The sparsity of dimensions of irreducible representations of finite simple groups

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Abstract

We show that the set of natural numbers which are dimensions of irreducible complex representations of finite quasisimple groups (excluding the natural representations of alternating groups) has density zero. We also determine the exact asymptotics for this set, showing that it has $(7+o(1))\frac{x}{\log x}$ elements less than x. Our tools combine representation theory and number theory. An application to finite subgroups of classical Lie groups is given.

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1 Introduction

In 1982, Cameron, Neumann and Teague [2] studied the set S of degrees of finite primitive permutation groups excluding A_n and S_n in their natural actions. They showed that S has density zero in $\mathbb{N}$, and indeed that $|S \cap [1,x]| = 2\pi(x) + O(x^{1/2})$, where $\pi(x)$ is the number of primes in the interval $[1,x]$. The core of their proof is the case of almost simple groups, where they count indices of maximal subgroups. This was extended in [5], where the asymptotic behaviour of the set of indices of arbitrary subgroups of almost simple groups is determined.

In this paper we study a linear analogue of these problems. In this context it is natural to consider quasisimple groups, namely perfect groups G such that $G/Z(G)$ is simple.

The alternating group A_n has an irreducible complex representation ρ_n of dimension $n-1$ (the nontrivial irreducible constituent of the natural permutation representation). Define D to be the set of dimensions $\dim \rho$ of irreducible complex representations ρ of finite quasisimple groups with $\rho \neq \rho_n$ for any n. Let

$$D(x) = |D \cap [1,x]|.$$

Our main result is

Theorem 1.1 *We have*

$$D(x) = 2\pi(2x) + 3\pi(x) + O(\frac{x}{\log^2 x}).$$

Consequently

$$D(x) = 7\frac{x}{\log x} + O(\frac{x}{\log^2 x}) = (7 + o(1))\frac{x}{\log x}.$$

The second estimate follows from the first using the Prime Number Theorem $\pi(x) = \frac{x}{\log x} + O(\frac{x}{\log^2 x})$.

The leading term $2\pi(2x) + 3\pi(x)$ of $D(x)$ comes from the 5 character degrees $p \pm 1$, $(p \pm 1)/2$, p of the groups $SL_2(p)$ (p a prime). It is intriguing that the error term relies on classical bounds, going back to Selberg and others, for the number of twin primes, and also the number of solutions of equations $2p_1 + b = p_2$ with p_i prime and $b \in \{\pm 1, \pm 3\}$ (see [4, Chapter 3]).

The main part of our proof of Theorem 1.1 is to show that the quasisimple groups apart from $SL_2(p)$ contribute only marginally to $D(x)$; indeed, we show that they contribute $2\sqrt{2}x^{1/2} + O(\frac{x^{1/2}}{\log x})$. This leads to an upper bound on $D(x)$ which is sharper than that stated in the theorem, namely

$$D(x) \leq 2\pi(2x) + 3\pi(x) + 2\sqrt{2}x^{1/2} + c\frac{x^{1/2}}{\log x} \tag{1}$$

where c is an absolute constant.

Corollary 1.2 *For almost all positive integers n, the only irreducible complex representation of dimension n of any finite quasisimple group is the representation ρ_{n+1} of A_{n+1}.*

Here, "for almost all n" means "for all n in some density 1 subset of $\mathbb{N}$", where the density of a subset B of $\mathbb{N}$ is defined to be $\limsup |B \cap [1, x]|/x$. The corollary is immediate from Theorem 1.1, since this shows that D has density zero.

Theorem 1.1 also has an application to the theory of finite subgroups of classical Lie groups. Define a *Lie primitive* subgroup of a simple algebraic group over $\mathbb{C}$ to be a subgroup which is contained in no proper closed subgroup of positive dimension.

Corollary 1.3 *For almost all positive integers n, $SL_n(\mathbb{C})$ has no finite Lie primitive subgroups.*

In fact, we show that the set of n for which there exist finite Lie primitive subgroups of $SL_n(\mathbb{C})$ is contained in D. The proof is given at the end of the paper, where analogous results for the other classical groups over $\mathbb{C}$ are also discussed.

Notation For any subset B of $\mathbb{N}$, define $B(x) = |B \cap [1, x]|$. For a finite group G, write $k(G)$ for the number of conjugacy classes of G, and $Irr(G)$ for the set of irreducible (complex) characters of G. In particular $k(S_n) = p(n)$, the partition function, and $p(n) < c^{\sqrt{n}}$ for some constant c (see [1, 6.3]).

