Maximal subgroups of large rank in exceptional groups of Lie type

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1 Introduction

Let G be a simple adjoint algebraic group of exceptional type over $K = \bar{\mathbb{F}}_p$, the algebraic closure of the prime field $\mathbb{F}_p$, where p is prime, and let σ be a Frobenius endomorphism of G. If G_σ denotes the fixed point group $\{g \in G : g^\sigma = g\}$, then $G_0 = (G_\sigma)'$ is a finite simple exceptional group of Lie type, with the exceptions of $G_2(2)' \cong U_3(3)$ and ${}^2G_2(3)' \cong L_2(8)$, which we exclude from consideration.

The main result of this paper represents a contribution to the study of the maximal subgroups of almost simple groups with socle G_0 as above. Let L be such an almost simple group (i.e. $F^*(L) = G_0$), and let M be a maximal subgroup of L not containing G_0. In the case where M is not almost simple, the possibilities for M up to conjugacy are completely determined by [11, Theorem 2]. Hence we assume that M is almost simple, and write $M_0 = F^*(M)$, a simple group.

Denote by Lie(p) the set of finite quasisimple groups of Lie type in characteristic p. In the case where $M_0 \notin$ Lie(p), the possibilities for M_0 are given up to isomorphism in [15] (although the problem of determining them up to conjugacy remains largely open).

Our main result focusses on the case where $M_0 \in$ Lie(p); say $M_0 = M(q)$, a simple group of Lie type over the finite field $\mathbb{F}_q$. There are several re-

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sults in the literature concerning this case. Two of the main ones are [16, Corollary 5], which determines the possibilities for M up to conjugacy under the assumption that q is not too small (usually $q > 9$ suffices); and [19, Theorem 3], which gives the possibilities assuming that $q > 2$, that $\mathrm{rank}(M(q)) > \frac{1}{2}\mathrm{rank}(G)$, and also that $(M(q), G) \neq ({}^2A_5(5), E_8)$ or $({}^2D_5(3), E_8))$. Here $\mathrm{rank}(M(q))$ denotes the untwisted Lie rank of $M(q)$ (i.e. the rank of the corresponding untwisted group); and we write just E_8 for the algebraic group $E_8(K)$.

It is apparent that the above results say nothing in the case where $q = 2$, a fact which frequently causes difficulties when applying them. Our main purpose is to extend the results of [19] to this case, and to settle the exceptional cases mentioned above. This requires a different approach to that of [19], one reason being that the latter is based on the vanishing of H^1 groups for $M(q)$ acting on various modules, and such conclusions are either false or unknown for many of the groups $M(2)$ (see for example [8]).

Here is our main result on maximal subgroups.

Theorem 1 *Let L be a finite almost simple group with $F^*(L) = (G_\sigma)'$ and G of exceptional type in characteristic p, as above. Suppose M is a maximal subgroup of L such that $F^*(M) = M(q)$, a simple group of Lie type in characteristic p, with rank$(M(q)) > \frac{1}{2}$rank(G). Then one of the following holds:*

(i) *$M(q)$ is a subgroup of maximal rank (determined in [18]);*

(ii) *$M(q)$ is of the same type as G, possibly twisted (determined by [12, 5.1]);*

(iii) *$F^*(L) = E_6^\epsilon(q)$ and $M(q) = F_4(q)$ or $C_4(q)$ (q odd) (two G_σ-classes of each, interchanged by a graph automorphism);*

(iv) *$F^*(L) = E_7(q)$ (q odd) and $M(q) = {}^3D_4(q)$ (one G_σ-class).*

The case $q = 2$ of this theorem is proved here; the case $q > 2$ is covered by [19, Theorem 3], apart from the ${}^2A_5(5)$, ${}^2D_5(3)$ cases mentioned above. Unfortunately the maximal subgroups in part (iv) were omitted in error in [19, Theorem 3]. These arise as follows: let $p > 2$, take $\bar{M}$ to be a maximal closed σ-stable local subgroup $(2^2 \times D_4).Sym_3$ of G (see [4] for a construction), and take σ to act on $\bar{M}$ as $\sigma_q w$, where σ_q is a standard field morphism and $w \in Sym_3$ has order 3 (note that $\sigma_q w$ is G_σ-conjugate to σ_q by Lang's theorem); then $M_\sigma = {}^3D_4(q).3$, and these are the subgroups in (iv). (The error in [19, Theorem 3] is in the penultimate sentence of the proof, which is precisely where the above subgroups should have arisen.)

We shall deduce Theorem 1 from the following result describing embeddings of arbitrary (not necessarily maximal) subgroups $X(q)$ of Lie type of rank more than $\frac{1}{2}\mathrm{rank}(G)$. This is done in [19, Theorem 2] for $q > 2$ (and also excluding the two cases $(M(q), G) = ({}^2A_5(5), E_8)$ or $({}^2D_5(3), E_8)$) mentioned above). Here we cover the remaining cases.

Write V_{adj} for the nontrivial composition factor of the adjoint module for G, excluding $(G, p) = (F_4, 2)$ or $(G_2, 3)$; and write V_{min} for one of the irreducible modules for E_7, E_6, F_4, G_2 of dimension $56, 27, 26 - \delta_{p,3}, 7 - \delta_{p,2}$ and high weight $\lambda_7, \lambda_1, \lambda_4, \lambda_1$ respectively (also λ_1 for $F_4, p = 2$). Note that $V_{adj} = L(G)$, except for $(G, p) = (E_7, 2)$ or $(E_6, 3)$, in which cases it has codimension 1 in $L(G)$ (see [14, 1.10]). Note also that for $(G, p) = (F_4, 2)$, V_{adj} is undefined, while there are two choices for V_{min}.

Theorem 2 *Let $X = X(q)$ be a simple group of Lie type in characteristic p, and suppose that $X < G$, where G is a simple adjoint algebraic group of type F_4, E_6, E_7 or E_8, also in characteristic p. Assume that rank$(X) > \frac{1}{2}$rank(G), and also that either*

(i) $q = 2$, *or*

(ii) $G = E_8$ *and* $X = {}^2A_5(5)$ *or* ${}^2D_5(3)$.

Then, with the exception of $(X, G) = (L_4(2), F_4)$, there is a closed connected subgroup $\bar{X}$ of G containing X, such that for at least one module $V \in \{V_{adj}, V_{min}\}$, X and $\bar{X}$ stabilize exactly the same subspaces of V.

In the exceptional case $(X, G) = (L_4(2), F_4)$, the centralizer $C_G(X)$ contains a long or short root subgroup of G.

The layout of the paper is as follows. After some preliminaries in Section 2, we give the proof of Theorem 2 in Sections 3 and 4. Section 5 contains the deduction of Theorem 1.

2 Preliminaries

In this section we collect some preliminary results from the literature which will be needed in our proofs. We use standard notation. In particular, if X is a group of Lie type in characteristic p, $V_X(\lambda)$ denotes the irreducible X-module in characteristic p of high weight λ; often we just write λ intead of $V_X(\lambda)$. We may write $\lambda = \sum c_i\lambda_i$, where the c_i are non-negative integers and the sum is over fundamental dominant weights λ_i (see [2, p.250]). When all $c_i \leq p-1$ we say the weight λ and the module $V_X(\lambda)$ are restricted. For small ranks we usually denote the weight $\lambda = \sum c_i\lambda_i$ just by the sequence $c_1c_2\ldots$. Finally, for dominant weights $\mu_1,\ldots,\mu_k$, and positive integers $c_1,\ldots,c_k$, we write $\mu_1^{c_1}/\cdots/\mu_k^{c_k}$ to denote an X-module having the same composition factors as the module $V_X(\mu_1)^{c_1}\oplus\cdots\oplus V_X(\mu_k)^{c_k}$.

