PERMUTATION REPRESENTATIONS OF NONSPLIT EXTENSIONS INVOLVING ALTERNATING GROUPS

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ABSTRACT. L. Babai has shown that a faithful permutation representation of a nonsplit extension of a group by an alternating group A_k must have degree at least $k^2(\frac{1}{2} - o(1))$, and has asked how sharp this lower bound is. We prove that Babai's bound is sharp (up to a constant factor), by showing that there are such nonsplit extensions that have faithful permutation representations of degree $\frac{3}{2}k(k-1)$. We also reprove Babai's quadratic lower bound with the constant $\frac{1}{2}$ improved to 1 (by completely different methods).

Dedicated to our friend and colleague Alex Lubotzky

1. INTRODUCTION

Let A_k and S_k denote the alternating and symmetric groups of degree k. We consider finite group extensions H of the form

$$1 \to M \to H \to A_k \to 1, \tag{1}$$

where $M \neq 1$ and the extension is nonsplit.

In a November 2016 lecture at the Jerusalem conference "60 Faces to Groups" in honour of Alex Lubotzky's 60th birthday, Laci Babai discussed faithful permutation representations of such groups H, and noted that he had proved a lower bound of the form $k^2(\frac{1}{2} - o(1))$ for the degree of such a representation; this bound appears in [1]. He asked how close to best possible his bound is, suggesting (perhaps provocatively) that there might be an exponential lower bound for the degree of the form C^k for some constant $C > 1$.

In this note, we show that Babai's lower bound is in fact sharp (up to a constant factor), and we also give a different proof of his quadratic lower bound. Here are our two main results.

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Theorem 1.1. *Let $k > 20$. If H is a nonsplit extension as in* (1)*, and H embeds in S_ℓ for some ℓ, then $\ell \geq k(k-1)$.*

With significantly more work, it is likely that the lower bound $k(k-1)$ in this result could be replaced by $\frac{3}{2}k(k-1)$. This would be best possible, by the next result (taking $p = 3$ in part (i)).

Theorem 1.2. *Let $k \geq 10$ and let p be a prime.*

(i) *If p is odd and p divides k, then there is a nonsplit extension H as in* (1)*, with M an elementary abelian p-group, such that H has a faithful permutation representation of degree $\frac{1}{2}pk(k-1)$.*
(ii) *There is a nonsplit extension H as in* (1)*, with M an elementary abelian 2-group, such that H has a faithful transitive permutation representation of degree $2k(k-1)$.*

In §2 we prove Theorem 1.1, and in §3 and §4 we prove the two parts of Theorem 1.2.

Throughout we shall use the notation of [6] for modules for symmetric groups S_n: for a field F and a partition λ of n, S^λ denotes the Specht module, and D^λ the irreducible module for S_n over F corresponding to λ. Also, if H is a subgroup of G, and V is an FH-module, then V_H^G denotes the corresponding induced module for G.

2. Lower bounds: proof of Theorem 1.1

In this section we prove Theorem 1.1. Let $P(H)$ denote the minimal degree of a faithful permutation representation of a group H.

Theorem 2.1. *Let $k > 20$, and suppose H is a nonsplit extension*

$$1 \to M \to H \to A_k \to 1.$$

Then $P(H) \geq k(k-1)$.

Proof. Suppose false, and let H be a minimal counterexample (for a fixed k). Write $G = A_k$, and let π be the projection map $H \to G$. If there is a proper subgroup H_1 of H such that $\pi(H_1) = G$, then H_1 is still a nonsplit extension and so $P(H_1) \geq k(k-1)$ by the minimality of H. This is a contradiction, since $P(H) \geq P(H_1)$. Hence there is no such H_1, and so $M \leq \Phi(H)$, the Frattini subgroup. In particular, M is nilpotent.

As $P(H) < k(k-1)$, H has a subgroup H_0 such that $[H : H_0] < k(k-1)$ and H_0 is maximal subject to not containing M. If H_0 is not corefree in H, its core is a nontrivial normal subgroup N of H; but then H/N is a nonsplit extension of M/N by A_k, and $P(H/N) \leq [H/N : H_0/N] < k(k-1)$, contradicting the minimality of H. Hence H_0 is corefree.

