Riemannian geometry

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1 Introduction

These notes accompany the Riemannian geometry course at Imperial College London. Please email any typos or suggestions (of which I expect there to be many) to martin.taylor@imperial.ac.uk. Check the blackboard page for the most up to date version of the notes. Sections of the notes marked with a $*$ contain non-examinable material.

We begin with an informal introduction to the course.

1.1 What is Riemannian geometry?

The objects of concern in Riemannian geometry are *manifolds*. Informally (see Section 2.1 for a precise definition), a topological manifold is a topological space which, moreover, "looks locally like $\mathbb{R}^n$". Examples are $\mathbb{R}^n$ itself, the sphere S^n, products of these manifolds, appropriate quotients, etc.

If $\mathcal{M}$ is a topological manifold, then since $\mathcal{M}$ is, in particular, a topological space, one has a notion of what it means for a function $f : \mathcal{M} \to \mathbb{R}$ to be continuous. In order to make sense of the geometric notions which are introduced in this course it becomes necessary to perform calculus on such functions. In order to entertain the notion of the *differentiability* of such a function $f : \mathcal{M} \to \mathbb{R}$, one requires $\mathcal{M}$ to have additional structure. A *differentiable manifold* is a topological manifold equipped with, in addition, a *differentiable structure*. See Section 2.1 for a precise definition of a differentiable structure but, for now, one can think of a differentiable structure as the necessary structure required in order to perform classical calculus on $\mathcal{M}$.

The subject of differential topology is the study of differentiable manifolds and maps between them. But what defines geometry?

In the world of differential topology one cannot tell the difference between donuts and coffee cups. Roughly speaking, geometry is the study of properties, such as lengths, distances, angles, volume, etc., which allow one to distinguish donuts from coffee cups.

Consider, for example, the notion of length. What structure is required on $\mathcal{M}$ in order to make sense of the length of a curve $\gamma : (a, b) \to \mathcal{M}$?

Suppose first that $\mathcal{M} \subset \mathbb{R}^n$ is an embedded manifold (and note that the embedding of $\mathcal{M}$ into $\mathbb{R}^n$ gives $\mathcal{M}$ a "shape" or "geometry"). Given a curve in $\mathcal{M}$, $\gamma : (a, b) \to \mathcal{M} \subset \mathbb{R}^n$, one can simply forget about $\mathcal{M}$ and consider γ merely as a curve in $\mathbb{R}^n$, $\gamma : (a, b) \to \mathbb{R}^n$, for which one has the familiar formula for its length

$$L(\gamma) = \int_a^b \sqrt{\dot{\gamma}(s) \cdot \dot{\gamma}(s)} ds. \tag{1}$$

Here $\dot{\gamma}(s) \in \mathbb{R}^n$ is the tangent vector to the curve γ at time $s \in (a, b)$, and $\cdot$ denotes the Euclidean dot product on $\mathbb{R}^n$.

How does the definition (1) fail when $\mathcal{M}$ is no longer embedded in $\mathbb{R}^n$? If $\mathcal{M}$ is no longer embedded then the tangent vector $\dot{\gamma}(s)$ is no longer an element of $\mathbb{R}^n$, but now lives in the *tangent space* to $\mathcal{M}$ at $\gamma(s)$, denoted $T_{\gamma(s)}\mathcal{M}$ (see Section 3.2 for a precise definition of tangent space). The definition (1) then fails due to the fact that the Euclidean dot product is not (canonically) defined on the tangent space $T_{\gamma(s)}\mathcal{M}$.

One then sees that, in order to abstract the definition (1), the relevant structure one requires is an analogue of the Euclidean dot product on each tangent space of $\mathcal{M}$. Such a structure is known as a *Riemannian metric*. To be (slightly) more precise (see Section 5 for a concrete definition), a Riemannian metric on $\mathcal{M}$ is a "smooth assignment" of an inner product to each tangent space $T_p\mathcal{M}$ of $\mathcal{M}$. Given such a Riemannian metric on $\mathcal{M}$, the length of $\gamma : (a, b) \to \mathcal{M}$ is then defined as

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(s)}(\dot{\gamma}(s), \dot{\gamma}(s))} ds. \tag{2}$$

It turns out that a Riemannian metric gives rise not only to a notion of length, but also other geometric notions such as distances, angles, volume, etc.

A differentiable manifold $\mathcal{M}$ together with a Riemannian metric g is known as a *Riemannian manifold* $(\mathcal{M}, g)$. The subject of Riemannian geometry is the study of Riemannian manifolds.

1.2 Why the need for abstraction?

The whole motivating discussion of the previous section was driven by the desire to define geometric notions on $\mathcal{M}$, without invoking an embedding of $\mathcal{M}$ into $\mathbb{R}^n$. It is worth taking a moment to address the obvious question — why the need for abstraction? Why not define a Riemannian manifold to be a manifold $\mathcal{M}$ along with an embedding $\iota : \mathcal{M} \to \mathbb{R}^n$, for some large n? Why are we so reluctant to invoke an embedding of $\mathcal{M}$ when doing so would absolve us of many of our difficulties? These questions become even more pertinent in view of the following two theorems (whose proofs are outside of the scope of the course).

Theorem 1.1 (Whitney 1936)**.** *If $\mathcal{M}$ is a smooth manifold of dimension n, there exists an embedding (an immersion which is homeomorphic onto its image) $F\colon \mathcal{M} \to \mathbb{R}^{2n+1}$.*

In other words, in studying smooth manifolds one loses no generality in restricting to submanifolds of $\mathbb{R}^n$.

Theorem 1.2 (Nash 1956)**.** *If $(\mathcal{M}, g)$ is a Riemannian manifold of dimension n, there exists an* isometric embedding $F\colon \mathcal{M} \to \mathbb{R}^{\frac{(n+2)(n+3)}{2}}$.[1]

An isometric embedding will be defined later in the course. Until then you should view Theorem 1.2 as the statement that, in the study of Riemannian manifolds, one loses no generality by restricting to submanifolds of $\mathbb{R}^n$.

It should be obvious by now that the reason for considering abstract manifolds is not because one loses any generality in only considering embedded manifolds. The point is — *it is typically the case that the embeddings of Theorem 1.1 and Theorem 1.2 are very unnatural.* This is perhaps difficult to appreciate fully at the moment, when most of the manifolds one has encountered (such as $\mathbb{R}^n$ itself, or S^n) arise so naturally as submanifolds of $\mathbb{R}^n$. Most of the manifolds we are typically interested in, however, do not embed so naturally. (One such example of a smooth manifold, which arises very naturally and will be defined in Section 3.3, is the tangent bundle of the sphere TS^n. Other examples are quotients of familiar spaces, such as the projective plane. *Hyperbolic space* is a Riemannian manifold which does not embed naturally, neither do many very naturally arising Riemannian manifolds in branches of physics such as classical mechanics or general relativity.) One should also add that the proof of Theorem 1.2 is rather involved. It would be very strange if one required such a heavy result in order just to make sense of the basic geometric notions introduced in this course.

After a bit of (what can at first seem like, very) hard work in giving, and becoming familiar with, the abstract definition of a Riemannian manifold, there is a huge payoff later when many of the objects which we will introduce are defined in a way that is *much more natural* than they would be had we insisted on using an embedding into $\mathbb{R}^n$ from the beginning. In other words, were one to insist on only considering embedded manifolds, far from simplifying matters, the embeddings would become a major hinderance.

1.3 Outline of the course

The following is a brief outline of the topics covered in the course.

- Review of notions from differential topology: topological and smooth manifolds, tangent and cotangent spaces, vector bundles, tensor bundles, Lie bracket, Lie derivative.
- Riemannian metrics, lengths, angles, existence of metrics.
- Derivatives of tensor fields, Lie derivative, affine connections, the Levi-Civita connection, parallel transport.
- Geodesics, Riemannian distance, exponential map, Hopf–Rinow Theorem.
- Curvature: Riemann and Ricci curvature tensors, scalar curvature, sectional curvatures.
- Submanifolds: the second fundamental form and the Gauss equation.
- Jacobi fields and second variation of curves.
- Classical theorems in Riemannian geometry: Bonnet–Myers Theorem, Cartan–Hadamard Theorem.
- Time permitting: Lorentzian geometry, Penrose and Hawking singularity theorems.

[1] In the original theorem of Nash, the dimension of the target, $\mathbb{R}^N$, of the embedding was much larger.

1.4 Books

Though the course will not strictly follow a textbook, the most relevant textbooks are [6] and [10]. The relevant background on differential manifolds can be found in the book [11]. The books [2] and [14] were very helpful in preparing the course, though they cover far more material than will be considered here. There is no shortage of other good books on Riemannian geometry. See, for example, [3], [7], [9], [12], [13]. The lecture notes [1], [5], [4] and the book [8] were helpful in preparing Section 13.

2 Smooth manifolds

The basic objects of study in this course are smooth manifolds.

2.1 Topological and smooth manifolds

Recall the following notions from topology.

Definition 2.1 (Hausdorff, second countable, and locally Euclidean topological spaces)**.** *A topological space X is called:*

- Hausdorff *if, for all $x, y \in X$ with $x \neq y$ there exists open sets $U, V \subset X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$;*
- Second countable *if there exists a countable basis for the topology of X;*
- Locally Euclidean of dimension n *for some $n \in \mathbb{N}$ if, for all $x \in X$, there exists an open set $U \subset X$ such that $x \in U$, an open set $V \subset \mathbb{R}^n$, and a homeomorphism $\phi : U \to V$.*

These notions are used to define a topological manifold.

Definition 2.2 (Topological manifolds)**.** *Given $n \in \mathbb{N}$, a* topological manifold of dimension n *is defined to be a topological space $\mathcal{M}$ which is Hausdorff, second countable, and locally Euclidean of dimension n.*

By far the most important component of Definition 2.2 is that $\mathcal{M}$ be locally Euclidean. The other two conditions are imposed to rule out certain pathologies.

Exercise 2.3 (Non-Hausdorff and non-second countable spaces)**.**

1. *Give an example of a topological space X and $n \in \mathbb{N}$ such that X is second countable and locally Euclidean of dimension n, but is not Hausdorff.*
2. *Give an example of a topological space X and $n \in \mathbb{N}$ such that X is Hausdorff and locally Euclidean of dimension n, but is not second countable.*

Given a topological manifold $\mathcal{M}$, one has a notion of what it means for a function $f : \mathcal{M} \to \mathbb{R}$ to be continuous. There is, however, no way to entertain the notion of such a function being *smooth* without introducing additional structure.

Definition 2.4 (Charts and smooth atlases)**.** *Given $n \in \mathbb{N}$, a* smooth n-atlas *of a topological space X is a collection $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ where $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is an open cover of X, there exist open sets $V_\alpha \subset \mathbb{R}^n$ such that $\phi_\alpha : U_\alpha \to V_\alpha$ are homeomorphisms, and for all $\alpha, \beta \in \mathcal{A}$ such that $U_\alpha \cap U_\beta \neq \emptyset$, the map*

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta),$$

is a smooth diffeomorphism.

Each element (U_α, ϕ_α) of the atlas is called a chart*.*

Throughout this course, smooth will always mean C^∞. Though one can, of course, introduce less (or more) regular notions of smooth atlases and smooth manifolds, for simplicity we will not do so here. We often refer to a "smooth atlas", by which we mean a "smooth n-atlas for some $n \in \mathbb{N}$".

Remark 2.5 (Remarks on Definition 2.4)**.**

1. *Note that there is no sense in which the maps ϕ_α and ϕ_β of Definition 2.4 are individually smooth. The composition $\phi_\alpha \circ \phi_\beta^{-1}$, however, is a bonafide map between two open subsets of $\mathbb{R}^n$, and we all know very well what it means for a map between open subsets of $\mathbb{R}^n$ to be smooth.*

2. *There is no restriction on the size of the index set $\mathcal{A}$ in Definition 2.4. It could have one element or could be uncountably infinite.*

Definition 2.6 (Maximal atlases)**.** *A smooth atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ of a topological space X is called* maximal *if, for any (U, ϕ) such that $U \subset X$ is open and $\phi : U \to V$ is a homeomorphism, for some $V \subset \mathbb{R}^n$ open, and the map*

$$\phi \circ \phi_\alpha^{-1} \colon \phi_\alpha(U_\alpha \cap U) \to \phi(U_\alpha \cap U),$$

is a smooth diffeomorphism for all $\alpha \in \mathcal{A}$ such that $U_\alpha \cap U \neq \emptyset$, then $(U, \phi) \in \{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$.