Lemma 2.1 ([8]). *Let $X = L_n(2)$ with $n \geq 4$.*

(i) *For $n \geq 5$ we have $H^1(X, V_X(\lambda_i)) = 0$ for all fundamental dominant weights λ_i.*

(ii) *For $n = 4$, $H^1(X, V_X(\lambda_1)) = 0$ and $H^1(X, V_X(\lambda_2))$ has dimension 1.*

(iii) *$\dim H^1(X, V_X(\lambda_1 + \lambda_{n-1})$ is 0 if n is odd, and is 1 if n is even.*

Lemma 2.2 *Let $G = F_4, E_6$ or E_7 and let $V = V_{min}$. Let α be a 1-space in V.*

(i) *If $G = E_7$ then $(G_\alpha)^0$ is contained in an E_6-parabolic or a D_6-parabolic subgroup of G.*

(ii) *If $G = E_6$ then $(G_\alpha)^0$ is contained either in a D_5-parabolic, or in a subgroup F_4 of G.*

(iii) *If $G = F_4$ then $(G_\alpha)^0$ is contained either in a maximal parabolic, or in a subgroup B_4 or $C_4\,(p = 2)$ of G.*

Proof For parts (i) and (ii), the orbits of the corresponding groups and modules over finite fields are classified completely in [10, 4.3, 5.4]; there are 5 orbits in case (i) and 4 orbits in case (ii). Hence the same holds for the action of G on $P_1(V)$, by [6, 2.10], and the conclusion follows from the lists of finite stabilizers in [10].

For (iii), since $\dim G = 52$ we have $\dim G_\alpha \geq 27$. Any such subgroup either lies in a parabolic or in a maximal rank subgroup B_4 or $C_4\,(p = 2)$, by [21]. ■

Lemma 2.3 *Let $X < \bar{X} < G$ with $X = X(q)$ a quasisimple group in* Lie(p) *and $\bar{X}$ a simple connected subsystem subgroup of the same type as X. Suppose that V is a KG-module such that $V \downarrow \bar{X}$ is completely reducible with all composition factors restricted. Then X and $\bar{X}$ stabilize precisely the same subspaces of V.*

Proof By hypothesis each irreducible summand of $V \downarrow \bar{X}$ is restricted, and hence (by [22, 13.3]) is irreducible upon restriction to X. Moreover, non-isomorphic irreducible $\bar{X}$-submodules are also nonisomorphic upon restriction to X (again by [22]). The assertion follows.

■

Table 1

n	G	$C_G(t)^0$	$\chi_n(t)$
248	E_8	A_8	-4
		A_2E_6	5
		D_7T_1	14
		E_7T_1	77
$132, 56$	E_7 (adj., s.c.)	A_6T_1	$6, -7$ (resp.)
		E_6T_1	$51, -25$
		A_5A_2	$-3, 2$
		$A_1D_5T_1$	$6, 2$
		D_6T_1	$33, 20$
$78, 27$	E_6 (adj., s.c.)	A_5T_1	$15, 9$ (resp.)
		$A_2A_2A_2$	$-3, 0$
		D_4T_2	$6, 0$
26	F_4	C_3T_1	8
		B_3T_1	-1
		A_2A_2	-1

Lemma 2.4 *Assume $p = 2$, let $V = V_{adj}$ or V_{min} (taking $V_{min} = V(\lambda_4)$ if $G = F_4$), and let $n = \dim V$. Let $t \in G$ be an element of order 3; moreover, if G is adjoint of type E_6, assume that t lifts to an element of order 3 in the simply connected group. Then the possibilities for $C_G(t)^0$ and the values of the trace $\chi_n(t)$ of t on V are as recorded in Table 1.*

Proof Most of this information can be found in [15, 1.2]; the rest can, as in the proof of that result, easily be deduced from the corresponding information for the case $K = \mathbb{C}$ found in [5, 3]. ∎

Lemma 2.5 *If $X < G$, where X is a finite quasisimple group of Lie type in characteristic p, then $\mathrm{rank}(X) \leq \mathrm{rank}(G)$.*

Proof This is [9, 1.4]. ∎

Lemma 2.6 *Let $X < G$, where X is a finite quasisimple group, and suppose that $X < \bar{X} < G$, where $\bar{X}$ is closed of positive dimension and $X \cap \bar{X}^0 \not\leq Z(X)$. Then $X < \bar{X}^0$.*

Proof Since X is quasisimple and $X \cap \bar{X}^0$ is a normal subgroup of X not contained in $Z(X)$, we have $X \cap \bar{X}^0 = X$, as required. ∎

3 Proof of Theorem 2, part I: the $q = 2$ cases

In this section let G be an exceptional adjoint algebraic group over an algebraically closed field K of characteristic 2, and let $X = X(2)$ be a subgroup of G which is simple of Lie type over $\mathbb{F}_2$, where $\text{rank}(X) > \frac{1}{2}\text{rank}(G)$. By Lemma 2.5, we have $\text{rank}(X) \leq \text{rank}(G)$.

We begin by handling the case where $G = G_2$.

Lemma 3.1 *Theorem 2 holds when $G = G_2$.*

Proof Here $X = L_3(2)$ or $Sp_4(2)$. Let $V = V_G(\lambda_1)$, a symplectic module of dimension 6. If $X = Sp_4(2)$ then as X has no irreducibles in characteristic 2 of dimension 6, X must fix a 1-space $\langle v \rangle$ of V, hence lie in G_v, which is a parabolic of G; this is clearly not possible, as it would force $Sp_4(2)$ to embed in a Levi factor A_1. And if $X = L_3(2)$ then $V \downarrow X = 10/01$. An element $t \in X$ of order 7 has distinct eigenvalues on V, hence fixes the same subspaces of V as any torus T containing t. This gives the conclusion with $\bar{X} = \langle X, T \rangle$ (note that $X < \bar{X}^0$ by Lemma 2.6). ∎

Assume from now on that $G \neq G_2$.

3.1 Subgroups $X = L_n(2)$

Suppose $X = L_n(2) = A_{n-1}(2) < G$, with $n - 1 = \text{rank}(X) > \frac{1}{2}\text{rank}(G)$.

Lemma 3.2 *Theorem 2 holds for $X = L_n(2)$, $G = E_6, E_7$ or E_8. In each case X lies in a subsystem subgroup A_{n-1} of G.*

Proof Suppose first that $G = E_6$. We begin by establishing the result for $X = L_5(2)$. For this, we consider the action of X on $V_{min} = V_{27}$. By [20], the only nontrivial irreducible modules for X in characteristic 2 of dimension 27 or less are $V(\lambda)$ for either $\lambda = \lambda_i$ $(1 \leq i \leq 4)$, of dimension 5 or 10, or $\lambda = \lambda_1 + \lambda_4$, of dimension 24. Lemma 2.1 shows that $H^1(X, \lambda) = 0$ for all these λ, and it follows that X fixes a 2-space in V_{27}. Since X does not lie in a point-stabilizer in F_4 (note that $L_5(2) \not\leq B_4$), it follows from Lemma 2.2

that X lies in a D_5-parabolic of G. Any subgroup $L_5(2)$ of D_5 must have two composition factors of high weights λ_1, λ_5 on the orthogonal 10-dimensional module, and hence we have $X < QA_4$, where Q is unipotent and $A_4 < D_5$. The nontrivial composition factors of A_4 acting on Q have the structure of irreducible KA_4-modules (see [1]), and each has high weight λ_i for some i (see [13, 3.1]). Hence, as $H^1(X, \lambda_i) = 0$, an inductive argument shows that there is just one class of complements to Q in QX. One such complement lies in a Levi A_4, hence we have $X < A_4$. The conclusion now follows, using Lemma 2.3 together with [13, 2.1].

To complete the proof for $G = E_6$, we deduce the conclusion for $X = L_n(2)$, $n \geq 6$. By the above, a subgroup $Y = L_5(2)$ of X lies in A_4, a subsystem subgroup of G. Then Y and A_4 fix the same subspaces of $L(G)$ by Lemma 2.3, so X and $\bar{X} := \langle X, A_4 \rangle$ also fix the same subspaces. By [13, Theorem 4], X is reducible on $L(G)$, so $\bar{X} < G$. If M is a maximal connected subgroup of G containing $\bar{X}^0$, then by [21], M is either parabolic or reductive of maximal rank. It then follows that $X = L_6(2)$ and $X < Q_1A_5$, where Q_1 is a unipotent group normalized by the subsystem group A_5. As above, the high weights of the composition factors of A_5 on Q_1 are fundamental weights λ_i, so Lemma 2.1 shows that $X < A_5$, completing the proof.