Throughout the paper c_i are absolute constants.

2 Proofs

For a finite group G and a positive integer d, denote by $r_d(G)$ the number of irreducible complex representations of G of dimension d (up to equivalence). There has been considerable recent interest in studying the representation growth (i.e. the behaviour of the function $r_d(G)$) for both finite and infinite groups, with diverse applications – see [13],[10],[11],[12],[6]. Set

$$R_d(G) = \sum_{k=1}^{d} r_k(G),$$

the number of irreducibles of dimension at most d. Define D_{Alt} (D_{Lie}) to be the subset of D consisting of the dimensions of irreducibles of alternating groups (simple groups of Lie type, resp.) and their covers.

We first study the function $D_{Alt}(x)$, starting with a result which may be of independent interest. It was shown in [10, 1.1] that $r_d(S_n) = d^{o(1)}$. Here we improve this by proving a better bound on the larger function $R_d(S_n)$.

Proposition 2.1 *There is an absolute constant c such that for any $n, d \in \mathbb{N}$,*

$$R_d(S_n) \leq c^{\sqrt{\log d}}.$$

Proof. For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n, let χ_λ be the corresponding irreducible character of S_n, and $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ the conjugate partition. Recall that $\chi_{\lambda'} = \chi_\lambda \otimes \mathrm{sgn}$. We may therefore count characters $\chi = \chi_\lambda$ of degree at most d, assuming that $\lambda_1 \geq \lambda'_1$.

Let $\chi = \chi_\lambda$ be an irreducible character of degree at most d.

Suppose first that $\lambda_1 \leq 2n/3$. Then by [10, 2.4], $\chi_\lambda(1) > c_1^n$, where $c_1 > 1$. Hence $n \leq c_2 \log d$. Therefore the contribution of such χ_λ to $R_d(S_n)$ is at most

$$k(S_n) \leq c_3^{\sqrt{n}} \leq c_3^{\sqrt{c_2 \log d}} = c_4^{\sqrt{\log d}}. \tag{2}$$

Now suppose that $\lambda_1 > 2n/3$, and let $k = n - \lambda_1 < n/3$. Then by [10, 2.1], we have (assuming that n is large)

$$d \geq \chi_\lambda(1) \geq \binom{n-k}{k} \geq 2^k.$$

Hence $k \leq \log d$. Given k, the number of possible λ with $\lambda_1 = n - k$ is $p(k)$.

Therefore the contribution of such χ_λ to $R_d(S_n)$ is at most

$$\sum_{k \le \log d} p(k) \le \log d \cdot c_3^{\sqrt{\log d}} \le c_5^{\sqrt{\log d}}. \tag{3}$$

The result follows from (2) and (3). ■

The bound in Proposition 2.1 is almost tight – indeed, for $d = (n!)^{1/2}$ we have $R_d(S_n) = p(n) \ge c^{\sqrt{\log d / \log \log d}}$.

In the next result, $2A_n$ denotes the double cover of A_n.

Corollary 2.2 *There is an absolute constant c such that for any $n, d \in \mathbb{N}$,*

$$R_d(A_n) \le R_d(2A_n) \le c^{\sqrt{\log d}}.$$

Proof. For each irreducible character χ of S_n, either $\chi \downarrow A_n$ is irreducible, or $\chi \downarrow A_n = \chi_1 + \chi_2$, a sum of two irreducible characters of degree $\chi(1)/2$. All irreducible characters of A_n occur in this way. Hence

$$R_d(A_n) \le 2R_{2d}(S_n).$$

By [15], every faithful irreducible character of $2A_n$ has degree at least $2^{(n - \log_2 n - 2)/2} \ge c_6^n$, where $c_6 > 1$. So the contribution of such characters to $R_d(2A_n)$ is at most

$$k(2A_n) \le 2k(A_n) \le 4p(n) \le c_7^{\sqrt{n}} \le c_8^{\sqrt{\log d}}.$$

The result follows from this, together with Proposition 2.1. ■

The next result determines the contribution of the alternating groups to

$D(x)$.

