On a conjecture of G.E. Wall

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Abstract

We prove that if G is a finite almost simple group, having socle of Lie type of rank r, then the number of maximal subgroups of G is at most $Cr^{-2/3}|G|$, where C is an absolute constant. This verifies a conjecture of Wall for groups of sufficiently large rank. Using this we prove that any finite group G has at most $2C|G|^{3/2}$ maximal subgroups.

1 Introduction

For a finite group G, let $\max(G)$ denote the number of maximal proper subgroups of G. In [33], Wall proves that $\max(G) \leq |G|$ for soluble groups G, and conjectures that this is true for all finite groups G. In [22, 4.6] this conjecture was verified for sufficiently large symmetric groups. In this paper we establish new bounds on $\max(G)$ for almost simple groups, and also for general finite groups. Our result for almost simple groups gives something stronger than Wall's conjecture, namely that $\max(G)/|G| \to 0$

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as $\mathrm{rank}(G) \to \infty$, where we define the rank of a group of Lie type to be the rank of the overlying simple algebraic group.

Theorem 1.1 *There exists an absolute constant C such that if G is a finite almost simple group, having socle of Lie type of rank r, then*

$$\max(G) \le Cr^{-2/3}|G|.$$

Corollary 1.2 *Finite simple groups of sufficiently large rank satisfy Wall's conjecture.*

Observe that when r is bounded Theorem 1.1 is best possible up to improvement of the constant: for example, for infinitely many values of q, $G = L_2(q)$ has a maximal subgroup isomorphic to A_5, and hence $\max(G) \ge |G|/60$ for such q. Also for r unbounded, the exponent $-2/3$ is probably not best possible, but cannot be improved by too much: as we will show in Section 3 of the paper, there are families of classical groups of rank r with $\max(G) \ge r^{-2-\epsilon}|G|$ for arbitrarily small $\epsilon > 0$.

We apply Theorem 1.1 and other tools to bound $\max(G)$ for arbitrary finite groups G. An unpublished result of Pyber (see [24, 11.3.4]) states that $\max(G) \le |G|^c$ for some absolute constant c. Here we improve this as follows.

Theorem 1.3 *For any finite group G, we have* $\max(G) \le 2C|G|^{3/2}$, *where C is the constant in Theorem* 1.1.

We also show that groups G with non-abelian socle have at most $(C+1)|G|$ core-free maximal subgroups (see Theorem 4.13).

Along the way we establish the following result, which may be of independent interest. It is an improvement of [25, Corollary 2].

Theorem 1.4 *There is an absolute constant c such that every finite group has at most $cn^{1/2}$ faithful primitive permutation representations of degree n (up to equivalence), where n is an arbitrary positive integer.*

What we actually show in the proof of Theorem 1.4 is the equivalent fact that any finite group has at most $cn^{3/2}$ core-free maximal subgroups of index n.

Our results on $\max(G)$ have applications to some questions in number theory (see [5]).

The paper is divided into three further sections. In the first, Theorem 1.1 is proved. The following section establishes a lower bound on the number of maximal subgroups of certain classical groups, showing that Theorem 1.1 is close to best possible. The final section contains the proof of Theorems 1.3 and 1.4.

2 Proof of Theorem 1.1

Let G be a finite almost simple group having socle of Lie type of rank r over a field $\mathbb{F}_q$, where $q = p^e$ and p is prime. For each prime divisor f of e, G has a subfield subgroup of type $G(q^{1/f})$.

Lemma 2.1 (i) *The number of conjugacy classes of maximal subgroups of G which are not subfield subgroups is at most $c(r)$.*

(ii) *For each divisor f of e, G has at most r conjugacy classes of subfield subgroups of type $G(q^{1/f})$.*

Proof (i) This follows from [17, Theorem 1.3] and the remark following it.

(ii) This is a consequence of [20, 5.1]. ∎

Lemma 2.2 *Theorem* 1.1 *holds if the rank r is bounded.*

Proof Since each conjugacy class of maximal subgroups has size at most $|G|$, the previous lemma implies that

$$\max(G) \leq c(r)|G| + r \sum_{f|e} |G : G(q^{1/f})| < c_1(r)|G|,$$

and the conclusion follows. ∎

Now assume that the rank r is unbounded, so the socle of G is a classical group — denote it by $Cl_n(q)$, where n is the dimension of the natural module V. Thus

$$\mathrm{soc}(G) = Cl_n(q) = Cl(V).$$

A well known theorem of Aschbacher [1] classifies all maximal subgroups of G into eight families $\mathcal{C}_i$ ($1 \leq i \leq 8$) of well understood subgroups (see [14] for detailed descriptions), together with a family $\mathcal{S}$ consisting of almost simple subgroups M whose socle has (projective) representation on V which is absolutely irreducible and is not realised over a proper subfield of $\mathbb{F}_q$.

Define $\mathcal{C}$ to be the union of the families $\mathcal{C}_i$.