Now assume $G = E_7$. Again it is enough to prove the result for $X = L_5(2)$. We consider the action on $V_{min} = V_{56}$. By [20], the only nontrivial self-dual irreducible X-module in characteristic 2 of dimension 56 or less is $V(\lambda_1 + \lambda_4)$, of dimension 24; and the non-self-dual irreducibles of dimension 28 or less are $V(\lambda_i)$ $(1 \leq i \leq 4)$, of dimension 5 or 10. Hence, using Lemma 2.1, it is clear that X fixes a 1-space in V_{56}. Then by Lemma 2.2, X lies in either an E_6-parabolic or a D_6-parabolic subgroup of G. In either case we deduce as in the previous paragraph that $X < A_4$, a subsystem subgroup, and the conclusion follows.

Finally, assume that $G = E_8$. Here it suffices to consider $X = L_6(2)$. Take a parabolic subgroup UR of X with $U = 2^5, R = L_5(2)$. This lies in a parabolic $P = QL$ of G with Levi subgroup L. If L has a factor E_6 or E_7, we deduce from the previous paragraphs that $R < QA_4$, where A_4 is a subsystem subgroup; otherwise, L is a product of classical groups and the same conclusion follows, using Lemma 2.1. Thus $R < QA_4$. By [13, 3.1], the nontrivial composition factors of A_4 on Q have fundamental high weights λ_i, so we deduce as before that $R < A_4$. Now the argument in the second paragraph of this proof gives $X < A_6$, completing the proof. ∎

Lemma 3.3 *Theorem* 2 *holds for* $X = L_n(2)$, $G = F_4$.

Proof First assume that $n = 5$ and consider $X = L_5(2) < G < E_6$. By the previous proof, there is a subsystem subgroup A_4 of E_6 containing X such that X and A_4 fix the same subspaces of $L(E_6)$. In particular, X, and hence A_4, fix the subspace $L(G)$. However, the stabilizer S of $L(G)$ in E_6 is F_4, and hence $A_4 < F_4$, which is a contradiction. Thus $L_5(2) \not\leq F_4$.

It remains to prove the result for $X = L_4(2)$. We consider the action on V_{26}. The nontrivial irreducibles for X in characteristic 2 of dimension at most 26 (at most 13 for non-self-dual modules) are those of high weights $100, 001, 010$ and 101, of dimensions 4, 4, 6 and 14, respectively (see [20]). Write χ_8 for the Brauer character of the X-module $V(100) \oplus V(001)$, and χ_6, χ_{14} for the Brauer characters of the other irreducibles. Let v denote an element of order 3 in $L_2(2)$, and write $t = \mathrm{diag}(v, I_2)$, $u = \mathrm{diag}(v, v) \in X$. Now a graph morphism of G interchanges $V(\lambda_1)$ and $V(\lambda_4)$ and also the subsystems B_3T_1 and C_3T_1. Hence by Lemma 2.4, replacing X by its image under a graph morphism of G if necessary, we may take $\chi(u) = -1$, where χ is the Brauer character of X on V. Write $\chi \downarrow X = a\chi_1 + b\chi_8 + c\chi_6 + d\chi_{14}$. Then evaluating χ on the elements $1, t$ and u, we obtain the equations

$$\begin{aligned} a + 8b + 6c + 14d = &\ 26 \\ a + 2b - d = &\ 8 \text{ or } -1 \\ a - 4b + 3c + 2d = &\ -1 \end{aligned}$$

The only solution is $(a, b, c, d) = (4, 2, 1, 0)$, i.e. $V \downarrow X = 100^2/001^2/010/000^4$. By Lemma 2.1 this forces $C_V(X) \neq 0$. Hence by Lemma 2.2, X lies in either B_4, C_4 or a B_3- or C_3-parabolic of G. In the latter two cases X centralizes a long or short root group in G, giving the conclusion of Theorem 2. And if $X < B_4$ then either X lies in a B_3- or C_3-parabolic, giving the result again, or it lies in an A_3-parabolic; in the latter case using Lemma 2.1 we see that X lies in a subsystem A_3, which centralizes a long or short A_1. ∎

3.2 Subgroups $X = U_n(2)$

We begin by handling one of the base cases for E_8.

Lemma 3.4 *Suppose $G = E_8$ $(p = 2)$. Then G has no subgroup isomorphic to $U_6(2)$.*

Proof Suppose $X = U_6(2) < G$. We consider the restriction of $V = L(G)$ to X. By [20], the nontrivial irreducible modules for X (as opposed to

$SU_6(2)$) in characteristic 2 of dimension at most 248 (at most 124 for non-self-dual modules) are the modules $V_X(\lambda)$ listed below (up to duals):

λ	00100	10001	11000	01010
$\dim V_X(\lambda)$	20	34	70	154

Let χ denote the Brauer character of X on V. We may write

$$\chi = a\chi_1 + b\chi_{20} + c\chi_{34} + d\chi_{140} + e\chi_{154},$$

where each χ_i is the Brauer character of the above module of dimension i, except for χ_{140}, which is the Brauer character of $V(11000) \oplus V(00011)$.

We now calculate the values of χ on elements of order 3 in X. Let $\omega \in \mathbb{F}_4$ be a cube root of 1, and define the following elements of order 3 in X (relative to an orthonormal basis of the natural 6-dimensional unitary module $W = V_6(4)$):

$$t = \mathrm{diag}(\omega, \omega^{-1}, 1^{(4)}),\ u = \mathrm{diag}(\omega^{(2)}, \omega^{-1\,(2)}, 1^{(2)}),\ v = \mathrm{diag}(\omega^{(3)}, \omega^{-1\,(3)})$$

where the bracketed superscripts denote multiplicities. If $ch(\lambda)$ denotes the character of $V(\lambda)$, then $ch(11000) = ch(10000) \cdot ch(01000) - ch(00100)$ and $ch(01010) = ch(01000) \cdot ch(00010) - 2ch(10001) - 3ch(00000)$. Hence we calculate the following values:

i	$\chi_i(t)$	$\chi_i(u)$	$\chi_i(v)$
20	2	2	-7
34	7	-2	7
140	14	-4	-22
154	-8	1	19

Evaluating χ at the elements $1, t, u, v$, and using Lemma 2.4, we obtain the following equations:

$$\begin{array}{lllllllllll}
(1) & a & + & 20b & + & 34c & + & 140d & + & 154e & = 248 \\
(2) & a & + & 2b & + & 7c & + & 14d & - & 8e & = -4, 5, 14 \text{ or } 77 \\
(3) & a & + & 2b & - & 2c & - & 4d & + & e & = -4, 5, 14 \text{ or } 77 \\
(4) & a & - & 7b & + & 7c & - & 22d & + & 19e & = -4, 5, 14 \text{ or } 77
\end{array}$$

Suppose first that $e = 0$. Then subtraction of (4) from (1) shows that the right hand side of (4) must be 5 (i.e. $\chi(v) = 5$), and gives $b + c + 6d = 9$. Subtraction of (2) and (3) from (1) yields the equations

$$2b + 3c + 14d = 28, 27, 26 \text{ or } 19,\ \ 2b + 4c + 16d = 28, 27, 26 \text{ or } 19.$$

Combining these with the equation $b+c+6d = 9$, we get $c+2d = 10, 9, 8$ or 1 and $2c + 4d = 10, 9, 8$ or 1. These are clearly contradictory.

Hence $e = 1$, from which it is readily seen that the only solution to the equations (1)-(4) is $(a, b, c, d, e) = (0, 3, 1, 0, 1)$. It follows that $\chi(t) = \chi(u) = \chi(v) = 5$; that is, all elements of X of order 3 are conjugate in G, with G-centralizer A_2E_6.