Given any atlas, there clearly exists a unique maximal atlas containing it. Indeed, take the union of the atlas with all such (U, ϕ) as in Definition 2.6. Given a smooth atlas A, let $\max(A)$ denote the unique maximal smooth atlas containing A.

Definition 2.7 (Smooth structures and smooth manifolds)**.** *A* smooth structure *on a topological manifold is a maximal atlas. A* smooth manifold of dimension n *is a topological manifold of dimension n together with a smooth structure.*

The manifolds considered in this course will always be assumed to be connected. From now on, "$\mathcal{M}^n$ is a manifold" means "$\mathcal{M}$ is a connected smooth manifold of dimension n".

For each $i = 1, \ldots, n$, let $\pi^i : \mathbb{R}^n \to \mathbb{R}$ denote the projection onto the i-th component:

$$\pi^i(x^1, \ldots, x^n) := x^i.$$

Definition 2.8 (Local coordinates)**.** *If $\mathcal{M}^n$ is a manifold and (U, ϕ) is a chart, the collection $\{\pi^i \circ \phi \mid i = 1, \ldots, n\}$ is called a* system of local coordinates of $\mathcal{M}$*. We typically abuse notation and write $(x^1, \ldots, x^n)$ as the local coordinates, with the understanding that $x^i = \pi^i \circ \phi$.*

For brevity, one often says "let (U, ϕ, x^i) be a chart of $\mathcal{M}$" to mean that (U, ϕ) is a chart of $\mathcal{M}$ and $x^1, \ldots, x^n$ are the associated system of local coordinates.

2.2 Examples of smooth manifolds

The most basic example of a smooth manifold is $\mathbb{R}^n$ with the Euclidean topology. The identity map $Id\colon \mathbb{R}^n \to \mathbb{R}^n$ is a global chart. Hence $\{(\mathbb{R}^n, Id)\}$ is an atlas. The standard smooth structure on $\mathbb{R}^n$ is the maximal atlas containing $\{(\mathbb{R}^n, Id)\}$.

Consider now the topological manifold

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

with the subspace topology. Let $A_1 = \{(U, \phi), (V, \psi)\}$, where

$$U = \{(\cos\theta, \sin\theta) \mid \theta \in (0, 2\pi)\}, \quad V = \{(\cos\theta, \sin\theta) \mid \theta \in (-\pi, \pi)\},$$

and $\phi\colon U \to (0, 2\pi) \subset \mathbb{R}$ is defined by $\phi(\cos\theta, \sin\theta) = \theta$ (meaning the unique value of $\theta \in (0, 2\pi)$) and $\psi\colon V \to (-\pi, \pi) \subset \mathbb{R}$ is defined by $\psi(\cos\theta, \sin\theta) = \theta$ (meaning the unique value of $\theta \in (-\pi, \pi)$). Clearly

$\{U, V\}$ is an open cover of S^1 and ϕ, ψ are homeomorphisms onto $(0, 2\pi)$ and $(-\pi, \pi)$ respectively. Now $\phi(U \cap V) = (0, \pi) \cup (\pi, 2\pi)$ and $\psi(U \cap V) = (-\pi, 0) \cup (0, \pi)$, and the transition map

$$\psi \circ \phi^{-1}(\theta) = \begin{cases} \theta & \text{if } \theta \in (0, \pi), \\ \theta - 2\pi & \text{if } \theta \in (\pi, 2\pi), \end{cases}$$

is a smooth diffeomorphism. It follows that A_1 is an atlas on S^1, which induces a unique smooth structure on S^1.

Remark 2.9 (Slight abuse of notation)**.** *In accordance with the abuse of notation of Definition 2.8, when it is clear from the context which chart is under consideration, we write $\theta \in S^1$ when in fact we really mean $\theta \in (0, 2\pi)$ and the point in S^1 under consideration is $\phi^{-1}(\theta) \in S^1$. Though this abuse may seem strange at first, it becomes very cumbersome to always include explicit reference to the chart ϕ.*

Consider now $A_2 = \{(U_N, \phi_N), (U_S, \phi_S)\}$ where $U_N = S^1 \setminus \{(0, 1)\}$, $U_S = S^1 \setminus \{(0, -1)\}$ and ϕ_N, ϕ_S denote stereographic projection from $(0, 1)$ and $(0, -1)$ respectively,

$$\phi_N : U_N \to \mathbb{R}, \qquad \phi_S : U_S \to \mathbb{R}.$$

Exercise 2.10 (Atlas defined by stereographic projection)**.**

1. *Derive expressions for $\phi_N(x, y)$ and $\phi_S(x, y)$.*

2. *Show that A_2 is an atlas.*

One can similarly show that

$$S^n = \{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} \mid (x^1)^2 + \ldots + (x^{n+1})^2 = 1\},$$

naturally admits a smooth n-manifold structure for all $n \in \mathbb{N}$.

2.3 Smooth maps and diffeomorphisms

Let $\mathcal{M}^m$ and $\mathcal{N}^n$ be manifolds. It is now possible to define what it means for a function $f : \mathcal{M} \to \mathbb{R}$ to be smooth.

Definition 2.11 (Smooth functions)**.** *A function $f : \mathcal{M} \to \mathbb{R}$ is* smooth at $p \in \mathcal{M}$ *if there exists a chart (U, ϕ) of $\mathcal{M}$ such that $p \in U$ and $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth at $\phi(p) \in \mathbb{R}^m$.*

The function f is smooth *if it is smooth at every $p \in \mathcal{M}$.*

More generally, it is now possible to define what it means for a function $F : \mathcal{M} \to \mathcal{N}$ to be smooth.

Definition 2.12 (Smooth functions between smooth manifolds)**.** *A function $F : \mathcal{M} \to \mathcal{N}$ is* smooth at $p \in \mathcal{M}$ *if there exist charts (U, ϕ) and (V, ψ) of $\mathcal{M}$ and $\mathcal{N}$ respectively such that $p \in U$, $F(p) \in \mathcal{N}$, $F(U) \subset V$,*

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V),$$

is smooth at $\phi(p) \in \mathbb{R}^m$.

The function F is smooth *if it is smooth at every $p \in \mathcal{M}$.*

Note that $\phi(U) \subset \mathbb{R}^m$ and $\psi(V) \subset \mathbb{R}^n$ are open subsets and so the compositions $f \circ \phi^{-1}$ and $\psi \circ F \circ \phi^{-1}$ of Definitions 2.11 and 2.12 respectively are maps between open subsets of Euclidean space and so it indeed makes sense to talk about their smoothness.

Exercise 2.13 (Smoothness of a function is well defined)**.** *Show that smoothness of F is well defined, i.e. that the definition does not depend on the choice of charts: if (U_1, ϕ_1), (U_2, ϕ_2) are charts of $\mathcal{M}$ and (V_1, ψ_1), (V_2, ψ_2) are appropriate charts of $\mathcal{N}$, then $\psi_1 \circ F \circ \phi_1^{-1}$ is smooth at $\phi_1(p)$ if and only if $\psi_2 \circ F \circ \phi_2^{-1}$ is smooth at $\phi_2(p)$.*

Exercise 2.14 (Composition of smooth functions is a smooth function)**.** *Suppose that $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ are smooth manifolds and $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{P}$ are smooth. Show that $G \circ F\colon \mathcal{M} \to \mathcal{P}$ is smooth.*

Definition 2.15 (Smooth diffeomorphisms)**.** *A map $F\colon \mathcal{M} \to \mathcal{N}$ is called a* smooth diffeomorphism *if it is a smooth bijection and $F^{-1}\colon \mathcal{N} \to \mathcal{M}$ is smooth.*

Define

$$C^\infty(\mathcal{M}, \mathcal{N}) = \{\text{smooth maps from } \mathcal{M} \text{ to } \mathcal{N}\},$$

and

$$C^\infty(\mathcal{M}) = C^\infty(\mathcal{M}, \mathbb{R}).$$

As a brief aside, one can ask the following question:

How may smooth structures are there, up to diffeomorphism, on $\mathbb{R}^n$?

As long as $n \neq 4$ the answer, perhaps unsurprisingly, is only one — the standard smooth structure defined in Section 2.2. Surprisingly, on $\mathbb{R}^4$ there are uncountably many non-diffeomorphic smooth structures (the names associated with these facts are Donaldson, Freedman, Taubes).

One can ask a similar question about S^n. In 1956 Milnor constructed non-diffeomorphic smooth structures on S^7, a work for which he was awarded the Fields Medal in 1962. For most values of n it is still unknown whether there exists a non-standard smooth structure on S^n. Such spheres with non-standard smooth structures are known as *exotic spheres.*

Note that the words "up to diffeomorphism" in the above question are important.

Exercise 2.16 (Two atlases on $\mathbb{R}$ such that the identity map is not smooth)**.** *Find two atlases A_1, A_2 on $\mathbb{R}$ such that $(\mathbb{R}, \max(A_1))$ and $(\mathbb{R}, \max(A_2))$ are diffeomorphic, but such that the identity map*

$$Id\colon (\mathbb{R}, \max(A_1)) \to (\mathbb{R}, \max(A_2)),$$

is not a smooth map. Here $\max(A_1)$ and $\max(A_2)$ denote the unique maximal atlases containing A_1 and A_2 repsectively.

Smooth curves are particular examples of smooth maps.

Definition 2.17 (Curves)**.** *For $a, b \in \mathbb{R}$, $a < b$, a smooth map $\gamma\colon (a, b) \to \mathcal{M}$ is call a (smooth)* curve*.*

3 Tangent spaces and the tangent bundle

In the previous section we defined what it means for a function between two manifolds to be differentiable. In this section we discuss what type of object the derivative of such a function is. Understanding this requires first introducing tangent spaces.

3.1 Submanifolds of $\mathbb{R}^n$

If $\mathcal{M} \subset \mathbb{R}^n$ is an embedded manifold, one pictures the tangent space to $\mathcal{M}$ at the point $p \in \mathcal{M}$ as "the set of directions at p" or "the best linear approximation to $\mathcal{M}$ at p". Such definitions do not lend themselves well to abstract manifolds. In this informal section we motivate the definition of tangent space given in Section 3.2 by examining more closely the situation for embedded manifolds.

Note instead that there is a correspondence between "directions at $p \in \mathcal{M}$" and "directional derivatives at p". Indeed, one associates the direction v with the directional derivative in the direction v and vice versa.

For example, consider a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$, a point $p \in \mathbb{R}^n$ and a direction $v \in \mathbb{R}^n$ (note that the tangent space to $\mathbb{R}^n$ is canonically identified with $\mathbb{R}^n$ itself). Define the directional derivative of f at p in the direction v as

$$D_v f|_p = \frac{d}{dt} f(p + tv)|_{t=0}.$$

The directional derivative D_v satisfies the product rule

$$D_v(fg)|_p = D_v f|_p g(p) + f(p) D_v g|_p,$$

for all smooth functions f and g.

Define a *derivation* at $p \in \mathbb{R}^n$ to be a linear map

$$X|_p \colon \{\text{smooth functions at } p\} \to \mathbb{R},$$

which satisfies the product rule

$$X|_p(fg) = X|_p(f) \cdot g(p) + f(p) \cdot X|_p(g),$$

for all smooth functions f, g.

Exercise 3.1 (Derivation/direction correspondence for submanifolds of $\mathbb{R}^n$)**.** *Show that, in $\mathbb{R}^n$, for every derivation $X|_p$ at p there exists a vector $v \in \mathbb{R}^n$ whose associated directional derivative at p is $X|_p$, i.e. such that $D_v|_p = X|_p$.*

Show, moreover, that the set of derivations at p admits a natural vector space structure and is vector space isomorphic to $\mathbb{R}^n$.

The above discussion suggests that, rather than thinking of the tangent space to an embedded manifold at a point p as being the "set of all directions at p", one could equally view the tangent space as the "set of all derivations at p". The latter lends itself much better to generalisation to abstract manifolds.

3.2 Tangent spaces and the differential of a smooth map

Consider now a smooth manifold $\mathcal{M}$. Given $p \in \mathcal{M}$, define the space

$$C^\infty(p) = \{\text{functions } f \text{ defined on an open neighbourhood } U \text{ of } p \text{ such that } f \colon U \to \mathbb{R} \text{ is smooth at } p\}.$$

Note that the space $C^\infty(p)$ is an *algebra* with respect to pointwise multiplication, i.e. if $f, g \in C^\infty(p)$ then $fg \in C^\infty(p)$.

