Subgroups of exceptional algebraic groups which are irreducible on an adjoint or minimal module

Martin W. Liebeck
Department of Mathematics
Imperial College
London SW7 2BZ
England

Gary M. Seitz
Department of Mathematics
University of Oregon
Eugene, Oregon 97403
USA

1 Introduction

Let G be a simple algebraic group of exceptional type over an algebraically closed field K of characteristic p. The purpose of this paper is to determine all maximal closed subgroups of G which act irreducibly on either the adjoint G-module or one of the well-known "minimal" modules of dimension $56, 27, 26 - \delta_{p,3}$ or $7 - \delta_{p,2}$ for G of type E_7, E_6, F_4 or G_2 respectively. (The adjective "minimal" here refers to the minimality of the dimension.) This greatly extends [21, Theorem 4] for exceptional groups.

A typical application of this sort of result is the following. We are given a finite subgroup X of G and from given information one determines a subgroup $Y < X$ and a subgroup $\bar{Y} < G$ of positive dimension such that Y and $\bar{Y}$ leave invariant precisely the same subspaces of V. If W is any X-invariant subspace of V, we conclude that W is stabilized by $\langle X, \bar{Y} \rangle$, a group of positive dimension containing X. The maximal subgroups of positive dimension in G are known explicitly, so this yields information on the embedding of X in G, except when X acts irreducibly on V. Hence a list of irreducible subgroups is necessary for understanding the subgroups of G. Our result has already been applied in this way in [24].

In some small characteristics the adjoint module can be reducible for G (see [21, 1.10]): namely, it has a composition factor of codimension 1 for $(G, p) = (E_7, 2)$ or $(E_6, 3)$, and is also reducible for $(G, p) = (F_4, 2)$ or $(G_2, 3)$ (two composition factors of dimension 26, 26 or 7, 7 respectively).

We write V_{adj} for the nontrivial composition factor of the adjoint module for G, excluding $(G, p) = (F_4, 2)$ or $(G_2, 3)$. And write V_{min} for one of the irreducible modules for E_7, E_6, F_4, G_2 of dimension $56, 27, 26 - \delta_{p,3}, 7 - \delta_{p,2}$ and high weight $\lambda_7, \lambda_1, \lambda_4, \lambda_1$ respectively; for $(G,p) = (F_4, 2)$ or $(G_2, 3)$, we include also $V_G(\lambda_1)$ or $V_G(\lambda_2)$ as possibilities for V_{min}, of dimension 26 or 7. The acting group on V_{adj} is of adjoint type, and on V_{min} is of simply connected type, and we take G to be the acting group.

If H is a finite subgroup of G, we say that H is *Lie primitive* if H normalizes no proper nontrivial connected subgroup of G, and we say that *H has the same type as G* if $F^*(H) = G'_\sigma$ for some Frobenius morphism σ of G.

Theorem 1 *Let H be a proper closed subgroup of the exceptional algebraic group G. If $\dim H > 0$, assume H is maximal among proper closed subgroups of G; and if H is finite, assume that it is a Lie primitive subgroup, and is not of the same type as G.*

(i) *If H has positive dimension, then H is reducible on V_{adj}.*

(ii) *If H is finite and is irreducible on V_{adj}, then H is as in Table* 1.1 *below.*

(iii) *If H has positive dimension and is irreducible on V_{min}, then H is as in Table* 1.2 *below.*

(iv) *If H is finite and is irreducible on V_{min}, then H is as in Table* 1.3 *below.*

Each entry for H in Table 1.2 *corresponds to exactly one conjugacy class of subgroups in* Aut*(G). In Tables* 1.1, 1.3, *for each entry there is at least one subgroup H which is irreducible on the relevant module (but we make no uniqueness assertion).*

We remark that the proof shows that the same conclusion holds if we allow $G = \mathrm{Aut}(E_6) = E_6.2$ when $V = V_{adj}$.

The proof of Theorem 1 is given in Section 2 for positive dimensional subgroups H, and in Sections 3 and 4 for finite subgroups.

A consequence of the proof of Theorem 1 is the following result concerning irreducible subgroups of finite exceptional groups.

Corollary 2 *Let σ be a Frobenius morphism of the exceptional algebraic group G, so that the fixed point group G_σ is a finite exceptional group of Lie type over $\mathbb{F}_r$ ($r = p^a$), and let M be a maximal subgroup of G_σ. Suppose M is irreducible on $V \in \{V_{adj}, V_{min}\}$. Then one of the following holds:*

(i) *M is of the same type as G (possibly twisted) over a subfield of $\mathbb{F}_r$;*

(ii) *$V = V_{min}$ and $M = H_\sigma$ with H as in Table* 1.2*;*

(iii) *$F^*(M)$ is as in Table* 1.1 *(with $V = V_{adj}$) or in Table* 1.3 *(with $V = V_{min}$).*

We justify the corollary in Section 5 at the end of the paper.

Notation For X a simple algebraic group over K and λ a dominant weight, we denote by $V_X(\lambda)$ (or just λ) the rational irreducible KX-module of high weight λ, and by $W_X(\lambda)$ the corresponding Weyl module. Also, if $V_1, \ldots, V_k$ are X-modules then $V_1/\ldots/V_k$ denotes an X-module having the same composition factors as $V_1 \oplus \ldots \oplus V_k$.

Table 1.1: Lie primitive finite subgroups H irreducible on an adjoint module

G	V	$F^*(H)$ (H finite irreducible Lie prim.)
E_8	V_{248}	$2^{5+10}.L_5(2)\,(p \neq 2)^*,\ 5^3.L_3(5)\,(p \neq 5)^*,$ $L_4(5)\,(p=2),\ Th\,(p=3),\ {}^2F_4(2)'\,(p=3)^{**}$
E_7	$V_{133-\delta_{p,2}}$	$U_3(8)\,(p \neq 2),\ M_{22}\,(p=5),\ Ru\,(p=5),\ HS\,(p=5)$
E_6	$V_{78-\delta_{p,3}}$	$3^{3+3}.L_3(3)\,(p \neq 3)^*,\ {}^2F_4(2)'\,(p \neq 2),\ G_2(3)\,(p=2),$ $\Omega_7(3)\,(p=2),\ J_3\,(p=2),\ Fi_{22}\,(p=2),\ M_{12}\,(p=5)$
F_4	$V_{52}\,(p \neq 2)$	$3^3.L_3(3)\,(p \neq 3)^*,\ {}^3D_4(2)$
G_2	$V_{14}\,(p \neq 3)$	$2^3.L_3(2)\,(p \neq 2)^*,\ U_3(3),\ L_2(13)\,(p \neq 3,13),$ $J_2\,(p=2),\ J_1\,(p=11)$

* For these local subgroups, H rather than $F^*(H)$ is listed; moreover, only the maximal versions are listed – some subgroups may also be irreducible.

** Here $H = {}^2F_4(2) = {}^2F_4(2)'.2$ and $V_{248} \downarrow {}^2F_4(2)' = V_{124} \oplus V'_{124}$.

Table 1.2: maximal subgroups H of positive dimension irreducible on a minimal module

G	V	H (max. irred. of positive dim.)	$V \downarrow H^0$	Comment
E_7	V_{56}	$A_7.2$	$\lambda_2 \oplus \lambda_6$	
		$(A_1)^7.L_3(2)$	see 2.1	
		$(T_7).W(E_7)$	see 2.1	
		$A_2.2\,(p > 5)$	$60 \oplus 06$	
		$(2^2 \times D_4).S_3\,(p \neq 2)$	$(\lambda_2)^2$	
E_6	V_{27}	$(A_2)^3.S_3$	see 2.1	
		$(T_6).W(E_6)$	see 2.1	
		$C_4\,(p \neq 2)$	λ_2	
		$G_2\,(p \neq 2,7)$	20	
		$A_2.2\,(p \neq 2,5)$	22	
F_4	$V_{26-\delta_{p,3}}$	$D_4.S_3\,(p=2)$	λ_2	D_4 short if $V = V(\lambda_4)$, long if $V = V(\lambda_1)$
		$C_4\,(p=2)$	λ_2	$V = V(\lambda_4)$
		$B_4\,(p=2)$	λ_2	$V = V(\lambda_1)$
		$G_2\,(p=7)$	20	
G_2	$V_{7-\delta_{p,2}}$	$A_2.2\,(p=2,3)$	$10 \oplus 01\,(p=2)$	
			$11\,(p=3)$	A_2 short if $V = V(\lambda_1)$, long if $V = V(\lambda_2)$
		$A_1\,(p \geq 7)$	6	

Table 1.3: Lie primitive finite subgroups H irreducible on a minimal module

G	V	$F^*(H)$ (H finite irreducible Lie prim.)
E_7	V_{56}	$U_3(8)\ (p \neq 2)$
E_6	V_{27}	$3^{3+3}.L_3(3)\ (p \neq 3)^*,\ L_2(8)\ (p \neq 2)^{**},\ {}^2F_4(2)'\ (p \neq 2),$ $L_3(3)\ (p \neq 2, 13)^{***},\ U_3(3)\ (p \neq 2, 7)^{***},$ $G_2(3)\ (p = 2),\ \Omega_7(3)\ (p = 2),$ $Fi_{22}\ (p = 2),\ J_1\ (p = 11)$
F_4	$V_{26-\delta_{p,3}}$	$3^3.L_3(3)\ (p \neq 3)^*,\ L_2(25)\ (p \neq 5),\ L_2(27)\ (p \neq 3, 7),$ $L_3(3)\ (p \neq 3),\ {}^3D_4(2)\ (p \neq 2),\ Alt_9\ (p = 2),\ Alt_{10}\ (p = 2),$ $L_4(3)\ (p = 2),\ U_3(3)\ (p = 7)$
G_2	$V_{7-\delta_{p,2}}$	$2^3.L_3(2)\ (p \neq 2)^*,\ L_2(7)\ (p \neq 2),\ L_2(8)\ (p \neq 2),$ $L_2(13)\ (p \neq 13),\ U_3(3)\ (p \neq 3),\ J_1\ (p = 11),\ J_2\ (p = 2)$

* As in Table 1.1, for these local subgroups, H rather than $F^*(H)$ is listed, and only the maximal versions are listed – some subgroups may also be irreducible.

** Here $H = L_2(8).3$ and $V \downarrow L_2(8) = V_9 \oplus V_9' \oplus V_9''$.

*** We have left in the possibility that $p = 3$ for $F^*(H) = L_3(3)$ or $U_3(3)$. There exist irreducible such subgroups lying in a connected subgroup A_2, but we have not determined whether there exist Lie primitive examples.

2 Proof of Theorem 1, I : positive-dimensional subgroups

Suppose that G is a simple algebraic group of exceptional type, and H is a maximal closed subgroup of positive dimension. The possibilities for H are given by [23, Corollary 2].

Observe that part (i) of the Theorem is trivial, as H stabilizes the subspace $L(H)$ of the adjoint module $L(G)$.

Now consider part (iii). Let $V = V_{min}$, one of the minimal modules for $G = E_7, E_6, F_4, G_2$, of dimension $56, 27, 26-\delta_{p,3}, 7-\delta_{p,2}$, respectively. Write also $V = V_d$, where $d = \dim V$. Suppose H acts irreducibly on V. Then H is not a parabolic subgroup.