Now choose a subgroup $S = SU_3(2) \circ SU_3(2) < X$. Then $S \cong 3^{1+4}.(Q_8 \times Q_8)$, where the normal subgroup $E = 3^{1+4}$ is extraspecial of exponent 3, and is the central product E_1E_2 of two subgroups E_1, E_2, both extraspecial of order 27, and both normal in S. Write $Z(E) = \langle z \rangle$. Then

$$S \leq C_G(z) = A_2E_6.$$

Choose $x, y \in E_1$ with $\langle x, y \rangle = E_1$ and define $F = \langle z, x \rangle \cong 3^2$. Calculation with characters gives

$$\dim C_G(F) = \frac{1}{9}(248 + (8 \times 5)) = 32,$$

and similarly $\dim C_G(E_1) = 14, \dim C_G(E) = 6$.

Consider the embedding $E < C_G(z) = A_2E_6$. We have $E \not\leq E_6$, since otherwise E would centralize the A_2 factor, whereas $\dim C_G(E) = 6$. Also $E \cap E_6 \lhd S$, so $E \cap E_6 = E_1$ or E_2, say the former. Now $C_G(F) = C_G(z, x) = A_2C_{E_6}(x)$ has dimension 32. From the possible 3-element centralizers in E_6 given by Lemma 2.4, we see that $C_G(F)^0 = A_2^4$ and $C_{E_6}(x)^0 = A_2^3$. The element $y \in E_6$ has order 3 and satisfies $[x, y] = z^{\pm 1}$, and hence y permutes the three A_2 factors of $C_{E_6}(x)$ cyclically. Consequently $C_{E_6}(x, y) \geq A_2$. It follows that $C_G(x, y) = C_G(E_1) \geq A_2A_2$. However, $C_G(E_1)$ has dimension 14, which is a contradiction. ■

Most of the rest of the proof for $X = U_n(2)$ concerns the case where $G = E_7$. For this case we shall make heavy use of the subgroups $SU_3(2) \cong 3^{1+2}.Q_8$ of X (where as before, 3^{1+2} denotes an extraspecial group of order 27 and exponent 3). To this end, we classify the extraspecial subgroups 3^{1+2} of E_7 in the next lemma. Note that if such a subgroup lies in a subgroup $SU_3(2)$ of G, then all of its non-central elements of order 3 are conjugate.

Lemma 3.5 *The group $G = E_7\,(p = 2)$ has exactly 4 conjugacy classes of subgroups isomorphic to 3^{1+2} in which all non-central elements are fused. Representatives $E_i\,(1 \leq i \leq 4)$ of these classes have the following properties, where $Z(E_i) = \langle z_i \rangle$:*

(i) $E_1 < M_1 = A_2$, *a subsystem subgroup of* G;

(ii) $E_2 < M_2 < A_2A_2$, *where* M_2 *is a diagonal* A_2 *in the subsystem* A_2A_2; *we have* $C_G(M_2)^0 = A_2A_1$; z_2 *has* G*-centralizer* A_2A_5, *with* M_2 *acting on the natural* A_5*-module as* $10 + 10$; *the other order 3 elements in* E_2 *have* G*-centralizer* $A_1D_5T_1$;

(iii) $E_3 < M_3 < A_2A_2$, *where* M_3 *is a diagonal* A_2 *in the subsystem* A_2A_2; *we have* $C_G(M_3)^0 = G_2T_1$; z_3 *has* G*-centralizer* E_6T_1, *and the other order 3 elements in* E_3 *have centralizer* $A_1D_5T_1$;

(iv) $E_4 < M_4 < A_2A_2A_2$, *where* M_4 *is a diagonal* A_2 *in the subsystem* $A_2A_2A_2$; *we have* $C_G(M_4)^0 = A_1$; *all order 3 elements in* E_4 *have centralizer* A_2A_5.

Proof Let $E < G$ with $E \cong 3^{1+2}$, and let $Z(E) = \langle z \rangle$. The possibilities for $C_G(z)$ are listed in Lemma 2.4. Since $z \in C_G(z)'$, the centralizer $C_G(z)$ must be E_6T_1 or A_2A_5. Choose x, y such that $E = \langle x, y \rangle$, and write $F = \langle z, x \rangle \cong 3^2$. Also let χ be the Brauer character of E on $L(G)$, and write $a = \chi(x), b = \chi(z)$. We have

$$\dim C_{L(G)}(F) = (133 + 2b + 6a)/9, \ \dim C_{L(G)}(E) = (133 + 2b + 24a)/27.$$

Suppose now that $C_G(z) = E_6T_1$. Then $E \cap E_6 \geq \langle z \rangle$, so we may assume that $F \leq E_6$. We have $b = 52$, so $(\dim C_G(F), \dim C_G(E))$ is $(31, 15), (25, 7), (49, 39)$ or $(61, 55)$, according as $a = 7, -2, 34$ or 52, respectively. The centralizers of order 3 elements in E_6 are A_2^3, D_4T_2, A_5T_1, so $\dim C_G(F)$ cannot be 49 or 61. If $C_G(F)$ has dimension 25, then $C_G(F)^0 = A_2^3T_1$ with $F \leq Z(A_2^3)$, so y must cycle the three A_2 factors. Consequently $C_G(E) \geq A_2$, whereas $\dim C_G(E) = 7$ in this case, a contradiction.

We are left with the case where $a = 7$: here $C_G(F)^0 = D_4T_3$ and y acts as a triality on D_4, giving $C_G(E)^0 = G_2T_1$. Now $N_G(D_4)^0 = D_4A_1^3$ and y acts on this with centralizer $G_2\bar{A}_1$, where the second term is diagonal in A_1^3. Also $z \in T_1 < \bar{A}_1$, so that y centralizes an involution t which inverts T_1. Then $y \in C_{E_6T_1}(t) < E_6$. So $E < E_6$ and hence $E < C_{E_6}(G_2) = A \cong A_2$. Now $C_G(G_2) = C_3$ and the C_3 lies in a subsystem A_5 with A diagonal in a subsystem A_2A_2 of this A_5. Calculation of $L(G) \downarrow A$ shows that there are precisely 15 trivial composition factors, and hence we have $C_G(A) = G_2T_1$, giving the conclusion of part (iii) of the lemma (note that $C_G(y) = A_1D_5T_1$ rather than A_6T_1, since in the latter case $\dim C_{V_{56}}(E)$ would be $(56 - 50 - 24 \cdot 7)/9$, which is ridiculous).

Now suppose that $C_G(z) = A_2A_5$. Here $b = -2$, and $(\dim C_G(F), \dim C_G(E))$ is $(19, 11), (13, 3), (37, 35)$ or $(49, 51)$, according as $a = 7, -2, 34$ or 52. The last case is clearly absurd, as $\dim C_G(F) \leq \dim C_G(z) = 43$.

In the third case we have $C_G(E) = A_5$, so $E \leq A_2$, a subsystem group, as in part (i).

Now consider the second case: $a = -2$ and $(\dim C_G(F), \dim C_G(E)) = (13, 3)$. As $|x| = 3$, the only 13-dimensional possibility for $C_G(F)^0 = C_{A_2A_5}(x)^0$ is $A_1^3T_4$. As $[x, y] = z^{\pm 1}$, y must act nontrivially on T_4 and must cycle the three A_1 factors. Hence $A_1^3 < A_5$ and $C_G(E)^0 = A_1$. This A_1, call it A, is diagonal in $A_1^3 < A_5$, so from the construction of the maximal subgroup A_1F_4 of G in [21], we see that $C_G(A) = F_4$. Thus $E < F_4$, indeed, $E < C_{F_4}(z) = A_2\tilde{A}_2 < A_2A_2A_2$, a subsystem subgroup of G, as in (iv).

