A survey of maximal subgroups of exceptional groups of Lie type

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The object of this survey is to bring the reader up to date with recent results concerning the maximal subgroups of finite and algebraic groups of exceptional Lie type. The first section deals with algebraic groups, and the second with finite groups.

1 Maximal subgroups of exceptional algebraic groups

Let G be a simple algebraic group of exceptional type G_2, F_4, E_6, E_7 or E_8 over an algebraically closed field K of characteristic p. The analysis of maximal subgroups of exceptional groups has a history stretching back to the fundamental work of Dynkin [3], who determined the maximal connected subgroups of G in the case where K has characteristic zero. The flavour of his result is that apart from parabolic subgroups and reductive subgroups of maximal rank, there are just a few further conjugacy classes of maximal connected subgroups, mostly of rather small dimension compared to $\dim G$. In particular, G has only finitely many conjugacy classes of maximal connected subgroups.

The case of positive characteristic was taken up by Seitz [15], who determined the maximal connected subgroups under some assumptions on p, obtaining conclusions similar to those of Dynkin. If $p > 7$ then all these assumptions are satisfied. This result was extended in [7], where all maximal closed subgroups of positive dimension in G were classified, under similar assumptions on p.

In the years since [15, 7], the importance of removing the characteristic assumptions in these results has become increasingly clear, in view of applications to both finite and algebraic group theory (see for example Section 2 below for some such applications). This has finally been achieved in [11]. Here is a statement of the result.

Theorem 1 ([11]) *Let M be a maximal closed subgroup of positive dimension in the exceptional algebraic group G. Then one of the following holds:*

(a) *M is either parabolic or reductive of maximal rank;*

(b) *$G = E_7$, $p \neq 2$ and $M = (2^2 \times D_4).Sym_3$;*

(c) *$G = E_8$, $p \neq 2, 3, 5$ and $M = A_1 \times Sym_5$;*

(d) *M^0 is as in Table 1 below.*

The subgroups M in (b), (c) *and* (d) *exist, are unique up to conjugacy in* $\mathrm{Aut}(G)$, *and are maximal in G.*

Table 1

G	M^0 simple	M^0 not simple
G_2	$A_1\ (p \geq 7)$	
F_4	$A_1\ (p \geq 13)$, $G_2\ (p = 7)$,	$A_1G_2\ (p \neq 2)$
E_6	$A_2\ (p \neq 2, 3)$, $G_2\ (p \neq 7)$, $C_4\ (p \neq 2)$, F_4	A_2G_2
E_7	A_1 (2 classes, $p \geq 17, 19$ resp.), $A_2\ (p \geq 5)$	$A_1A_1\ (p \neq 2, 3)$, $A_1G_2\ (p \neq 2)$, A_1F_4, G_2C_3
E_8	A_1 (3 classes, $p \geq 23, 29, 31$ resp.), $B_2\ (p \geq 5)$	$A_1A_2\ (p \neq 2, 3)$, $A_1G_2G_2\ (p \neq 2)$, G_2F_4

For notational convenience in the table, we set $p = \infty$ if K has characteristic zero; thus, for example, the condition $p \geq 7$ includes the characteristic zero case.

In fact [11] has a somewhat more general version of Theorem 1, which allows the presence of Frobenius and graph morphisms of G.

A few remarks are in order concerning the subgroups occurring in the conclusion of Theorem 1.

The subgroups of G of type (a) in the theorem are well understood. Maximal parabolic subgroups correspond to removing a node of the Dynkin diagram. Subgroups which are reductive of maximal rank are easily determined. They correspond to various subsystems of the root system of G, and a complete list of those which are maximal in G can be found in [11, Table 10.3].

The subgroups under (b) and (c) of Theorem 1 were constructed in [2, 7]: in (b), the connected component $M^0 = D_4$ lies in a subsystem A_7 of G, and in (c), $M^0 = A_1$ lies in a subsystem A_4A_4, with restricted irreducible embedding in each factor.