According to [23, Corollary 2], the maximal subgroup H satisfies one of the following conditions:

(a) H is a subgroup of maximal rank; the possibilities are as follows:

G	H^0	H/H^0
G_2	$A_1\tilde{A}_1,\ A_2$	1, 2
$F_4\,(p \neq 2)$	$B_4,\ D_4,\ A_1C_3,\ A_2\tilde{A}_2$	1, S_3, 1, 2
$F_4\,(p = 2)$	$B_4, C_4, D_4, \tilde{D}_4, A_2\tilde{A}_2$	$1, 1, S_3, S_3, 2$
E_6	$A_1A_5,\ A_2^3,\ D_4T_2,\ T_6$	1, S_3, S_3, $W(E_6)$
E_7	$A_1D_6,\ A_7,\ A_2A_5,\ A_1^3D_4,$	1, 2, 2, S_3,
	$A_1^7,\ E_6T_1,\ T_7$	$L_3(2)$, 2, $W(E_7)$

(where $\tilde{A}_r, \tilde{D}_4$ denote subgroups generated by short root groups).

(b) H is as in Table 1 of [23, Theorem 1]; the possibilities are:

G	H^0
G_2	$A_1\,(p \geq 7)$
F_4	$A_1\,(p \geq 13),\ G_2\,(p = 7),\ A_1G_2\,(p \neq 2)$
E_6	$A_2\,(p \neq 2, 3),\ G_2\,(p \neq 7),\ A_2G_2,\ C_4\,(p \neq 2),\ F_4$
E_7	$A_1\,(2 \text{ classes}, p \geq 17, 19 \text{ resp.}),\ A_2\,(p \geq 5),\ A_1A_1\,(p \neq 2, 3),$
	$A_1G_2\,(p \neq 2),\ A_1F_4,\ G_2C_3$

(c) $H = (2^2 \times D_4).S_3 < E_7$.

Lemma 2.1 *Theorem* 1(*iii*) *holds if* H *is as in* (a) *above.*

Proof Consider first $G = E_7$. The composition factors of the restriction of V to various maximal rank subgroups are given in [20, 2.3], from which it follows that $H^0 \neq A_1D_6, A_2A_5, A_1^3D_4$ or E_6T_1.

When $H = A_7.2$, we have $V \downarrow A_7 = \lambda_2 \oplus \lambda_6$, and the two summands are interchanged by the outer involution, hence $V \downarrow H$ is irreducible, as in Table 1.2.

Now consider $H^0 = A_1^7$. This is contained in a subsystem A_1D_6, which has composition factors $1 \otimes \lambda_1/0 \otimes \lambda_5$ on V. Hence there is a subproduct A_1^3 which has a composition factor $1 \otimes 1 \otimes 1$ of dimension 8 in $1 \otimes \lambda_1$. The group $H/H^0 \cong L_3(2)$ permutes 7 such summands transitively, so $V \downarrow H$ is irreducible in this case too.

Now suppose $H^0 = T_7$, a maximal torus. The weights spaces of T_7 on V all have dimension 1 and are permuted transitively by $W(E_7)$. Hence $V \downarrow H$ is irreducible in this case as well.

Next let $G = E_6$. Here [20, 2.3] shows that $H^0 \neq A_1A_5$ or D_4T_2. When $H^0 = T_6$, $V \downarrow H$ is irreducible as above. And when $H^0 = A_2^3$, [20, 2.3] shows that $V \downarrow A_2^3$ is the sum of three 9-dimensional irreducible summands, permuted transitively by $H/H^0 \cong S_3$, giving irreducibility here as well.

Now let $G = F_4$. When $p \neq 2$ it follows from [20, 2.3] that $V \downarrow H$ is reducible in all cases. So assume $p = 2$. Then $L(G)$ has two G-composition factors λ_1/λ_4, both of dimension 26. One of these is the ideal generated by short root elements, which affords $V_G(\lambda_4)$, and on which the subsystem group C_4 (generated by short root groups) has highest weight λ_2. As $V_{C_4}(\lambda_2)$ has dimension 26, $V_G(\lambda_4) \downarrow C_4$ is irreducible. Likewise, so are $V_G(\lambda_1) \downarrow B_4$ and $V_G(\lambda_i) \downarrow D_4$ ($i = 1, 4$, D_4 short if $i = 4$, long if $i = 1$).

Finally, when $G = G_2$ and $p = 2$, the subsystem subgroup A_2 acts on $V = V_6$ as the sum of two 3-dimensional summands interchanged by an outer involution; and when $p = 3$, subsystem subgroups A_2 (long or short) act irreducibility on the 7-dimensional G-modules of high weight λ_2 or λ_1, respectively. ■

Lemma 2.2 *Theorem* 1(*iii*) *holds if* H *is as in* (b) *above.*

Proof Let H be as in (b). The action of H^0 on V is given in [23, Table 10.2], from which we see that H is as in Table 1.2. The only case where $V \downarrow H^0$ is reducible is $H = A_2.2 < E_7$, where $V \downarrow H^0 = 60 \oplus 06$; here the outer involution in H interchanges the two irreducible summands, so $V \downarrow H$ is irreducible. ■

Lemma 2.3 *Theorem* 1(*iii*) *holds if* H *is as in* (c) *above.*

Proof This subgroup was constructed in [6, 2.15], and lifts to a subgroup $(Q_8 * D_4).S_3$ in simply connected E_7. The connected component $H^0 = D_4$ lies in a subsystem A_7 of G, and hence $V \downarrow D_4 = (\lambda_2)^2$. It follows that $V \downarrow Q_8 * D_4 = V_2 \otimes \lambda_2$, where V_2 is an irreducible 2-dimensional Q_8-module. In particular $V \downarrow H$ is irreducible, as in Table 1.2. ■

This completes the proof of Theorem 1(iii).

3 Proof of Theorem 1, II : finite subgroups not in Lie(p)

Let G be a simple algebraic group of exceptional type, and $\hat{H}$ a finite Lie primitive subgroup of G. Suppose $\hat{H}$ acts irreducibly on one of the G-modules V_{adj}, V_{min} defined in the Introduction. Write H for the image of $\hat{H}$ in the adjoint group of type G.

For convenience we list the dimensions of the modules V_{adj}, V_{min}:

G	$\dim V_{adj}$	$\dim V_{min}$
E_8	248	–
E_7	$133 - \delta_{p,2}$	56
E_6	$78 - \delta_{p,3}$	27
F_4	$52\,(p \neq 2)$	$26 - \delta_{p,3}$
G_2	$14\,(p \neq 3)$	$7 - \delta_{p,2}$

Lemma 3.1 *Theorem* $1(ii, iv)$ *holds if* H *is not almost simple.*

Proof Suppose H is not almost simple. By assumption, H is Lie primitive. Hence, according to a result of Borovik (see [2]), one of the following holds:

(i) H is contained in one of the following finite local subgroups:

$G = G_2$: $H = 2^3.L_3(2)\,(p \neq 2)$
$G = F_4$: $H = 3^3.L_3(3)\,(p \neq 3)$
$G = E_6$: $H = 3^{3+3}.L_3(3)\,(p \neq 3)$
$G = E_8$: $H = 5^3.L_3(5)\,(p \neq 5)$ or $2^{5+10}.L_5(2)\,(p \neq 2)$

(ii) $H = (Alt_5 \times Alt_6).2^2 < G = E_8\,(p \neq 2, 3, 5)$.

Each of the subgroups listed in (i) acts irreducibly on $L(G) = V_{adj}$, by [3], while the subgroup in (ii) clearly does not.

Finally, the subgroups in (i) (with $G \neq E_8$) also act irreducibly on V_{min}, since in each case $\dim V_{min}$ is the minimal dimension of a faithful module for $\hat{H}$ in characteristic p: this is clear in all cases except $G = E_6$, where $\hat{H} = 3^{1+3+3}.L_3(3)$. Here $\hat{H}$ has a subgroup $3^3.L_3(3)$ lying in a subgroup F_4, hence has a subgroup $3 \times 3^3.L_3(3)$ leaving invariant a 1-space of V; therefore $V \downarrow \hat{H}$ is an induced module and is irreducible. ■

In view of the previous lemma, assume from now on that H is almost simple, and let $H_0 = F^*(H)$, a non-abelian finite simple group.

We next record some well known information concerning the action of G on the modules V_{adj}, V_{min}. Recall that by our definition, V_{adj} is not defined when $(G,p) = (F_4, 2)$ or $(G_2, 3)$.

Lemma 3.2 (i) *G acts as an adjoint group on V_{adj}, and preserves a non-degenerate bilinear form.*

(ii) *G acts as a simply connected group on V_{min}, preserving bilinear forms as follows:*

G	V	*type of form*
E_7	V_{56}	*symplectic*
E_6	V_{27}	*not self-dual*
F_4	$V_{26-\delta_{p,3}}$	*orthogonal*
G_2	V_7, $p \neq 2$	*orthogonal*
	V_6, $p = 2$	*symplectic*

Lemma 3.3 *Let $K = \mathbb{C}$, $V = V_{adj}$ or V_{min}, let $n = \dim V$ and let χ_n be the character of G on V. Let $t \in G$ be an element satisfying one of the following conditions:*

(i) *$t^2 \in Z(G)$;*

(ii) *t has order 3; moreover, if G is adjoint of type E_6, then t lifts to an element of order 3 in the simply connected group;*

(iii) *t has order 5 and is a rational element (i.e. G-conjugate to all its nontrivial powers).*

Then the possibilities for $C_G(t)^0$ and the values of $\chi_n(t)$ are recorded in in Table 1 below.

Table 1

n	G	order of t	$C_G(t)^0$	$\chi_n(t)$
248	E_8	2	A_1E_7	24
			D_8	-8
		3	A_8	-4
			A_2E_6	5
			D_7T_1	14
			E_7T_1	77
		5	A_4A_4	-2
			D_6T_2	23
$133, 56$	E_7 (adj., s.c.)	$2, 2$	A_1D_6	$5, \pm 8$
		$2, 4$	A_7	$-7, 0$
		$2, 4$	E_6T_1	$25, 0$
		$3, 3$	A_6T_1	$7, -7$
			E_6T_1	$52, -25$
			A_5A_2	$-2, 2$
			$A_1D_5T_1$	$7, 2$
			D_6T_1	$34, 20$
		$5, 5$	$A_1D_4T_2$	$8, 6$
$78, 27$	E_6 (adj., s.c.)	$2, 2$	A_5A_1	$-2, 3$
			D_5T_1	$14, -5$
		$3, 3$	A_5T_1	$15, 9$
			$A_2A_2A_2$	$-3, 0$
			D_4T_2	$6, 0$
		5	A_3T_3	$3, 2$
$52, 26$	F_4	2	C_3A_1	$-4, 2$
			B_4	$20, -6$
		3	C_3T_1	$7, 8$
			B_3T_1	$7, -1$
			A_2A_2	$-2, -1$
		5	B_2T_2	$2, 1$
$14, 7$	G_2	2	A_1A_1	$-2, -1$
		3	A_2	$5, -2$
			A_1T_1	$-1, 1$

Proof As in [22, 1.2], this can be read off from [5, 4]. ∎

Note that if H is a finite subgroup of G, then the Brauer character of H

on V is the restriction of χ_n to H (where as above $n = \dim V$).