Definition 3.2 (Tangent vectors and tangent space)**.**

- *A* derivation *or* tangent vector X *at $p \in \mathcal{M}$ is a linear functional*

  $$X \colon C^\infty(p) \to \mathbb{R},$$

  which satisfies the Leibniz rule

  $$X(fg) = Xf \cdot g(p) + f(p) \cdot Xg, \tag{3}$$

  for all $f, g \in C^\infty(p)$.

- *The* tangent space *to $\mathcal{M}$ at p, denoted $T_p\mathcal{M}$, is the set of all tangent vectors to $\mathcal{M}$ at p.*

Note the reason for including the Leibniz rule in the definition of a derivation.

Exercise 3.3 (Why the Leibniz rule?)**.** *Why is the Leibniz rule* (3) *included in Definition 3.2? More precisely, where is the Leibniz rule used in Exercise 3.1? What additional types of objects would be tangent vectors to $\mathbb{R}^n$ if* (3) *was not included in the definition?*

Given $X, Y \in T_p\mathcal{M}$ and $\lambda \in \mathbb{R}$, define

$$(X + Y)f = Xf + Yf, \qquad (\lambda X)f = \lambda Xf,$$

for all $f \in C^\infty(p)$.

Exercise 3.4 (The tangent space is a vector space)**.** *Show that the tangent space $T_p\mathcal{M}$ is a vector space with the above addition and scalar multiplication operations.*

Any curve $\gamma\colon (-\varepsilon,\varepsilon)\to\mathcal{M}$ such that $\gamma(0)=p\in\mathcal{M}$ defines a derivation

$$\dot\gamma(0)\colon C^\infty(p)\to\mathbb{R},$$

by

$$\dot\gamma(0)f := \frac{d}{ds}\Big|_{s=0} f\circ\gamma = \frac{df(\gamma(s))}{ds}\Big|_{s=0}.$$

(Note that $f\circ\gamma$ is a map $f\circ\gamma\colon(-\varepsilon,\varepsilon)\to\mathbb{R}$, and so we indeed know how to define $\frac{d}{ds}$ of such a composition.) Indeed, if $f,g\in C^\infty(p)$ then $f\circ\gamma, g\circ\gamma\colon(-\varepsilon,\varepsilon)\to\mathbb{R}$ and

$$\dot\gamma(0)fg = \frac{df(\gamma(s))g(\gamma(s))}{ds}\Big|_{s=0} = \dot\gamma(0)f\cdot g(p) + f(p)\cdot\dot\gamma(0)g,$$

by the product rule on $\mathbb{R}$.

Recall now that a chart (U,ϕ) of $\mathcal{M}$ defines a local coordinate system $(x^1,\dots,x^n)$ (see Definition 2.8).

Definition 3.5 (Coordinate vectors)**.** *Given a chart (U,ϕ,x^i) of $\mathcal{M}$ and $p\in U$, define the* coordinate vectors

$$\frac{\partial}{\partial x^1}\Big|_p,\dots,\frac{\partial}{\partial x^n}\Big|_p\in T_p\mathcal{M},$$

by

$$\frac{\partial}{\partial x^i}\Big|_p f := \frac{\partial f\circ\phi^{-1}}{\partial x^i}\Big|_{\phi(p)}, \tag{4}$$

for all $f\in C^\infty(p)$.

Note that on the right hand side of (4) x^i denotes the i-th standard coordinate on $\mathbb{R}^n$, and that $f\circ\phi^{-1}\colon\phi(U)\to\mathbb{R}^n$, where $\phi(U)\subset\mathbb{R}^n$ is an open set.

Let e_i denote the standard i-th basis vector and define the curves $\alpha_i\colon(-\varepsilon,\varepsilon)\to U\subset\mathcal{M}$ for $i=1,\dots,n$ by

$$\alpha_i(s) = \phi^{-1}\big(\phi(p)+se_i\big).$$

Exercise 3.6 (Coordinate vectors as tangent vectors to curves)**.** *Show that the tangent vectors to the curves $\alpha_1,\dots,\alpha_n$ at $s=0$ are the coordinate vectors of Definition 3.5,*

$$\dot\alpha_i(0) = \frac{\partial}{\partial x^i}\Big|_p.$$

Exercise 3.7 (Coordinate vectors form a basis)**.** *Show that the coordinate vectors $\frac{\partial}{\partial x^1}\big|_p,\dots,\frac{\partial}{\partial x^n}\big|_p$ form a basis of $T_p\mathcal{M}$.*

Exercise 3.7 in particular implies that any $X\in T_p\mathcal{M}$ can be uniquely written as

$$X = \sum_{i=1}^n X^i\frac{\partial}{\partial x^i}\Big|_p, \tag{5}$$

for some $X^1,\dots,X^n\in\mathbb{R}$.

Throughout this course Einstein's greatest contribution to differential geometry, namely the Einstein summation convention, will be adopted, whereby the summation sign $\sum$ is omitted and repeated indices indicate a summation.[2] For example (5) is simply written as

$$X = X^i\frac{\partial}{\partial x^i}\Big|_p.$$

[2] Einstein once joked to a friend *"I have made a great discovery in mathematics; I have suppressed the summation sign every time that the summation must be made over an index which occurs twice ...."*

Note in particular that, for any vector $X \in T_p\mathcal{M}$, the curve $\gamma_X : (-\varepsilon, \varepsilon) \to \mathcal{M}$ defined by

$$\gamma_X(t) = \phi^{-1}(\phi(p) + tX^i e_i),$$

satisfies $\dot{\gamma}_X(0) = X$, i.e. every tangent vector arises as a tangent vector to some curve.

Exercise 3.8 (Change of coordinate vectors under change of coordinates)**.** *Consider two charts (U, ϕ) and $(\widetilde{U}, \widetilde{\phi})$ with associated local coordinates $(x^1, \dots, x^n)$ and $(\widetilde{x}^1, \dots, \widetilde{x}^n)$ respectively. If $p \in U \cap \widetilde{U}$ then*

$$\frac{\partial}{\partial \widetilde{x}^i}\Big|_p = \frac{\partial x^j}{\partial \widetilde{x}^i} \frac{\partial}{\partial x^j}\Big|_p,$$

where

$$\frac{\partial x^j}{\partial \widetilde{x}^i}(p) := \partial_i(\pi^j \circ \phi \circ \widetilde{\phi}^{-1})(\widetilde{\phi}(p)).$$

In particular, if $p \in U \cap \widetilde{U}$ and $X \in T_p\mathcal{M}$, then

$$X = X^i \frac{\partial}{\partial x^i}\Big|_p = \widetilde{X}^i \frac{\partial}{\partial \widetilde{x}^i}\Big|_p,$$

where

$$X^i = \widetilde{X}^j \frac{\partial x^i}{\partial \widetilde{x}^j}(p).$$

Given a smooth function $F : \mathcal{M} \to \mathcal{N}$, the derivative of F at $p \in \mathcal{M}$ is defined to be a linear map from $T_p\mathcal{M}$ to $T_{F(p)}\mathcal{N}$.

Definition 3.9 (Pushforward)**.** *Given $p \in \mathcal{M}$, the map $F_{*p} : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$, also denoted $dF|_p$, defined by*

$$(F_{*p}X)(h) := X(h \circ F),$$

for all $X \in T_p\mathcal{M}$ and $h \in C^\infty(F(p))$, is called the differential *or* derivative *of F at p. The vector $F_{*p}X$ is called the* pushforward *of X by F.*

Exercise 3.10 (Pushforward of tangent vector of a curve)**.** *Show that F_{*p} is a linear map and that, if $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ is such that $\dot{\gamma}(0) = X$, then*

$$F_{*p}X(h) = \frac{dh(F(\gamma(s)))}{ds}\Big|_{s=0},$$

for all $h \in C^\infty(F(p))$.

Exercise 3.11 (Properties of pushforward)**.** *Suppose $F \in C^\infty(\mathcal{M}, \mathcal{N})$. Show the following properties of the differential of F.*

1. *If F is constant then $F_{*p} \equiv 0$ for all $p \in \mathcal{M}$.*
2. *The differential satisfies the chain rule: if $G \in C^\infty(\mathcal{N}, \mathcal{P})$ then*

   $$(G \circ F)_{*p} = G_{*F(p)} \circ F_{*p} : T_p\mathcal{M} \to T_{G \circ F(p)}\mathcal{P},$$

   for all $p \in \mathcal{M}$.
3. *The identity map satisfies $(Id_{\mathcal{M}})_{*p} = Id_{T_p\mathcal{M}} : T_p\mathcal{M} \to T_p\mathcal{M}$ for all $p \in \mathcal{M}$.*
4. *If F is a diffeomorphism then F_{*p} is a vector space isomorphism.*
5. *Given charts (U, ϕ) and (V, ψ) of $\mathcal{M}^m$ and $\mathcal{N}^n$ respectively with coordinate systems $(x^1, \dots, x^m)$ and $(y^1, \dots, y^n)$ respectively, with $p \in U$, $F(p) \in V$, then*

   $$F_{*p}\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \frac{\partial(\psi \circ F \circ \phi^{-1})^j}{\partial x^i}(\phi(p)) \frac{\partial}{\partial y^j}\Big|_{F(p)}.$$

3.3 The tangent bundle and vector fields

The goal of this section is to introduce vector fields. One should view a vector field as a "smooth assignment of a vector at each $p \in \mathcal{M}$". Note then that a vector field should in particular be a map

$$X: \mathcal{M} \to \bigcup_{p\in\mathcal{M}} \{p\} \times T_p\mathcal{M},$$

which has the property that $X_p \in T_p\mathcal{M}$ for all $p \in \mathcal{M}$. In order to call such a map "smooth" it is necessary to define a smooth manifold structure on the space $\bigcup_{p\in\mathcal{M}}\{p\} \times T_p\mathcal{M}$.

Definition 3.12 (Tangent bundle)**.** *The* tangent bundle *of a manifold $\mathcal{M}$, denoted $T\mathcal{M}$, is defined to be the disjoint union of the tangent spaces at each point in $\mathcal{M}$,*

$$T\mathcal{M} = \bigcup_{p\in\mathcal{M}} \{p\} \times T_p\mathcal{M}.$$

Points in $T\mathcal{M}$ are typically denoted (p, v) or $v_p \in T\mathcal{M}$. The projection map $\pi: T\mathcal{M} \to \mathcal{M}$ defined by $\pi(p, v) = p$ is surjective and $\pi^{-1}(p) = \{p\} \times T_p\mathcal{M}$ for all $p \in \mathcal{M}$.

The smooth structure on $\mathcal{M}$ can be used to define a topology and smooth structure on $T\mathcal{M}$ so that π is continuous and smooth. If $\dim\mathcal{M} = n$ then $\dim T\mathcal{M} = 2n$. Indeed, given a chart (U, ϕ) for $\mathcal{M}$, define $V_U := \cup_{p\in U}\{p\} \times T_p\mathcal{M}$ and define $\Phi_\phi : V_U \to \phi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ by

$$\Phi_\phi(p, v^i\partial_{x^i}|_p) := (\phi(p), v^1, \dots, v^n).$$

Clearly Φ_ϕ is a bijection onto its image. The topology of $T\mathcal{M}$ is defined so that each Φ_ϕ is a homeomorphism onto its image.

If $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathcal{A}}$ is an atlas for $\mathcal{M}$ then $\{(V_{U_\alpha}, \Phi_{\phi_\alpha})\}_{\alpha\in\mathcal{A}}$ is an atlas for $T\mathcal{M}$. Indeed, $\{V_{U_\alpha}\}_{\alpha\in\mathcal{A}}$ is clearly an open cover of $T\mathcal{M}$ since $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover of $\mathcal{M}$. Let U, $\widetilde{U}$ be such that $U \cap \widetilde{U} \neq \emptyset$ and consider the corresponding Φ_ϕ, $\Phi_{\widetilde{\phi}}$. If $p \in U \cap \widetilde{U}$ and $(v^1, \dots, v^n) \in \mathbb{R}^n$ then, by Exercise 3.8,

$$v^i\partial_{\widetilde{x}^i}|_p = v^i\frac{\partial x^j}{\partial\widetilde{x}^i}\partial_{x^j}|_p,$$

and so

$$\Phi_\phi \circ \Phi_{\widetilde{\phi}}^{-1}(q, v^1, \dots, v^n) = (\phi \circ \widetilde{\phi}^{-1}(q), v^i\frac{\partial x^1}{\partial\widetilde{x}^i}(p), \dots, v^i\frac{\partial x^n}{\partial\widetilde{x}^i}(p)),$$

with $q = \widetilde{\phi}(p)$. The smoothness of this map then follows from the smoothness of $\phi \circ \widetilde{\phi}^{-1}$.