Finally, consider the case where $a = 7$ and $(\dim C_G(F), \dim C_G(E)) = (19, 11)$. Here $C_G(x) = A_1D_5T_1$ as above, and $C_G(z) = A_2A_5$. Looking at order 3 elements in $A_1D_5T_1$, we see that the 19-dimensional group $C_G(F)^0$ is A_3T_4 or $A_2A_1^3T_2$.

In the latter case we have $C_G(F)^0 = A_2A_1^3T_2 < A_2A_5$. Then $x \in A_5$ and $C_{A_5}(x) = A_1^3T_2$. Now $y \in A_2A_5$ and $[y, x] = z^{\pm 1}$, so y cycles the three A_1 factors and as $\dim C_G(E) = 11$, this gives $C_G(E) = A_2A_1$. So here $E < C_G(A_1A_2) = C_{A_5}(A_1) = A_2$ (a factor of a tensor product subgroup $A_1 \otimes A_2 < A_5$). Thus we have conclusion (ii).

Now assume that $C_G(F)^0 = A_3T_4$. Then $C_G(E)^0$ must be A_2T_3, so $E \leq C_G(A_2T_3) = C_{A_5}(T_3) = T_3A_2$. Consider the action of E on the natural module for this A_5. The space decomposes under the action of E as an irreducible of dimension 3 and three linear representations. Choose $s \in E - \langle z \rangle$. On the nonlinear part s has eigenvalues $1, \omega, \omega^{-1}$. On the linear part s either has eigenvlues $1, \omega, \omega^{-1}$ or δ, δ, δ for $\delta \in \{1, \omega, \omega^{-1}\}$. The latter must occur for at least one such element s. But then $C_{A_5}(z, s) \geq A_2$ and $C_{A_2A_5}(z, s) \geq A_2A_2$. However, having settled all other cases we may assume $F = \langle z, s \rangle$ and obtain a contradiction, since A_3T_4 does not contain A_2A_2. ■

Lemma 3.6 *Let $G = E_7\,(p = 2)$, and suppose $S = SU_3(2) \cong 3^{1+2}.Q_8 < G$. Let $E = O_3(S) \cong 3^{1+2}$, and suppose that $E = E_i$ ($i = 1$ or 2) is as in* (i) *or* (ii) *of Lemma 3.5, so that $E < M_i < G$ with $M_i \cong A_2$. Then every S-invariant subspace of V_{56} is also M_i-invariant.*

Proof Consider $E = E_1 < M_1$, a subsystem A_2. The restriction of V_{56} to M_1 is completely reducible, with summands $10, 01$ and 00. Evidently E acts irreducibly on each 3-dimensional summand, and $10 \downarrow E \not\cong 01 \downarrow E$. Therefore E and M_1 fix exactly the same subspaces of V_{56} in this case.

Now consider $E = E_2 < M_2 < A_2A_2 < A_5$. Proposition 2.3 of [13] shows that $V_{56} \downarrow A_5 = V_{A_5}(\lambda_1)^3 \oplus V_{A_5}(\lambda_5)^3 \oplus V_{A_5}(\lambda_3)$. Using Lemma 3.5(ii) we see from this that $V_{56} \downarrow M_2 = 10^6 \oplus 01^6 \oplus 11^2 \oplus 00^4$ (see [13, Table 8.6]). Observe that $11 \downarrow E$ is a sum of eight 1-spaces corresponding to the nontrivial linear characters of E. These are permuted transitively by $S/E \cong Q_8$; hence any M_2-submodule of V_{56} isomorphic to 11 is S-invariant and S-irreducible. The conclusion follows. ■

Lemma 3.7 *Theorem 2 holds when $X = U_5(2)$, $G = E_7$ for both $V = V_{min}$ and $V = V_{adj}$.*

Proof Suppose $X < G$ with $X = U_5(2)$, $G = E_7$. We first prove the result for $V = V_{min} = V_{56}$. Consider the restriction $V_{56} \downarrow X$. By [20], the nontrivial irreducible X-modules in characteristic 2 of dimension at most 56 (at most 28 for non-self-dual modules) are $V_X(\lambda)$ for $\lambda = 1000, 0100, 0010, 0001$ and 1001. Let χ be the Brauer character of X on V_{56}, and write

$$\chi = a\chi_1 + b\chi_{10} + c\chi_{20} + d\chi_{24},$$

where χ_{10}, χ_{20} are the characters of $V(1000) \oplus V(0001)$, $V(0100) \oplus V(0010)$ respectively, and χ_{24} is the character of $V(1001)$.

Now choose a subgroup $S = SU_3(2) < X$, and let $E = O_3(S) \cong 3^{1+2}$ and $Z(E) = \langle z \rangle$, so z acts as $\mathrm{diag}(\omega^{(3)}, 1^{(2)})$ on the natural 5-dimensional X-module. Easy calculation gives $\chi_{10}(z) = 1$, $\chi_{20}(z) = -7$, $\chi_{24}(z) = 6$.

If $E = E_i$ ($i = 1$ or 2) as in Lemma 3.5, then Lemma 3.6 gives the conclusion, taking $\bar{X} = \langle X, M_i \rangle$.

Now assume $E = E_3$. Here $C_G(z) = E_6T_1$, so $\chi(z) = -25$ (see Lemma 2.4), giving the equations

$$a + 10b + 20c + 24d = 56, \quad a + b - 7c + 6d = -25.$$

These clearly have no solutions with a, b, c, d non-negative integers.

Finally, consider the case where $E = E_4$. Here all the elements of order 3 in E have G-centralizer A_2A_5. Let $x \in E - Z(E)$. Then $\chi(x) = \chi(z) = 2$. Moreover, x acts on the natural X-module as $\mathrm{diag}(\omega, \omega^{-1}, 1^{(3)})$, from which we calculate that $\chi_{10}(x) = 4$, $\chi_{20}(x) = 2$, $\chi_{24}(x) = 3$. Thus we have the equations

$$\begin{aligned} a + 10b + 20c + 24d &= 56 \\ a + b - 7c + 6d &= 2 \\ a + 4b + 2c + 3d &= 2 \end{aligned}$$

Again these have no solutions with a, b, c, d non-negative integers.

This completes the proof of the lemma for $V = V_{56}$. We now prove it for $V = V_{adj}$. Since $X = U_5(2)$ is not irreducible on V_{56} (see [20]), it follows from the above that $X < \bar{X}$, where $\bar{X}$ is a proper connected subgroup of G fixing the same subspaces of V_{56} as X. By [21], if $\bar{M}$ is a maximal connected subgroup of G containing $\bar{X}$, then either $\bar{M} = A_1F_4$, or $\bar{M}$ is parabolic or reductive of maximal rank. If $\bar{X}$ is of the form QE_6 or QF_4 with Q a (possibly trivial) unipotent normal subgroup, then it has a composition factor of dimension 26 or 27 on V_{56}; however by [20], X has no irreducibles of dimension 26 or 27, so this is impossible. It follows that $\bar{X}$, hence also X, lies in a connected group QD, where Q is unipotent and D is a subsystem group which is a product of classical groups. Using this it is easy to see that $X < Q_1A_4$, where Q_1 is unipotent and the A_4 is a subsystem group. If $S = SU_3(2) < X$ and $E = O_3(S)$, this means that E lies in a subsystem subgroup A_2 of G. Now $V_{adj} \downarrow A_2$ is completely reducible, with composition factors $10, 01, 11$ and 00. Hence we see as in Lemma 3.6 that every S-invariant subspace of V_{adj} is also fixed by A_2, and so X and $\bar{X} := \langle X, A_2\rangle$ fix the same subspaces. Note finally that $X < \bar{X}^0$ by Lemma 2.6, giving the conclusion of Theorem 2. ■

Lemma 3.8 *Theorem 2 holds for $X = U_4(2)$, $G = F_4$.*

Proof Suppose $X = U_4(2) < G$. Take a subgroup $S = SU_3(2)$ of X and let $E = O_3(S) \cong 3^{1+2}$. Let $Z(E) = \langle z\rangle$, $x \in E - Z(E)$ and $F = \langle z, x\rangle$. As $z \in C_S(z)'$, we must have $C_G(z) = A_2A_2$. If χ is the Brauer character of X on $L(G)$, then $\chi(z) = -2$ and $\chi(x) = -2$ or 7 (see Lemma 2.4).