Lemma 2.3 *Any maximal subgroup of G has order at least $n^2/2$. Moreover, any subgroup in $\mathcal{S}$ has socle of order at least $n^2/2$.*

Proof Inspection show that this is true for maximal subgroups in $\mathcal{C}$. (In fact a lower bound of n^4 can easily be seen to hold for these.)

Now consider a maximal subgroup $M \in \mathcal{S}$. Let L be the simple socle of M. Then some covering group $\hat{L}$ of L has an absolutely irreducible representation of dimension n, and hence $|\hat{L}| \geq n^2$ (since $|\hat{L}|$ is at least the sum of the squares of the degrees of its absolutely irreducible representations in characteristic p). Now $|\hat{L}| \leq |L|M(L)$, where $M(L)$ is the order of the Schur multiplier of L. If L is an alternating group the result follows since $M(L) = 2$ (recall that n is unbounded). If L is of Lie type, then $n \leq 2|L : B|$ where B is a Borel subgroup, by [28, 2.2], and hence $n^2 \leq 4|L : B|^2$. One checks that this is at most $2|L|$. The conclusion follows. ∎

The next result is a slight refinement of [8, 2.1].

Lemma 2.4 *The number of conjugacy classes of maximal non-subfield subgroups in $\mathcal{C}$ is at most $4n \log n / \log\log n$.*

Proof This follows from [14, Chapter 4]. Lemma 2.1 of [8] gives bounds on the number of Δ-classes of subgroups in each $\mathcal{C}_i$, where Δ is the full isometry group associated with G. Each Δ-class corresponds to a bounded number of G-classes unless G is linear or unitary and the subgroups are in the families $\mathcal{C}_4$, $\mathcal{C}_6$ or $\mathcal{C}_7$ (note that we are excluding $\mathcal{C}_5$ as these are subfield subgroups), so the result follows from [8, 2.1] except in these cases. In the remaining cases each Δ-class corresponds to at most $\sqrt{n}$, n, n G-classes respectively (see [14, 4.4.10, 4.6.5, 4.7.3]), so by [8, 2.1] the numbers of G-classes are at most $2nd(n)$, n, $3n \log n$, respectively. In fact it is easily seen that the number of Δ-classes in $\mathcal{C}_7$ is at most $3 \log n / \log\log n$ (rather than the $3 \log n$ in [8, 2.1]). The conclusion follows. ∎

Lemma 2.5 *The number of maximal subgroups of G lying in $\mathcal{C}$ is at most*

$$\frac{9 \log n}{n \log\log n}|G|.$$

Proof By Lemma 2.3, any conjugacy class of maximal subgroups of G has size at least $2|G|/n^2$. Hence by Lemma 2.4, the number of maximal non-subfield subgroups in $\mathcal{C}$ is at most

$$(2|G|/n^2) \cdot (4n \log n / \log\log n) = \frac{8 \log n}{n \log\log n}|G|.$$

Finally, Lemma 2.1 shows that the number of maximal subfield subgroups of G is at most $\sum_{f|e} |G : G(q^{1/f})| \leq |G| \sum_{f|e} q^{-n/f} \leq |G|n^{-2}$. The result follows. ∎

It remains to bound the number of subgroups of G in the family $\mathcal{S}$. Recall that V is the natural n-dimensional module for G, and denote by $\mathcal{S}_0$ the set of isomorphism types of simple groups which occur as socles of groups in $\mathcal{S}$. For $S \in \mathcal{S}_0$, denote by $\hat{S}$ the universal covering group of S, and let $k(\hat{S})$ be the number of conjugacy classes of $\hat{S}$. Note that each subgroup in $\mathcal{S}$ arises from an absolutely irreducible n-dimensional representation of some group $\hat{S}$ with $S \in \mathcal{S}_0$.

Lemma 2.6 *Let $S = X_t(s)$ be a simple group of Lie type of rank t over $\mathbb{F}_s$, and suppose that S has an absolutely irreducible (projective) representation of dimension n.*

(i) *There is a constant $c(t)$ such that*

$$\frac{k(\hat{S})}{|S|} < c(t)n^{-2}.$$

(ii) *Given $\epsilon > 0$, there is $N = N(\epsilon)$ such that for $n > N$ we have*

$$\frac{k(\hat{S})}{|S|} < n^{-2+\epsilon}.$$

Proof By [28, 2.2], we have $n \le 2|S : B|$, where B is a Borel subgroup of S. For some d we have $|S| \sim s^d$ and $|B| \sim s^{(d-t)/2}$ (where $\sim$ denotes equality up to multiplicative constants depending only on t). Moreover, we have $k(\hat{S}) \sim s^t$ by [18, Theorem 1]. Part (i) follows.

To prove part (ii) it suffices to show that $c(t) < |S : B|^\epsilon$ for sufficiently large t. This follows since the implied constants in the above estimates for $|S|, |B|$ and $k(\hat{S})$ are at most exponential in t, whereas $|S : B|$ is at least exponential in t^2. ■

Lemma 2.7 *For $S \in \mathcal{S}_0$, the number of subgroups in $\mathcal{S}$ with socle isomorphic to S is at most*

$$|\Delta : G \cap \Delta| \cdot |G| \cdot \frac{k(\hat{S})}{|S|},$$

where Δ is the full (projective) isometry group associated with G. Moreover, $|\Delta : G \cap \Delta| \le n$.