We next claim that $\pi(H_0) = Y \cong A_{k-1}$. For if not, then $[A_k : \pi(H_0)] \geq \frac{1}{2}k(k-1)$ (see [9, 1.1]), and so $[H : H_0] \geq k(k-1)$, a contradiction. Any nontrivial module for Y has dimension at least $k-3$ (see [7, 5.3.7]). Hence if M has no trivial Y-quotient, then $[H : H_0] \geq k \cdot 2^{k-3} > k(k-1)$, a contradiction. Thus Y acts trivially on M/M_0, where $M_0 = M \cap H_0$. By the maximality of H_0, it follows that M_0 has prime index p in M. Since M_0 is corefree, this implies that M is an elementary abelian p-group and hence is an $\mathbb{F}_pG$-module.

There is a surjective Y-homomorphism from M to $\mathbb{F}_p$. By Frobenius reciprocity and the fact that M_0 is corefree, it follows that as an $\mathbb{F}_pG$-module, M injects into the induced module $(\mathbb{F}_p)_Y^G$.

Suppose first that p does not divide k. Then $(\mathbb{F}_p)_Y^G \cong \mathbb{F}_p \oplus S^{(k-1,1)}$. If $p \neq 2$, then $H^2(G, (\mathbb{F}_p)_Y^G) \cong H^2(Y, \mathbb{F}_p) = 0$ and also $H^2(G, \mathbb{F}_p) = 0$ (see [8]) whence $H^2(G, M) = 0$, a contradiction. If $p = 2$, then the same computation shows that $H^2(G, (\mathbb{F}_2)_Y^G)$ is 1-dimensional and similarly $H^2(Y, \mathbb{F}_2)$ is 1-dimensional (again by [8]), whence $H^2(G, S^{(k-1,1)}) = 0$. Thus, M is the trivial module and so H is the double cover of G. However, the double cover of A_{k-1} embeds in the double cover of A_k, whence $\pi(H_0)$ is not Y, a contradiction.

Now suppose that p divides k. Then $(\mathbb{F}_p)_Y^G$ is a uniserial module with trivial socle and head, and heart equal to the irreducible module $D^{(k-1,1)}$ of dimension $k-2$ (notation of [6, p.39]). If $p \neq 2$, it follows by [8] that $H^2(G, M) = 0$ for any submodule M of $(\mathbb{F}_p)_Y^G$, a contradiction. Hence $p = 2$. It is still true by [8] that $H^2(G, D^{(k-1,1)}) = 0$, whence $D^{(k-1,1)}$ is not a quotient of M. Thus, either $M = \mathbb{F}_2$ or $M = (\mathbb{F}_2)_Y^G$. In the first case, we note as above that $\pi(H_0)$ cannot be Y, a contradiction. Hence $M = (\mathbb{F}_2)_Y^G$, of dimension k. Since M is uniserial, it follows that H has a unique minimal normal subgroup.

Note that $[H : H_0] = 2k$. This implies that there is a faithful irreducible $\mathbb{C}H$-module W of dimension less than $2k$. Now any H-orbit on the set of nontrivial linear characters of M has size 1, k or at least $k(k-1)/2$. Since W_M must have a linear constituent that is not fixed by H, by Clifford's theorem there are therefore precisely k distinct linear characters of M occurring in W, and since $\dim W < 2k$, each occurs with multiplicity 1. Hence $\dim W = k$, and so H embeds in $\mathrm{GL}_k(\mathbb{C})$. Since H is perfect (it is perfect modulo the Frattini subgroup), in fact, H embeds in $\mathrm{SL}_k(\mathbb{C})$. But the largest elementary abelian 2-subgroup of $\mathrm{SL}_k(\mathbb{C})$ has rank $k-1$, whereas $M \cong \mathbb{F}_2^k$, a contradiction. This completes the proof. $\square$

We can obtain a better lower bound in certain cases:

Theorem 2.2. *Let $k > 22$, and let H be a nonsplit extension*

$$1 \to M \to H \to A_k \to 1$$

such that $\gcd(2k, |M|) = 1$. *Then* $P(H) \geq \frac{1}{2}k(k-1)(k-2)$.