Lemma 3.4 *Suppose $H_0 \notin \mathrm{Lie}(p)$. Then the possibilities for G, V, H_0 are among those listed in Table 2 below.*

Table 2

G	V	H_0 ($\notin \mathrm{Lie}(p)$)	p	Remarks
E_8	V_{248}	$PSp_4(5)$, $L_4(5)$, Th	$2, 2, 3$	
		${}^2F_4(2)'$	3	$V \downarrow H_0 = V_{124} \oplus V'_{124}$,
				$H = H_0.2$
E_7	V_{133}, $p \neq 2$	$U_3(8)$		
		J_2, Alt_9, Alt_{10}, $Sp_6(2)$,	$3, 5, 5, 5,$	
		M_{22}, HS, Ru	$5, 5, 5$	
	V_{56}	$U_3(8)$		
		J_2	$p \neq 2, 3$	
		$L_2(27)$, $L_2(29)$		$V \downarrow H_0 = V_{28} \oplus V'_{28}$,
				$H = H_0.2$
E_6	$V_{78-\delta_{p,3}}$	Alt_9, $\Omega_7(3)$, $G_2(3)$, J_3,	$2, 2, 2, 2,$	
		Fi_{22}, M_{12}	$2, 5$	
		${}^2F_4(2)'$		
		${}^3D_4(2)$	$p \neq 3$	$V \downarrow H_0 = V_{26} \oplus V'_{26} \oplus V''_{26}$,
				$H = H_0.3$
	V_{27}	$G_2(3)$, $\Omega_7(3)$, Fi_{22}, J_1	$2, 2, 2, 11$	
		$L_2(27), U_3(3)$	$p \neq 2, 7$	
		$L_3(3)$	$p \neq 2, 13$	
		${}^2F_4(2)'$		
		$L_2(8)$		$V \downarrow H_0 = V_9 \oplus V'_9 \oplus V''_9$,
				$H = H_0.3$
F_4	V_{52}, $p \neq 2$	${}^3D_4(2)$		
	V_{26}, $p \neq 3$	Alt_9, Alt_{10}, $L_4(3)$, $U_3(3)$	$2, 2, 2, 7$	
		$L_2(25)$, $L_2(27)$, $L_3(3)$, ${}^3D_4(2)$		
	V_{25}, $p = 3$	${}^3D_4(2)$, $L_2(25)$		
G_2	V_{14}, $p \neq 3$	J_1, J_2	$11, 2$	
		$U_3(3)$, $L_2(13)$		
	V_7, $p \neq 2$	J_1	11	
		$U_3(3)$, $L_2(7)$, $L_2(8)$, $L_2(13)$		
	V_6, $p = 2$	$L_2(13)$, $U_3(3)$, J_2		

Proof The absolutely irreducible characteristic p representations of quasisimple groups not in $\mathrm{Lie}(p)$ of dimension up to 250 are listed in [9], together with their Schur indicators. Those which have an irreducible such representation of dimension $\dim V_{min}$ or $\dim V_{adj}$ fixing an appropriate bilinear form, and also lying in the appropriate exceptional algebraic group (see [22, Tables 10.1-10.4]), are listed in Table 2. In addition, it could be that $V \downarrow H_0$ is not irreducible, but is a direct sum of $t > 1$ irreducibles of equal dimension, where t divides $|\mathrm{Out}(H_0)|$. Such possibilities are also included in Table 2, except for those where $t = 2$, $p \neq 2$ and $H \backslash H_0$ contains an involution, say u, in which case $\chi_n(u) = 0$, which is impossible by Lemma 3.3. ■

Lemma 3.5 *Theorem* 1 *holds when* $H_0 \notin \mathrm{Lie}(p)$.

Proof The proof is in two parts:

(i) we establish the existence of the irreducible examples in Tables 1.1 and 1.3, and

(ii) we show that the remaining entries in Table 2 above do not give irreducible subgroups.

Part (i) First consider $G = E_8$. From the construction given in [6, 5.1], a subgroup $L_4(5)$ of $E_8\,(p = 2)$ contains the local subgroup $5^3.L_3(5)$ seen in the proof of Lemma 3.1; this local subgroup is irreducible on V_{adj} by [3], hence so is $L_4(5)$. Also, a subgroup Th of $E_8\,(p = 3)$ must be irreducible on V_{adj}, since 248 is the smallest dimension of a nontrivial module for Th (see [9]). Finally, an irreducible subgroup ${}^2F_4(2)$ of $E_8\,(p = 3)$ is constructed by Ryba in [27].

Next consider $G = E_7$. The irreducibility of the subgroups M_{22}, HS, Ru on V_{133} $(p = 5)$ is given by [13, 14]. And as for $U_3(8)$, the smallest degree of a nontrivial representation in any characteristic is 56 (see [10]), so it is irreducible on V_{56}. Now consider $H = U_3(8)$, and assume that $V_{133} \downarrow H$ is reducible. From [10] we see that $\chi_{133} \downarrow H = 2\chi_{56} + 21\chi_1$ or $\chi_{57} + \chi'_{57} + 19\chi_1$. If $p \neq 3$, then letting $u \in H$ be a $3C$-element, we see that $\chi_{133}(u) = 25$, contrary to Lemma 3.3. Now let $p = 3$. If $t \in H$ is of order 4, then $\chi_{133}(t) = 21$ and $\chi_{133}(t^2) = 5$. Hence t^2 acts on V_{133} as $\mathrm{diag}(-1^{(64)}, 1^{(69)})$, and t acts as $\mathrm{diag}(i^{(32)}, -i^{(32)}, -1^{(24)}, 1^{(45)})$. This means that $C_G(t)$ has dimension 45; however there is no such semisimple element centralizer in E_7, so this is a contradiction.

Now let $G = E_6$. First consider $V = V_{78-\delta_{p,3}}$. The smallest nontrivial representations in characteristic 2 of the simple groups $\Omega_7(3), Fi_{22}$ and J_3 have dimension 78 (see [9]), so these groups are irreducible on V_{78}. A glance at the 2-modular character table of $G_2(3)$ in [10] shows that $G_2(3)$ has no reducible Brauer character of degree 78 which is compatible with the values of χ_{78} in Lemma 3.3 which preserves an orthogonal form; hence $G_2(3)$ is irreducible on V_{78} $(p = 2)$. Similarly, we see that ${}^2F_4(2)'$ is irreducible on V for $p \neq 2$. Finally, M_{12} is irreducible on V_{78} $(p = 5)$ by [15].

Next consider $V = V_{27}$ (still with $G = E_6$). For $p = 2$, we have $3.G_2(3) < 3.\Omega_7(3) < 3.Fi_{22} < E_6$ (simply connected), and the smallest nontrivial 2-modular representations of these groups have dimension 27 (see [10]), so these are irreducible on V. The smallest nontrivial representations of ${}^2F_4(2)'$ in characteristic $p \neq 2$ have dimensions 26, 27 (see [10]). Since ${}^2F_4(2)'$ does not lie in a 1-space stabilizer in G (the latter are either F_4 or contained in a parabolic), the subgroup ${}^2F_4(2)'$ is irreducible on V. Moreover, ${}^2F_4(2)'$ has a subgroup $L_3(3)$ (see [7]), and for $p \neq 2, 3, 13$ (i.e. for p not dividing $|L_3(3)|$), the irreducible degree 27 characters of ${}^2F_4(2)'$ remain irreducible on restriction to $L_3(3)$.

It remains to justify the examples $U_3(3), J_1$ and $L_2(8).3$ in Table 1.3. For these, observe first that for $p \neq 2, 7$, E_6 possesses a maximal connected subgroup G_2 which acts irreducibly on V_{27} with high weight 20 (see [29]). This module has codimension 1 in the symmetric square $S^2(V_7)$, where $V_7 = V_{G_2}(10)$, a 7-dimensional irreducible G_2-module. It is well known (see [1] for example) that G_2 possesses subgroups $L_2(8)\,(p \neq 2), U_3(3)\,(p \neq 2, 3, 7)$ and $J_1\,(p = 11)$ acting irreducibly on V_7, and from the tables in [7, 10] for these groups, we see that $U_3(3), J_1$ also act irreducibly on V_{27}, while $V_{27} \downarrow L_2(8) = V_9 + V_9' + V_9''$, a direct sum of 3 irreducible 9-dimensional submodules permuted transitively by an outer automorphism of order 3. The existence of $L_2(8).3 < G$ was originally established in an unpublished article by M. Aschbacher (29.18 of "The maximal subgroups of E_6"), and has recently been constructed in a different way in [27]. This subgroup acts irreducibly on V_{27}, as required.

Next we consider $G = F_4$. The subgroup ${}^3D_4(2)\,(p \neq 2)$ is irreducible on V_{52}, as can be seen using the character tables [7, 10] together with Lemma 3.3. So consider now $V = V_{26-\delta_{p,3}}$.

The subgroups $L_4(3)$, ${}^3D_4(2)$ have smallest nontrivial character degree equal to $26 - \delta_{p,3}$ (see [10]), so these are irreducible on V.

Next we justify the Alt_9, Alt_{10} examples with $p = 2$. Note that when $p = 2$, G has a maximal rank subgroup C_4 acting irreducibly on V_{26} with high weight 0100; this module is the nontrivial irreducible constituent of $\wedge^2 V_8$, where V_8 is the natural module for C_4. If we embed Alt_9 and Alt_{10} irreducibly in C_4 via the usual permutation module, an easy calculation using the Alt_9 table in [10] shows that these subgroups are irreducible on

V_{26}.

By [29], for $p = 7$, G has a maximal connected subgroup G_2 acting irreducibly on V_{26} with high weight 20. As above, G_2 ($p = 7$) has a subgroup $U_3(3)$ acting irreducibly on $V_7 = V_{G_2}(10)$, and hence we see that this $U_3(3)$ is irreducible on V_{26}.

Next, use of the $L_3(3)$ character tables in [7, 10], together with Lemma 3.3, show that a subgroup $L_3(3)$ of G ($p \neq 3$) acts irreducibly on V_{26}.

We have already seen that E_6 ($p \neq 2$) has a subgroup ${}^2F_4(2)'$ which acts irreducibly on V_{27}. This contains a subgroup $L_2(25)$ (see [7]), and a glance at character tables shows that the Brauer character χ_{27} restricts to $L_2(25)$ as $1+\chi_{26}$ if $p \neq 3$, and as $1+1+\chi_{25}$ if $p = 3$, where χ_{25}, χ_{26} are irreducible. Hence $L_2(25)$ lies in a 1-space stabilizer in E_6, which must be F_4 (the others are in parabolics, which do not act irreducibly on a 25- or 26-dimensional section in V_{27}).

It remains to justify the example $L_2(27)$ in Table 1.3. In [5, 6.5], a subgroup $L_2(27)$ of $F_4(\mathbb{C})$ is constructed which acts irreducibly on V_{26}. Now [28, Corollaire, p.546] shows that there is a subgroup $L_2(27)$ of $F_4(K)$ (K algebraically closed of characteristic p), in which semisimple elements have the same eigenvalues on V_{26} as for the embedding in $F_4(\mathbb{C})$. Now a glance at the tables for $L_2(27)$ in [10] shows that provided $p \neq 3, 7$, this $L_2(27) < F_4(K)$ acts irreducibly on V_{26}. When $p = 7$ this is not the case, as the 7-modular irreducible of degree 26 for $L_2(27)$ takes the value -2 on an involution, so cannot be the Brauer character of $V_{26} \downarrow L_2(27)$ by Lemma 3.3.