Proof By [14, 2.10.4(iii)], the conjugacy class of an absolutely irreducible subgroup in Δ is determined by its representation on V up to equivalence. Given $S \in \mathcal{S}_0$, there are at most $k(\hat{S})$ such representations, each giving rise to at most $|\Delta : G \cap \Delta|$ G-classes of subgroups, and each of these classes has size at most $|G|/|S|$. Hence the number of subgroups in $\mathcal{S}$ with socle isomorphic to S is at most $|\Delta : G \cap \Delta| \cdot k(\hat{S}) \cdot |G|/|S|$, as required. ■

Lemma 2.8 *Fix $c > 0$. Then given any m and any field K, there are at most $f(c)$ absolutely irreducible $K\hat{A}_m$-modules have dimension less than m^c.*

Proof Note that for $m > 7$, $\hat{A}_m = 2.A_m$. For faithful $K\hat{A}_m$-modules, the result follows from [32], which shows that such modules of dimension at least $2^{[(m-\log m-1)/2]}$.

Now consider KA_m-modules. Theorem 5 of [13] shows that for sufficiently large m, every irreducible KS_m-module of dimension less than m^{c+1} lies in a set $R_m(c+1)$, parametrised by (p-regular if $\mathrm{char}(K) = p$) partitions whose first part is at least $m - c - 1$; clearly there are at most $f(c)$ such partitions. For an irreducible KS_m-module W, the restriction $W \downarrow A_m$ is the direct sum of at most 2 conjugate irreducible KA_m-modules, and every irreducible KA_m-module arises in this way. Hence the number of irreducible KA_m-modules of dimension less than m^c is bounded by the number of irreducible KS_m-modules of dimension at most $2m^c$ ($< m^{c+1}$), which is at most $2f(c)$. ■

Lemma 2.9 *Given $\gamma > 0$ there is $N = N(\gamma)$ such that if $n \geq N$, then the number of maximal subgroups in $\mathcal{S}$ with alternating socle is at most $|G|n^{-1+\gamma}$.*

Proof Let $S = A_m \in \mathcal{S}_0$, and fix $\epsilon > 0$. Let $c = 1/\epsilon$. Observe that $m \leq n + 2$ (see [14, 5.3.5]). Also we have $n^2 \leq m!$ by Lemma 2.3, so in particular m is unbounded.

First consider the case where $m > n^\epsilon$. Then $n < m^c$. By the previous lemma, there are at most $f(c)$ irreducible n-dimensional representations of $\hat{A}_m$, and hence the number of subgroups in $\mathcal{S}$ with socle A_m is at most

$$|\Delta : G \cap \Delta| f(c) |G| / |A_m| < 2n f(c) |G| / m!$$

It follows that the number of subgroups in $\mathcal{S}$ with socle A_m, $m > n^\epsilon$, is at most

$$2n f(c) |G| \sum_{n^\epsilon < m \leq n+2} 1/m! < 2n f(c) |G| 2^{-n^\epsilon}. \tag{1}$$

Now consider $m \leq n^\epsilon$. By Lemma 2.7, the number of subgroups in $\mathcal{S}$ with socle A_m, $m \leq n^\epsilon$, is at most

$$|\Delta : G \cap \Delta| \cdot |G| \sum_{m \leq n^\epsilon} \frac{k(\hat{A}_m)}{|A_m|}.$$

Now $k(\hat{A}_m) \leq 2k(A_m) \leq 4k(S_m) = 4P(m) < b^{\sqrt{m}}$ for some constant b. Also $b^{\sqrt{m}} < (m!)^\delta$ for arbitrarily small δ, and $m! \geq n^2$. Hence the above number is at most

$$2n|G| \sum_{m \leq n^\epsilon} (m!)^{-1+\delta} \leq 2n|G| \sum_{m \leq n^\epsilon} n^{2(-1+\delta)} \leq 2n|G| \cdot n^\epsilon \cdot n^{2(-1+\delta)}$$

$$= 2|G|n^{-1+\epsilon+2\delta}.$$

Together with (1), this gives the conclusion. ■

Lemma 2.10 *The number of maximal subgroups in $\mathcal{S}$ with socle of Lie type in p'-characteristic at most $c|G|n^{-2/3}$.*

Proof Let $\mathcal{S}_{p'}$ be the set of maximal subgroups in the statement of the lemma. Let S be the socle of a group in $\mathcal{S}_{p'}$, and assume $S \neq L_2(s)$, ${}^2B_2(s)$ or ${}^2G_2(s)$.

We claim that, given t, the number of possible isomorphism types for S of rank at least t is at most $c_2 n^{1/t}$. To see this, observe that by [15] we have $n \geq \frac{1}{2}(s^t - 1)$. Hence $s \leq c_3(n^{1/t} + n^{1/(t+1)} + \cdots) < c_4 n^{1/t}$, and the claim follows.