Proof. Suppose false, and let H be a minimal counterexample. We copy the previous proof. In particular, we deduce that $M \le \Phi(H)$, M is nilpotent and H has a corefree subgroup H_0 of index less than $\frac{1}{2}k(k-1)(k-2)$. By [9, 1.1], H_0M/M must contain $X := A_{k-2}$. As in the previous result, H_0 must normalize a subgroup of prime index r in M. Hence M is an elementary abelian r-group for some odd prime r with r not dividing k. It follows that M embeds in $(\mathbb{F}_r)_X^{A_k}$. In particular, the composition factors of M are among $D^{(k)}$, $D^{(k-1,1)}$, $D^{(k-2,2)}$ and $D^{(k-2,1,1)}$. By [10, Thm. 2] and [4, Thm 4.1, Prop. 5.4], it follows that $H^2(A_k, M) = 0$, a contradiction. □

Remark One can construct examples with M a 3-group such that H has a faithful permutation representation of degree $\frac{3}{2}k(k-1)(k-2)$. The construction is very similar to those given in the next section.

3. Proof of Theorem 1.2 – the odd case

In this section we prove Theorem 1.2(i). Let p be an odd prime and assume that $k \ge 10$ and p divides k. Let $G = A_k$ and let $Y \cong S_{k-2}$ be a Young subgroup stabilizing a subset of size 2.

Let $S = S^{(k-2,1,1)}$ be the Specht module, and $D = D^{(k-2,1,1)}$ and $L = D^{(k-1,1)}$ be irreducible modules for G over $\mathbb{F}_p$ (the restrictions to $G = A_k$ of the corresponding irreducibles for S_k, using the notation of [6]). We need the following relatively easy results about these modules. The first follows from [10, Theorem 2].

Lemma 3.1. (i) $\dim S = \frac{1}{2}(k-1)(k-2)$, $\dim L = k-2$ *and* $\dim D = \frac{1}{2}(k-2)(k-3)$.

(ii) *S is indecomposable with socle isomorphic to L and head isomorphic to D.*

Let θ be the nontrivial 1-dimensional $\mathbb{F}_pY$-module.

Lemma 3.2. (i) *The induced module θ_Y^G has socle and head isomorphic to L, and the maximal submodule modulo the socle is isomorphic to $\mathbb{F}_p \oplus D$.*

(ii) *Let M be the submodule of θ_Y^G with composition factors D and L. Then*

(a) *$H^2(G, M) \ne 0$, and*

(b) *the restriction $M_Y \cong \theta \oplus M_0$ for some $\mathbb{F}_pY$-module M_0.*

Proof. (i) In characteristic 0, the Littlewood-Richardson rule shows that θ_Y^G has composition factors $S^{(k-2,1,1)}$ and $S^{(k-1,1)}$. Hence using Lemma 3.1(ii), we see that in characteristic p, the composition factors are D, $\mathbb{F}_p$ and L (twice).

Let V be the $\mathbb{F}_pG$-module θ_Y^G. We now compute the socle of V. By Frobenius Reciprocity, $\mathrm{Hom}(\mathbb{F}_p, V) = 0$. Also $L_Y = \theta \oplus L_0$ with L_0 irreducible, so $\mathrm{Hom}(L, V) \cong \mathrm{Hom}(L_Y, \theta)$ is 1-dimensional. Finally, $D \cong \wedge^2 L$, so $D_Y \cong (\theta \otimes L_0) \oplus \wedge^2 L_0$, and both summands are irreducible. Hence $\mathrm{Hom}(D, V) \cong \mathrm{Hom}(D_Y, \theta) = 0$. It follows that $\mathrm{soc}(V) \cong L$. Now the conclusion of (i) follows from the fact that V is self-dual.

(ii) By the previous paragraph, θ is the only 1-dimensional composition factor of M_Y, and it appears in the socle. On the other hand, $M \subseteq V = \theta_Y^G$, so by Frobenius reciprocity M_Y surjects onto θ. Hence $M_Y = \theta \oplus M_0$ as claimed.