Proposition 2.3 *We have*

$$D_{Alt}(x) = 2\sqrt{2}x^{1/2} + O(x^{1/3}).$$

Proof. Let $\chi \in Irr(2A_n)$ with $\chi(1) \leq x$, and suppose that $\chi(1) > n-1$ (so that χ is not the character of the representation ρ_n referred to in the Introduction). It follows from [7, Theorem 5] (see also [14]) for A_n, and from [15] for $2A_n$, that either $\chi(1) \in \{\frac{1}{2}n(n-3), \frac{1}{2}(n-1)(n-2)\}$, or $\chi(1)$ is one of three cubic polynomials in n, or $\chi(1) > c_9 n^4$ with $c_9 > 0$.

The numbers $\frac{1}{2}n(n-3)$, $\frac{1}{2}(n-1)(n-2)$ are the degrees of the characters $\chi_{(n-2,2)}$, $\chi_{(n-2,1^2)}$ (note these remain irreducible for A_n). There are $\sqrt{2x}+O(1)$ values of n with $\chi_{(n-2,2)}(1) \leq x$ and similarly for $\chi_{(n-2,1^2)}(1) \leq x$. Also $\chi_{(n-2,2)}(1) \neq \chi_{(m-2,1^2)}(1)$ for any $m, n \geq 5$. Hence the contribution of these characters to $D_{Alt}(x)$ is $2\sqrt{2}x^{1/2} + O(1)$.

Similarly, the contribution of the characters of cubic degree is $O(x^{1/3})$.

Now consider characters χ with $x \geq \chi(1) > c_9 n^4$. There are $O(x^{1/4})$ choices for n. Fixing n, we apply Corollary 2.2, which shows that the contribution of such character degrees of $2A_n$ to $D_{Alt}(x)$ is at most $c^{\sqrt{\log x}}$. Putting everything together, we deduce that

$$D_{Alt}(x) = 2\sqrt{2}x^{1/2} + O(1) + O(x^{1/3}) + O(x^{1/4}c^{\sqrt{\log x}}) = 2\sqrt{2}x^{1/2} + O(x^{1/3}),$$

as required. ∎

The next results determine the contribution of the groups of Lie type to $D(x)$. Our main tool is the following result from [12].

Lemma 2.4 ([12, Theorem 1.5]) *There is an absolute constant c such that every finite quasisimple group of Lie type of rank r has at most $d(r) = c^{r^{5/6}(\log r)^{1/3}}$ distinct irreducible character degrees.*

Notice that this bound does not depend on the field over which the group is defined, and is sub-exponential in the rank.

Write

$$D_{Lie} = E_{Lie} \cup F_{Lie},$$

where E_{Lie} is the set of dimensions of irreducible complex representations of groups $SL_2(p)$ ($p \geq 5$ prime), and F_{Lie} is the corresponding set for all the other groups of Lie type.

Proposition 2.5 *We have*

$$F_{Lie}(x) = O(\frac{x^{1/2}}{\log x}).$$

Proof. Let $G = X_r(q)$ be a group of universal Lie type of untwisted rank r over $\mathbb{F}_q$, and assume $G \neq SL_2(p)$. Let χ be an irreducible character of G with $\chi(1) \leq x$.

Suppose first that the rank $r \geq 3$. Then $\chi(1) \geq c_{10}q^r \geq c_{10}q^3$ by [8], so $r \leq c_{11}\log x$ and $q < c_{12}x^{1/3}$. Given q and r, there are at most 10 possibilities for $X_r(q)$, and each contributes at most $d(r) \leq d(c_{11}\log x)$ to $F_{Lie}(x)$. The number of possibilities for (r,q) is at most $c_{11}\log x \cdot c_{12}x^{1/3} \leq c_{13}x^{1/3}\log x$. Hence the total contribution of groups of rank $r \geq 3$ to $F_{Lie}(x)$ is at most

$$c_{14} \cdot x^{1/3} \cdot \log x \cdot c^{(c_{11}\log x)^{5/6}(\log(c_{11}\log x))^{1/3}} = o(x^{1/3+\epsilon})$$

for any positive ϵ.

Now suppose $r \leq 2$ and $G \neq SL_2(q)$, ${}^2B_2(q)$. Then $x \geq \chi(1) \geq c_{15}q^2$ by [8]. Given q, there are at most c_{16} possible character degrees (see Lemma 2.4). The number of possibilities for the prime power $q \leq c_{17}x^{1/2}$ is bounded by $O(x^{1/2}/\log x)$ by the Prime Number Theorem, and this is also the contribution to $F_{Lie}(x)$.