We now apply the claim for $t = 4$. Recall that by Lemma 2.6, $k(\hat{S})/|S| < n^{-2+\epsilon}$. Hence by by Lemma 2.7, the number of subgroups in $\mathcal{S}_{p'}$ having socle of rank at least 4 is at most

$$n|G| \cdot c_2 n^{1/4} \cdot n^{-2+\epsilon} = c_2|G|n^{-3/4+\epsilon}.$$

Next, for S of rank 3, or $S = G_2(s)$, we have $n \geq (s^3 - 1)/2$ by [15], so there are at most $c_5 n^{1/3}$ possible isomorphism types for S. For such S we have $k(\hat{S})/|S| < c_6 n^{-2}$ by Lemma 2.6, and so the number of subgroups in $\mathcal{S}_{p'}$ having such socles is at most

$$c_7 n|G| \cdot n^{1/3} \cdot n^{-2} = c_7|G|n^{-2/3}.$$

Now consider $S = L_3^\epsilon(s)$ or $PSp_4(s)$. For these groups, it is shown in [9] (for $L_3(s)$), [12] (for $U_3(s)$), [10] (for $PSp_4(s)$, s odd) and [15] (for $PSp_4(s)$, s even), that there is a bounded set of quadratic polynomials $f_i(x)$, and a positive constant c, such that, in any given characteristic coprime to s, the degree of every irreducible representation of $\hat{S}$ either takes one of the values $f_i(s)$, or is greater than cs^3. Hence again, given n, there are at most $c_8 n^{1/3}$ possible isomorphism types for S, and the conclusion follows as before.

It remains to consider subgroups in $\mathcal{S}_{p'}$ with socle $L_2(s)$, ${}^2B_2(s)$ or ${}^2G_2(s)$. The degrees of the irreducible modular characters of these groups in characteristic coprime to s are determined by the results in [3] (for $L_2(s)$),in [4] (for ${}^2B_2(s)$), and in [11],[16] (for ${}^2G_2(s)$). In particular, there is a bounded number of degrees of irreducible representations of $\hat{S} = SL_2(s), {}^2B_2(s), {}^2G_2(s)$ in any given characteristic coprime to s, given by linear polynomials in s (respectively, polynomials of degree at most 4 in $\sqrt{2s}$, polynomials of degree at most 5 in $\sqrt{3s}$). Hence, given n, there is a bounded number of possibilities for s. Using Lemma 2.6 again, it follows that the number of subgroups in question is at most

$$c_8 n|G| \cdot n^{-2} = c_8|G|n^{-1}.$$

This completes the proof. ■

Finally we estimate the number of subgroups in $\mathcal{S}$ having socle of Lie type in characteristic p. Denote this set by $\mathcal{S}_p$.

Lemma 2.11 *The number of isomorphism types of socles of subgroups in $\mathcal{S}_p$ is at most $c_{15}n^{1/4}$.*

Proof Let $S = X_t(s)$ (of rank t over $\mathbb{F}_s$) be such a socle. We apply results from [26, 27]. First, [26, Table 1B] provides a list of subgroups of classical groups, of the form $Cl_y(q^r) < Cl_{y^r}(q)$, embedded via a twisted tensor product representation of the form $W \otimes W^{(q)} \otimes \cdots \otimes W^{(q^{r-1})}$, where $W = V_y(q^r)$. Then [27, Corollary 6] and its proof imply that either S is one of these subgroups $Cl_y(q^r)$ (with $n = y^r$), or $\mathbb{F}_s$ is a subfield of $\mathbb{F}_q$ of index at most 3. There are at most $c \log n/ \log\log n$ possibilities for $Cl_y(q^r)$, so we may assume that the latter possibility holds.

If $n < t^3/8$, the possibilities for the irreducible representations of $S = X_t(s)$ of dimension n are given by [23, 5.1] (they are just the natural module, its symmetric and alternating squares, the adjoint module, and tensor products of the natural module with a Frobenius twist of itself). It follows that in this case the number of possibilities for S up to isomorphism is bounded by a constant.

In fact, the method of proof of [23, 5.1] shows easily that there is a positive constant c, such that there is just a bounded number of dimensions of irreducible representations of $X_t(s)$ less than ct^4. Hence if $n < ct^4$ the number of possibilities for S up to isomorphism is bounded by a constant.

Finally, when $n \geq ct^4$, the number of isomorphism types is clearly bounded by $c'n^{1/4}$. ■

Lemma 2.12 *Given $\epsilon > 0$, there is $N = N(\epsilon)$ such that for $n > N$ we have*

$$|\mathcal{S}_p| < c|G|n^{-3/4+\epsilon}.$$

Proof Let S be the socle of a subgroup in $\mathcal{S}_p$. Then by Lemmas 2.7 and 2.6, the number of subgroups in $\mathcal{S}_p$ having socle isomorphic to S is at most

$$n|G| \cdot k(\hat{S})/|S| < n|G| \cdot n^{-2+\epsilon} = |G|n^{-1+\epsilon}.$$

Now multiplying by the number of possible isomorphism types for S, bounded by the previous lemma, we obtain the conclusion. ■

Theorem 1.1 follows from Lemmas 2.5, 2.9, 2.10 and 2.12.