When one refers to the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$, one typically does not just mean the set $T\mathcal{M}$ as defined in Definition 3.12, but rather the set $T\mathcal{M}$ equipped with this smooth manifold structure.

Definition 3.13 (Vector fields)**.** *Let $\mathcal{M}$ be a manifold. A* vector field *on $\mathcal{M}$ is a smooth map $X : \mathcal{M} \to T\mathcal{M}$ such that $\pi \circ X = Id_{\mathcal{M}}$.*

Denote the set of all vector fields on $\mathcal{M}$ by

$$\mathfrak{X}(\mathcal{M}) = \{\text{vector fields on } \mathcal{M}\}.$$

The space $\mathfrak{X}(\mathcal{M})$ is a vector space with the operations

$$(aX + bY)|_p = aX|_p + bY|_p,$$

for all $p \in \mathcal{M}$, $X, Y \in \mathfrak{X}(\mathcal{M})$ and $a, b \in \mathbb{R}$. Moreover, given a smooth function $f \in C^\infty(\mathcal{M})$, define $fX: \mathcal{M} \to T\mathcal{M}$ by $fX|_p := f(p)X|_p$, so that $fX \in \mathfrak{X}(\mathcal{M})$. The space $\mathfrak{X}(\mathcal{M})$ is then linear over $C^\infty(\mathcal{M})$ (note that $C^\infty(\mathcal{M})$ is a ring, but not a field).

Example 3.14 (Coordinate vector fields). *Given a chart (U,ϕ) of $\mathcal{M}$ with local coordinates $(x^1,\dots,x^n)$, the maps*

$$\frac{\partial}{\partial x^i}\colon U \to \pi^{-1}(U),$$

for $i=1,\dots,n$, defined by $\frac{\partial}{\partial x^i}(p) = \frac{\partial}{\partial x^i}|_p \in T_p\mathcal{M}$ (see Definition 3.5) are vector fields on U called coordinate vector fields.

Given $X \in \mathfrak{X}(\mathcal{M})$ and a chart (U,ϕ), since $\frac{\partial}{\partial x^1}|_p,\dots,\frac{\partial}{\partial x^n}|_p$ form a basis of $T_p\mathcal{M}$ for each $p \in U$, one can write in U

$$X = X^i(p)\frac{\partial}{\partial x^i}\Big|_p,$$

for some smooth functions $X^i\colon U \to \mathbb{R}$, for $i=1,\dots,n$, called the *components of X with respect to* (U,ϕ).

Given a chart (U,ϕ) of $\mathcal{M}$ and $f \in C^\infty(U)$, note that $\frac{\partial f}{\partial x^i} \in C^\infty(U)$ for $i=1,\dots,n$, where

$$\frac{\partial f}{\partial x^i}(p) := \frac{\partial}{\partial x^i}\Big|_p f.$$

3.4 The Lie bracket

Finally, there is a natural way of "multiplying" two vector fields to produce a third. This "multiplication" operation is known as the Lie bracket.

Definition 3.15 (Lie bracket). *The* Lie bracket *of $X,Y \in \mathfrak{X}(\mathcal{M})$ is defined as*

$$[X,Y]f := X(Yf) - Y(Xf),$$

for all $f \in C^\infty(\mathcal{M})$.

Exercise 3.16 (Properties of Lie bracket).

1. *Show that $[X,Y] \in \mathfrak{X}(\mathcal{M})$.*
2. *Show that, if $X = X^i\frac{\partial}{\partial x^i}$ and $Y = Y^i\frac{\partial}{\partial x^i}$ in some local coordinates then*

$$[X,Y]f = \Big(X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\Big)\frac{\partial f}{\partial x^j}.$$

Exercise 3.17 (Further properties of Lie bracket). *Suppose $X,Y,Z \in \mathfrak{X}(\mathcal{M})$, $a,b \in \mathbb{R}$ and $f,g \in C^\infty(\mathcal{M})$. Show that the Lie bracket has the following properties.*

- $[X,Y] = -[Y,X]$ *(anti-commutitivity);*
- $[aX+bY,Z] = a[X,Z]+b[Y,Z]$ *($\mathbb{R}$ linearity);*
- $[[X,Y],Z]+[[Z,X],Y]+[[Y,Z],X] = 0$ *(Jacobi identity);*
- $[fX,gY] = fg[X,Y]+fX(g)Y-gY(f)X$.

One in particular sees from the latter property that $[\cdot,\cdot]$ is not linear over the ring $C^\infty(\mathcal{M})$.

Though introduced as way of "multiplying" two vector fields, one should really view the Lie bracket as a way of *differentiating* vector fields. We will return to this comment in Section 6.3.

4 Tensors, vector bundles and tensor fields

Recall from Section 1.1 that a Riemannian metric, i.e. the relevant object for entertaining *geometric* concepts, on a manifold $\mathcal{M}$ should be, for each $p \in \mathcal{M}$, a map $T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$, *which moreover depends smoothly on p.* In order to fulfil the latter requirement, it is necessary to introduce a smooth manifold structure on the space

$$\bigcup_{p \in \mathcal{M}} \{p\} \times \{\text{Bilinear maps: } T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}\}. \tag{6}$$

Such a space with such a smooth structure is an example of a *vector bundle.* For fixed p, such an element of such a vector space is an example of a *tensor.* Such a smooth assignment of an element of each such space is an example of a *tensor field.*

4.1 Cotangent spaces and the cotangent bundle

A simpler example of a vector bundle than (6) is the *cotangent bundle* which is introduced in this section.[3]

Given a finite dimensional vector space V, define the *dual space* of V as

$$V^* = \{L : V \to \mathbb{R} \mid L \text{ is linear}\}.$$

The dual space V^* is a vector space with the obvious addition and scalar multiplication operations. Elements of V^* are called *covectors.*

Throughout Section 4, vector spaces which are denoted V are always assumed to be finite dimensional.

Exercise 4.1 (Identification of a vector space with its bidual)**.** *Given a (finite dimensional) vector space V and an element $v \in V$, define $\delta_v : V^* \to \mathbb{R}$ by $\delta_v(L) := L(v)$. Show that the map $\Phi : V \to (V^*)^*$ defined by*

$$\Phi(v) := \delta_v,$$

is a vector space isomorphism.

Remark 4.2 (Another abuse of notation)**.** *Exercise 4.1 implies that $V \cong (V^*)^*$* in a canonical way*. It is worth emphasising the latter point. We know very well that any two finite dimensional vector spaces of the same dimension are isomorphic. In fact there are infinitely many isomorphisms and, for two general vector spaces of the same dimension, none of these isomorphisms is preferred. By Exercise 4.1 there is one preferred isomorphism between V and $(V^*)^*$. As such, we view V and $(V^*)^*$* as the same space *and write*

$$V = (V^*)^*,$$

with this slight abuse being understood as an identification of the two spaces using the canonical isomorphism Φ of Exercise 4.1.

Remark 4.3 (Yet another abuse of notation)**.** *In the spirit of Remark 4.2, here is a natural place to make the following observation regarding another slight abuse of notation which will be used frequently. Given $p \in \mathbb{R}^n$, the existence of the canonical (global) Cartesian coordinate system on $\mathbb{R}^n$ means that there is a* canonical *isomorphism between the tangent space $T_p\mathbb{R}^n$ and $\mathbb{R}^n$ itself. Indeed, any $X \in T_p\mathbb{R}^n$ can be uniquely written as*

$$X = X^i \frac{\partial}{\partial x^i}\Big|_p,$$

where $(x^1, \dots, x^n)$ are the standard Cartesian coordinates on $\mathbb{R}^n$. One then defines a canonical isomorphism $\Phi : T_p\mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi\Big(X^i \frac{\partial}{\partial x^i}\Big|_p\Big) := (X^1, \dots, X^n).$$

One easily checks that Φ is a vector space isomorphism. In view of this canonical *isomorphism, we again abuse notation slightly and view $T_p\mathbb{R}^n$ and $\mathbb{R}^n$ as the same space, implicitly meaning they are identified via Φ.*

[3] An even simpler example is the tangent bundle, introduced in Section 3.3.

Definition 4.4 (Cotangent space)**.** *Given a manifold* $\mathcal{M}$ *and a point* $p \in \mathcal{M}$*, define the* cotangent space *at* p*, denoted* $T_p^*\mathcal{M}$*, to be the dual of the tangent space,*

$$T_p^*\mathcal{M} := (T_p\mathcal{M})^*.$$

Elements of $T_p^*\mathcal{M}$ *are called* cotangent vectors *or* covectors *at* p*.*

Given a smooth function $f\colon \mathcal{M} \to \mathbb{R}$ and $p \in \mathcal{M}$, its differential at p, f_{*p}, is also denoted df_p and, using the abuse of Remark 4.3, is a map

$$df_p\colon T_p\mathcal{M} \to \mathbb{R}.$$

Exercise 4.5 (Coordinate covectors)**.** *Given a chart* (U, ϕ) *of* $\mathcal{M}$ *with local coordinates* $x^i\colon U \to \mathbb{R}$*, for* $i = 1, \dots, n$*, show that, if* $p \in U$ *then*

$$dx_p^1, \dots, dx_p^n,$$

forms a basis of $T_p^*\mathcal{M}$*. Moreover,* $dx_p^1, \dots, dx_p^n$ *is the dual basis of the basis* $\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p$ *of* $T_p\mathcal{M}$*, i.e.*

$$dx_p^i\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \delta_j^i.$$

Exercise 4.6 (Differential of a function in local coordinates)**.** *If* $f\colon \mathcal{M} \to \mathbb{R}$ *is a smooth function and* (U, ϕ, x^i) *is a chart for* $\mathcal{M}$*, show that*

$$\frac{\partial}{\partial x^i}\Big|_p f = df_p\Big(\frac{\partial}{\partial x^i}\Big|_p\Big),$$

and hence

$$df_p = \frac{\partial f}{\partial x^i}\Big|_p dx_p^i,$$

where

$$\frac{\partial f}{\partial x^i}\Big|_p := \frac{\partial}{\partial x^i}\Big|_p f.$$

Definition 4.7 (Cotangent bundle)**.** *The* cotangent bundle *of a manifold* $\mathcal{M}$*, denoted* $T^*\mathcal{M}$*, is the disjoint union of the cotangent space at each point in* $\mathcal{M}$*:*

$$T^*\mathcal{M} := \bigcup_{p \in \mathcal{M}} \{p\} \times T_p^*\mathcal{M}.$$

The map $\pi\colon T^*\mathcal{M} \to \mathcal{M}$ *defined, for* $p \in \mathcal{M}$*,* $\omega \in T_p^*\mathcal{M}$*, by* $\pi(p, \omega) = p$ *is called the* natural projection*.*

Similar to $T\mathcal{M}$, the cotangent bundle $T^*\mathcal{M}$ admits a topology and smooth structure (of a $2n$ dimensional manifold) such that π is smooth.

Exercise 4.8 (Topology and smooth structure of cotangent bundle)**.** *Fill in the details of the definition of the topology and smooth structure of* $T^*\mathcal{M}$*.*

The analogue of a vector field (see Definition 3.13) for the cotangent bundle is called a *one form*.

Definition 4.9 (One forms)**.** *A* one form *on* $\mathcal{M}$ *is a smooth map* $\xi\colon \mathcal{M} \to T^*\mathcal{M}$ *such that* $\pi \circ \xi = Id_{\mathcal{M}}$*.*

If $f\colon \mathcal{M} \to \mathbb{R}$ is a smooth function, then df, defined by

$$df(p) := df_p,$$

is a one form on $\mathcal{M}$.

Define the space of one forms on $\mathcal{M}$

$$\Gamma(T^*\mathcal{M}) = \{\text{one forms on } \mathcal{M}\}.$$

Clearly $\Gamma(T^*\mathcal{M})$ is a vector space, with the obvious operations.

Note that d can be viewed as a map

$$d\colon C^\infty(\mathcal{M}) \to \Gamma(T^*\mathcal{M}). \tag{7}$$

This comment will be briefly returned to in Section 4.4.