If $\chi(x) = 7$ then $\dim C_G(F) = 10, \dim C_G(E) = 8$. Therefore $C_G(F)^0 = A_2T_2$ and $C_G(E) = A_2$, a long or short subsystem group. Then $E \leq C_G(A_2) = J$, where J is also a subsystem A_2. Then E and J fix the same subspaces of either $V_G(\lambda_4)$ or $V_G(\lambda_1)$.

Now suppose that $\chi(x) = -2$, so that $E - \{1\}$ is fused. Then $C_G(E)^0 = 1$ and also $C_V(E) = 0$, where V is either of the 26-dimensional modules $V_G(\lambda_4), V_G(\lambda_1)$. Consider the monomial subgroup $3^3.S_4$ of X, and let H be the normal elementary abelian 3^3 subgroup. Then $H - \{1\}$ has 20 elements which are X-conjugate to z or x; let h be one of the remaining 6 elements (so h is conjugate to $\mathrm{diag}(\omega^{(2)}, \omega^{-1\,(2)})$). Then $\dim C_G(H) = 2$ or 0, according as $\chi(h) = 7$ or -2, respectively.

If $\chi(h) = 7$, then $C_G(h) = B_3T_1$ or C_3T_1. As $C_{B_3}(h')$ (respectively $C_{C_3}(h')$) is connected for all $h' \in H$, it follows that H lies in a torus of

$C_G(h)$, so $C_G(H)$ contains a maximal torus of G, contradicting the fact that $\dim C_G(H) = 2$.

Hence $\chi(h) = -2$, and so all order 3 elements of X have G-centralizer A_2A_2. Let χ_{26} be the Brauer character of X on the 26-dimensional module $V_G(\lambda_4)$. Then $\chi_{26}(u) = -1$ for all elements $u \in X$ of order 3. Referring to [7, p.60], we can write

$$\chi_{26} = a\chi_1 + b\chi_8 + c\chi_6 + d\chi_{14},$$

where $\chi_8, \chi_6, \chi_{14}$ are the Brauer characters of the X-modules $V(100) \oplus V(001), V(010), V(101)$ respectively. Evaluating at the elements 1, $(\omega, \omega^{-1}, 1, 1)$, $(\omega, \omega, \omega, 1)$ and $(\omega, \omega, \omega^{-1}, \omega^{-1})$, we obtain the following equations:

$$\begin{aligned} a + 8b + 6c + 14d &= \ \ 26 \\ a + 2b - d &= \ -1 \\ a - b - 3c + 5d &= \ -1 \\ a - 4b + 3c + 2d &= \ -1 \end{aligned}$$

These have no non-negative integer solutions. ∎

Lemma 3.9 *Theorem 2 holds for $X = U_n(2)$.*

Proof Suppose $X = U_n(2) = {}^2A_{n-1}(2) < G$, with $n - 1 = \mathrm{rank}(X) > \frac{1}{2}\mathrm{rank}(G)$. If $G = E_6$ or E_7 then X contains a subgroup $U = U_5(2)$, and by Lemma 3.7, there is a connected subgroup $\bar{U}$ of E_7 containing U and fixing the same subspaces of V_{56} as U. Then X and $\bar{X} := \langle X, \bar{U} \rangle$ fix the same subspaces of V_{56}. As $X < \bar{X}^0$ by Lemma 2.6, the result follows for $G = E_7$. For $G = E_6$, note that if $X < E_6$ then X fixes a pair of 27-dimensional subspaces of V_{56}, of which the stabilizer is E_6. Hence $\bar{X}$ also fixes this pair, so that $\bar{X} \leq E_6$.

If $G = F_4$, the result follows from Lemma 3.8 for $X = U_4(2)$. For $X = U_5(2)$, the previous paragraph gives a connected subgroup $\bar{X}$ of E_6 containing X and fixing the same subspaces of V_{27}. Since F_4 is the stabilizer in E_6 of a 1-space of V_{27} it follows that $X < \bar{X} < F_4$, giving the result.

Now consider $G = E_8$. By Lemma 3.4, we have $n \geq 7$, so X has a subgroup $V = U_7(2)$. Pick an element $t \in V$ of order 3 such that $C_V(t) \geq SU_6(2)$. As $t \in C(t)'$, it follows from Lemma 2.4 that $C_G(t) = A_8$ or A_2E_6. In the former case the group A_8 is $SL_9/\mathbb{Z}_3$, so t must lift to an element of order 9 in the preimage of $SU_6(2)$ in SL_9, which is not possible as $SU_6(2)$ is the full covering group of $U_6(2)$. Hence $C_G(t) = A_2E_6$, and we have

$SU_6(2) < E_6 < E_7 < G$. This $SU_6(2)$ contains a subgroup $U = U_5(2)$, and from the last paragraph of the proof of Lemma 3.7, if $S = SU_3(2) < U$ and $E = O_3(S) \cong 3^{1+2}$, then $E < A_2$, a subsystem subgroup of E_7. This A_2 is also a subsystem group in G, and so every S-invariant subspace of $L(G)$ is also A_2-invariant. The result follows, taking $\bar{X} = \langle X, A_2 \rangle$. ■

3.3 Subgroups $X = D_n^\epsilon(2)$

In this section $X = D_n^\epsilon(2)$, where $n \geq 4$, $\epsilon = \pm$, and also for $n = 4$, ϵ can be 3 in which case $D_4^\epsilon(2)$ denotes the twisted group ${}^3D_4(2)$.

Lemma 3.10 *Theorem* 2 *holds for* $X = D_n^\epsilon(2)$, $G = E_8$.

Proof Suppose $X = D_n^\epsilon(2) < G = E_8$, with $n > \frac{1}{2}\mathrm{rank}(G) = 4$. Then X contains a subgroup $D = D_5^\epsilon(2)$.

If $\epsilon = +$ then D has a parabolic subgroup $P_D = 2^{10}.L_5(2)$, and this lies in a proper parabolic subgroup P of G. Using Lemma 3.2 if the Levi factor of P contains E_7 or E_6, we see that $P_D < QA_4$, where Q is a unipotent group and A_4 is a subsystem subgroup of G. The composition factors of A_4 acting on Q have high weight λ_i for some i (see [13, 3.1]), so by Lemma 2.1, the Levi subgroup $L = L_5(2)$ of P_D lies in a subsystem group A_4. Since $L(G) \downarrow A_4$ is completely reducible with all composition factors restricted (see [13, 2.1]), Lemma 2.3 implies that L and A_4 fix the same subspaces of $L(G)$, and this gives the conclusion taking $\bar{X} = \langle X, A_4 \rangle$.

Now suppose $\epsilon = -$. Then D has an element t of order 3 such that $C_D(t) \geq U_5(2) = U$. By Lemma 2.4, $C_G(t) = A_8, A_2E_6, D_7T_1$ or E_7T_1. Hence $U < A_8, D_7$ or E_7. In the first two cases, clearly $U < QA_4$, where Q is unipotent and A_4 is a subsystem group; the same holds when $U < E_7$, arguing as in the last paragraph of the proof of Lemma 3.7. Now we complete the argument as at the end of that proof. ■

Lemma 3.11 *Theorem* 2 *holds for* $X = D_n^\epsilon(2)$, $G = E_7, E_6, F_4$.

Proof We deal with $n = 4, G = E_7$; the result will follow from this, by the argument of the first two paragraphs of the proof of Lemma 3.9. So suppose that $X = D_4^\epsilon(2) < G = E_7$. Then X has a subgroup $S = SU_3(2)$. Let $E = O_3(S) \cong 3^{1+2}$, $Z(E) = \langle z \rangle$ and $x \in E - Z(E)$. By Lemma 3.6, we may take $E = E_3$ or E_4 in the notation of Lemma 3.5.