The subgroups in Table 1 are constructed in [15, 16, 17], apart from a few cases in small characteristic which can be found in [11].

Theorem 1 has a number of consequences. The first is the following, which applies to all types of simple algebraic groups, both classical and exceptional.

Corollary 2 *If H is a simple algebraic group over an algebraically closed field, then H has only finitely many conjugacy classes of maximal closed subgroups of positive dimension.*

Another major consequence of Theorem 1 is that sufficiently large maximal subgroups of finite exceptional groups of Lie type are known. We shall discuss this in the next section.

Also determined in [11] are the precise actions of maximal subgroups X in Table 1 on the adjoint module $L(G)$, as a sum of explicit indecomposable modules. An interesting feature of these actions is that very few types of indecomposables arise. Indeed, with one exception, each restriction $L(G) \downarrow X$ is the sum of indecomposables of one of the following three types: an irreducible module $V(\lambda)$; an indecomposable tilting module $T(\lambda)$; or an indecomposable module $\Delta(\lambda;\gamma)$ of shape $\mu|(\lambda \oplus \gamma)|\mu$ arising in the following way: suppose λ, γ, μ are dominant weights for X such that $T(\lambda) = \mu|\lambda|\mu$ and $T(\gamma) = \mu|\gamma|\mu$ (where μ denotes the irreducible $V(\mu)$, etc.). Then $\Delta(\lambda;\gamma)$ denotes an indecomposable module of shape $\mu|(\lambda \oplus \gamma)|\mu$ with socle and cosocle both of type μ, and which is obtained as a section of $T(\lambda) \oplus T(\gamma)$, by taking a maximal submodule and then factoring out a diagonal submodule of the socle. The one exception to the above is $X = G_2 < E_6$ with $p = 3$, in which case $L(G)' \downarrow X$ is uniserial with series $10|01|11|01|10$.

Finally, we mention that as a consequence of Theorem 1, together with work on finite subgroups of exceptional groups described in the next section, all closed subgroups of G which act irreducibly on either the adjoint module for G, or on one of the irreducible modules of dimension $26 - \delta_{p,3}$, 27 or 56 for $G = F_4, E_6$ or E_7 respectively, have been determined in [12].

2 Maximal subgroups of finite exceptional groups

In this section let G be an adjoint simple algebraic group of exceptional type over $K = \bar{\mathbb{F}}_p$, the algebraic closure of the prime field $\mathbb{F}_p$, where p is a prime, and let σ be a Frobenius morphism of G. Denote by G_σ the fixed point group $\{g \in G : g^\sigma = g\}$. Then $G_0 := G_\sigma'$ is a finite simple exceptional group (exclude the cases $G_2(2)' \cong U_3(3)$ and ${}^2G_2(3)' \cong L_2(8)$).

Throughout the section, let H be a maximal subgroup of G_σ; all the results below apply more generally to maximal subgroups of any almost simple group with socle G_0, but we restrict ourselves to G_σ for notational convenience. The ultimate aim is of course to determine completely all the possiblities for H up to conjugacy. This task is by no means finished, but there has been a great deal of recent progress, and our aim is to bring the reader up to date with this.

First, we present a "reduction theorem", reducing considerations to the case where H is almost simple. In the statement reference is made to the following *exotic* local subgroups of G_σ (one G_σ-class of each):

$2^3.SL_3(2)$	$<$	$G_2(p)$ $(p > 2)$
$3^3.SL_3(3)$	$<$	$F_4(p)$ $(p \geq 5)$
$3^{3+3}.SL_3(3)$	$<$	$E_6^\epsilon(p)$ $(p \equiv \epsilon \bmod 3, p \geq 5)$
$5^3.SL_3(5)$	$<$	$E_8(p^a)$ $(p \neq 2, 5; a = 1$ or 2, as $p^2 \equiv 1$ or $-1 \bmod 5)$
$2^{5+10}.SL_5(2)$	$<$	$E_8(p)$ $(p > 2)$

Note that these local subgroups exist for $p = 2$ in lines 2, 3 and 4, but are non-maximal because of the containments $3^3.SL_3(3) < L_4(3) < F_4(2)$, $3^{3+3}.SL_3(3) < \Omega_7(3) < {}^2E_6(2)$ and $5^3.SL_3(5) < L_4(5) < E_8(4)$ (see [2]).