3 A lower bound for $\max(G)$

In this section we establish a lower bound

$$\max(G) \geq n^{-2-\epsilon}|G|,$$

where G is a classical group of suitable (large) dimension n, and $\epsilon > 0$ is arbitrary.

To see this, let p be a prime, let $d \geq 3$, and let $H = L_d(p)$. There is an irreducible $\mathbb{F}_p H$-module V of dimension $p^{d(d-1)/2}$, namely the Steinberg module. This embeds H as an absolutely irreducible subgroup of a classical simple group $G = Cl(V) = Cl_n(p)$, where $n = p^{d(d-1)/2}$ and G is the stabilizer of the non-degenerate bilinear form on V preserved by H.

Proposition 3.1 *For sufficiently large p, $N_G(H)$ is a maximal subgroup of G.*

The desired conclusion is a consequence of this result, since it implies that for large p,

$$\max(G) \geq |G : N_G(H)| \geq |G|/|\mathrm{Aut}H| > |G|p^{-d^2},$$

which, given $\epsilon > 0$, is greater than $|G|n^{-2-\epsilon}$ for sufficiently large d.

Proof of Proposition 3.1 Suppose $N_G(H)$ is not maximal; say $N_G(H) < M < G$, where M is a maximal subgroup of G. Then by [1], M lies in one of the families $\mathcal{C}_i$ $(1 \leq i \leq 8)$ or $\mathcal{S}$ of subgroups of G discussed in the previous section.

Suppose first that $N_G(H) < M \in \mathcal{C}_i$ for some i. The subgroups in $\mathcal{C}_1$ are reducible on V, so $i \neq 1$. The subgroups in $\mathcal{C}_2$ are imprimitive on V; however the representation of H on V is primitive, by [29, Theorem 2]. Hence $i \neq 2$. Subgroups in $\mathcal{C}_3$ have socles which are not absolutely irreducible on V, so $i \neq 3$. The socles of subgroups in $\mathcal{C}_4$, $\mathcal{C}_7$ preserve a tensor decomposition of V, whereas H does not (see [30, 1.6]); so $i \neq 4, 7$. Since G is defined over $\mathbb{F}_p$, there are no subgroups in $\mathcal{C}_5$. Finally, members of $\mathcal{C}_6$ are normalizers of extraspecial-type groups, while those in $\mathcal{C}_8$ are full classical groups on V, and it is clear that M cannot be either of these.

Therefore M must lie in the collection $\mathcal{S}$. Recall that $\mathcal{S}$ consists of almost simple subgroups whose socle has (projective) representation on V which is absolutely irreducible. Let M_0 be the socle of M, so $H < M_0 < G$. Then the main theorem of [19] implies that either M_0 is a sporadic group, or it is a group of Lie type in characteristic p. Hence, taking p sufficiently large, we may assume the latter.

We now have $H < M_0 < G$, where $H = L_d(p)$, $M_0 = M_0(q)$ and $G = Cl_n(p)$ are all groups of Lie type in characteristic p. At this point we can apply [21, Theorem 11]: this implies that, provided p is sufficiently large (in fact $p > 7$ suffices), the embedding $H < M_0 < G$ lifts to an embedding $\bar{H} < \bar{M}_0 < \bar{G}$ of connected simple algebraic groups over $\bar{\mathbb{F}}_p$ of the same types. Such triples of irreducible algebraic groups are classified in [30], [31], and the full list is given in [30, Table 1]. Inspecting this table, we see that there are no triples in which the embedding of $\bar{H}$ in $\bar{G}$ restricts to H as the Steinberg representation.

This final contradiction completes the proof of the proposition. ■

4 Proof of Theorems 1.3 and 1.4

In this section we prove Theorems 1.3 and 1.4. The proofs are given at the end of the section, after a considerable amount of preparation. The main ingredients of the proof are Theorem 1.1, the results of Aschbacher and Scott [2], and the following result of Guralnick and Hoffman [7].

Proposition 4.1 ([7]) *If G is a finite group, k a field with $\mathrm{char}(k) = p$ and V an irreducible kG-module which is faithful as a G-module, then*

$$\dim H^1(G, V) \leq \frac{1}{2} \dim V.$$

Corollary 4.2 *Let G be a finite group with a unique minimal normal subgroup A, such that A is abelian. Then G has at most $|A|^{3/2}$ core-free maximal subgroups M. The index of these subgroups is $|A|$.*

Proof If M is a core-free maximal subgroup of G, then it is a complement of A. Moreover, A is a faithful irreducible $\mathbb{F}_p M$-module. By Proposition 4.1, the number of conjugacy classes of such complements M in G is at most $|A|^{\frac{1}{2}}$. Hence the total number of choices for M is at most $|A|^{\frac{3}{2}}$. ■

We denote by $n_p(G)$ the product of the orders of C_p composition factors in a composition series of G.