4.2 Multi-linear algebra

Before defining more general vector bundles, it is convenient to first recall some notions from (multi-) linear algebra.

Definition 4.10 (Multi-linear maps)**.** *If $V_1, \ldots, V_k$ are (finite dimensional) vector spaces, a function $T\colon V_1 \times \ldots \times V_k \to \mathbb{R}$ is called* multi-linear *if*

$$T(v_1, \ldots, v_{i-1}, av_i + b\tilde{v}_i, v_{i+1}, \ldots, v_k) = aT(v_1, \ldots, v_i, \ldots, v_k) + bT(v_1, \ldots, \tilde{v}_i, \ldots, v_k)$$

for all $v_1 \in V_1, \ldots, v_i, \tilde{v}_i \in V_i, \ldots, v_k \in V_k$ and $a, b \in \mathbb{R}$.

Definition 4.11 (Tensor product of vector spaces)**.** *The* tensor product *of vector spaces $V_1, \ldots, V_k$ is defined as*

$$V_1 \otimes \ldots \otimes V_k = \{T\colon V_1^* \times \ldots \times V_k^* \to \mathbb{R} \mid T \text{ is multi-linear}\}.$$

Note that if $k = 1$ then $\{T\colon V_1^* \to \mathbb{R} \mid T \text{ is linear}\} = (V_1^*)^*$ and so the notation of Definition 4.11 is consistent, provided one uses the slight abuse of notation of Remark 4.2.

Definition 4.12 ((r, s) tensors)**.** *Given a vector space V, elements of the tensor product*

$$V_s^r := \underbrace{V \otimes \ldots \otimes V}_{r \text{ times}} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s \text{ times}},$$

are called (r, s) tensors, or r-contravariant, s-covariant tensors.

The space V_s^r is a vector space with the obvious operations.

Definition 4.13 (Tensor product of tensors)**.** *Given a vector space V and $T \in V_s^r$, $\tilde{T} \in V_{\tilde{s}}^{\tilde{r}}$, for some $r, \tilde{r}, s, \tilde{s} \geq 0$, define the* tensor product *of T and $\tilde{T}$, denoted $T \otimes \tilde{T}$, to be the element of $V_{s+\tilde{s}}^{r+\tilde{r}}$ defined by*

$$(T \otimes \tilde{T})(L_1, \ldots, L_{r+\tilde{r}}, v_1, \ldots, v_{s+\tilde{s}}) := T(L_1, \ldots, L_r, v_1, \ldots, v_s)\tilde{T}(L_{r+1}, \ldots, L_{r+\tilde{r}}, v_{s+1}, \ldots, v_{s+\tilde{s}}),$$

for all $L_1, \ldots, L_{r+\tilde{r}} \in V^*$, $v_1, \ldots, v_{s+\tilde{s}} \in V$.

The tensor product operation is

- Linear:

$$\begin{aligned}(T_1 + T_2) \otimes T_3 &= T_1 \otimes T_3 + T_2 \otimes T_3, \\ T_1 \otimes (T_2 + T_3) &= T_1 \otimes T_2 + T_1 \otimes T_3, \\ (\lambda T_1) \otimes T_2 &= T_1 \otimes (\lambda T_2) = \lambda(T_1 \otimes T_2),\end{aligned}$$

  for all $\lambda \in \mathbb{R}$;

- Associative:

$$(T_1 \otimes T_2) \otimes T_3 = T_1 \otimes (T_2 \otimes T_3),$$

- Not commutative:

$$T_1 \otimes T_2 \neq T_2 \otimes T_1 \qquad \text{in general,}$$

for all appropriate tensors T_1, T_2, T_3.

Remark 4.14 (Index convention)**.** *Given a vector space V, we adopt the convention that subscripts are used for the indices labelling elements of a basis of V and superscripts are used for the indices labelling elements of a basis of V^*. Given bases $\{e_i\}$ of V and $\{\theta^i\}$ of V^*, we can then write any $v \in V$ as $v = v^i e_i$ and any $L \in V^*$ as $L_i \theta^i$.*

Exercise 4.15 (Basis for V^r_s). *If V is a vector space, $\{e_i\}_{i=1}^n$ is a basis for V and $\{\theta^i\}_{i=1}^n$ is its dual basis for V^*, then*

$$\{e_{i_1}\otimes\ldots\otimes e_{i_r}\otimes\theta^{j_1}\otimes\ldots\otimes\theta^{j_s}\}_{\substack{1\le i_1,\ldots,i_r\le n,\\ 1\le j_1,\ldots,j_s\le n}}$$

is a basis for V^r_s. In particular, $dim V^r_s = n^{r+s}$.

Given any such bases for V and V^*, any (r,s) tensor T can uniquely written as

$$T = T^{i_1\ldots i_r}{}_{j_1\ldots j_s} e_{i_1}\otimes\ldots\otimes e_{i_r}\otimes\theta^{j_1}\otimes\ldots\otimes\theta^{j_s}.$$

The constants $T^{i_1\ldots i_r}{}_{j_1\ldots j_s}$ are called the *components of* T with respect to the basis $\{e_{i_1}\otimes\ldots\otimes e_{i_r}\otimes\theta^{j_1}\otimes\ldots\otimes\theta^{j_s}\}$.

When defining certain operations on tensors, it is often convenient to first define them on *reducible tensors*, and then extend to all tensors by linearity.

Definition 4.16 (Reducible tensor). *If V is a vector space, a tensor $T\in V^r_s$ is called* reducible *if it can be written as*

$$T = v_1\otimes\ldots\otimes v_r\otimes L^1\otimes\ldots\otimes L^s, \tag{8}$$

for some $v_1,\ldots,v_r\in V$, $L^1,\ldots,L^s\in V^*$.

Example 4.17 (Examples of reducible tensors). *Given a vector space V and $v_1,v_2,v_3,v_4\in V$, in the space V^2_0 the tensor $v_1\otimes v_2$ is reducible, $v_1\otimes v_2+v_1\otimes v_3$ is reducible to $v_1\otimes(v_2+v_3)$, but $v_1\otimes v_2+v_3\otimes v_4$ is not, in general, reducible.*

Clearly any tensor can be written as a linear combination of reducible tensors.

Definition 4.18 (Tensor contraction). *Given a vector space V, $r,s\ge 1$, and $1\le i\le r$, $1\le j\le s$, for any reducible tensor $T\in V^r_s$ of the form* (8) *define the element of V^{r-1}_{s-1}*

$$c_i{}^j(T) := L^j(v_i)v_1\otimes\ldots\otimes v_{i-1}\otimes v_{i+1}\otimes\ldots\otimes v_r\otimes L^1\otimes\ldots\otimes L^{j-1}\otimes L^{j+1}\otimes\ldots\otimes L^s,$$

and then extend $c_i{}^j$ to act on the general $T\in V^r_s$ by linearity. The map $c_i{}^j\colon V^r_s\to V^{r-1}_{s-1}$ is called tensor contraction.

For any vector space V define

$$V^0 := \mathbb{R},\qquad (V^*)^0 := \mathbb{R},\qquad V^0_0 := \mathbb{R}\otimes\mathbb{R} = \mathbb{R}.$$

Example 4.19 (Example of a tensor contraction). *If V is a vector space and $T\in V^1_1 = V\otimes V^*$, and if $\{e_i\}$ a basis for V with dual basis $\{\theta^i\}$ for V^*, then $T = T^i{}_j e_i\otimes\theta^j$ and*

$$c_1{}^1(T) = \theta^j(e_i)T^i{}_j = \delta^j_i T^i{}_j = T^i{}_i \in\mathbb{R}.$$

4.3 Vector bundles and tensor bundles

A vector bundle over a manifold $\mathcal{M}$ is, roughly speaking, a smooth assignment of isomorphic vector spaces to each point $p\in\mathcal{M}$.

Definition 4.20 (Vector bundles). *Consider some $k\ge 0$. A* vector bundle of rank k *is a triple $(E,\mathcal{M},\pi)$ where $\mathcal{M}$ is a manifold of dimension n, E is a manifold of dimension $n+k$ and $\pi\colon E\to\mathcal{M}$ is a smooth surjection such that $\pi^{-1}(p)$ is a vector space for all $p\in\mathcal{M}$, with the following properties.*

1. *There exists an open cover $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ of $\mathcal{M}$ and a family of diffeomorphisms $\{\psi_\alpha\}_{\alpha\in\mathcal{A}}$,*

   $$\psi_\alpha\colon \pi^{-1}(V_\alpha)\to V_\alpha\times\mathbb{R}^k.$$

   Here $V_\alpha\times\mathbb{R}^k$ is equipped with the product smooth structure.

2. *For any $p \in \mathcal{M}$ and any $\alpha \in \mathcal{A}$,*
$$\psi_\alpha\big(\pi^{-1}(p)\big) = \{p\} \times \mathbb{R}^k,$$
and
$$\psi_\alpha\big|_{\pi^{-1}(p)} \colon \pi^{-1}(p) \to \{p\} \times \mathbb{R}^k,$$
is a vector space isomorphism.

3. *If $\alpha, \beta \in \mathcal{A}$ are such that $V_\alpha \cap V_\beta \neq \emptyset$, then the diffeomorphisms*
$$\psi_\alpha \circ \psi_\beta^{-1} \colon (V_\alpha \cap V_\beta) \times \mathbb{R}^k \to (V_\alpha \cap V_\beta) \times \mathbb{R}^k,$$
take the form
$$\psi_\alpha \circ \psi_\beta^{-1}(p, a) = (p, A_{\alpha\beta}(p)a),$$
where $A_{\alpha\beta} \colon V_\alpha \cap V_\beta \to GL(k, \mathbb{R})$ is smooth.

The manifold $\mathcal{M}$ is called the base space*, the manifold E is called the* total space*, and the map π is called the projection map. For each $p \in \mathcal{M}$, the vector space $E_p := \pi^{-1}(p)$ is called the fibre of E at p. The pairs (V_α, ψ_α) are called* local trivialisations*. The maps $A_{\alpha\beta}$ are called* transition maps*.*

We often say "let $\pi \colon E \to \mathcal{M}$ be a vector bundle" to mean "let $(E, \mathcal{M}, \pi)$ be a vector bundle".

Remark 4.21 (Smoothness of transition maps)**.** *In order to make sense of the smoothness of $A_{\alpha\beta} \colon V_\alpha \cap V_\beta \to GL(k, \mathbb{R})$ in Definition 4.20, it is necessary to introduce a smooth manifold structure on the general linear group $GL(k, \mathbb{R})$.*

Exercise 4.22 (Smooth structure on $GL(n, \mathbb{R})$)**.** *Given a smooth manifold M and an open subset $\Sigma \subset M$, show that Σ is itself a smooth manifold. Deduce that the space $GL(n, \mathbb{R})$ of real valued $n \times n$ invertible matrices can be given the structure of a smooth n^2 dimensional manifold.*

Remark 4.23 (Local trivialisations vs charts)**.** *The local trivialisations of a vector bundle are not to be confused with the charts of E!*

Example 4.24 (Examples of vector bundles)**.**

1. *Given a manifold $\mathcal{M}$ and $k \geq 0$, the* trivial bundle *over $\mathcal{M}$ is the vector bundle $(\mathcal{M} \times \mathbb{R}^k, \mathcal{M}, \pi)$, where $\pi \colon \mathcal{M} \times \mathbb{R}^k \to \mathcal{M}$ is defined by $\pi(p, a) = p$ for $p \in \mathcal{M}$, $a \in \mathbb{R}^k$. There is only one local trivialisation $(\mathcal{M}, Id_{\mathcal{M} \times \mathbb{R}^k})$, and so the compatibility condition trivially holds.*

2. *Given a manifold $\mathcal{M}$, the tangent bundle $(T\mathcal{M}, \mathcal{M}, \pi)$, where $\pi \colon T\mathcal{M} \to \mathcal{M}$ is the natural projection, is a vector bundle. Indeed, let $\{(V_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ be an atlas of $\mathcal{M}$. Define local trivialisations $\{(V_\alpha, \psi_\alpha)\}_{\alpha \in \mathcal{A}}$ by*
$$\psi_\alpha \colon \pi^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^n, \qquad \psi_\alpha\Big(p, a^i \frac{\partial}{\partial x^i}\Big|_p\Big) = (p, a^1, \ldots, a^n).$$
To check the compatibility condition, consider $\alpha, \beta \in \mathcal{A}$ with $V_\alpha \cap V_\beta \neq \emptyset$. Let $\{\tilde{x}^i\}$ denote the coordinates of (V_α, ϕ_α) and $\{x^i\}$ denote the coordinates of (V_β, ϕ_β). Given
$$\Big(p, a^i \frac{\partial}{\partial x^i}\Big|_p\Big) = \Big(p, \tilde{a}^i \frac{\partial}{\partial \tilde{x}^i}\Big|_p\Big) \in \pi^{-1}(V_\alpha \cap V_\beta),$$
it follows from Exercise 3.8 that
$$\tilde{a}^i = a^j \frac{\partial \tilde{x}^i}{\partial x^j}\Big|_p.$$
Hence the transition map
$$A_{\alpha\beta}(p) = \Big(\frac{\partial \tilde{x}^i}{\partial x^j}\Big|_p\Big)_{i,j=1}^n,$$
is smooth.

