G.M. Seitz, The local maximal subgroups of exceptional groups of Lie type, finite and algebraic, *Proc. London Math. Soc.* **64** (1992), 21–48.

[8] D. Gorenstein, R. Lyons and R. Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A. Almost simple K-groups.* Mathematical

Table 7: $G = E_8$: finite subgroups F with irreducible centralizer

F	$C_G(F)^0$	elements of F	maximal rank overgp. of $C_G(F)^0$
2	A_1E_7	$2A$	
	D_8	$2B$	
2^2	D_4^2	$2B^3$	
	$A_1^2D_6$	$2A^2, 2B$	
2^3	$A_1^4D_4$	$2A^4, 2B^3$	
	A_1^8	$2B^7$	
2^4	A_1^8	$2A^8, 2B^7$	
4	A_1A_7	$2A, 4A^2$	
	A_3D_5	$2B, 4B^2$	
4×2	$A_1^2A_3^2$	$2A^2, 2B, 4B^4$	
Dih_8	$B_1^2B_4$	$2A^4, 2B, 4B^2$	D_8
	$B_1^2B_2^2$	$2B^5, 4B^2$	D_8
	$B_1B_2B_3$	$2A^2, 2B^3, 4B^2$	D_8
Q_8	$\bar{A}_1D_4$	$2A, 4A^6$	A_1A_7
	B_2^3	$2B, 4B^6$	D_8
	$B_1^3B_3$	$2B, 4B^6$	D_8
$Dih_8 \times 2$	$\bar{A}_1^2B_1^2B_2$	$2A^6, 2B^5, 4B^4$	D_8
$4 \circ Dih_8$	$A_3B_1^3$	$2A^6, 2B, 4B^8$	D_8
$Q_8 \times 2$	$\bar{A}_1^2B_1^4$	$2A^2, 2B, 4B^{12}$	D_8
2_-^{1+4}	B_1^5	$2A^{10}, 2B, 4B^{20}$	D_8
3	A_2E_6	$3B^2$	
	A_8	$3A^2$	
3^2	A_2^4	$3B^8$	
5	A_4^2	$5A$	
6	$A_1A_2A_5$	$2A, 3B^2, 6A^2$	
Dih_6	A_1F_4	$2A^3, 3B^2$	A_2E_6
	A_1C_4	$2B^3, 3B^2$	A_2E_6
	B_4	$2B^3, 3A^2$	A_8
Dih_{12}	$\bar{A}_1A_1C_3$	$2A^4, 2B^3, 3B^2, 6A^2$	$A_1A_2A_5$
G_{12}	$\bar{A}_1A_1A_3$	$2A, 3B^2, 4A^6, 6A^2$	$A_1A_2A_5$
Alt_4	A_2G_2	$2B^3, 3B^8$	D_4^2
	A_2A_2	$2B^3, 3A^8$	D_4^2
Sym_4	A_1G_2	$2A^6, 2B^3, 3B^8, 4B^6$	D_4^2
	A_1A_1	$2B^9, 3A^8, 4B^6$	D_4^2
$Alt_4 \times 2$	$\bar{A}_1A_1A_2$	$2A^4, 2B^3, 3B^8, 6A^8$	$A_1^4D_4$
$Sym_4 \times 2$	$\bar{A}_1A_1A_1$	$2A^{12}, 2B^7, 3B^8, 4B^{12}, 6A^8$	$A_1^4D_4$
$SL_2(3)$	$\bar{A}_1A_2$	$2A, 3B^8, 4A^6, 6A^8$	A_1A_7
$3^2.2$	A_1^4	$2B^9, 3B^8$	A_2^4
	$A_1A_2A_2$	$2A^3, 3B^8, 6A^6$	A_2^4
$3^2.4$	A_1^2	$2B^9, 3B^8, 4B^{18}$	A_2^4
$3^2.2^2$	$A_1^2A_1$	$2A^6, 2B^9, 3B^8, 6A^{12}$	A_2^4
$3^2.Dih_8$	A_1^2	$2A^{12}, 2B^9, 3B^8, 4B^{18}, 6A^{24}$	A_2^4
Dih_{10}	B_2^2	$2B^5, 5A^4$	A_4^2
$Frob_{20}$	B_2	$2B^5, 4B^{10}, 5A^4$	A_4^2
Alt_5	A_1	$2B^{15}, 3B^{20}, 5A^{24}$	A_4^2
Sym_5	A_1	$2A^{10}, 2B^{15}, 3B^{20}, 4B^{30}, 5A^{24}, 6A^{20}$	A_4^2