Lemma 4.3 *Let A be an abelian minimal normal subgroup of a finite group G. If A has order p^r, then the number of maximal subgroups of G not containing A is at most*

$$n_p(G)|A|^{\frac{1}{2}}.$$

Proof Every maximal subgroup M not containing A is a complement of A in G. If A is the unique minimal normal subgroup of G, then our statement follows from Corollary 4.2.

In the general case we proceed by induction on $|G|$. Assume that G has another minimal normal subgroup B, such that A and B are non-isomorphic. We claim that in this case every maximal subgroup M not containing A contains B. For let $P \cong G/\mathrm{core}_G(M)$ be the primitive permutation group obtained by considering the action of G on the (right) cosets of M. If

$\mathrm{core}_G(M)$ does not contain B, then P has two non-isomorphic minimal normal subgroups, which is impossible.

It follows that the maximal subgroups of G not containing A are in one-to-one correspondence with the maximal subgroups of G/B not containing AB/B. By induction in this case the number of possibilities for M is at most

$$n_p(G/B)|A|^{\frac{1}{2}} \leq n_p(G)|A|^{\frac{1}{2}}.$$

It remains to consider the case when B and all other minimal normal subgroups of G are isomorphic to A.

Consider again the primitive group $P \cong G/\mathrm{core}_G(M)$. The images of A and B coincide in P, hence they coincide with the image of $A \times B$. It follows that the intersection of $A \times B$ with $\mathrm{core}_G(M)$ is a normal subgroup of order p^r, hence it is a minimal normal subgroup intersecting A trivially.

Let $A = N_0, N_1, \ldots, N_t$ be the minimal normal subgroups of G contained in $A \times B$. The N_i are intersect trivially pairwise, hence their number is at most $(|A \times B| - 1)/(|A| - 1) = p^r + 1$. That is, we have $t \leq p^r$.

By induction each N_i different from A is contained in at most

$$n_p(G/N_i)|A|^{\frac{1}{2}} \leq n_p(G)|A|^{-\frac{1}{2}}$$

maximal subgroups not containing A. Hence the total number of such maximal subgroups in this case is at most

$$t\, n_p(G)|A|^{-\frac{1}{2}} \leq n_p(G)|A|^{\frac{1}{2}}$$

as required. ∎

The above result essentially reduces the proof of Theorem 1.3 to considering groups G with no abelian minimal normal subgroups. We have to handle the case where G has a core-free maximal subgroup M. In [2, Theorem 1], Aschbacher and Scott give a detailed structural description of such pairs M and G. In the various cases we will quote much simplified versions of their actual results.

We also need the following fact, which can easily be deduced using the information given in [14, 5.1].

Proposition 4.4 *If L is a non-abelian finite simple group, then* $|\mathrm{Out}(L)| \leq |L|^{\frac{1}{4}}$.

The following is an extract from [2, Theorem 1(B)]. For a subgroup K of a group G, denote by K^G the set of G-conjugates of K.

Proposition 4.5 ([2]) *Let G be a finite group with a core-free maximal subgroup M. Assume that G has two non-abelian minimal normal subgroups A and B. Then there exist simple components K and L of G and a full diagonal subgroup D of $K \times L$ such that $A = \langle K^G \rangle$, $B = \langle L^G \rangle$ and $M \cap (A \times B)$ is the direct product of the M-conjugates of D. The groups A, B and $M \cap (A \times B)$ are isomorphic to L^t for some t.*

The number of conjugacy classes of core-free maximal subgroups M of G is at most $t\,|\mathrm{Out}(L)|$.

Corollary 4.6 *Let G be as in Proposition 4.5. The number of core-free maximal subgroups M of G is at most $|A|^{5/4} \leq |G|$. The index of these maximal subgroups M is $|A|$.*

Proof The index of the subgroups M is clearly $|A|$. By Proposition 4.5 the number of choices for M up to conjugacy is at most $t\,|\mathrm{Out}\,(L)|$. It follows from Proposition 4.4 that $|\mathrm{Out}\,(L)|t \leq |A|^{\frac{1}{4}}$. Hence the number of choices for M is at most $|G : M|\ |A|^{\frac{1}{4}} \leq |A|^{\frac{5}{4}}$ as required. ∎

If H is a subgroup of G, then $\mathrm{Aut}_G(H)$ denotes the group of automorphisms of H induced in G; thus

$$\mathrm{Aut}_G(H) \cong N_G(H)/C_G(H).$$

The following is an extract from [2, Theorem 4].