Theorem 3 ([7, Theorem 2]) *Let H be a maximal subgroup of G_σ as above. Then one of the following holds:*

(i) *H is almost simple;*

(ii) $H = M_\sigma$, *where* M *is a maximal* σ*-stable closed subgroup of positive dimension in* G *as in Theorem* 1*;*

(iii) H *is an exotic local subgroup;*

(iv) $G = E_8, p > 5$ *and* $H = (Alt_5 \times Alt_6).2^2$.

A version of this theorem was also proved by Borovik [1], who in particular discovered the interesting maximal subgroup in part (iv).

In view of this result attention focusses on the case where H is an almost simple maximal subgroup of G_σ. Let H be such, and write $H_0 = F^*(H)$, a simple group. The analysis falls naturally into two cases: $H_0 \in \mathrm{Lie}(p)$, and $H_0 \notin \mathrm{Lie}(p)$, where $\mathrm{Lie}(p)$ denotes the set of finite simple groups of Lie type in characteristic p. We call these *generic* and *non-generic* subgroups, respectively.

We first discuss non-generic subgroups. Here we have the following result, which determines the possibilities for H_0 up to isomorphism; however the problem of determining them up to conjugacy remains open.

Theorem 4 ([10]) *Let* S *be a finite simple group, some cover of which is contained in the exceptional algebraic group* G *, and assume* $S \notin \mathrm{Lie}(p)$*. Then the possibilities for* S *and* G *are given in Table* 2*.*

Table 2

G	S
G_2	$\mathrm{Alt}_5, \mathrm{Alt}_6, L_2(7), L_2(8), L_2(13), U_3(3)$, $\mathrm{Alt}_7(p=5), J_1(p=11), J_2(p=2)$
F_4	above, plus: $\mathrm{Alt}_{7-10}, L_2(17), L_2(25), L_2(27), L_3(3), U_4(2), Sp_6(2), \Omega_8^+(2), {}^3D_4(2), J_2$ $\mathrm{Alt}_{11}(p=11), L_3(4)(p=3), L_4(3)(p=2), {}^2B_2(8)(p=5), M_{11}(p=11)$
E_6	above, plus: $\mathrm{Alt}_{11}, L_2(11), L_2(19), L_3(4), U_4(3), {}^2F_4(2)', M_{11}$, $\mathrm{Alt}_{12}(p=2,3), G_2(3)(p=2), \Omega_7(3)(p=2), M_{22}(p=2,7)$, $J_3(p=2), Fi_{22}(p=2), M_{12}(p=2,3,5)$
E_7	above, plus: $\mathrm{Alt}_{12}, \mathrm{Alt}_{13}, L_2(29), L_2(37), U_3(8), M_{12}$, $\mathrm{Alt}_{14}(p=7), M_{22}(p=5), Ru(p=5), HS(p=5)$
E_8	above, plus: $\mathrm{Alt}_{14-17}, L_2(16), L_2(31), L_2(32), L_2(41), L_2(49)$, $L_2(61), L_3(5), PSp_4(5), G_2(3), {}^2B_2(8)$, $\mathrm{Alt}_{18}(p=3), L_4(5)(p=2), Th(p=3), {}^2B_2(32)(p=5)$

This is actually a condensed version of the main result of [10], which also determines precisely which simple groups (rather than just covers thereof) embed in adjoint exceptional groups.