Exercise 4.25 (The cotangent bundle is a vector bundle). *Show that the cotangent bundle $(T^*\mathcal{M}, \mathcal{M}, \pi)$ of a manifold $\mathcal{M}$, where $\pi\colon T^*\mathcal{M} \to \mathcal{M}$ is the natural projection, is a vector bundle.*

Note that the local trivialisations of a vector bundle make it look locally like the trivial bundle of Example 4.24. Hence the nomenclature.

Given a vector bundle $(E, \mathcal{M}, \pi)$ over a manifold of $\mathcal{M}$, a smooth assignment to each point of $\mathcal{M}$ of an element of each fibre of E is called a section.

Definition 4.26 (Sections of vector bundles). *A* section *of a vector bundle $(E, \mathcal{M}, \pi)$ is a smooth map $s\colon \mathcal{M} \to E$ such that $\pi \circ s = Id_{\mathcal{M}}$, i.e. $s(p) \in E_p$ for all $p \in \mathcal{M}$.*

The space of all sections of a vector bundle $(E, \mathcal{M}, \pi)$ is denoted $\Gamma(E)$. Clearly $\Gamma(E)$ is a vector space with the obvious operations.

Example 4.27 (Vector fields are sections of $T\mathcal{M}$). *Vector fields are sections of the tangent bundle, and $\Gamma(T\mathcal{M}) = \mathfrak{X}(\mathcal{M})$.*

Definition 4.28 (Local and global frames). *Suppose a vector bundle $(E, \mathcal{M}, \pi)$ of rank k admits k sections which are everywhere linearly independent. Such a collection of sections is called a* global frame *of $(E, \mathcal{M}, \pi)$. If these k sections are only locally defined, we call such a collection a* local frame.

Remark 4.29 (Existence and non-existence of local and global frames). *Global frames of vector bundles do not always exist. For example, there is no nowhere vanishing vector field on S^2. This statement is the content of the* Hairy Ball Theorem.

Local frames do always exist. If $(E, \mathcal{M}, \pi)$ is a vector bundle with local trivialisations $\{(V_\alpha, \psi_\alpha)\}_{\alpha \in \mathcal{A}}$ then, for each α, $\{\psi_\alpha^{-1}(p, e_i)\}_{i=1}^k$ is a local frame, where e_i is the i-th standard basis vector of $\mathbb{R}^k$.

If $\mathcal{M}$ is a manifold and (U, ϕ) is a chart with local coordinates $\{x^i\}$, then $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ is a local frame for $T\mathcal{M}$.

Given a vector bundle, the operations of the following two definitions provide ways of generating many more examples of vector bundles.

Definition 4.30 (Dual bundle). *Given a vector bundle $(E, \mathcal{M}, \pi)$, recall that $E = \bigcup_{p \in \mathcal{M}} E_p$. Define the* dual bundle *of $(E, \mathcal{M}, \pi)$, denoted $(E^*, \mathcal{M}, \pi^*)$, by*

$$E^* = \bigcup_{p \in \mathcal{M}} \{p\} \times (E_p)^*, \qquad \pi^*\colon E^* \to \mathcal{M}, \qquad \pi^*(p, \theta) = p.$$

Exercise 4.31 (Dual bundle is a vector bundle). *Show that $(E^*, \mathcal{M}, \pi^*)$ is a vector bundle. Use the local trivialisations (V_α, ψ_α) for $(E, \mathcal{M}, \pi)$ to define local trivialisations for $(E^*, \mathcal{M}, \pi^*)$. The transition maps for $(E^*, \mathcal{M}, \pi^*)$ are*

$$(A^*)_{\alpha\beta} = (A_{\alpha\beta}^{-1})^T.$$

Example 4.32 (Cotangent bundle). *The dual bundle of the tangent bundle of a manifold is the cotangent bundle.*

Definition 4.33 (Tensor product bundles). *Given vector bundles $(E, \mathcal{M}, \pi)$ and $(\tilde{E}, \mathcal{M}, \tilde{\pi})$ of ranks k and l respectively, define the* tensor product bundle

$$E \otimes \tilde{E} := \bigcup_{p \in \mathcal{M}} \{p\} \times (E_p \otimes \tilde{E}_p),$$

with the projection map $\pi \otimes \tilde{\pi}\colon E \otimes \tilde{E} \to \mathcal{M}$ defined in the obvious way.

Exercise 4.34 (Tensor product bundle is a vector bundle)**.** *Show that the tensor product bundle of* $(E, \mathcal{M}, \pi)$ *and* $(\tilde{E}, \mathcal{M}, \tilde{\pi})$ *(of ranks* k *and* l *respectively) is a vector bundle of rank* kl*. If* $A_{\alpha\beta}$ *and* $\tilde{A}_{\alpha\beta}$ *are transition maps for* E *and* $\tilde{E}$ *respectively, then*

$$(A^{E\otimes\tilde{E}})_{\alpha\beta}(a\otimes\tilde{a}) := A_{\alpha\beta}a\otimes\tilde{A}_{\alpha\beta}\tilde{a}\in\mathbb{R}^k\otimes\mathbb{R}^l\cong\mathbb{R}^{kl},$$

for $a\in\mathbb{R}^k$, $\tilde{a}\in\mathbb{R}^l$*, are transition maps for* $E\otimes\tilde{E}$*.*

Definition 4.35 (The (r, s) tensor bundles of a smooth manifold)**.** *Let* $\mathcal{M}$ *be a manifold. The* (r, s) tensor bundle *of* $\mathcal{M}$ *is the vector bundle*

$$T^r_s\mathcal{M} := \underbrace{T\mathcal{M}\otimes\ldots\otimes T\mathcal{M}}_{r\ times}\otimes\underbrace{T^*\mathcal{M}\otimes\ldots\otimes T^*\mathcal{M}}_{s\ times}.$$

An (r, s) tensor field T *is a section of* $T^r_s\mathcal{M}$*:* $T\in\Gamma(T^r_s\mathcal{M})$*.*

Given local coordinates $\{x^i\}_{i=1}^n$ of a chart (U, ϕ) of $\mathcal{M}$, any (r, s) tensor field T can be written locally as

$$T = T^{i_1\ldots i_r}{}_{j_1\ldots j_s}\frac{\partial}{\partial x^{i_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_s},$$

for some smooth functions $T^{i_1\ldots i_r}{}_{j_1\ldots j_s}\colon U\to\mathbb{R}$.

Example 4.36 (Tangent and cotangent bundles are (r, s) tensor bundles)**.** *For any smooth manifold* $\mathcal{M}$ *the tangent and cotangent bundles are tensor bundles. Indeed,* $T^1_0\mathcal{M} = T\mathcal{M}$ *and* $T^0_1\mathcal{M} = T^*\mathcal{M}$*.*

Recall that, given a function $F\colon\mathcal{M}\to\mathcal{N}$, there is a way of "pushing forward" vectors on $\mathcal{M}$ to vectors on $\mathcal{N}$ (see Definition 3.9). There is similarly a way of "pulling back" one forms, and more generally $(0, k)$ tensor fields on $\mathcal{N}$ to one forms or $(0, k)$ tensor fields on $\mathcal{M}$.

Definition 4.37 (Pullback of a $(0, k)$ tensor field)**.** *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be smooth manifolds,* $F\colon\mathcal{M}\to\mathcal{N}$ *a smooth function, and* $\omega\in\Gamma(T^0_k\mathcal{N})$ *a* $(0, k)$ *tensor field on* $\mathcal{N}$*. The* pullback *of* ω *with respect to* F *is a* $(0, k)$ *tensor field on* $\mathcal{M}$*, denoted* $F^*\omega\in\Gamma(T^0_k\mathcal{M})$*, defined by*

$$(F^*\omega)_p(v_1,\ldots,v_k) = \omega_{F(p)}(F_{*p}v_1,\ldots F_{*p}v_k),$$

for all $v_1,\ldots,v_k\in T_p\mathcal{M}$ *and all* $p\in\mathcal{M}$*.*

If $f\in C^\infty(\mathcal{N})$*, define*

$$F^*f = f\circ F.$$

Exercise 4.38 (Properties of pullback)**.** *Show that pullback has the following properties. Let* $\mathcal{M}$*,* $\mathcal{N}$*,* $\mathcal{Q}$ *be manifolds,* $F\colon\mathcal{M}\to\mathcal{N}$ *and* $G\colon\mathcal{N}\to\mathcal{Q}$ *be smooth, and consider* $\omega\in\Gamma(T^0_k\mathcal{N})$*,* $\eta\in\Gamma(T^0_l\mathcal{N})$*. Then*

1. $(G\circ F)^* = F^*\circ G^*$*.*
2. $F^*(\omega\otimes\eta) = F^*\omega\otimes F^*\eta$*. In particular, if* $f\in C^\infty(\mathcal{N}) = \Gamma(T^0_0\mathcal{N})$*, then* $F^*(f\omega) = f\circ F\cdot F^*\omega$*.*
3. *If* $f\in C^\infty(\mathcal{N})$ *then* $F^*(df) = d(F^*f) = d(f\circ F)$*.*
4. *If* $p\in\mathcal{M}$ *and* $\{y^i\}$ *are the local coordinates of a chart for* $\mathcal{N}$ *containing* $F(p)\in\mathcal{N}$*, then*

$$F^*\big(\omega_{j_1\ldots j_k}dy^{j_1}\otimes\ldots\otimes dy^{j_k}\big) = \omega_{j_1\ldots j_k}\circ F\cdot d(y^{j_1}\circ F)\otimes\ldots\otimes d(y^{j_k}\circ F).$$

4.4 *Other important vector bundles

There are many other important examples of vector bundles, which do not form part of the examinable material of this course. Most notable among these examples are the vector bundles of *differential forms.* Before introducing the vector bundle of differential forms, consider first the following notions from linear algebra.

Definition 4.39 (Alternating tensors)**.** *Let V be a vector space. A $(0,k)$ tensor $T \in V^0_k$ is called* antisymmetric *or* alternating *if*

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k),$$

for all $1 \leq i < j \leq k$. The space of antisymmetric $T \in V^0_k$ is denoted $\Lambda^k V^*$,

$$\Lambda^k V^* := \{T \in V^0_k \mid T \text{ is antisymmetric}\}.$$

Two antisymmetric tensors can be multiplied to produce another antisymmetric tensor using an operation known as *wedge product.*

Definition 4.40 (The alternating map)**.** *Let V be a vector space. Define the* alternating map *$Alt\colon V^0_k \to \Lambda^k V^*$ by*

$$Alt(T) := \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\sigma(T),$$

where S_k is the symmetric group of order k (the group of permutations of the set $\{1, \ldots, k\}$), where

$$\sigma(T)(v_1, \ldots, v_k) := T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

for all $T \in V^0_k$.