Proposition 4.7 ([2]) *Let G be a finite group with a unique minimal normal subgroup D, which is a power of some non-abelian simple group L, say $D = L_1 \times \ldots \times L_t$ (where $L_i \cong L$). Assume that G has a core-free maximal subgroup M such that*

$$\mathrm{Aut}_M(L_1) = \mathrm{Aut}_G(L_1) \text{ and } M \cap D = 1.$$

Let $\mathcal{E}$ be the set of normal subgroups E of $N_G(L_1)$ such that $D \leq E \leq DC_G(L_1)$ with $DC_G(L_1)/E \cong L$. Then the number of G-conjugacy classes of such maximal subgroups M is at most $|\mathcal{E}|\,|\mathrm{Out}(L)|$.

Corollary 4.8 *If G is as in Proposition 4.7, then the number of core-free maximal complements M of D is at most $\min\left(\frac{|G|}{20}, |D|^{\frac{5}{4}}\right)$. The index of these maximal subgroups M is $|D|$.*

Proof From the definition of $\mathcal{E}$ it follows that the group $DC_G(L_1)/D$ has a quotient isomorphic to L^e where $|\mathcal{E}| = e$. Hence $\mathrm{Aut}(D)/\mathrm{Inn}(D)$ has a section isomorphic to L^e. Since $\mathrm{Aut}(D)/\mathrm{Inn}(D) \leq \mathrm{Out}(L)\,wr\,\mathrm{Sym}(t)$ and

$\mathrm{Out}(L)$ is solvable, in fact $\mathrm{Sym}(t)$ has a section isomorphic to L^e. It follows that $\mathrm{Sym}(t)$ has an elementary abelian section of order p^e for some p and hence that $e \leq t$.

As in the proof of Corollary 4.6 we see that the number of choices for M is at most $t|\mathrm{Out}(L)|\,|G:M| \leq |D|^{\frac{5}{4}}$.

Also, from the first paragraph we have $|M| = |G/D| \geq |L|^e$, hence $e|\mathrm{Out}(L)| \leq |M|^{1/4} < \frac{|M|}{20}$. Hence

$$e|\mathrm{Out}(L)| \cdot |G:M| \leq \frac{|G|}{20}.$$

■

Let D be a group with a direct product decomposition $D = L_1 \times \ldots \times L_t$. A *diagonal subgroup* A of D (with respect to this decomposition) is a subgroup for which each projection $A \to L_i$ is injective; if these maps are in fact isomorphisms, then A is a *full diagonal* subgroup of D. For $J \subseteq \{1, \ldots, t\}$ we write $D_J = \prod_{j \in J} L_j$.

The following is an extract from [2, Theorem 1.C.2] and [2, 6.4.3].

Proposition 4.9 *Let G be a finite group with a unique minimal normal subgroup D, which is a power of some non-abelian simple group L, say $D = L_1 \times \ldots \times L_t$ (where $L_i \cong L$). Assume that G has a core-free maximal subgroup M such that* $\mathrm{Aut}_M(L_1) = \mathrm{Aut}_G(L_1)$ *and* $M \cap D \neq 1$.

Consider the action of G on the set $L_1, \ldots, L_t$ as a permutation group. Let $\mathcal{P}^*(G, L_1)$ denote the set of minimal non-trivial blocks of imprimitivity containing L_1 under this action. Then $M \cap D$ is the direct product of the M-conjugates of some full diagonal subgroup $M \cap D_\Gamma$ of D_Γ for $\Gamma \in \mathcal{P}^*(G, L_1)$. The number of conjugacy classes of such maximal subgroups M is at most $|\mathcal{P}^*(G, L_1)|\,|\mathrm{Out}(L)|$.

Corollary 4.10 *Let G be as in Proposition 4.9. The number of maximal subgroups M as in 4.9 of a given index n is at most $n^{\frac{3}{2}}$. The total number of such maximal subgroups is at most $\frac{|G|}{20}$.*

Proof Denote by $\widetilde{G}$ the transitive permutation group defined by the action of G on the components L_i and let $\widetilde{H}$ be the stabilizer of L_1 in $\widetilde{G}$. The minimal blocks Γ in $\mathcal{P}^*(G, L_1)$ are in one-to-one correspondence with the minimal subgroups of $\widetilde{G}$ containing $\widetilde{H}$ (see [6, Theorem 1.5.A]). Each of these minimal subgroups can be generated by $\widetilde{H}$ and some coset $g\widetilde{H}$ of $\widetilde{H}$ in $\widetilde{G}$. Therefore the number of these subgroups is at most t. Hence the number of possibilities for M up to conjugacy is at most $t\,|\mathrm{Out}(L)| \leq |D|^{\frac{1}{4}}$.

Now $M \cap D$ is a product of at most $\frac{t}{2}$ groups isomorphic to L, which implies that $|L| \leq |M \cap D| \leq |D|^{\frac{1}{2}}$. Therefore $|D|^{\frac{1}{2}} \leq |G : M| \leq \frac{|G|}{|L|t}$, hence the total number of maximal subgroups M considered is at most

$$t\,|\mathrm{Out}(L)|\frac{|G|}{|L|t} \leq \frac{|G|}{|L|^{\frac{3}{4}}} \leq \frac{|G|}{20}.$$

Finally, since $|G : M| \geq |D|^{\frac{1}{2}}$, the number of maximal subgroups M of index n is at most $n|D|^{\frac{1}{4}} \leq n^{\frac{3}{2}}$. ■

The following is an extract from [2, Theorem 1.C.3].