We now move on to discuss generic maximal subgroups H of G_σ - namely, those for which $H_0 = F^*(H)$ lies in $\mathrm{Lie}(p)$. The expectation in this case is that in general H is of the form M_σ, where M is a maximal closed σ-stable subgroup of positive dimension in G, given by Theorem 1. This is proved in the next result, under some assumptions on the size of the field over which H_0 is defined. In [9], a certain constant $t(G)$ is defined, depending only on the root system of G; and R. Lawther has computed the values of $t(G)$ for all exceptional groups except E_8: we have $t(G) = u(G) \cdot (2, p-1)$, where $u(G)$ is as follows

G	G_2	F_4	E_6	E_7
$u(G)$	12	68	124	388

Theorem 5 ([9]) *Let H be a maximal subgroup of the finite exceptional group G_σ such that $F^*(H) = H(q)$, a simple group of Lie type over $\mathbb{F}_q$, a field of characteristic p. Assume that*

$$\begin{array}{ll} q > t(G), & \text{if } H(q) = L_2(q),\ {}^2B_2(q) \text{ or } {}^2G_2(q) \\ q > 9 \text{ and } H(q) \neq A_2^\epsilon(16), & \text{otherwise.} \end{array}$$

Then one of the following holds:

(i) *$H(q)$ has the same type as G (possibly twisted);*

(ii) *$H = M_\sigma$ for some maximal closed σ-stable subgroup M of positive dimension in G (given by Theorem 1).*

Writing $G_\sigma = G(q_1)$, the subgroups in (i) are subgroups of the form $G(q)$ or a twisted version, where $\mathbb{F}_q$ is a subfield of $\mathbb{F}_{q_1}$; they are unique up to G_σ-conjugacy, by [8, 5.1].

One of the points of this result is that it excludes only finitely many possibilities for $F^*(H) = H(q)$, up to isomorphism. Since there are also only finitely many non-generic simple subgroups up to isomorphism, the following is an immediate consequence.

Corollary 6 *There is a constant c, such that if H is a maximal subgroup of G_σ with $|H| > c$, then either $F^*(H) = H(q)$ has the same type as G, or $H = M_\sigma$ where M is maximal closed σ-stable of positive dimension in G.*

This is all very well, but in practice one needs more information concerning the generic almost simple maximal subgroups which are not covered by Theorem 5. A useful result in this direction is the following, which determines generic maximal subgroups of rank more than half the rank of G. For a simple group of Lie type $H(q)$, let $\mathrm{rk}(H(q))$ denote the untwisted Lie rank of $H(q)$.

Theorem 7 ([5, 13]) *Suppose H is a maximal subgroup of G_σ such that $F^*(H) = H(q)$, a simple group of Lie type in characteristic p, with $rk(H(q)) > \frac{1}{2}rk(G)$. Then either $H(q)$ has the same type as G, or $H = M_\sigma$ where M is maximal closed σ-stable of positive dimension in G. In the latter case, the possibilities are as follows:*

(i) *M is a subgroup of maximal rank (possibilities determined in [6]);*

(ii) *$G_\sigma = E_6^\epsilon(q)$ and $H(q) = F_4(q)$ or $C_4(q)$ (q odd);*

(iii) *$G'_\sigma = E_7(q)$ and $H(q) = {}^3D_4(q)$ (with M as in Theorem 1(b)).*

This is proved in [5, Theorem 3] assuming that $q > 2$, and in [13] for $q = 2$. The maximal subgroups in part (iii) were omitted in error in [5]. They arise when M is the maximal closed subgroup $(2^2 \times D_4).Sym_3$ in Theorem 1(b) and σ acts on M as $\sigma_q w$, where σ_q is the standard field morphism and $w \in Sym_3$ has order 3 (so that $M_\sigma = {}^3D_4(q).3$).

For the reader's convenience, we now present a compendium result which summarises almost all of the work above on maximal subgroups of G_σ.