It remains to show that $H^2(G, M) \neq 0$. First note that for $i = 1, 2$ we have

$$H^i(G, V) = H^i(G, \theta_Y^G) = H^i(Y, \theta) = 0 \tag{2}$$

(here we are using the assumptions that p is not 2 and $k > 9$). Consider

$$0 \to M \to V \to V/M \to 0.$$

Then (2), together with the long exact sequence in cohomology, gives $H^2(G, M) \cong H^1(G, V/M)$. Since V/M is isomorphic to the codimension 1 submodule of the k-dimensional permutation module for G which is uniserial, $H^1(G, V/M) \neq 0$. This completes the proof. □

Proof of Theorem 1.2(i) Let M be as in the previous lemma, and consider the group H defined by a nonsplit extension as follows:

$$1 \to M \to H \to A_k \to 1.$$

(such a nonsplit extension exists, by Lemma 3.2(ii)(a)). Since M is uniserial and $H^2(G, L) = 0$, M is contained in the Frattini subgroup of H. Let M_0 the Y-invariant hyperplane of M, as in Lemma 3.2(ii)(b), and let $E = N_G(M_0)$. Then $E/M = Y$. This gives rise to the sequence

$$1 \to M/M_0 \to E/M_0 \to Y.$$

As observed in (2), $H^2(Y, M/M_0) = H^2(Y, \theta) = 0$, and so the above sequence splits. Thus E contains a subgroup E_0 of index p. The action of H on the cosets of E_0 maps H into the symmetric group of degree $\frac{1}{2}pk(k-1)$. Since the core of E_0 is trivial, this is an embedding of H. This proves Theorem 1.2(i).

4. Proof of Theorem 1.2 – the even case

In this section we prove Theorem 1.2(ii), the case where $p = 2$. In this case, unlike part (i) of the theorem, there is no restriction on residue of k modulo p. . Let $k \geq 10$, let $G = A_k$ acting on $\Omega = \{1, \ldots, k\}$, and let $Y \cong S_{k-2}$ be the stabilizer in G of the subset $\{1, 2\}$. Write $F = \mathbb{F}_2$, and define $P = (F)_Y^G$, the permutation module over F of G acting on the set of pairs in Ω. Define the fixed point space $C_P(Y) = \{v \in P : vy = v \ \forall y \in Y\}$.

Lemma 4.1. (i) *The fixed point space $C_P(Y)$ has dimension* 3.

(ii) $\dim H^2(G, P) = 2$.

Proof. The first statement holds since the action of G on pairs has rank 3. The second follows by the Eckmann–Shapiro Lemma [5, Theorem 4]: $\dim H^2(G,P) = \dim H^2(G,(F)_Y^G) = \dim H^2(Y,F) = 2$. □

Let e_{ij} denote a basis element of P corresponding to the subset $\{i,j\}$ of Ω, where $i<j$. Note that this is an orthonormal basis with respect to the standard inner product $(\ ,\ )$ on P (which is preserved by G). Now define the following elements of P:

$$x_i = \sum_j e_{ij} \quad (1 \le i \le k)$$
$$f = \sum_{i,j} e_{ij}$$
$$u = \sum_{2<i<j} e_{ij}.$$

Note that a basis for the fixed point space $C_P(Y)$ is $\{u,f,y\}$, where $y = x_1 + x_2$. Since P is self dual, it follows also that $P/[Y,P]$ is 3-dimensional, i.e. there is a 3-dimensional trivial quotient of P as an FY-module.

Assume now that $k \equiv 3 \pmod 4$. Then $P = P_1 \oplus P_2 \oplus P_3 \cong F \oplus S^{(k-1,1)} \oplus S^{(k-2,2)}$ with each summand irreducible (where we identify each $S^{(k-i,i)}$ with its reduction modulo 2) . We can identify $P_1 = Ff$, $P_2 = Fx_1 + \ldots + Fx_k$ (of dimension $k-1$) and $P_3 = (P_1+P_2)^{\perp}$. By Frobenius reciprocity, $C_{P_i}(Y)$ has dimension 1 for each i.