We shall consider the actions of X on $V_{min} = V_{56}$, and $V_{adj} = V_{132}$, with Brauer characters χ_n $(n = 56, 132)$. Using [7], we see that

$$(*) \quad \chi_n = a\chi_1 + b\chi_8 + c\chi_{26} + d\chi_{48},$$

where χ_i $(i = 8, 26, 48)$ is the Brauer character of an irreducible X-module of dimension i; we do not distinguish here between the three irreducibles of dimension 8 (or 48), as we shall evaluate χ on order 3 elements $z, x \in X$ which have the same trace on all of these modules: namely, $\chi_8(z) = -1, \chi_8(x) = 2$ $\chi_{48}(z) = 3, \chi_{48}(x) = 0$. The values of χ_{26} on z, x are both -1.

Suppose that $E = E_3$. Then $C_G(z) = E_6T_1$, so $\chi_{56}(z) = -25$ by Lemma 2.4, so evaluating $(*)$ for $n = 56$ on the elements $1, z$ gives the equations

$$a + 8b + 26c + 48d = 56, \quad a - b - c + 3d = -25.$$

This is clearly impossible.

Now suppose $E = E_4$. Here $C_G(z) = A_2A_5$ and $E - \{1\}$ is fused, so $\chi_{132}(z) = \chi_{132}(x) = -3$ by 2.4. Evaluating $(*)$ for $n = 132$ on $1, z, x$ gives the equations

$$\begin{aligned} a + 8b + 26c + 48d = &\ 132 \\ a - b - c + 3d = &\ -3 \\ a + 2b - c = &\ -3 \end{aligned}$$

We easily see that the only solution is $(a, b, c, d) = (2, 0, 5, 0)$: in other words,

$$V_{132} \downarrow X = 0100^5/0000^2.$$

From [7] we see that X has a rational element u of order 7 such that $\chi_{26}(u) = -2$, hence $\chi_{132}(u) = -8$. This means that u acts on $L(G)$ with eigenvalues $(1^{(13)}, \lambda^{(20)}, \ldots, \lambda^{6\,(20)})$, where λ is a 7th root of 1. Hence $\dim C_G(u) = 13$. Then $C_G(u) = A_1^3T_4$ or A_2T_5. In the latter case $u \in C(A_2) = A_5$ and $u = \mathrm{diag}(\lambda, \lambda^2, \ldots, \lambda^6) \in A_5 = SL_6$. But $(L(G)/L(A_5)) \downarrow A_5 = \lambda_2^3/\lambda_4^3/0^8$ (see [13, Table 8.2]), from which it follows easily that $\dim C_{L(G)}(u) > 13$, a contradiction. So suppose $C_G(u) = A_1^3T_4$. We have $C_G(A_1) = D_6$, so $u \in D_6$ with $C_{D_6}(u) = A_1^2T_4$. However D_6 has no such element of order 7, a contradiction. ■

3.4 Remaining subgroups over $\mathbb{F}_2$

Suppose $X = X(2) < G$ with $G = F_4, E_6, E_7$ or E_8 and $\mathrm{rank}(X) > \frac{1}{2}\mathrm{rank}(G)$. The possibilities not already dealt with are $X = C_n(2)$, $F_4(2)$,

$E_6^\epsilon(2)$,$E_7(2)$ or $E_8(2)$. These groups contain a subgroup $D_n^-(2)$, $D_4(2)$, $D_5^\epsilon(2)$, $D_5(2)$, $D_5(2)$, respectively. Call this subgroup Y. By what we have proved, there is a connected subgroup $\bar{Y}$ of G containing Y such that Y and $\bar{Y}$ fix the same subspaces of some $V \in \{V_{min}, V_{adj}\}$. Then X and $\bar{X} := \langle X, \bar{Y}\rangle$ fix the same subspaces, as required.

This completes the proof of Theorem 2 for the subgroups $X = X(2)$.

4 The exceptional cases ${}^2A_5(5)$, ${}^2D_5(3)$ in E_8

Lemma 4.1 *Theorem* 2 *holds for $X = U_6(5)$, $G = E_8$ $(p = 5)$.*

Proof Suppose $X = U_6(5) < G = E_8$. Pick an involution $t \in X$ such that $C_X(t) \geq C = SU_2(5) \circ SU_4(5)$.

First we handle the case where $C_G(t) = A_1E_7$. If the factor $SU_2(5)$ lies in the A_1, then as this is a fundamental A_1, it fixes the same subspaces of $L(G)$ as the $SU_2(5)$, and the conclusion follows by defining $\bar{X} = \langle X, A_1\rangle$. So suppose the $SU_2(5)$ does not lie in A_1. Then C projects into the adjoint group $E_7/\langle t\rangle$ as $L_2(5) \times U_4(5) = L_1 \times L_2$, say.

Let $A < L_1$ with $A \cong Alt_4$. Let $O_2(A) = \langle a, b\rangle \cong 2^2$. Then a, b lift to elements of order 4 in simply connected E_7, so have connected E_7-centralizer A_7 or E_6T_1. If the centralizer is A_7, we see as in the proof of [4, 2.15] that $C_{E_7}(a,b)^0 = D_4$, and an element $v \in A$ of order 3 acts as a triality automorphism of this D_4. Thus we have $U_4(5) < C_{D_4}(v)$, which is impossible as the latter group is G_2 or A_2.

Hence $C_{E_7}(a)^0 = E_6T_1$. Moreover, b acts as a graph automorphism of the E_6 factor, so $C_{E_7}(a,b)^0 = C_4$ or F_4 (see [4, 2.7]). Since $L(E_7) \downarrow E_6T_1 = L(E_6T_1) + V(\lambda_1) + V(\lambda_6)$ with b interchanging $V(\lambda_1)$ and $V(\lambda_6)$, we have $\dim C_{E_7}(b) = 27 + \dim C_{E_6}(b)$; as b is conjugate to a, it follows that $C_{E_7}(a,b)^0 = C_{E_6}(b)^0 = F_4$. Thus $L_2 = U_4(5) < F_4$. But this is impossible, as the derived group of the preimage of L_2 in the simply connected group E_7 is $SU_4(5)$ (with centre $\langle t\rangle$), whereas the derived group of the preimage of F_4 has trivial centre.

Now consider the case where $C_G(t) = D_8$. Here $L_2(5) \times U_4(5) = L_1 \times L_2$ embeds in $D_8/\langle t\rangle = PSO_{16}$. Let V_{16} be the corresponding 16-dimensional orthogonal space.

Let $\hat{L} = \hat{L}_1\hat{L}_2$ be the preimage in SO_{16} of $L_1 \times L_2$. Suppose $\hat{L}$ acts on V_{16} as $1 \otimes 100/1 \otimes 001$. Then $\hat{L}$ lies in a parabolic subgroup QA_7 of D_8. The unipotent radical Q is an A_7-module of high weight λ_2 or λ_6, so the composition factors of $\hat{L}_2$ on Q have high weights $\lambda = 200$, 002 or 010. Since $H^1(SU_4(5), \lambda) = 0$ for both of these weights λ (see [19, 1.8]), it follows that $\hat{L}_2 = SU_4(5)$ lies in a Levi subgroup A_7, indeed $\hat{L}_2 < E < A_3A_3 < A_7$, where E is a diagonal subgroup A_3 of the subsystem A_3A_3. Now we see using Lemma 2.3, along with [13, Table 8.1] and the table in [13, p.109], that $\hat{L}_2$ and E fix the same subspaces of $L(G)$, which gives Theorem 2, taking $\bar{X} = \langle X, E\rangle$.