Finally, let $G = G_2$. The irreducible examples on $V_{7-\delta_{p,2}}$ are immediate from [1, Theorem 9], and those on V_{14} ($p \neq 3$) using the character tables in [7, 10] together with Lemma 3.3.

Part (ii) We now show that the remaining entries in Table 2 above do not give irreducible subgroups. This amounts to eliminating the possibilities in the following table.

G	V	H_0
E_8	V_{248}	$PSp_4(5)$
E_7	V_{133}	J_2, Alt_9, Alt_{10}, $Sp_6(2)$
	V_{56}	J_2, $L_2(27)$, $L_2(29)$
E_6	V_{78}	Alt_9, ${}^3D_4(2)$
	V_{27}	$L_2(27)$

Consider first $G = E_8$. Here $H_0 = PSp_4(5)$ and $p = 2$. Choose a transvection $u \in H_0$ (in class $5A$ of $PSp_4(5)$ in the notation of [7]). The

2-modular table for H_0 in [10] shows that $\chi_{248}(u)$ is irrational. However, $C_{H_0}(u)'$ contains u, hence $C_G(u)$ has no nontrivial central torus, and so $C_G(u) = A_4A_4$. This means that $\chi_{248}(u) = -2$ by Lemma 3.3, a contradiction.

Next consider $G = E_7$. For the V_{133} possibilities, a glance at the tables in [10] shows that H_0 possesses an involution t with $\chi_{133}(t) = 1, -3, -3$ or -27, which is impossible by Lemma 3.3. For V_{56}, observe from [10] that none of the groups $SL_2(27), GL_2(27), SL_2(29)$ or $GL_2(29)$ has an irreducible representation of degree 56, so $H_0 \neq L_2(27), L_2(29)$.

It remains to rule out $H_0 = J_2$ on V_{56} with $p \neq 2, 3$. We use the character tables for J_2 in [7, 10]. The acting group is $\hat{H} = 2.J_2 < \hat{E}_7$; suppose this is irreducible on V_{56}. A $2A$ involution in J_2 lifts to an involution $t_1 \in \hat{H}$, hence $C_G(t_1) = A_1D_6$; and a $2B$ involution lifts to an element $t_2 \in \hat{H}$ of order 4, hence $C_G(t_2)^0 = A_7$ or T_1E_6. Now consider the action on $L(G) = V_{133}$, and let $\chi = \chi_{133}$. By Lemma 3.3 we have $\chi(t_1) = 5$ and $\chi(t_2) = -7$ or 25. Also, if $u \in \hat{H}$ is an element of order 3 then $\chi(u) = 7, 52, -2$ or 34.

Suppose $p = 5$. We have

$$\chi \downarrow H_0 = m\chi_1 + a\chi_{14} + b\chi_{21} + c\chi_{41} + d\chi_{70} + e\chi_{85} + f\chi_{90}$$

where $m, a, \ldots, f$ are non-negative integers. Evaluating at the elements $1, t_1, t_2$ we have

$$\begin{array}{lrcl} (1) & m + 14a + 21b + 41c + 70d + 85e + 90f & = & 133 \\ (2) & m - 2a + 5b + 9c - 10d + 5e + 10f & = & 5 \\ (3) & m + 2a - 3b + c - 2d + 5e + 6f & = & -7 \text{ or } 25 \end{array}$$

Subtracting (2) from (1) and also (3) from (2) gives the equations $a + b + 2c + 5d + 5e + 5f = 8$ and $-a + 2b + 2c - 2d + f = 3$ or -5, and adding these gives

$$(4) \qquad 3b + 4c + 3d + 5e + 6f = 11 \text{ or } 3.$$

If the right hand side of (4) is 3, then either $b = 1$ and $c, d, e, f = 0$ or $d = 1$ and $b, d, e, f = 0$. In the first case $a = 7, m = 14$; taking u to be in the class $3A$ of H, this implies that $\chi(u) = 52$. This means that $\chi_{56}(u) = -25$ by Lemma 3.3. However, the 5-modular table for J_2 in [10] shows that $\chi_{56}(u) = 2$, so this is a contradiction. In the second case ($d = 1$), we have $a = 3, m = 21$, and so $\chi(u) = 43$, which is impossible.

Now suppose the right hand side of (4) is 11. If $f = 1$ then by (1), $d = e = 0$, so $3b + 4c = 5$, which is impossible. Hence $f = 0$. If $e = 1$ then $d = 0$ and $3b + 4c = 6$, hence $b = 2, c = 0$; then $a = 1$, but this forces

m to be negative by (1). Hence also $e = 0$, and $3b + 4c + 3d = 11$. Then $c = 2, b + d = 1$, from which we get $b = 1, d = 0, a = 3$; this again gives $m < 0$ by (1). This completes the $p = 5$ case. The $p = 7$ and $p = 0$ cases are similar.

Now let $G = E_6$. If $H_0 = Alt_9\,(p = 2)$, and $h \in H_0$ is an element of order 5, then by [10], $\chi_{78}(h) = -2$. On the other hand, h is a rational element, so by Lemma 3.3, $\chi_{78}(h) = 3$, a contradiction. Also , H_0 cannot be ${}^3D_4(2)$, since in this case $V_{78} \downarrow H_0 = V_{26} \oplus V'_{26} \oplus V''_{26}$ with $H = H_0.3$, whereas by [7, 10], ${}^3D_4(2)$ has only one irreducible representation of degree 26. Finally, if $H_0 = L_2(27)$, irreducible on V_{27}, then by [10], H_0 has an involution t such that $\chi_{27}(t) = -1$, contrary to Lemma 3.3.

This completes the proof. ■

4 Proof of Theorem 1, III : finite subgroups in Lie(p)

Continue to let G be a simple adjoint algebraic group over K of exceptional type, and H an almost simple finite subgroup, which normalizes no proper nontrivial connected subgroup of G, and is irreducible on $V = V_{adj}$ or V_{min}.

In this section we complete the proof of Theorem 1 by handling the case where $H_0 = F^*(H)$ is a simple group in Lie(p). Say $H_0 = H(q)$, a group of Lie type over $\mathbb{F}_q$, where $q = p^e$. Assume H_0 is not of the same type as G. We shall prove the following:

Proposition 4.1 *Under the above assumptions, the only possibility is that $G = E_6, p = 3$ and $H_0 = A_2^{\epsilon}(3)$ or $G_2(3)$, as in Table* 1.3.

We shall use the following result, taken from [21].

Lemma 4.2 (i) *Assume $G \neq E_8$ and $H(q) = A_1(q)$, ${}^2B_2(q)$ or ${}^2G_2(q)$. Then $q \leq t(G) \cdot (2, p-1)$, where $t(G)$ is as follows:*

G	G_2	F_4	E_6	E_7
$t(G)$	12	68	124	388

(ii) *Assume $H(q) \neq A_1(q)$, ${}^2B_2(q)$, ${}^2G_2(q)$. Then either $q \leq 9$, or $H(q) = A_2^{\epsilon}(16)\,(\epsilon = \pm)$.*

(iii) *$V_{adj} \downarrow H_0$ is reducible.*

Proof Corollary 5 of [21] shows that if q violates the bounds in (i) or (ii), then H normalizes a proper nontrivial connected subgroup of G, contrary to assumption. The given values of $t(G)$ were computed by R. Lawther, and are listed in [21, p.3411]. Finally, part (iii) follows from [21, Theorem 4]. ■

We prove Proposition 4.1 in a series of lemmas.

Lemma 4.3 *Suppose that $V \downarrow H_0$ is reducible. Then it is not the case that H_0 lies in a proper closed subgroup $\bar{H}$ of positive dimension G such that H_0 and $\bar{H}$ fix exactly the same subspaces of V.*

Proof Suppose $H_0 < \bar{H}$ as in the hypothesis. Let $\mathcal{D}$ be the set of subspaces of V fixed by H_0, and let Y be the stabilizer in G of $\mathcal{D}$. Then $\bar{H} \leq Y < G$, and Y is normalized by H. This contradicts our assumption of Lie primitivity for H. ■

Lemma 4.4 *If $V = V_{adj}$ then $H_0 \neq A_1(q)$.*

Proof Suppose for a contradiction that $H_0 = A_1(q)$ and H is irreducible on $V = V_{adj}$. By Lemma 4.2, H_0 is reducible on V, so $V \downarrow H_0 = \bigoplus_{i=1}^{t} W_i$, where $t > 1$, the W_i are irreducible H_0-modules, all of dimension m, say, and $mt = \dim V$. Moreover, every irreducible $SL_2(q)$-module over K extends to $GL_2(q)$, and hence t divides $|Aut(H_0) : PGL_2(q)| = \log_p q$.

If $G = E_7$ and $p \neq 2$, then $\dim V = 133 = 7 \cdot 19$, hence $t \geq 7$ and $q \geq 3^7$, contrary to Lemma 4.2. And if $p = 2$ then $\dim V = 132$, and since irreducible H_0-modules have dimension a power of 2 we have $\dim W_i = 4$ or 2 and $t \geq 33$, again contradicting Lemma 4.2.

Next consider $G = E_6$. If $p = 3$ then $\dim V = 77$ and so $t \geq 7$, $q \geq 3^7$ contrary to 4.2. Now assume $p \neq 3$, so $\dim V = 78$. Clearly $t < 13$ by 4.2, so $13 | \dim W_i$, and it follows that $p \geq 13$; hence $t = 2$ by 4.2, and $\dim W_i = 39$. The only possibility is that $q = p^2$ and $V \downarrow H_0 = (2 \otimes 12^{(p)}) \oplus (2^{(p)} \otimes 12)$. If $u \in H_0$ is an element of order 3, then u acts on each summand of V as $\mathrm{diag}(1^{(13)}, \omega^{(13)}, (\omega^2)^{(13)})$, where ω is a cube root of 1 and superscripts denote the multiplicity of each eigenvalue. Moreover, u lifts to an element of order 3 in the simply connected group $\hat{E}_6$. This is impossible, by Lemma 3.3.

If $G = F_4$, $\dim V = 52$ and we see as before that $t = 2, \dim W_i = 26$. But there is no irreducible $L_2(q)$-module over K of dimension 26 (note that $V(25)$ and $1 \otimes 12^{(p^i)}$ admit $SL_2(q)$).

If $G = G_2$, $\dim V = 14$ and $q \geq 7^2$, again contradicting 4.2.

It remains to deal with the case where $G = E_8$. Here $\dim V = 248 = 31 \cdot 8$. Now t is not divisible by 31, since otherwise involutions and elements of order 3 in H_0 would have trace on V equal to a multiple of 31, contrary to Lemma 3.3. Hence 31 divides $\dim W_i$. It follows that $p \geq 31$ and either $W_i \cong V(30)$ with $t = 8$, or $W_i \cong V(30) \otimes V(1)^{(r)} \otimes V(1)^{(s)}$ with $t = 2$, where r, s are powers of p. An element $u \in H_0$ of order 3 has trace 1 on both of these modules, hence has trace 8 or 2 on V. This again contradicts Lemma 3.3. ■

Lemma 4.5 *If $V = V_{min}$ then $H_0 \neq A_1(q)$.*

Proof Suppose $H_0 = A_1(q)$ and $V = V_{min}$. Consider first $G = E_7, V = V_{56}$, a symplectic module. If H_0 is irreducible on V then up to a field twist, $V \downarrow H_0$ is one of the following modules:

$$55,\ 13 \otimes 1^{(r)} \otimes 1^{(s)},\ 6 \otimes 7^{(r)},\ 6 \otimes 1^{(r)} \otimes 1^{(s)} \otimes 1^{(t)},$$

where r, s, t are distinct powers of p. In all cases we calculate that an element of order 3 in H_0 has trace -1 on V, contrary to Lemma 3.3.