Now u is orthogonal to $P_1 + P_2$, since $(u,u) = 0$ (as $k \equiv 3 \pmod 4$), whence $(u,f) = 0$ and $(u,x_i) = k-3 = 0$ for $i>2$ and $(u,x_i) = 0$ for $i = 1,2$. Let $P_0 = (Fu)^{\perp}$. Then P_0 is a Y-invariant hyperplane containing $P_1 + P_2$.

Since $H^2(A_s,F)$ is 1-dimensional for $s \ge 4$, it follows by Frobenius reciprocity that $\dim H^2(G,P_1) = 1$. Also $H^2(G,P_2) = 0$ by [8], and so $\dim H^2(G,P_3) = 1$ by Lemma 4.1(ii).

Define the following two elements in G:

$$g_1 = (1\ 2)(3\ 4),\ g_2 = (3\ 4)(5\ 6),$$

so that $g_1 \in Y \cong S_{k-2}$ and $g_2 \in X$, where $X := G_{12} \cong A_{k-2}$.

We now consider 2-cocycles. We assume that all 2-cocycles δ are normalized so that $\delta(1,h) = \delta(h,1) = 1$ for all h.

By the Eckmann-Shapiro Lemma, as we noted, we have an isomorphism from $H^2(Y,F)$ to $H^2(G,P)$ and these have dimension 2. For each of the four elements in $H^2(Y,F)$, we can choose a 2-cocycle ϵ representing that element, and ϵ is completely determined (up to a coboundary) by $\epsilon(g_1,g_1)$ and $\epsilon(g_2,g_2)$.

Let δ be a 2-cocyle representing an element of $H^2(G,P)$ which is nontrivial in $H^2(G,P_3)$. Let $\epsilon \in H^2(Y,F)$ correspond to δ via the isomorphism given by the Eckmann-Shapiro Lemma. Choose coset representatives g_{ij} for Y in

G as follows:

$$\begin{aligned} g_{ij} &= (1\ i)(2\ j), \text{ if } 2 < i < j, \\ g_{1j} &= (2\ 1\ j) \text{ if } j > 2, \\ g_{2j} &= (1\ 2\ j) \text{ if } j > 2, \\ g_{12} &= 1. \end{aligned}$$

Note that g_{ij} sends $\{1,2\} \to \{i,j\}$ for all i,j.

We need some information about $\delta(g_i, g_i)$ for $i = 1, 2$.

Lemma 4.2. (i) *$\delta(g_1, g_1)$ is not contained in P_0.*
(ii) *$\delta(g_2, g_2)$ is contained in P_0.*

Proof. We proceed by induction on k (assuming as above that $k \equiv 3 \pmod 4$). If $k = 7$ or 11, the conclusion follows by direct computation using Magma [3]. So assume that $k \geq 15$.

Let $\Omega_0 = \{1, \ldots, k-4\}$, and let $G(\Omega_0) \cong A_{k-4}$ be the subgroup of G acting trivially on the complement of Ω_0. Note that the permutation module $P(\Omega_0)$ for $G(\Omega_0)$ acting on pairs in Ω_0 is a $G(\Omega_0)$-summand of P.

Let $m_s = \delta(g_s, g_s)$ for $s = 1, 2$ and write $m_s = \sum \alpha_{ij} e_{ij}$. By the isomorphism given in the Eckmann-Shapiro Lemma (cf. [5, p. 488] or [2, p. 43]), the determination of α_{ij} depends only on $\epsilon(g_s, g_s)$ and the coset representatives g_{ij} given above that take $\{1,2\}$ to $\{i,j\}$. Note that if $i, j \in \Omega_0$, then g_{ij} is in $G(\Omega_0)$. Thus, in computing α_{ij} for $i, j \in \Omega_0$, we can work in $G(\Omega_0)$. It follows that the projection of $\delta(g_s, g_s)$ in $P(\Omega_0)$ is precisely $\delta'(g_s, g_s)$, where δ' is the 2-cocycle corresponding to δ, viewed as a function on $G(\Omega_0) \times G(\Omega_0)$ with values in $P(\Omega_0)$ (i.e. δ' corresponds to ϵ in the isomorphism given by the Eckmann-Shapiro Lemma for the smaller group).