Theorem 8 *Let H be a maximal subgroup of the finite exceptional group G_σ over $\mathbb{F}_q$, $q = p^a$. The one of the following holds:*

(I) *$H = M_\sigma$ where M is maximal closed σ-stable of positive dimension in G; the possibilities are as follows:*

(a) *M (and H) is a parabolic subgroup;*

(b) *M is reductive of maximal rank: the possibilities for H are determined in* [6]*;*

(c) *$G = E_7$, $p > 2$ and $H = (2^2 \times P\Omega_8^+(q).2^2).Sym_3$ or ${}^3D_4(q).3$;*

(d) *$G = E_8$, $p > 5$ and $H = PGL_2(q) \times Sym_5$;*

(e) *M is as in Table* 1*, and $H = M_\sigma$ as in Table* 3 *below.*

(II) *H is of the same type as G;*

(III) *H is an exotic local subgroup;*

(IV) *$G = E_8$, $p > 5$ and $H = (Alt_5 \times Alt_6).2^2$;*

(V) *$F^*(H) = H_0$ is simple, and not in* Lie(p)*: the possibilities for H_0 are given up to isomorphism by* [10] *(see also Theorem* 4 *above);*

(VI) *$F^*(H) = H(q_0)$ is simple and in* Lie(p)*; moreover $rk(H(q_0)) \leq \frac{1}{2} rk(G)$, and one of the following holds:*

(a) *$q_0 \leq 9$;*

(b) *$H(q_0) = A_2^\epsilon(16)$;*

(c) *$q_0 \leq t(G)$ and $H(q_0) = A_1(q_0)$, ${}^2B_2(q_0)$ or ${}^2G_2(q_0)$.*

In cases (I)-(IV)*, H is determined up to G_σ-conjugacy.*

Table 3

G'_σ	possibilities for $F^*(M_\sigma), M$ in Table 1
$G_2(q)$	$A_1(q)\,(p \geq 7)$
$F_4(q)$	$A_1(q)\,(p \geq 13),\ G_2(q)\,(p = 7),\ A_1(q) \times G_2(q)\,(p \geq 3, q \geq 5)$
$E_6^\epsilon(q)$	$A_2^\epsilon(q)\,(p \geq 5),\ G_2(q)\,(p \neq 7),\ C_4(q)\,(p \geq 3),\ F_4(q),$ $A_2^\epsilon(q) \times G_2(q)\,((q,\epsilon) \neq (2,-))$
$E_7(q)$	$A_1(q)\,(2 \text{ classes}, p \geq 17, 19),\ A_2^\epsilon(q)\,(p \geq 5),\ A_1(q) \times A_1(q)\,(p \geq 5),$ $A_1(q) \times G_2(q)\,(p \geq 3, q \geq 5),\ A_1(q) \times F_4(q)\,(q \geq 4),\ G_2(q) \times C_3(q)$
$E_8(q)$	$A_1(q)\,(3 \text{ classes}, p \geq 23, 29, 31),\ B_2(q)\,(p \geq 5),\ A_1(q) \times A_2^\epsilon(q)\,(p \geq 5),$ $G_2(q) \times F_4(q),\ A_1(q) \times G_2(q) \times G_2(q)\,(p \geq 3, q \geq 5),$ $A_1(q) \times G_2(q^2)\,(p \geq 3, q \geq 5)$

We remind the reader that, as mentioned before, the above results apply more generally to maximal subgroups of all almost simple groups whose socle is a finite exceptional group of Lie type.

Bounds for the orders of maximal subgroups of finite groups of Lie type have proved useful in a variety of applications. For exceptional groups, the first such bounds appeared in [4]; using some of the above results, these were improved as follows in [14].

Theorem 9 ([14, 1.2]) *Let H be a maximal subgroup of the finite exceptional group G_σ over $\mathbb{F}_q$, $q = p^a$. Assume that $|H| \geq 12aq^{56}$, $4aq^{30}$, $4aq^{28}$ or $4aq^{20}$, according as $G = E_8, E_7, E_6$ or F_4, respectively. Then H is as in conclusion* (I)(a),(b) or (e) *of Theorem* 8.