Hence $V \downarrow H_0$ is reducible. Then $V \downarrow H_0 = \bigoplus_1^t W_i$, where $t > 1$, the W_i are irreducible, all of dimension m, say, and $mt = 56$; moreover the W_i are permuted by a field automorphism of order t, so in particular, $q \geq p^t$. Note also that the W_i are each self-dual and hence nondegenerate symplectic modules.

If $t \geq 4$, then we easily see that one of the following holds: $t = 4$ with $q \geq 7^4$; $t = 7$ with $q = 3^7$; or $t = 7$ with $q = 2^7$ and (up to a field twist) $W_1 = 1 \otimes 1^{(r)} \otimes 1^{(s)}$ for some distinct powers r, s of 2. Hence the latter occurs, by Lemma 4.2. Take a Borel subgroup $B = RT\langle\phi\rangle$ of $H = L_2(2^7).7$, where $R = 2^7$, $T = 2^7 - 1$ and ϕ is a field automorphism of order 7. This lies in a parabolic subgroup P of G, and the presence of $T\langle\phi\rangle = (2^7 - 1).7$ forces P to be an A_6-parabolic (since the corresponding Weyl group must have an element of order 7). Then T lies in a 1-dimensional torus in $A_6 = SL_7$ of the form $T_1 = \{\mathrm{diag}(c, c^2, c^4, c^8, c^{16}, c^{32}, c^{-63}) : c \in K^*\}$. The composition factors of A_6 on V have high weights $\lambda_i\,(i = 1, 2, 5, 6)$ (see [20, p.106]), from which one checks that distinct weights of T_1 on V remain distinct on restriction to T. In other words, T and T_1 fix exactly the same subspaces of V. It follows that $Y = \langle H_0, T_1\rangle$ is a subgroup of positive dimension fixing exactly the same subspaces of V as H_0. This is a contradiction by Lemma 4.3.

Thus we have $t = 2$. Here the H_0-module W_1 is 27 or $6 \otimes 3^{(r)}$ (up to a field twist). In the first case we have $q \geq 29^2$, contrary to Lemma 4.2. Hence $W_1 = 6 \otimes 3^{(r)}$. Since $p \geq 7$, Lemma 4.2 gives $q = p^2$. Moreover, the preimage of H in $\hat{E}_7$ has no outer involutions, as these would have trace zero on V, contrary to Lemma 3.3. Thus the preimage of H is a group $SL_2(p^2).2$ possessing no outer involutions, where the outer elements induce field automorphisms. However, we claim that there is no such group: for suppose there is, and let τ be an outer element of order 4, squaring to the central involution in $SL_2(p^2)$. As τ induces a field automorphism, its centralizer contains $SL_2(p)$, which has an element t of order 4 squaring to the central involution. But then τt has order 2, contradicting the assumption that the group has no outer involutions. This completes the proof for $G = E_7$.

Now let $G = E_6, V = V_{27}$. If $V \downarrow H_0$ is irreducible, then up to a field twist $V = 26$, $2 \otimes 8^{(r)}$ or $2 \otimes 2^{(r)} \otimes 2^{(s)}$. In each case an involution in H_0 has trace -1 on V, contrary to Lemma 3.3.

Thus $V \downarrow H_0 = \bigoplus_1^t W_i$, where the W_i are irreducible of dimension m and $mt = 27$, $t > 1$. If $t \geq 9$ then $q \geq 3^9$, which is not so by 4.2. Hence $t = 3$, and (up to a field twist) $W_1 = 8$ or $2 \otimes 2^{(r)}$. In the first case, an element of order p in H_0 acts on V with Jordan form J_9^3; however, there is no such p-element in E_6, by [16, Table 5].

Hence we may assume $W_1 = 2 \otimes 2^{(r)}$ with $t = 3$. By Lemma 4.2 we have $q = 3^3$ or 5^3, and

$$V \downarrow H_0 = (2 \otimes 2^{(p)}) \oplus (2^{(p)} \otimes 2^{(p^2)}) \oplus (2^{(p^2)} \otimes 2).$$

As before, take a Borel subgroup $B = RT\langle\phi\rangle$ of H, where $|R| = q, |T| = (q-1)/2$ and ϕ is a field automorphism of order 3. Then B lies in a parabolic $P = QL$ of G, with $R < Q$ and $T\langle\phi\rangle < L$. From the action of H_0 on V we see that $C_V(R)$ has dimension 3 and $T\langle\phi\rangle$ acts irreducibly on $C_V(R)$. Hence $C_V(R) = C_V(Q)$, and the Levi subgroup L must have a factor A_2 acting. The same considerations apply to actions on the dual V^*, so L has two A_2 factors and $L = A_2A_2T_2$ or $A_2A_2A_1T_1$. The action of ϕ implies that $T_o = [T, \phi] < A_2A_2$. By [20, 2.3], with a suitable labelling of the A_2 factors, we have

$$V \downarrow A_2A_2 = (10 \otimes 01) \oplus (01 \otimes 00)^3 \oplus (00 \otimes 10)^3.$$

Now the actions on $C_V(Q)$ and $C_{V^*}(Q)$ show that T_o is contained in the 1-dimensional torus

$$T_1 = \{(\mathrm{diag}(c, c^p, c^{-1-p}), \mathrm{diag}(c, c^p, c^{-1-p}) : c \in K^*\} < A_2A_2,$$

where these actions are on the modules 10, 10 for the A_2 factors. Therefore, the distinct T_1-weights on V are $0, \pm 1, \pm(p-1), \pm p, \pm(p+1), \pm(p+2), \pm(2p+1)$. When $p \geq 3$, no two distinct T_1-weights are congruent modulo $|T_o| = (p^3-1)/(p-1)$, and hence T_o and T_1 fix exactly the same subspaces of V. Then H_0 and $\langle H_0, T_1 \rangle$ fix the same subspaces of V, contrary to Lemma 4.3.

It remains only to treat the case where $G = F_4, V = V_{26-\delta_{p,3}}$. Here, $V \downarrow H_0$ is reducible, as $L_2(q)$ has no irreducible module over K of dimension $26-\delta_{p,3}$. In the usual way, Lemma 4.2 implies that $t = 2$ (and $V = V_{26}$). But then W_1 has high weight 12, so $p \geq 13$ and $q \geq 13^2$, contrary to Lemma 4.2. ■

Lemma 4.6 *H_0 is not ${}^2B_2(q)$ or ${}^2G_2(q)$.*

Proof Irreducible representations of ${}^2B_2(q)$ over K (in characteristic 2) have dimension a power of 2, so it is clear that $H_0 \neq {}^2B_2(q)$.

Now suppose $H_0 = {}^2G_2(q)$ with $q = 3^{2e+1} > 3$. By [25], the irreducible KH_0-modules of dimension 248 or less have dimension 7, 27, 49 or 189. The cases $V_{77} \downarrow H_0 = (V_7)^{11}$ and $V_{133} \downarrow H_0 = V_7^{19}$ have $q \geq 3^{11}$, contrary to Lemma 4.2. The only other possibility is that $G = E_6$ and $V = V_{27}$; moreover, $V \downarrow H_0$ has high weight 20, and is a submodule of codimension 1 in S^2V_7, where V_7 is the irreducible 7-dimensional KH_0-module of high weight 10. Let $t \in H_0$ be an involution. Then t acts on V_7 as $\mathrm{diag}(-1^4, 1^3)$, hence on V as $\mathrm{diag}(-1^{12}, 1^{15})$, and it follows that $C_G(t) = A_1A_5$. Also, $C_{H_0}(t) = \langle t \rangle \times L$ with $L \cong L_2(q)$. Applying a suitable Frobenius twist, we can take it that L embeds in A_1A_5 via one of the following projections:

$0^2,\ 2/2^{(3^a)}\ (a < 2e)$
$0^2,\ 2/0^3$
$1^{(3^a)},\ 2 \otimes 1^{(3^b)}\ (b < 2e)$
$1^{(3^a)}, 2^{(3)} \otimes 1$
$1^{(3^a)},\ 1/1^{(3^b)}/1^{(3^c)}$

where $a, b, c < 2e+1$. We can choose a connected subgroup $\bar{L} = A_1$ of A_1A_5 containing L, and with one of the above listed projections. By [20, 2.1], we have $L(G) \downarrow A_1A_5 = L(A_1A_5)/1 \otimes \lambda_3$. Hence we check that all the weights of $\bar{L}$ on $L(G)$ are less than q, and so by [21, 1.5], L and $\bar{L}$ fix exactly the same subspaces of $L(G)$. Then as usual, H_0 and $\langle H_0, \bar{L} \rangle$ fix the same subspaces of $L(G)$, contradicting Lemma 4.3. ■

Lemma 4.7 (i) *Either $H_0 = H(q)$ with $q \leq 9$, or $H_0 = A_2^{\epsilon}(16)$.*

(ii) *If* $q > 2$ *then* $\mathrm{rank}(H_0) \leq \frac{1}{2}\mathrm{rank}(G)$ *(where* $\mathrm{rank}(H_0)$ *denotes the untwisted Lie rank of* H_0*).*

Proof Part (i) follows from the previous three lemmas. By [18, Theorem 2], either the conclusion of part (ii) holds, or $G = E_8$ and $H_0 = {}^2A_5(5)$ or ${}^2D_5(3)$. In the latter cases [25] shows that there is no suitable irreducible for H_0 of dimension m dividing 248. ■

Write

$$V \downarrow H_0 = \bigoplus_1^t W_i,$$

where $t \geq 1$, the W_i are irreducible, all of dimension m, and t divides $|\mathrm{Out}(H_0) : \mathrm{Inndiag}(H_0)|$.

Lemma 4.8 *We have* $t \leq 2$.

Proof If $t = 5$ or $t \geq 7$, the only possibility is that $H_0 = A_2^\epsilon(16)$ with $t = 8$. Then $m = 7$ or 31, but there are no irreducibles of this dimension for A_2 with $p = 2$.

Now suppose $t = 3$ or 6. Then by Lemma 4.7 and the fact that t divides $|\mathrm{Out}(H_0)|$, one of the following holds:

(i) $q = 8$ and $\mathrm{rank}(H_0) \leq \frac{1}{2}\mathrm{rank}(G)$;

(ii) $H_0 = D_4^\epsilon(q)$, and moreover, $q = 2$ if $G \neq E_8$.

For $t = 6$ we have $m = 13$ or $22\,(p = 2)$; and for $t = 3$ we have $m = 9, 26$ or $44\,(p = 2)$. Using [25], it follows that either $H_0 = A_2^\epsilon(8)$ with $m = 9, t = 3$, or $H_0 = D_4^\epsilon(2)$ with $m = 26, t = 3$. In the latter case the irreducible 26-dimensional module for H_0 has high weight λ_2, so is fixed by triality, and so this case is out.