We may now assume that $1 \otimes 100, 1 \otimes 001$ do not appear in $V_{16} \downarrow \hat{L}$. Then $\hat{L}$ must be $L_2(5) \times U_4(5)$. The only possible composition factors for $L_2 = U_4(5)$ on V_{16} are 000, 010, 101 and 200. The latter is impossible as V_{16} is self-dual, and 101 (of dimension 15) is impossible as L_2 centralizes $L_1 = L_2(5)$. Hence $V_{16} \downarrow L_2 = 010^2/000^4$ or $010/000^{10}$. Moreover, $H^1(SU_4(5), 010) = 0$ by [8], so $V_{16} \downarrow L_2$ is completely reducible. It follows that $L_2 = SU_4(5) < D = A_3$, where D is either a subsystem subgroup of G, or a diagonal subgroup of a subsystem A_3A_3. In either case we see as usual using [13] that L_2 and D fix the same subspaces of $L(G)$, giving the result by taking $\bar{X} = \langle X, D\rangle$. ∎

Lemma 4.2 *Theorem* 2 *holds for* $X = P\Omega_{10}^-(3)$, $G = E_8$ $(p = 3)$.

Proof Suppose $X = P\Omega_{10}^-(3) < G = E_8$. There is an involution $t \in X$ such that $C_X(t) \geq D = \Omega_8^+(3)$.

Suppose $C_G(t) = A_1E_7$. Then $D < E_7$, and by [19], there is a connected subgroup D_4 of E_7 containing D, and this D_4 is either a subsystem group or contained in a subsystem A_7. As usual using [13], we see that D and D_4 fix the same subspaces of $L(G)$, giving the result with $\bar{X} = \langle X, D_4\rangle$.

Now suppose $C_G(t) = D_8$, and let V_{16} be the associated othogonal 16-space. By [8] as usual, $V_{16} \downarrow D$ is completely reducible, showing that D lies in a connected subgroup $E = D_4$ of a subsystem D_4D_4, and now we see in the usual way using [13] that D and E fix the same subspaces of $L(G)$. This completes the proof. ∎

5 Deduction of Theorem 1

Assume the hypotheses of Theorem 1, and write $X = F^*(M) = M(q)$. Suppose that X is not of the same type as G (this is conclusion (ii) of Theorem 1). Theorem 1 is already established in [19, Theorem 3] (see also the comment after our statement of Theorem 1), except for the cases where

$q = 2$ or $(M(q), G) = ({}^2A_5(5), E_8)$ or $({}^2D_5(3), E_8)$; so suppose we are in one of these cases.

By Theorem 2, with the possible exception of $(G, X) = (F_4, L_4(2))$, there is a connected subgroup $\bar{X}$ of G containing X, such that X and $\bar{X}$ fix the same subspaces of some $V \in \{V_{min}, V_{adj}\}$.

Assume for the moment that $G \neq F_4$, and also that if $G = E_6$ and $V = V_{27}$ then the almost simple group L does not contain a graph or graph-field automorphism of $F^*(L) = (G_\sigma)'$. Define $\mathrm{Aut}^+(G)$ to be the group generated by inner automorphisms and field morphisms of G. Then $\mathrm{Aut}^+(G)$ acts semilinearly on V. Moreover, by the above assumption L is contained in $\mathrm{Aut}^+(G)$ (where we identify an automorphism of $F^*(L)$ with its extension to $\mathrm{Aut}^+(G)$).

Now [17, Corollary 2] determines all maximal subgroups of G_σ which are irreducible on either V_{min} or V_{adj}. It follows from this result that X acts reducibly on V. Let $\mathcal{M}$ be the set of all subspaces of V which are X-invariant. Define $Y = G_{\mathcal{M}}$. Then $\bar{X} \leq Y$. Moreover Y is $N_{Aut^+(G)}(X)$-invariant (see the proof of [14, 1.12]). In particular Y is L- and σ-invariant. From the maximality of M, we deduce that $M \cap G_\sigma = Y_\sigma$, that $((Y^0)_\sigma)' = X$, and that Y^0 is a maximal connected $N_{Aut^+(G)}(X)$-invariant subgroup in G. At this point [11, Theorem 1] applies to give the possibilities for Y. Now Y is not a parabolic subgroup, so it is reductive. Moreover, since $\mathrm{rank}(X) > \frac{1}{2}\mathrm{rank}(G)$, it follows that Y^0 is simple of rank greater than $\frac{1}{2}\mathrm{rank}(G)$. It follows that either Y is of maximal rank in G, or $(G, Y) = (E_6, F_4)$ (note that by hypothesis, $p = 2$ when $G \neq E_8$). Hence conclusion (i) or (iii) of Theorem 1 holds.

Now suppose that $G = E_6$, $V = V_{27}$ and the almost simple group L contains a graph or graph-field automorphism τ of $F^*(L) = (G_\sigma)'$. Again X is reducible on V by [17]. Now τ interchanges the G-modules V and V^*. Moreover X and $\bar{X}$ fix the same subspaces of both V and V^* (as the latter are the annihilators of the former). Hence if we define $\mathcal{M}$ to be the set of all X-invariant subspaces of both V and V^*, then the argument of the previous paragraph goes through, yielding conclusion (i) or (iii) of Theorem 1.

Finally, suppose $G = F_4$. Here $X = D_4^\epsilon(2), C_4(2), C_3(2)$ or $A_3^\epsilon(2)$ (note that $\mathrm{rank}(X) \leq 4$ by Lemma 2.5). Consider $X = D_4^\epsilon(2)$ or $C_4(2)$. If X is reducible on $V_{26} = V_G(\lambda_4)$ or $V_G(\lambda_1)$, then $\bar{X}$ is proper in G, and clearly $\bar{X} = D_4$, B_4 or C_4. Defining $\mathcal{M}$ as above, we see that $(G_{\mathcal{M}})^0 = \bar{X}$, which is therefore σ-invariant (note that σ is not an exceptional isogeny since ${}^2F_4(q)$ does not contain $D_4^\epsilon(2)$), and hence $X = \bar{X}_\sigma$ is of maximal rank. If

X is irreducible on V_{26}, then [17] shows that X again lies in a connected subgroup $\bar{X} = D_4, B_4$ or C_4 of G. Further, $\bar{X}$ is σ-invariant: for $X < \bar{X}$ and $X < \bar{X}^\sigma = \bar{X}^g$ $(g \in G)$, so $X, X^{g^{-1}} < \bar{X}$, whence $X^{g^{-1}} = X^n$ $(n \in \bar{X})$, giving $ng \in N_G(X)$. Now $N_G(\bar{X})$ induces the full group $\mathrm{Aut}(X)$ on X, and $C_G(X) = 1$. It follows that $N_G(X) \le N_G(\bar{X})$, so $ng \in N_G(\bar{X})$. Therefore $\bar{X}^\sigma = \bar{X}^g = \bar{X}$, as asserted. It follows as before that $X = \bar{X}_\sigma$ is of maximal rank.

To conclude, consider $X = C_3(2)$ or $A_3^\epsilon(2)$. Then X is reducible on V_{26}. If $X = L_4(2)$ then Lemma 3.3 shows that $C_G(X)$ contains a root group. In the other cases we have $X < \bar{X} < G$, and [21] forces $\bar{X}$ either to lie in a B_3- or C_3-parabolic, or to be a maximal rank subgroup D_4, C_4 or B_4. In the latter two cases, X cannot fix the same subspaces of V as $\bar{X}$. Therefore X lies in a B_3- parabolic or C_3-parabolic of G, hence centralizes a root group.

We have established that in all cases, X centralizes a root subgroup of G. But this means that $C_G(X)_\sigma \neq 1$, which contradicts the maximality of M.

This completes the proof of Theorem 1.

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Abstract

Let $G = G(q)$ be a finite almost simple exceptional group of Lie type over the field of q elements, where $q = p^a$ and p is prime. The main result of this paper determines all maximal subgroups M of $G(q)$ such that M is an almost simple group which is also of Lie type in characteristic p, under the condition that $\mathrm{rank}(M) > \frac{1}{2}\mathrm{rank}(G)$. The conclusion is that either M is a subgroup of maximal rank, or it is of the same type as G over a subfield of $\mathbb{F}_q$, or (G, M) is one of $(E_6^\epsilon(q), F_4(q))$, $((E_6^\epsilon(q), C_4(q))$, $(E_7(q), {}^3D_4(q))$. This completes work of the first author with J. Saxl and D. Testerman, in which the same conclusion was obtained under some extra assumptions.