It should be possible to improve these bounds substantially.

Despite the progress reported above, there remain some substantial problems to tackle in the theory of maximal subgroups of finite exceptional groups. The most obvious ones are the determination of the conjugacy classes of non-generic simple subgroups (Theorem 8(IV)), and of generic simple subgroups over small fields (Theorem 8(V)). Of these, perhaps the most challenging and important is to reduce substantially the $t(G)$ bound for subgroups of rank 1 in Theorem 8(V(c)), especially for $G = E_8$, where $t(G)$ is currently unknown (and in any case is known to be quite large).

References

[1] A.V. Borovik, "The structure of finite subgroups of simple algebraic groups" (Russian), *Algebra i Logika* **28** (1989), no. 3, 249-279; translation in *Algebra and Logic* **28** (1989), no. 3, 163-182 (1990).

[2] A.M. Cohen, M.W. Liebeck, J. Saxl and G.M. Seitz, "The local maximal subgroups of exceptional groups of Lie type, finite and algebraic", *Proc. London Math. Soc.* **64** (1992), 21-48.

[3] E.B. Dynkin, “Semisimple subalgebras of semisimple Lie algebras”, *Translations Amer. Math. Soc.* **6** (1957), 111-244.

[4] M.W. Liebeck and J. Saxl, “On the orders of maximal subgroups of the finite exceptional groups of Lie type”, *Proc. London Math. Soc.* **55** (1987), 299-330

[5] M.W. Liebeck, J. Saxl and D. Testerman, “Subgroups of large rank in groups of Lie type”, *Proc. London Math. Soc.* **72** (1996), 425-457.

[6] M.W. Liebeck, J. Saxl and G.M. Seitz, “Subgroups of maximal rank in finite exceptional groups of Lie type”, *Proc. London Math. Soc.* **65** (1992), 297-325.

[7] M.W. Liebeck and G.M. Seitz, “Maximal subgroups of exceptional groups of Lie type, finite and algebraic”, *Geom. Dedicata* **36** (1990), 353-387.

[8] M.W. Liebeck and G.M. Seitz, “Subgroups generated by root elements in groups of Lie type”, *Annals of Math*, **139**, (1994), 293-361.

[9] M.W. Liebeck and G.M. Seitz, “On the subgroup structure of exceptional groups of Lie type”, *Trans. Amer. Math. Soc.* **350** (1998), 3409-3482.

[10] M.W. Liebeck and G.M. Seitz, “On finite subgroups of exceptional algebraic groups”, *J. reine angew. Math.* **515** (1999), 25-72.

[11] M.W. Liebeck and G.M. Seitz, “The maximal subgroups of positive dimension in exceptional algebraic groups”, submitted to *Mem. Amer. Math. Soc.*.

[12] M.W. Liebeck and G.M. Seitz, “Subgroups of exceptional algebraic groups which are irreducible on an adjoint or minimal module”, to appear.

[13] M.W. Liebeck and G.M. Seitz, “Subgroups of large rank in exceptional groups of Lie type”, to appear.

[14] M.W. Liebeck and A. Shalev, “The probability of generating a finite simple group”, *Geom. Dedicata* **56** (1995), 103-113.

[15] G.M. Seitz, “Maximal subgroups of exceptional algebraic groups”, *Mem. Amer. Math. Soc.* Vol. 90, No. 441, 1991.

[16] D.M. Testerman, “A construction of certain maximal subgroups of the algebraic groups E_6 and F_4”, *J. Algebra* **122** (1989), 299-322.

[17] D.M. Testerman, “The construction of the maximal A_1’s in the exceptional algebraic groups”, *Proc. Amer. Math. Soc.* **116** (1992), 635-644.