Hence we have $H_0 = A_2^\epsilon(8)$, $G = E_6$, $V = V_{27}$, and $t = 3$. First consider $\epsilon = 1$, $H_0 = L_3(8)$. In this case

$$\begin{aligned} V \downarrow H_0 = \; & (10 \otimes 10^{(2)}) \oplus (10^{(2)} \otimes 10^{(4)}) \oplus (10^{(4)} \otimes 10), \text{ or} \\ & (10 \otimes 01^{(4)}) \oplus (10^{(2)} \otimes 01) \oplus (10^{(4)} \otimes 01^{(2)}), \end{aligned}$$

or the dual of one of these. Let $x \in H_0$ be an element of order 73. This acts on the module 10 as $\mathrm{diag}(\omega, \omega^8, \omega^{64})$ for some $\omega \in K^*$ of order 73. Hence we check that x acts on V with 27 distinct eigenvalues, and it follows that there is a 1-dimensional torus $T_1 < G$ containing x such that T_1 and x fix exactly the same subspaces of V. Then H_0 and $\langle H_0, T_1\rangle$ fix the same subspaces, contrary to Lemma 4.3.

Now consider $\epsilon = -1$, $H_0 = U_3(8)$. In the simply connected group $\hat{E}_6$, H_0 lifts to $\hat{H}_0 = U_3(8)$ or $SU_3(8)$. Then the preimage of H contains a subgroup $(\langle a\rangle \times L).3$, where $L = SU_2(8) \cong L_2(8)$ and the element a has order 3 or 9, respectively. As above, $V \downarrow H_0$ is the direct sum of three 9-dimensional modules, each of which is the tensor product of twists of 10 or 01. In particular, $V \downarrow L$ is completely reducible.

If $|a| = 9$ then the outer 3 acts on $C_G(a)'$, and hence $C_G(a)' = D_4$ or A_1^3; and if $|a| = 3$ then by Lemma 3.3, $C_G(a)' = A_5, D_4$ or A_2^3. Thus in any case we have $L.3 < (A_1^3).3, (A_2^3).3, D_4.3$ or A_5. By [20, 2.3], the nontrivial composition factors of D_4 on V are the 8-dimensional modules of high weight $\lambda_i\,(i = 1, 3, 4)$; those of A_5 are of high weight λ_1, λ_4; while those of A_1^3 and A_2^3 are tensor products of two twists of natural 2- or 3-dimensional modules. Provided $C_G(a)' \neq A_2^3$, it follows that there is a connected subgroup A_1 of $C_G(a)'$ containing L having highest weight on V less than 8. By [21, 1.5] this A_1 fixes exactly the same subspaces of V as L. Hence H_0 and $\langle H_0, A_1\rangle$ fix the same subspaces, contrary to Lemma 4.3 again.

Now assume $C_G(a)' = A_2^3$. If L embeds completely reducibly in a factor A_2, we get a connected A_1 containing L fixing the same subspaces, as above. So assume L embeds in each A_2 indecomposably. Then the restriction of the 9-dimensional summands of $V \downarrow A_2^3$ to L is of the form $(1/0)\otimes(2/0)$ (or some twist of this), where $1/0, 2/0$ denote 3-dimensional indecomposables for L with the indicated composition factors. However, it is clear that $(1/0) \otimes (2/0)$ is not completely reducible for L. Indeed, this tensor product has either a submodule or quotient module of the form $1/0 \otimes 0 \cong 1/0$, which is not completely reducible. Hence, $V \downarrow L$ is not completely reducible, a contradiction.

Finally, suppose $t = 4$. Then either $q = 4$ or $H_0 = A_2^\epsilon(16)$, and moreover $\mathrm{rank}(H_0) \leq \frac{1}{2}\mathrm{rank}(G)$. Also $m = 13, 14, 62$ or $33\,(p = 2)$. A quick check using [25] shows that the only possibility is $H_0 = A_3^\epsilon(4)$ with $m = 14$, and the irreducible W_1 of high weight 101. But this is fixed by a graph automorphism of G, so the four W_i's are not all non-isomorphic, which is impossible. ■

At this point, a check using [25] gives the following.

Lemma 4.9 *The possibilities for H_0, t and $V \downarrow H_0$ are as in Table* 3 *below (up to Frobenius twists).*

Table 3

H_0	q	t	$V \downarrow H_0$	Case no.
$A_2^\epsilon(q)$	$3,7,9$	1	$V_{27} \downarrow H_0 = 22$	(1)
	$8,16$	1	$V_{27} \downarrow H_0 = V_3 \otimes V_3' \otimes V_3''$	(2)
	7	2	$V_{56} \downarrow H_0 = 60 \oplus 06$	(3)
	5	2	$V_{78} \downarrow H_0 = 32 \oplus 32$	(4)
$A_3^\epsilon(q)$	7	2	$V_{248} \downarrow H_0 = 302 \oplus 203$	(5)
$A_7^\epsilon(q)$	2	2	$V_{56} \downarrow H_0 = \lambda_2 \oplus \lambda_6$	(6)
$B_2(q)$	$3,9$	1	$V_{25} \downarrow H_0 = 12$	(7)
	9	1	$V_{25} \downarrow H_0 = 10 \otimes 10^{(3)}$	(8)
	9	1	$V_{56} \downarrow H_0 = 01 \otimes 20^{(3)}$	(9)
	5	1	$V_{52} \downarrow H_0 = 13$	(10)
	7	1	$V_{56} \downarrow H_0 = 05$	(11)
$B_3(q)$	$3,5,9$	1	$V_{27} \downarrow H_0 = 200$	(12)
	9	1	$V_{56} \downarrow H_0 = 100 \otimes 001^{(3)}$	(13)
	7	1	$V_{248} \downarrow H_0 = 021$	(14)
$B_4(q)$	2	1	$V_{26} \downarrow H_0 = 0100$	(15)
$C_3(q)$	$5,7$	1	$V_{56} \downarrow H_0 = 300$	(16)
$D_4^\epsilon(q)$	2	1	$V_{26} \downarrow H_0 = 0100$	(17)
$G_2(q)$	$3,5,9$	1	$V_{27} \downarrow H_0 = 20$	(18)
	7	1	$V_{26} \downarrow H_0 = 20$	(19)
	7	1	$V_{248} \downarrow H_0 = 12$	(20)

Lemma 4.10 *We have $t = 1$.*

Proof Consider the $t = 2$ cases in Table 3 above. In case (4), $H_0 = A_2^\epsilon(q)$, any extension $H_0.2$ splits, and so H possesses an involution u interchanging W_1 and W_2, hence acting as $\mathrm{diag}(-1^{39}, 1^{39})$, which has determinant -1, a contradiction.

Next consider (3). Here the outer involution in $A_2^\epsilon(7).2$ could (indeed, must) lift to an element of order 4 in the simply connected group $\hat{G} = \hat{E}_7$. Choose a subgroup $S = SL_2(7) < H_0$. The irreducible 60 for A_2 is the 6th symmetric power of the natural 3-dimensional module, so $60 \downarrow S = S^6(1 \oplus 0)$, which we calculate to be $6 \oplus 5 \oplus 4 \oplus 3 \oplus 2 \oplus 1 \oplus 0$. Hence if $u \in S$ is a unipotent element of order 7, then u acts on V_{56} as $J_7^2, J_6^2, \ldots, J_1^2$ (where J_i denotes a Jordan block of size i). Referring to [16, Table 7], we see that u is in the class $A_4 + A_1$ of G, and hence lies in a connected subgroup $A = A_1$ in a subsystem subgroup A_4A_1, embedded via the representations $4, 1$. The nontrivial composition factors of A_1A_4 on $L(G)$ are $1\otimes\lambda_1$, $1\otimes\lambda_4$ and $0\otimes\lambda_i$. Hence we see that the composition factors of A on $L(G)$ all have high weight less than 7. If U is a 1-dimensional unipotent subgroup of A containing u, it follows that u and U fix exactly the same subspaces of V_{56}. Hence H_0 and $\langle H_0, U\rangle$ fix the same subspaces, contrary to Lemma 4.3.

In case (5), $H_0 = A_3^\epsilon(7)$ with $V_{248} \downarrow H_0 = 203 \oplus 302$, the acting group has nontrivial centre (since $-I$ acts faithfully on 203), which is a contradiction.

Finally, consider case (6): $H = A_7^\epsilon(2).2$ with $V_{56} \downarrow H_0 = \lambda_2 \oplus \lambda_6$. For $\epsilon = -1, H_0 = U_8(2)$, pick an element $t \in H_0$ of order 3 with $C_{H_0}(t) = \langle t\rangle \times SU_7(2)$. Then t acts on the natural 8-dimensional H_0-module $V(\lambda_1)$ as $\mathrm{diag}(\omega^{-1}, \omega^{(7)})$ (where ω is a cube root of 1), hence on $V(\lambda_2)$ as $\mathrm{diag}(1^{(7)}, \omega^{-1\,(21)})$. Hence $\chi_{56}(t) = -7$, whence $C_G(t) = A_6T_1$ by Lemma 3.3. The subgroup $SU_7(2) = C_{H_0}(t)'$ lies in the factor A_6, and the composition factors of this A_6 on V_{56} are of high weight λ_i $(i = 1, 2, 5, 6)$. Hence $SU_7(2)$ and A_6 fix the same subspaces of V_{56}, and so H_0 and $\langle H_0, A_6\rangle$ also fix the same subspaces, giving a contradiction by Lemma 4.3.

For $\epsilon = +1$, $H_0 = L_8(2)$, pick an element $u \in H_0$ of order 3 with $C_{H_0}(u) = \langle u\rangle \times SL_6(2)$. Then u acts on the natural 8-dimensional H_0-module as $\mathrm{diag}(\omega, \omega^{-1}, 1^{(6)})$, from which we calculate that $\chi_{56}(u) = 20$, hence $C_G(u) = D_6T_1$. So we have $C_{H_0}(u)' = SL_6(2) < D_6$. By [11], $H^1(SL_6(2), V(\lambda_2)) = 0$, so this $SL_6(2)$ is completely reducible on the natural 12-dimensional D_6-module, and so we have $SL_6(2) < A_5 < D_6$. The composition factors of the A_5 on V_{56} are of high weight λ_i, so $SL_6(2)$ and A_5 fix the same subspaces of V_{56}, and now we complete the argument as before. ∎

Lemma 4.11 *Cases* (10), (13), (14) *of Table* 3 *do not occur.*

Proof In cases (10) and (14), the acting group has centre containing -1, which is impossible. In case (13), the module $100 \otimes 001^{(3)}$ for $B_3(q)$ admits a non-degenerate symmetric bilinear form, so does not embed H_0 in E_7. ∎

Lemma 4.12 *Cases* (2), (8), (9), (11), (12), (16) *of Table* 3 *do not occur.*

Proof Let $t \in H_0$ be an involution. In case (2), t acts on V_{27} as (J_2^{13}, J_1) (where J_i denotes an $i \times i$ Jordan block); but there is no such involution in E_6, by [16, Table 5]. In case (8) take t to act as $\mathrm{diag}(-1^4, 1)$ on the 5-dimensional orthogonal module 10; then t acts on V_{25} as $\mathrm{diag}(-1^8, 1^{17})$, contrary to Lemma 3.3. In case (9), take t to act as $\mathrm{diag}(-1^2, 1^2)$ on the