Now define a collection of subsets of Ω, as follows. Write $D = \{1, 2, \ldots, k-8\}$, and let

$$\begin{aligned} \Omega_0 &= D \cup \{k-7, k-6, k-5, k-4\}, \\ \Omega_1 &= D \cup \{k-3, k-2, k-1, k\}, \\ \Omega_2 &= D \cup \{k-7, k-6, k-3, k-2\}, \\ \Omega_3 &= D \cup \{k-5, k-4, k-3, k-2\}, \\ \Omega_4 &= D \cup \{k-7, k-6, k-1, k\}, \\ \Omega_5 &= D \cup \{k-5, k-4, k-1, k\}, \\ \Omega_6 &= D. \end{aligned}$$

Then $u = \sum_0^6 u(\Omega_i)$, where $u(\Omega_i) = \sum_{2<r,s\in\Omega_i} e_{rs}$.

Let $m_j = \delta(g_j, g_j)$ for $j = 1, 2$. By induction we have $(m_1, u(\Omega_i)) = 1$, $(m_2, u(\Omega_i)) = 0$ for all i, and hence $(m_1, u) = 1$ and $(m_2, u) = 0$. Both conclusions of the lemma follow. □

Corollary 4.3. *Let $k \geq 7$ be an integer such that $k \equiv 3 \pmod 4$, and let M be the $\mathbb{F}_2 A_k$-module that is the reduction modulo 2 of the Specht module $S^{(k-2,2)}$. Let H be a nonsplit extension*

$$1 \to M \to H \to A_k \to 1.$$

Then H has a faithful transitive permutation representation of degree $2k(k-1)$.

Proof. Identify M with $P/(P_1+P_2)$, and define M_0 to be the Y-invariant hyperplane $P_0/(P_1+P_2)$. Let $\pi : H \to A_k$ be the canonical map with kernel M, and set $J = \pi^{-1}(Y)$ and $L = \pi^{-1}(X)$.

We have $J = N_H(M_0)$. Consider the group L/M_0. By Lemma 4.2, $L/M_0 \cong \mathbb{Z}/2 \times A_{k-2}$. It follows that H contains a subgroup L_0 containing M_0 with $L_0/M_0 \cong A_{k-2}$. Thus, $[H : L_0] = 2k(k-1)$. Since M is the unique minimal normal subgroup of H, it follows that L_0 is corefree in H. Hence H has a faithful transitive permutation representation of degree $2k(k-1)$. □

We can now prove Theorem 1.2(ii).

Theorem 4.4. *Let $k \geq 7$. Then there is a nonsplit extension*

$$1 \to M \to H \to A_k \to 1,$$

with M an elementary abelian 2-group, such that H has a faithful transitive permutation representation of degree at most $2k(k-1)$.

Proof. If $k \equiv 3 \pmod 4$, the conclusion follows from the previous result, so assume this is not the case. Write $k = j - i$, where $0 < i \leq 3$ and $j \equiv 3 \pmod 4$. Let

$$1 \to M \to H \to A_j \to 1.$$

be the nonsplit extension constructed in Corollary 4.3, and let L_0 be a corefree subgroup of H of index $2j(j-1)$, with coset space $\Gamma = (H : L_0)$. Define J to be the subgroup of H containing M with $J/M = A_k$.

Observe that J is a nonsplit extension of M by A_k, since the coset xM for $x = (1\,2)(3\,4) \in H/M = A_j$ consists of elements of order 4 by Lemma 4.2. Also, the orbits of J on Γ have size 2, $2k$ or $2k(k-1)$. Let J_0 be a minimal subgroup of J with $J = J_0M$. Since J is a nonsplit extension, $N := J_0 \cap M \neq 1$. So J_0 is a nonsplit extension of N by A_k. Some orbit of J must be nontrivial for N and so the image of J on this orbit must be nonsplit. The conclusion follows. □

Remark It is not hard to see that in fact J_0 is faithful only on the orbit of size $2k(k-1)$ (for k sufficiently large, this follows by Theorem 1.1).

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