Table 8: $G = E_7$ (adjoint): finite subgroups F with irreducible centralizer

F	$C_G(F)^0$	elements of F	maximal rank overgp. of $C_G(F)^0$
2	A_1D_6	$2A$	
	A_7	$2B$	
2^2	$A_1^3D_4$	$2A^3$	
	D_4	$2B^3$	A_7
2^3	A_1^7	$2A^7$	
4	$A_1A_3^2$	$2A, 4A^2$	
Dih_8	$\bar{A}_1B_1^2B_2$	$2A^5, 4A^2$	A_1D_6
Q_8	$\bar{A}_1B_1^4$	$2A, 4A^6$	A_1D_6
3	A_2A_5	$3A$	
Dih_6	A_1C_3	$2A^3, 3A^2$	A_2A_5
	A_1A_3	$2B^3, 3A^2$	A_2A_5
Alt_4	A_1A_2	$2A^3, 3A^8$	$A_1^3D_4$
	A_2	$2B^3, 3A^8$	A_7
Sym_4	A_1A_1	$2A^9, 3A^8, 4A^6$	$A_1^3D_4$

Table 9: $G = E_6$: finite subgroups F with irreducible centralizer

F	$C_G(F)^0$	elements of F	maximal rank overgp. of $C_G(F)^0$
2	A_1A_5	$2A$	
3	A_2^3	$3A^2$	
Dih_6	A_1A_1	$2A^3, 3A^2$	A_2^3

Table 10: $G = \mathrm{Aut}\, E_6$: finite subgroups F with irreducible centralizer

F	$F \cap G^0$	$C_G(F)^0$	elements of F	maximal rank overgp. of $C_G(F)^0$
2	2	A_1A_5	$2A$	
	1	F_4	$2B$	
	1	C_4	$2C$	
4	2	$\bar{A}_1A_3$	$2A, 4A^2$	A_1A_5
2^2	2	$\bar{A}_1C_3$	$2A, 2B, 2C$	A_1A_5
3	3	A_2^3	$3A^2$	
Dih_6	Dih_6	A_1A_1	$2A^3, 3A^2$	A_2^3
Dih_{12}	Dih_6	A_1A_1	$2A^3, 2B, 2C^3, 3A^2, 6A^2$	A_2^3

Table 11: $G = F_4$: finite subgroups F with irreducible centralizer

F	$C_G(F)^0$	elements of F	maximal rank overgp. of $C_G(F)^0$
2	B_4	$2A$	
	A_1C_3	$2B$	
2^2	$A_1^2C_2$	$2A, 2B^2$	
	D_4	$2A^3$	
2^3	A_1^4	$2A^3, 2B^4$	
4	A_1A_3	$2A, 4A^2$	
Dih_8	B_1B_2	$2A^3, 2B^2, 4A^2$	D_4
Q_8	B_1^3	$2A, 4A^6$	B_4
3	A_2A_2	$3A^2$	
Dih_6	A_1A_1	$2B^3, 3A^2$	A_2A_2
Alt_4	A_2	$2A^3, 3A^8$	D_4
Sym_4	A_1	$2A^3, 2B^6, 3A^8, 4A^6$	D_4

Table 12: $G = G_2$: finite subgroups F with irreducible centralizer

F	$C_G(F)^0$	elements of F	maximal rank overgp. of $C_G(F)^0$
2	A_1A_1	$2A$	
3	A_2	$3A^2$	
Dih_6	A_1	$2A^3, 3A^2$	A_2

Surveys and Monographs, 40.3. American Mathematical Society, Providence, RI, 1998.

[9] P.B. Kleidman, The maximal subgroups of the finite 8-dimensional orthogonal groups $P\Omega_8^+(q)$ and of their automorphism groups, *J. Algbera* **110** (1987), 173–242.

[10] M.W. Liebeck and G.M. Seitz, Maximal subgroups of exceptional groups of Lie type, finite and algebraic, *Geom. Ded.* **36** (1990), 353–387.

[11] M.W. Liebeck and G.M. Seitz, Reductive subgroups of exceptional algebraic groups, Mem. Amer. Math. Soc. **121** (1996), no. 580.

[12] M.W. Liebeck and G.M. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, Mathematical Surveys and Monographs, Vol.180, American Math. Soc., Providence, RI, 2012.

[13] M.W. Liebeck and D.M. Testerman, Irreducible subgroups of algebraic groups, *Quarterly J. Math* **55** (2004), 47–55.

[14] R.V. Moody and J. Patera, Characters of elements of finite order in Lie groups, *SIAM J. Algebra Discrete Math.* **5** (1984), 359–383.

[15] J.-P. Serre, *Complète réductibilité*, Astérisque **299** (2005), Exp. No. 932, Séminaire Bourbaki. Vol. 2003-2004.

[16] A.R. Thomas, The irreducible subgroups of exceptional algebraic groups, preprint.

M.W. Liebeck, Imperial College, London SW7 2AZ, UK,
m.liebeck@imperial.ac.uk

A.R. Thomas, Heilbronn Institute for Mathematical Research, University of Bristol, Bristol, UK
adam.thomas@bristol.ac.uk