Proposition 4.11 *Let G be a finite group with a unique minimal normal subgroup D, which is a power of some non-abelian simple group L, say $D = L_1 \times \ldots \times L_t$ (where $L_i \cong L$). Assume that G has a core-free maximal subgroup M, such that $\mathrm{Aut}_M(L_1)$ is a core-free maximal subgroup in $A = \mathrm{Aut}_G(L_1)$. Then $G = MD$, $M \cap D$ is the direct product of the M-conjugates of $M \cap L_1$ and $\mathrm{Aut}_M(L_1) \cap \mathrm{Inn}(L_1) = \mathrm{Aut}_{M\cap L_1}(L_1)$.*

The map $M^G \to (\mathrm{Aut}_M(L_1))^A$ gives a bijection between G-conjugacy classes of such maximal subgroups M and A-conjugacy classes of core-free maximal subgroups of A.

Corollary 4.12 *Let G, M be as in Proposition 4.11.*

(i) *The total number of such maximal subgroups M is at most $C|G|$, where C is the absolute constant in Theorem 1.1.*

(ii) *Given $\varepsilon > 0$, there exists $c(\varepsilon)$ (not depending on G) such that for all n, the number of such maximal subgroups M of index n in G is at most $c(\varepsilon)n^{1+\varepsilon}$.*

Proof Let $x_i|A|$ denote the number of maximal subgroups of index i in A. The number of A-conjugacy classes of such maximal subgroups is $\frac{x_i}{i}|A|$.

By Theorem 1.1 we have $\sum x_i \leq C$. Let B be a core-free maximal subgroup of index i in A. We have $|L : B \cap L| = |A : B| = i$. If M is a maximal subgroup of G which corresponds via 4.11 to B, then we have $|L_1 : M \cap L_1| = i$ and hence $n = |G : M| = |D : D \cap M| = i^t$. Therefore the number of maximal subgroups M which correspond to all such subgroups B of index i is

$$\frac{x_i}{i}|A| \cdot i^t \leq x_i|G|.$$

Summing up we see that the total number of maximal subgroups M as above is at most $\sum x_i|G| \leq C|G|$ as required.

By [17, Theorem 1.1] for every $\varepsilon > 0$ there is a constant $c(\varepsilon)$ such that $x_i|A| \leq c(\varepsilon)i^{1+\varepsilon}$ independently of the choice of A. By the above argument this implies that the number of maximal subgroups M of index $n = i^t$ is at most $c(\varepsilon)i^{1+\varepsilon} \leq c(\varepsilon)n^{1+\varepsilon}$. ∎

By [2, Theorem 1] the groups G with a core-free maximal subgroup M and non-abelian socle are described by Propositions 4.5, 4.7, 4.9 and 4.11. Putting together the corresponding corollaries (4.6, 4.8, 4.10 and 4.12), together with Theorem 1.1, we obtain the following.

Theorem 4.13 *Let G be a group with a non-abelian socle. Then G has at most $(C+1)|G|$ core-free maximal subgroups.*

Similarly these corollaries together with Corollary 4.2 imply Theorem 1.4, stated in the Introduction.

Finally we can prove Theorem 1.3.

Proof of Theorem 1.3

We prove the result by induction on $|G|$. Assume first that G has an abelian minimal normal subgroup A. By induction A is contained in at most $2C|G/A|^{\frac{3}{2}}$ maximal subgroups. Using Lemma 4.3 we obtain that

$$\max(G) \leq 2C|G|^{\frac{3}{2}}|A|^{-\frac{3}{2}} + |G|\,|A|^{\frac{1}{2}} \leq C|G|^{\frac{3}{2}} + |G|^{\frac{3}{2}} \leq 2C|G|^{\frac{3}{2}}.$$

Assume now that G has at least three non-abelian minimal normal subgroups A, B and D. Each has order at least 60. Then any maximal subgroup contains A, B or D, hence by induction we have $\max(G) \leq 3\cdot 2C(|G|/60)^{\frac{3}{2}} \leq 2C|G|^{\frac{3}{2}}$.

Now suppose that G has exactly two (non-abelian) minimal normal subgroups A and B. Then by Corollary 4.6, G has at most $|G|$ core-free maximal subgroups. If M is a maximal subgroup of G with $\mathrm{core}_G(M) \neq 1$, then M contains A or B. Hence by induction we see that $\max(G) \leq |G| + 4C(|G|/60)^{\frac{3}{2}} \leq 2C|G|^{\frac{3}{2}}$.

Finally, assume that G has a unique non-abelian minimal normal subgroup D. Then by Theorem 4.13, G has at most $(C+1)|G|$ core-free maximal subgroups. The number of maximal subgroups M with $\mathrm{core}_G(H) \neq 1$ is at most $\max(G/D) \leq 2c\frac{|G|}{60}$. Theorem 1.3 follows.

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