Definition 4.41 (The wedge product)**.** *Let V be a vector space. For $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^l V^*$, define the* wedge product *or* exterior product *of ω and η to be the antisymmetric $(0, k+l)$ tensor,*

$$\omega \wedge \eta := \frac{(k+l)!}{k!\; l!} Alt(\omega \otimes \eta).$$

Explicitly,

$$\omega \wedge \eta(v_1, \ldots, v_{k+l}) = \frac{1}{k!\; l!} \sum_{\sigma \in S_{k+l}} \operatorname{sign}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).$$

Example 4.42 (Wedge product of two covectors)**.** *If V is a vector space and $\omega, \eta \in \Lambda^1 V^* = V^*$, then*

$$\omega \wedge \eta = \omega \otimes \eta - \eta \otimes \omega.$$

Exercise 4.43 (Properties of wedge product)**.** *Let V be a vector space. The wedge product satisfies the following properties.*

1. *The wedge product is bilinear and associative:*

   $$(aw_1 + bw_2) \wedge w_3 = a(w_1 \wedge w_3) + b(w_2 \wedge w_3),$$

   for all $w_1, w_2 \in \Lambda^k V^*$, $w_3 \in \Lambda^l V^*$ and $a, b \in \mathbb{R}$, and

   $$(w_1 \wedge w_2) \wedge w_3 = w_1 \wedge (w_2 \wedge w_3),$$

   for all $w_1 \in \Lambda^{k_1} V^*$, $w_2 \in \Lambda^{k_2} V^*$, $w_3 \in \Lambda^{k_3} V^*$ and $a, b \in \mathbb{R}$.

2.

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega,$$

for all $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^l V^*$.

3. *Given $\omega^1, \dots, \omega^k \in V^*$ and $v_1, \dots, v_k \in V$,*

$$(\omega^1 \wedge \ldots \wedge \omega^k)(v_1, \dots, v_k) = \det\big(\omega^i(v_j)\big).$$

4. *If $\{e_i\}_{i=1}^n$ is a basis of V and $\{\theta^j\}_{j=1}^n$ is its dual basis, then $\{\theta^{i_1} \wedge \ldots \wedge \theta^{i_k}\}_{1 \le i_1 < \ldots < i_k \le n}$ is a basis of $\Lambda^k V^*$. Any $\alpha \in \Lambda^k V^*$ can therefore be written as*

$$\alpha = \sum_{1 \le i_1 < \ldots < i_k \le n} \alpha(e_{i_1}, \dots, e_{i_k}) \theta^{i_1} \wedge \ldots \wedge \theta^{i_k}.$$

5. *Covectors $\omega_1, \dots, \omega_k \in V^*$ are linearly independent if and only if*

$$\omega_1 \wedge \ldots \wedge \omega_k \neq 0.$$

6. $\dim(\Lambda^k V^*) = \binom{n}{k}$. *In particular* $\dim(\Lambda^n V^*) = 1$, *and* $\dim(\Lambda^k V^*) = 0$ *for all $k \ge n+1$.*

Consider now a smooth manifold $\mathcal{M}$.

Definition 4.44 (Differential forms)**.** *A* (differential) k-form *on a manifold $\mathcal{M}$ is a section of the vector bundle $\Lambda^k \mathcal{M} := \Lambda^k T^* \mathcal{M}$. The space of k-forms is denoted $\Omega^k(\mathcal{M})$,*

$$\Omega^k(\mathcal{M}) := \Gamma(\Lambda^k \mathcal{M}).$$

In a local coordinate system $\{x^i\}$ for $\mathcal{M}$, any $\omega \in \Omega^k(\mathcal{M})$ can locally be written as

$$\omega = \omega_{i_1 \dots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

We adopt the convention that $\Omega^0(\mathcal{M}) = C^\infty(\mathcal{M})$. Recall the map $d\colon \Omega^0(\mathcal{M}) \to \Omega^1(\mathcal{M})$ defined in (7). This map can be extended to a map $d\colon \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ for any $k \ge 0$.

Definition 4.45 (The exterior derivative)**.** *The* exterior derivative *$d\colon \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ is defined as follows. If (U, ϕ, x^i) is a local chart of $\mathcal{M}$ then any $\omega \in \Omega^k(\mathcal{M})$ can be written as*

$$\omega = \omega_{i_1 \dots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

The exterior derivative of ω is defined in these local coordinates by

$$d\omega = d(\omega_{i_1 \dots i_k}) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

where $d(\omega_{i_1 \dots i_k})$ is the usual differential of $\omega_{i_1 \dots i_k} \in C^\infty(\mathcal{M})$.

One can check that the definition of d is globally well defined and independent of the choice of local chart.

Exercise 4.46 (Properties of exterior derivative)**.** *The exterior derivative has the following properties.*

1. *If $f \in C^\infty(\mathcal{M}) = \Omega^0(\mathcal{M})$ then df is the usual differential of f.*
2. *If $\omega \in \Omega^k(\mathcal{M})$ then $d(d\omega) = 0$.*
3. *$d(a\omega + b\eta) = a d\omega + b d\eta$ for all $\omega, \eta \in \Omega^k(\mathcal{M})$, $a, b \in \mathbb{R}$.*
4. *$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for all $\omega \in \Omega^k(\mathcal{M})$, $\eta \in \Omega^l(\mathcal{M})$.*

If $\mathcal{M}$ is an orientable manifold of dimension n and $\omega \in \Omega^n(\mathcal{M})$, then one can define the *integral* of ω, $\int_{\mathcal{M}} \omega$. See, for example, [11].

5 Riemannian metrics

We are now in a position to define a Riemannian metric, and some immediate geometric notions such an object gives rise to. First, recall the definition of an inner product.

Definition 5.1 (Inner product)**.** *An* inner product *on a vector space V is a function $\langle\cdot,\cdot\rangle\colon V\times V\to\mathbb{R}$ which is*

- *Symmetric: $\langle u,v\rangle=\langle v,u\rangle$ for all $u,v\in V$;*
- *Bilinear: $\langle au+bv,w\rangle=a\langle u,w\rangle+b\langle v,w\rangle$;*
- *Positive definite: $\langle u,u\rangle>0$ for all $u\neq 0$.*

Example 5.2 (Euclidean dot product)**.** *The Euclidean dot product is an inner product on $\mathbb{R}^n$.*

A Riemannian metric on a manifold $\mathcal{M}$ associates to each point $p\in\mathcal{M}$, in a smooth way, an inner product on the tangent space $T_p\mathcal{M}$.

Definition 5.3 (Riemannian metrics)**.** *A* Riemannian metric *on a manifold $\mathcal{M}$ is a* $(0,2)$ *tensor field $g\in\Gamma(T^0_2\mathcal{M})$ which is*

- *Symmetric: $g_p(X_p,Y_p)=g_p(Y_p,X_p)$ for all $X_p,Y_p\in T_p\mathcal{M}$ and all $p\in\mathcal{M}$;*
- *Positive definite $g_p(X_p,X_p)>0$ for all $X_p\in T_p\mathcal{M}$ with $X_p\neq 0$, and all $p\in\mathcal{M}$.*

Definition 5.4 (Riemannian manifolds)**.** *A pair $(\mathcal{M},g)$, where $\mathcal{M}$ is a smooth manifold and g is a Riemannian metric on $\mathcal{M}$, is called a* Riemannian manifold*.*

More generally, one can define a *semi-Riemannian metric* by relaxing the positive definiteness condition.

Definition 5.5 (Semi-Riemannian metrics)**.** *A* semi-Riemannian metric *or* pseudo-Riemannian metric *on a manifold $\mathcal{M}$ is a* $(0,2)$ *tensor field $g\in\Gamma(T^0_2\mathcal{M})$ which is*

- *Symmetric: $g_p(X_p,Y_p)=g_p(Y_p,X_p)$ for all $X_p,Y_p\in T_p\mathcal{M}$ and all $p\in\mathcal{M}$;*
- *Non-degenerate: for all $p\in\mathcal{M}$, if $X_p\in T_p\mathcal{M}$ is such that $g(X_p,Y_p)=0$ for all $Y_p\in T_p\mathcal{M}$, then $X_p=0$.*

Of particular note, for the role they play in Einstein's general theory of relativity, are *Lorentizian metrics.* A Lorentzian metric is a semi-Riemannian metric which everywhere has signature $(-,+,\dots,+)$ (i.e. the matrix $\big(g_{ij}(p)\big)_{i,j=1}^{n+1}$, defined below, has one negative eigenvalue and n positive eigenvalues for all $p\in\mathcal{M}$).

In this course we will mostly consider Riemannian metrics, but many of the definitions and results apply also to semi-Riemannian metrics. See the book [12] for more on semi-Riemannian geometry.

5.1 Expression in local coordinates

Given a Riemannian manifold $(\mathcal{M},g)$ and a chart (U,ϕ,x^i), consider the functions

$$g_{ij}\colon U\to\mathbb{R},\qquad g_{ij}(p):=g_p\Big(\frac{\partial}{\partial x^i}\Big|_p,\frac{\partial}{\partial x^j}\Big|_p\Big).$$

For each $p\in U$, $\big(g_{ij}(p)\big)_{i,j=1}^{n}$ is a symmetric positive definite $n\times n$ matrix. The functions g_{ij} are called the components of g with respect to $\{x^i\}$.

Recall $g\in\Gamma(T^0_2\mathcal{M})$ means that, using a local coordinate frame $\{dx^i\}$ for $T^*\mathcal{M}$, we can write

$$g=g_{ij}dx^i\otimes dx^j.$$

Since g is symmetric we often omit the $\otimes$ sign. To be more precise, define the *symmetric product* of one forms $\omega, \eta \in \Gamma(T^*\mathcal{M})$ by

$$\omega\eta := \frac{1}{2}\big(\omega \otimes \eta + \eta \otimes \omega\big).$$

Since (g_{ij}) is symmetric, it follows that

$$g = g_{ij} dx^i \otimes dx^j = \frac{1}{2}\big(g_{ij} dx^i \otimes dx^j + g_{ji} dx^i \otimes dx^j\big) = g_{ij} dx^i dx^j.$$

More generally, if $\{e_i\}$ is a local frame for $T\mathcal{M}$ and $\{\theta^i\}$ is its dual frame for $T^*\mathcal{M}$ (i.e. $\theta^j(e_i) = \delta_i^j$ for all p) then

$$g = g_{ij}\theta^i \otimes \theta^j,$$

where now $g_{ij} = g(e_i, e_j)$.

Note that, given two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, the inner product $g(X, Y)$ is a function on $\mathcal{M}$: $g(X, Y) \in C^\infty(\mathcal{M})$ where

$$g(X, Y)(p) := g_p(X_p, Y_p).$$

5.2 Lengths and angles

A Riemannian metric on a manifold immediately gives rise to notions of lengths of vectors and angles between vectors, and hence to notions of orthogonality and orthonormality of vectors.

Definition 5.6 (Lengths, angles, orthogonality and orthonormality of vectors)**.** *Let $(\mathcal{M}, g)$ be a Riemannian manifold and $p \in \mathcal{M}$.*

- *The* length *of $v \in T_p\mathcal{M}$ is defined to be*

$$|v|_g := \sqrt{g_p(v, v)}.$$

  If it is clear which metric g is meant, we often write $|v|$ for $|v|_g$.

- *The* angle *between two non-zero vectors $v, w \in T_p\mathcal{M}$ is the unique $\theta \in [0, \pi]$ such that*

$$\cos\theta = \frac{g_p(v, w)}{|v|_g |w|_g}.$$

- *Two vectors $v, w \in T_p\mathcal{M}$ are called* orthogonal *if $g_p(v, w) = 0$.*
- *Vectors $e_1, \dots, e_k \in T_p\mathcal{M}$ are called* orthonormal *if*

$$g_p(e_i, e_j) = \delta_{ij},$$

  for all $i, j = 1, \dots, k$.

Using the above, one can define the length of a curve and angles between two intersecting curves.

Definition 5.7 (The length of a curve and the angle between two intersecting curves)**.** *Let $(\mathcal{M}, g)$ be a Riemannian manifold. The* length *of a curve $\gamma\colon (a, b) \to \mathcal{M}$ is defined to be*

$$L(\gamma) := \int_a^b \sqrt{g_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))}ds.$$

If γ and $\tilde\gamma$ are curves in $\mathcal{M}$ such that $\gamma(c_1) = \tilde\gamma(c_2) = p \in \mathcal{M}$ for some $c_1, c_2 \in \mathbb{R}$, and $\dot\gamma(c_1) \neq 0$, $\dot{\tilde\gamma}(c_2) \neq 0$, define the angle *between γ and $\tilde\gamma$ at p as the unique $\theta \in [0, \pi]$ such that*

$$\cos\theta = \frac{g_p(\dot\gamma(c_1), \dot{\tilde\gamma}(c_2))}{|\dot\gamma(c_1)|_g |\dot{\tilde\gamma}(c_2)|_g},$$

i.e. the angle between $\dot\gamma(c_1)$ and $\dot{\tilde\gamma}(c_2)$.

One has the following expressions in local coordinates for the inner product of two vectors and of the length of a vector.