4-dimensional sympletic module 01; then t acts as $\mathrm{diag}(-1^4, 1^{10})$ on 20, hence as $\mathrm{diag}(-1^{28}, 1^{28})$ on V_{56}, contrary to Lemma 3.3. Likewise, in case (11), we have $V_{56} \downarrow H_0 = 05 = S^5(01)$, the 5th symmetric power of 01, from which we see that t again acts on V_{56} as $\mathrm{diag}(-1^{28}, 1^{28})$. In case (12) take t to act as $\mathrm{diag}(-1^2, 1^5)$ on the 7-dimensional orthogonal module 100. Since $V_{27} \downarrow H_0 = 200$ is of codimension 1 in $S^2(100)$, we see that t acts on V_{27} as $\mathrm{diag}(-1^{10}, 1^{17})$. But there is no such involution in E_6 by Lemma 3.3. Finally, in case (16) let u be an element of order 3 acting on the 6-dimensional symplectic module 100 as $\mathrm{diag}(\omega, \omega^{-1}, 1^4)$ (where ω is a cube root of unity). As $V_{56} \downarrow H_0 = 300 = S^3(100)$, u acts on V_{56} as $\mathrm{diag}(1^{(26)}, \omega^{(15)}, \omega^{-1\,(15)})$. But there is no such element in E_7, by Lemma 3.3. ■

Lemma 4.13 *Cases* (15), (17) *of Table* 3 *do not occur.*

Proof Here $H_0 = B_4(2)$ or $D_4^\epsilon(2)$ in $G = F_4$ with $p = 2$. The group G has two irreducible restricted 26-dimensional modules, which we shall denote by $V_{26} = V(\lambda_4)$ and $V'_{26} = V(\lambda_1)$ (in other characteristics this is the 52-dimensional adjoint module).

Assume now that $H_0 \neq {}^3D_4(2)$. Then we can choose an element $u \in H_0$ of order 3 such that $C_{H_0}(u)' = Sp_6(2)$ or $\Omega_6^\pm(2)$. The 3-element centralizers in G are listed in Lemma 3.3, and the only possibility is that $C_G(u) = T_1B_3$ or T_1C_3. Replacing H_0 by its image under a graph automorphism of G if necessary, we can take $C_G(u) = T_1C_3 < C_4$. Now $V_{26} \downarrow C_4 = V_{C_4}(\lambda_2)$, the irreducible constituent of $\wedge^2 V_8$ (where V_8 is the natural module for C_4). Moreover, T_1 acts on V_8 as $\{\mathrm{diag}(\alpha, \alpha^{-1}, 1^{(6)}) : \alpha \in K^*\}$, hence acts on V_{26} as $\{\mathrm{diag}(\alpha^{(6)}, \alpha^{-1\,(6)}, 1^{(14)}) : \alpha \in K^*\}$. On the other hand, $V'_{26} \downarrow C_4 = 2\lambda_1/\lambda_4/0^2$, on which T_1 acts as $\{\mathrm{diag}(\alpha^{(8)}, \alpha^{-1\,(8)}, \alpha^2, \alpha^{-2}, 1^{(8)}) : \alpha \in K^*\}$. Since H_0 contains $u \in T_1$, and H_0 has only one irreducible of dimension 26 (the nontrivial composition factor of the adjoint module) it follows that H_0 is irreducible on V_{26}, but reducible on V'_{26}.

First assume $C_{H_0}(u)' = Sp_6(2)$. One checks that $V'_{26} \downarrow C_3 = W \oplus \lambda_3^2 \oplus 0^2$, where W is uniserial of shape $0|2\lambda_1|0$ and we see that $Sp_6(2)$ and C_3 leave invariant the same subspaces of V'_{26}, contradicting Lemma 4.3. Now assume $C_{H_0}(u)' = D_4^\epsilon(2)$, so that $D_4^\epsilon(2)$ is contained in a subgroup D_3 of C_3. Then $V'_{26} \downarrow D_3 = 2\lambda_2 \oplus \lambda_1^2 \oplus \lambda_3^2 \oplus 0^4$. This time $C_{H_0}(u)'$ and D_3 leave invariant the same subspaces of V'_{26}, again contradicting Lemma 4.3.

Now let $H_0 = {}^3D_4(2)$. Pick $u \in H_0$ of order 3 with $C_{H_0}(u) = \langle u \rangle \times L_2(8)$. Then u acts as $\{\mathrm{diag}(\alpha^{(2)}, \alpha^{-1\,(2)}, 1^{(4)})$ on the orthogonal module and hence as $\{\mathrm{diag}(\alpha^{(9)}, \alpha^{-1\,(9)}, 1^{(8)})$ on the irreducible module of dimension 26. So this time, $V'_{26} \downarrow H_0$ is irreducible. On the other hand, u and T_1 leave invariant the same subspaces of V_{26}, so Lemma 4.3 yields a contradiction.

The only alternative is that $C_G(u) = A_2\tilde{A}_2$. We have

$$V_{26} \downarrow A_2\tilde{A}_2 = (10 \otimes 10) \oplus (01 \otimes 01) \oplus (00 \otimes 11).$$

Hence we see that u acts on both V_{26} and V'_{26} as $\mathrm{diag}(\omega^{(9)}, \omega^{-1\,(9)}, 1^{(8)})$ (where ω is a cube root of 1). If H_0 is reducible on either module, its restriction has only 8-dimensional nontrivial composition factors, and u acts on each such as $\mathrm{diag}(\omega^{(2)}, \omega^{-1\,(2)}, 1^{(4)})$, which is incompatible with the above action of u on the 26-dimensional modules. Hence H_0 is irreducible on both V_{26} and V'_{26} (and these are isomorphic as H_0-modules).

We can choose an element $v \in H_0$ of order 7 such that $C_{H_0}(v) = \langle v \rangle \times L_3(2)$ (see [7]). If $C_G(v) = T_1B_3$ or T_1C_3 then $v \in T_1$; but we showed above that T_1 acts differently on V_{26} and V'_{26}, so this is impossible. Hence $C_G(v) = T_2A_2$ or $T_1A_1A_2$, and applying a graph automorphism if necessary, we may take it that the A_2 factor is generated by long root groups. Then $C_{H_0}(v)' = L_3(2) < A_2$. However $V'_{26} \downarrow A_2$ has 11 as a composition factor, whereas $V_{26} \downarrow A_2$ has no such composition factor. This is a contradiction, as $L_3(2) < H_0$ acts isomorphically on V_{26} and V'_{26}. ∎

Lemma 4.14 *Cases* (7), (20) *of Table* 3 *do not occur.*

Proof First consider case (7): $G = F_4, H_0 = B_2(q)$. Here $H_0 \geq B_2(3) \cong U_4(2)$ and so has a subgroup $2^4.A_5$ with the A_5 acting on the 2^4 as $\Omega_4^-(2)$. We shall show that in general, for $p \neq 2$, F_4 does not contain such a subgroup $2^4.A_5$. Suppose then that $2^4.A_5 < G = F_4$, and let E be the normal subgroup 2^4. Then E is not fused in G (see [5, 3.4]), hence contains an involution e with $C_G(e) = B_4 = Spin_9$. Obviously $E < B_4$. Apart from e, the involutions in B_4 are those elements whose image in $SO_9 = B_4/\langle e \rangle$ is similar to either $\mathrm{diag}(-1^4, 1^5)$ or $\mathrm{diag}(-1^8, 1)$. The former have G-centralizer A_1C_3, the latter B_4.

Conjugating if necessary, we may assume that E consists of matrices whose image in SO_9 is diagonal with eigenvalues ± 1. The orbits of A_5 on E have sizes $5, 10$, so there are at least 5 B_4-involutions in E. Choose a B_4-involution $f \in E - \{e\}$ with image $\mathrm{diag}(-1^8, 1)$. As $\langle e, f \rangle$ has only 3 involutions, E contains a further B_4-involution, say g. Then the image of g in SO_9 has 8 eigenvalues -1, so the product fg has image with only 2 eigenvalues -1. But then fg is not an involution in B_4, which is a contradiction.

Now consider case (20): $G = E_8, H_0 = G_2(7), V_{248} \downarrow H_0 = 12$. Using the computer programme in [8], we find that the subdominant weights, weight space dimensions, and orbits sizes under $W = W(G)$ for this representation of G_2 are as follows:

weight λ	12	40	21	02	30	11	20	01	10	00
$\dim V_\lambda$	1	1	2	2	3	4	6	6	8	8
$\|W(\lambda)\|$	12	6	12	6	6	12	6	6	6	1

A Cartan subgroup of $G_2(7)$ has 3 involutions, namely $h_1(-1)$, $h_2(-1)$, $h_1(-1)h_2(-1)$, and these are all conjugate. Hence, evaluating the above weights on these involutions, we find that an involution in H_0 acts on V_{248} as $\text{diag}(-1^{124}, 1^{124})$. This contradicts Lemma 3.3. ∎

Lemma 4.15 *Cases* (1), (18), (19) *of Table* 3 *do not occur.*

Proof Consider case (1): $G = E_6$, $H_0 = A_2^\epsilon(q)$ ($q = 3, 7$ or 9), $V_{27} \downarrow H_0 = 22$. Let $t \in H_0$ be an involution, and let $S < C_{H_0}(t)$ with $S \cong SL_2(q)$. Note that for A_2 we have $20 \otimes 02 = W(22)/W(11)/W(00)$ (see [20, 2.14]). Hence we calculate that t acts on V_{27} as $\text{diag}(-1^{12}, 1^{15})$, so $C_G(t) = A_1A_5$ by Lemma 3.3. Moreover, $S < A_1A_5$ and $V_{27} \downarrow A_1A_5 = (1 \otimes \lambda_1) \oplus (0 \otimes \lambda_4)$, while $L(G) \downarrow A_1A_5 = L(A_1A_5)/1 \otimes \lambda_3$ (see [20, 2.1, 2.3]). The restriction of $V = V_{A_2}(22)$ to S has composition factors $W(4)/W(3)^2/W(2)^3/W(1)^2/0$. It follows that if $q \geq 7$, then S embeds in a connected subgroup $\bar{S} = SL_2(K)$ of A_1A_5 such that the composition factors of $\bar{S}$ on $L(G)$ are all less than q. By [21, 1.5] this means that S and $\bar{S}$ fix exactly the same subspaces of $L(G)$, and now we see in the usual way that H normalizes a proper nontrivial connected subgroup of G, a contradiction.

This leaves $q = 3$: $H_0 = A_2^\epsilon(3) < E_6$ with $V_{27} \downarrow H_0 = 22$. Note that this possibility is in Table 1.3: we know that there is are irreducible subgroups $A_2^\epsilon(3)$ lying in a connected subgroup A_2 of G, but we have not determined whether or not there are such subgroups $A_2^\epsilon(3)$ which are Lie primitive.

Next consider cases (18),(19): $H_0 = G_2(q)$ ($q = 3, 5, 7, 9$) with $V_{27-\delta_{p,7}} \downarrow H_0 = 20$. Let $t \in H_0$ be an involution. As above we see that $C_G(t) = A_1A_5$. Also, for $q \geq 5$, $C_{H_0}(t)' = S_1S_2$ with $S_1 \cong S_2 \cong SL_2(q)$, and we see as before that one of the S_i's lies in a connected SL_2 in A_1A_5 fixing the same subspaces of $L(G)$.