Finally suppose $G = SL_2(q)$ or ${}^2B_2(q)$. In the latter case we have $x \geq \chi(1) \geq c_{18}q^{3/2}$ by [8], and q is a power of 2, so the contribution to $F_{Lie}(x)$ is $O(\log x)$. And for $G = SL_2(q)$, we have $q = p^a$ with $a \geq 2$ (since $q \neq p$ by the definition of $F_{Lie}(x)$), and $x \geq \chi(1) \geq (q-1)/2$; hence the contribution to $F_{Lie}(x)$ is $O(x^{1/2}/\log x)$. The result follows. ∎

Proposition 2.6 (i) $E_{Lie}(x) \leq 2\pi(2x) + 3\pi(x) + O(1)$.

(ii) $E_{Lie}(x) \geq 2\pi(2x) + 3\pi(x) - O(\frac{x}{\log^2 x})$.

Proof. The nontrivial character degrees of $SL_2(p)$ $(p \geq 5)$ are

$$(p-1)/2,\ (p+1)/2,\ p-1,\ p,\ p+1$$

(see [3]). The number of degrees $(p \pm 1)/2 \leq x$ is at most $2\pi(2x) + O(1)$, and the number of degrees $p, p \pm 1 \leq x$ is at most $3\pi(x) + O(1)$. This proves (i).

For part (ii), we need to estimate the number of coincidences between the above expressions for different values of p. Write $f_1(p), \ldots, f_5(p)$ for $(p-1)/2, \ldots, p+1$ respectively. We require upper bounds for the number of solutions to $f_i(p_1) = f_j(p_2)$ with p_1, p_2 primes. For example, for $i, j = 1, 2$ or $3, 5$, this equation says that p_1, p_2 are twin primes. A classical number theoretic result [4, 3.12] shows that the number of twin primes less than or equal to x is $O(\frac{x}{\log^2 x})$, and that a similar bound holds for equations of the form $p_2 = ap_1 + b$ with a and b fixed integers. These estimates prove part (ii). $\blacksquare$

Proof of Theorem 1.1 By the classification of finite simple groups,

$$D(x) \leq D_{Alt}(x) + E_{Lie}(x) + F_{Lie}(x) + O(1).$$

Combining 2.3, 2.5 and 2.6(i), the upper bound in Theorem 1.1 follows, as does (1). The lower bound in Theorem 1.1 follows from the inequality $D(x) \geq E_{Lie}(x)$ and 2.6(ii).

Proof of Corollary 1.3 Let n be a positive integer, and suppose that $SL_n(\mathbb{C})$ has a finite Lie primitive subgroup G. We aim to show that n is in the set D defined in the Introduction. The result will then follow from Theorem 1.1.

According to [9, Theorem 1], either G is contained in a member of a certain family $\mathcal{C}$ of proper closed subgroups of $SL_n(\mathbb{C})$, or G has a unique quasisimple normal subgroup $E(G)$ which is irreducible on $V = \mathbb{C}^n$. In the former case, inspection shows that all the members of the family $\mathcal{C}$ have positive dimension except those in the sub-family $\mathcal{C}_5$, which only exist when the dimension n is a prime power. Any prime power q is in the set D: indeed, it is the dimension of an irreducible representation of $SL_2(q)$ when $q \geq 5$, and 2,3,4 are dimensions of irreducibles for covers of A_5, A_6. We conclude that $n \in D$ in this case.

Now consider the second case, in which $E(G)$ is quasisimple and irreducible on $\mathbb{C}^n$. By definition of D, this means that either $n \in D$, or

$E(G) = A_{n+1}$ and the representation of $E(G)$ on $\mathbb{C}^n$ is the nontrivial irreducible constituent of the natural permutation representation. However, the latter representation embeds A_{n+1} in $O_n(\mathbb{C})$, an orthogonal subgroup of $SL_n(\mathbb{C})$, contradicting the Lie primitivity of G. Hence $n \in D$ in this case as well, completing the proof. ■

Remark As far as the other classical Lie groups are concerned, the above proof gives the same conclusion for the symplectic groups: namely, if $Sp_{2n}(\mathbb{C})$ has a finite Lie primitive subgroup then $2n \in D$. However, for any $n \geq 4$ the orthogonal group $O_n(\mathbb{C})$ has finite Lie primitive subgroups A_{n+1} and also $O_1(\mathbb{C})\ wr\ S_n = 2^n.S_n$, as well as various subgroups of the latter.

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