This leaves $q = 3$. We complete the proof by showing that any $H_0 = G_2(3)$ in E_6, such that $V_{27} \downarrow H_0 = 20$, is not Lie primitive. Take a long root subgroup $L = L_3(3) < H_0$. Then writing $V_7 = V_{G_2}(10)$, we have

$V_7 \downarrow L = 10 + 01 + 00$. Since $V_{G_2}(20)$ is of codimension 1 in $S^2(10)$, we see that

$$V_{27} \downarrow L = T(11) + 20 + 02 + 10 + 01,$$

where $T(11)$ is the tilting module 00/11/00. Thus L fixes a 1-space in V_{27}. The stabilizer of this 1-space in E_6 is either F_4 or contained in a D_5-parabolic; however $V_{27} \downarrow D_5$ has a self-dual composition factor of dimension 10, which is not compatible with the above decomposition of $V_{27} \downarrow L$. Hence $L < F_4$.

Take an involution $t \in L$ and a subgroup $S = SL_2(3) < C_L(t)$. Restricting to S we have $V_{25} \downarrow S = 2^3 + 1^6 + 0^4$ (completely reducible). Hence $C_{F_4}(t) = A_1C_3$, where t generated the center of each factor. Since $V_{25} \downarrow A_1C_3 = (1 \otimes \lambda_1) \oplus (0 \otimes \lambda_2)$, it follows that the embedding of S in A_1C_3 must be 0^2, $1+1+1$. Hence we see that there is a connected subgroup $A = A_1$ in C_3 containing S and fixing the same subspaces as S. Setting $X = \langle L, A \rangle$, we have $L < X < F_4$, and L, X fix the same subspaces of V_{25}. Referring to the list of maximal connected subgroups of F_4 in [23] and the restrictions of V_{25} to these, we conclude that L lies in a maximal rank subgroup A_2A_2 of F_4, hence lies in a diagonal subgroup $B = A_2$ of a maximal rank subgroup A_2^3 of $G = E_6$. From [20, Table 8.3] we see that the composition factors of $L(G) \downarrow B$ are the irreducibles $00, 11, 10, 01, 20, 02, 21, 12$; moreover, $L(G) \downarrow B$ is a direct sum of irreducibles together with the indecomposable 00/11/00, which remains indecomposable for L. It follows that L and B fix the same subspaces of $V_{77} = L(G)'$. Then H_0 and $\langle H_0, B \rangle$ fix the same subspaces, and moreover, H_0 is reducible on $L(G)'$ (see [10]). This is a contradiction, by Lemma 4.3. ■

5 Proof of Corollary 2

Let σ be a Frobenius morphism of the exceptional algebraic group G, so that $G_\sigma = G(r)$ is a finite exceptional group over $\mathbb{F}_r$ ($r = p^a$). Let H be a maximal subgroup of G_σ which is irreducible on $V \in \{V_{adj}, V_{min}\}$, and assume that H is not of the same type as G.

Suppose first that $H = \bar{H}_\sigma$ with $\bar{H}$ a maximal closed σ-stable subgroup of positive dimension in G. If $\bar{H}$ is not of maximal rank, then [23, Theorem 1] shows that $\bar{H}$ must be as in the lists (b),(c) before Lemma 2.1, and now Theorem 1 shows that $\bar{H}$ must be as in Table 1.2 and hence conclusion (ii) of Corollary 2 holds. If $\bar{H}$ is of maximal rank, the possibilities are given by [17], and again the result follows using Theorem 1.

So assume from now on that H is not of the form $\bar{H}_\sigma$ for such a subgroup $\bar{H}$.

Next observe that if G_σ is of type ${}^2G_2$ or ${}^2F_4$, then the conclusion is immediate from [12, 26]. (Indeed, under our assumption that H is not of the same type as G, no further maximal subgroups of ${}^2G_2(q)\,(q > 3)$ are irreducible, and for $G_\sigma = {}^2F_4(q)$ only the maximal subgroups with socles $L_3(3), L_2(25)$ and $q = 2$ occur, with $V = V_{26}$.) So assume G_σ is not of one of these types, so that either σ is a field morphism or $G = E_6$ and σ is a graph-field morphism.

Suppose first that $F^*(H)$ is not simple. Then H is determined by [19, Theorem 2], from which it follows that $F^*(H)$ is one of the local subgroups in Tables 1.1, 1.3.

Thus we may now assume that $F^*(H)$ is simple. Write $H_0 = F^*(H)$. If $H_0 \notin \mathrm{Lie}(p)$ then the proofs of Lemmas 3.4 and 3.5 show that H_0 is as in Table 1.1 or 1.3.

Now assume $H_0 \in \mathrm{Lie}(p)$, say $H_0 = H(q)$, a group of Lie type over $\mathbb{F}_q$ $(q = p^b)$.

We next establish that the conclusion of Lemma 4.3 holds in our situation. Suppose $V \downarrow H_0$ is reducible and H_0 lies in a closed subgroup $\bar{H}$ of positive dimension in G such that H_0 and $\bar{H}$ fix the same subspaces of V. We apply the argument of the proof of [21, Theorem 6] (see p.3473 of [21]): if σ is a field morphism, let $\mathcal{M}$ be the set of all H_0-invariant subspaces of V, and define $Y = \bigcap_{W \in \mathcal{M}} G_W$. Then Y is $H\langle\sigma\rangle$-stable (see the proof of [21, 1.12]) and contains $\bar{H}$. If Z is a maximal $H\langle\sigma\rangle$-stable proper subgroup of G containing Y, then by maximality we have $H = Z_\sigma$, contrary to our assumption above. Finally, if $G = E_6$ and σ is a graph-field morphism, then σ interchanges the G-modules V and V^*, and moreover H_0 and $\bar{H}$ fix the same subspaces of both V and V^* (as the latter are the annihilators of the former). Thus if we define $\mathcal{M}$ to be the set of all H_0-invariant subspaces of both V and V^*, the above argument yields the same contradiction.

We have now proved that the conclusion of Lemma 4.3 holds in our situation. At this point the proof given in Section 4 establishes the result (namely, that the conclusion of Proposition 4.1 holds). This completes the proof of Corollary 2.

References

[1] M. Aschbacher, "Chevalley groups of type G_2 as the group of a trilinear form", *J. Algebra* **109** (1987), 193-259.

[2] A.V. Borovik, "The structure of finite subgroups of simple algebraic groups" (Russian), *Algebra i Logika* **28** (1989), no. 3, 249-279; translation in *Algebra and Logic* **28** (1989), no. 3, 163-182 (1990).

[3] A.V. Borovik, "Jordan subgroups and orthogonal decompositions" (Russian) *Algebra i Logika* **28** (1989), no. 4, 382–392; translation in *Algebra and Logic* **28** (1989), no. 4, 248–254 (1990).

[4] A.M. Cohen and R.L. Griess, "On finite simple subgroups of the complex Lie group E_8", *Proc. Symp. Pure Math.* **47** (1987), 367-405.

[5] A.M. Cohen and D.B. Wales, "Finite subgroups of $E_6(\mathbb{C})$ and $F_4(\mathbb{C})$", *Proc. London Math. Soc.* **74** (1997), 105-150.

[6] A.M. Cohen, M.W. Liebeck, J. Saxl and G.M. Seitz, "The local maximal subgroups of exceptional groups of Lie type, finite and algebraic", *Proc. London Math. Soc.* **64** (1992), 21-48.

[7] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of finite groups*, Oxford University Press, 1985.

[8] P.B. Gilkey and G.M. Seitz, "Some representations of exceptional Lie algebras", *Geom. Dedicata* **25** (1988), 407-416.

[9] G. Hiss and G. Malle, "Low-dimensional representations of quasi-simple groups", *LMS J. Comput. Math.* **4** (2001), 22-63.

[10] C. Jansen, K. Lux, R. Parker and R. Wilson, *An atlas of Brauer characters*, Oxford University Press, 1995.

[11] W. Jones and B. Parshall, "On the 1-cohomology of finite groups of Lie type", *Proc. Conf. on Finite Groups* (eds. W. Scott and F. Gross), Academic Press, New York, 1976, pp. 313-328.

[12] P.B. Kleidman, "The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, of the Ree groups ${}^2G_2(q)$, and of their automorphism groups", *J. Algebra* **117** (1988), 30–71.

[13] P.B. Kleidman, U. Meierfrankenfeld and A.J.E. Ryba, "HS $< E_7(5)$", *J. London Math. Soc.* **60** (1999), 95-107.

[14] P.B. Kleidman, U. Meierfrankenfeld and A.J.E. Ryba, "Ru $< E_7(5)$", *Comm. Algebra* **28** (2000), 3555-3583.

[15] P.B. Kleidman and R.A. Wilson, "$J_3 < E_6(4)$ and $M_{12} < E_6(5)$", *J. London Math. Soc.* **42** (1990), 555-561.

[16] R. Lawther, "Jordan block sizes of unipotent elements in exceptional algebraic groups", *Comm. Algebra* **23** (1995), 4125–4156; correction, *ibid*, **26** (1998), 2709.

[17] M.W. Liebeck, J. Saxl and G.M. Seitz, "Subgroups of maximal rank in finite exceptional groups of Lie type", *Proc. London Math. Soc.* **65** (1992), 297-325.

[18] M.W. Liebeck, J. Saxl and D.M. Testerman, "Subgroups of large rank in groups of Lie type", *Proc. London Math. Soc.* **72** (1996), 425-457.

[19] M.W. Liebeck and G.M. Seitz, "Maximal subgroups of exceptional groups of Lie type, finite and algebraic", *Geom. Dedicata* **36** (1990), 353-387.

[20] M.W. Liebeck and G.M. Seitz, "Reductive subgroups of exceptional algebraic groups", *Mem. Amer. Math. Soc.*, Vol. 121, No. 580, 1996.

[21] M.W. Liebeck and G.M. Seitz, "On the subgroup structure of exceptional groups of Lie type", *Trans. Amer. Math. Soc.* **350** (1998), 3409-3482.

[22] M.W. Liebeck and G.M. Seitz, "On finite subgroups of exceptional algebraic groups", *J. reine angew. Math.* **515** (1999), 25-72.

[23] M.W. Liebeck and G.M. Seitz, "The maximal subgroups of positive dimension in exceptional algebraic groups", *Mem. Amer. Math. Soc.*, to appear.

[24] M.W. Liebeck and G.M. Seitz, "Subgroups of large rank in exceptional groups of Lie type", to appear.

[25] F. Lübeck, "Small degree representations of finite Chevalley groups in defining characteristic", *LMS J. Comput. Math.* **4** (2001), 22-63.

[26] G. Malle, "The maximal subgroups of ${}^2F_4(q^2)$", *J. Algebra* **139** (1991), 52–69.

[27] A.J.E. Ryba, "Construction of some irreducible subgroups of E_8 and E_6", to appear.

[28] J-P. Serre, "Exemples de plongements des groups $PSL_2(F_p)$ dans des groupes de Lie simples", *Invent. Math.* **124** (1996), 525-562.

[29] D.M. Testerman, "A construction of certain maximal subgroups of the algebraic groups E_6 and F_4", *J. Algebra* **122** (1989), 299-322.