Exercise 5.8 (Inner product of two vectors in local coordinates)**.** *Let (U, ϕ, x^i) be a chart of $(\mathcal{M}, g)$ and consider $p \in U$. Show that, if*

$$X = X^i \frac{\partial}{\partial x^i}\Big|_p, \quad Y = Y^i \frac{\partial}{\partial x^i}\Big|_p \in T_p\mathcal{M},$$

then

$$g_p(X, Y) = g_{ij}(p) X^i Y^j,$$

where

$$g = g_{ij} dx^i dx^j.$$

In particular, the length of X takes the form

$$|X|_g = \sqrt{g_{ij}(p) X^i X^j}.$$

5.3 *Volume

A Riemannian metric also give rise to a notion of volume. Since volume is defined using differential forms, this section is non-examinable.

If $(\mathcal{M}^n, g)$ is an oriented Riemannian manifold, there exists a unique n-form dV_g such that

$$dV_g|_p(E_1, \ldots, E_n) = 1,$$

whenever $E_1, \ldots, E_n$ is an oriented orthonormal basis for $T_p\mathcal{M}$.

Exercise 5.9 (Volume form in local coordinates)**.** *Show that, in local coordinates $\{x^i\}$,*

$$dV_g = \sqrt{\det(g_{ij})} dx^1 \wedge \ldots \wedge dx^n.$$

The *volume* of $\mathcal{M}$ is defined to be

$$\mathrm{Vol}(M) := \int_{\mathcal{M}} dV_g.$$

More generally, if $f \in C^\infty(\mathcal{M})$ is compactly supported, then $f dV_g$ is an n-form and so one can define the *integral of f* to be

$$\int_{\mathcal{M}} f dV_g.$$

5.4 Examples of Riemannian metrics

The most basic example of a Riemannian metric is the Euclidean metric on $\mathbb{R}^n$.

- The Euclidean metric on $\mathbb{R}^n$ in Cartesian coordinates takes the form

$$g_{Eucl} = \delta_{ij} dx^i dx^j = (dx^1)^2 + \ldots + (dx^n)^2.$$

The following proposition gives rise to many other examples.

Proposition 5.10 (Pullback of a metric by an immersion)**.** *Let $(\mathcal{M}, g)$ be a Riemannian manifold, $\mathcal{N}$ a smooth manifold, and let $f \colon \mathcal{N} \to \mathcal{M}$ be an immersion (a smooth map such that $f_{*p} \colon T_p\mathcal{N} \to T_{f(p)}\mathcal{M}$ is injective for all $p \in \mathcal{N}$). The pullback $f^*g \in \Gamma(T^0_2\mathcal{N})$ defines a metric on $\mathcal{N}$.*

Proof. Recall that

$$(f^*g)_p(u,v) = g_{f(p)}(f_{*p}u, f_{*p}v),$$

for all $u, v \in T_p\mathcal{N}$ and all $p \in \mathcal{N}$. Clearly $(f^*g)_p$ is symmetric. Since f_{*p} is injective, if $f_{*p}u = 0$ then it follows that $u = 0$, and hence f^*g is positive definite. $\square$

- If $\mathcal{N}$ is a submanifold of a Riemannian manifold $(\mathcal{M}, g)$, the inclusion map $\iota \colon \mathcal{N} \hookrightarrow \mathcal{M}$ defines a metric on $\mathcal{N}$, ι^*g, called the *induced metric.*

In particular, the Euclidean metric induces a metric on every submanifold of $\mathbb{R}^n$.

- Consider the n sphere $S^n := \{(x^1)^2 + \ldots + (x^{n+1})^2 = 1\} \subset \mathbb{R}^{n+1}$ and the inclusion map $\iota \colon S^n \hookrightarrow \mathbb{R}^{n+1}$. The induced Euclidean metric $\iota^* g_{Eucl}$ on S^n is called the *round metric* (or *standard metric*) on S^n.

Proposition 5.11 (The round metric on S^2 in polar coordinates)**.** *In (θ, ϕ) polar coordinates for S^2, the round metric takes the form*

$$g_{S^2} = d\theta^2 + \sin^2\theta d\phi^2.$$

Proof. The inclusion map $\iota \colon S^2 \hookrightarrow \mathbb{R}^3$ in polar coordinates takes the form

$$\iota(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

First one computes the push forwards $\iota_{*(\theta,\phi)}\partial_\theta$ and $\iota_{*(\theta,\phi)}\partial_\phi$. Given $f \in C^\infty(\mathbb{R}^3)$,

$$\iota_{*(\theta,\phi)}(\partial_\theta)f = \partial_\theta\big(f \circ \iota(\theta,\phi)\big) = \partial_\theta\big(f(\sin\theta\cos\theta, \sin\theta\sin\phi, \cos\theta)\big) = \cos\theta\cos\phi\partial_x f + \cos\theta\sin\phi\partial_y f - \sin\theta\partial_z f.$$

It follows that

$$\iota_{*(\theta,\phi)}\partial_\theta = \cos\theta\cos\phi\partial_x + \cos\theta\sin\phi\partial_y - \sin\theta\partial_z,$$

and similarly

$$\iota_{*(\theta,\phi)}\partial_\phi = -\sin\theta\sin\phi\partial_x + \sin\theta\cos\phi\partial_y.$$

Now

$$g_{S^2}|_{(\theta,\phi)}(\partial_\theta, \partial_\theta) = g_{Eucl}|_{\iota(\theta,\phi)}(\iota_{*(\theta,\phi)}\partial_\theta, \iota_{*(\theta,\phi)}\partial_\theta) = \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta = 1,$$

$$g_{S^2}|_{(\theta,\phi)}(\partial_\theta, \partial_\phi) = -\cos\theta\cos\phi\sin\theta\sin\phi + \cos\theta\sin\phi\sin\theta\cos\phi = 0,$$

and

$$g_{S^2}|_{(\theta,\phi)}(\partial_\phi, \partial_\phi) = \sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi = \sin^2\theta,$$

and so it follows that

$$g_{S^2} = d\theta^2 + \sin^2\theta d\phi^2.$$

$\square$

- The n ball $B^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid (x^1)^2 + \ldots + (x^n)^2 < 1\}$ with the metric

$$g = \left(\frac{2}{1 - \big((x^1)^2 + \ldots + (x^n)^2\big)}\right)^2 \big((dx^1)^2 + \ldots + (dx^n)^2\big),$$

  is called *hyperbolic space.*

- If $(\mathcal{M}_1^{n_1}, g_1)$ and $(\mathcal{M}_2^{n_2}, g_2)$ are Riemannian manifolds, recall that the product $\mathcal{M}_1 \times \mathcal{M}_2$ admits the smooth structure of an $n_1 + n_2$ dimensional smooth manifold (the product smooth structure). The product manifold $\mathcal{M}_1 \times \mathcal{M}_2$ admits a Riemannian metric $g = g_1 \times g_2$ called the *product metric*

$$g_1 \times g_2|_{(p_1,p_2)}\big((u_1,u_2),(v_1,v_2)\big) = g_1|_{p_1}(u_1,v_1) + g_2|_{p_2}(u_2,v_2),$$

  for $(u_1,u_2),(v_1,v_2) \in T_{(p_1,p_2)}(\mathcal{M}_1 \times \mathcal{M}_2) \cong T_{p_1}\mathcal{M}_1 \times T_{p_2}\mathcal{M}_2$, $u_1, v_1 \in T_{p_1}\mathcal{M}_1$, $u_2, v_2 \in T_{p_2}\mathcal{M}_2$.

- Given a smooth function $f\colon \mathcal{M}_1 \to (0,\infty)$, define the *warped product metric* of $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ with respect to f on the product manifold $\mathcal{M}_1 \times \mathcal{M}_2$ as

$$g_1 \times_f g_2|_{(p_1,p_2)}\big((u_1,u_2),(v_1,v_2)\big) = g_1|_{p_1}(u_1,v_1) + f(p_1)g_2|_{p_2}(u_2,v_2).$$

  The warped product manifold is denoted $(\mathcal{M}_1, g_1) \times_f (\mathcal{M}_2, g_2)$.

Exercise 5.12 (Euclidean metric on $\mathbb{R}^2$ in polar coordinates)**.** *Show that, in polar coordinates, the Euclidean metric on* $\mathbb{R}^2 \smallsetminus \{0\}$ *takes the form*

$$g_{Eucl} = dr^2 + r^2 d\theta^2,$$

and hence $(\mathbb{R}^2 \smallsetminus \{0\}, g_{Eucl})$ *is isometric (see Definition 5.14 below) to the warped product*

$$(\mathbb{R}^+, g_{Eucl}) \times_{r^2} (S^1, g_{Round}).$$

- If $(\mathcal{M}, g)$ is a Riemannian manifold and $\lambda\colon \mathcal{M} \to (0,\infty)$ is a smooth function, then $(\mathcal{M}, \lambda g)$ is also a Riemannian manifold.

Exercise 5.13 (Angles for conformal metrics)**.** *For such a smooth function* $\lambda\colon \mathcal{M} \to (0,\infty)$*, show that the* g *angle between two vectors or curves is equal to the* λg *angle.*

The manifolds $(\mathcal{M}, g)$ and $(\mathcal{M}, \lambda g)$ are said to be *conformal.* If λ is a constant then $(\mathcal{M}, g)$ and $(\mathcal{M}, \lambda g)$ are said to be homothetic.

5.5 Isometries

In Riemannian geometry, the notion of "sameness" is that of isometry. One can call Riemannian geometry the study of properties which are invariant under isometry.

Definition 5.14 (Isometries)**.** *Two Riemannian manifolds* $(\mathcal{M}, g)$ *and* $(\mathcal{N}, h)$ *are called* isometric *if there exists a diffeomorphism* $f\colon \mathcal{M} \to \mathcal{N}$ *such that* $f^*h = g$*. Such a diffeomorphism* f *is called an* isometry*.*

An isometry $f\colon (\mathcal{M}, g) \to (\mathcal{M}, g)$ is called an *isometry of* $(\mathcal{M}, g)$. Think of such an f as describing a symmetry of $(\mathcal{M}, g)$.

Definition 5.15 (Local isometries)**.** *Two Riemannian manifolds* $(\mathcal{M}, g)$ *and* $(\mathcal{N}, h)$ *are called* locally isometric *if, for each* $p \in \mathcal{M}$*, there exists open sets* $U \subset \mathcal{M}$ *and* $V \subset \mathcal{N}$ *with* $p \in U$ *and an isometry* $f\colon U \to V$*.*

A Riemannian manifold $(\mathcal{M}, g)$ *is called* flat *if it is locally isometric to* $(\mathbb{R}^n, g_{Eucl})$*.*

Note that flat manifolds can have different topologies to $\mathbb{R}^n$.

One can ask the following naive question: *is every Riemannian manifold flat?* It is not yet obvious that the answer to this question is no. It will become clear after we discuss curvature that the answer is indeed no. Much of our efforts for the coming lectures will be directed towards introducing a notion of curvature. For now, one can take this question as a motivation for introducing curvature. The answer to this question is yes, however, when $n = 1$.

Exercise 5.16 (One dimensional Riemannian manifolds)**.** *Show that every one dimensional Riemannian manifold is flat.*

Recall that an immersion is a map between manifolds $f\colon \mathcal{M} \to \mathcal{N}$ such that f_{*p} is injective for all $p \in \mathcal{M}$. When a manifold $(\mathcal{M}, g)$ can be immersed into a larger manifold $(\mathcal{N}, h)$ in such a way that the geometry of $(\mathcal{M}, g)$ arises as the induced geometry of $(\mathcal{N}, h)$, we call such an immersion an *isometric immersion.*

Definition 5.17 (Isometric immersions)**.** *An immersion* $f\colon (\mathcal{M}, g) \to (\mathcal{N}, h)$ *is called* isometric *if* $f^*h = g$*.*

Recall the Nash Embedding Theorem, Theorem 1.2, which guarantees that every Riemannian manifold can be isometrically immersed (in fact, embedded) into $\mathbb{R}^N$ for some N.

5.6 Conformal maps

More generally, one can consider *conformal maps.*

Definition 5.18 (Conformal maps). *A diffeomorphism $f\colon (\mathcal{M}, g) \to (\mathcal{N}, h)$ between two Riemannian manifolds is called a* conformal map *with* conformal factor $\lambda\colon \mathcal{M} \to (0, \infty)$ *if*

$$f^*h = \lambda^2 g.$$

Two manifolds $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ are called conformal *if there is a conformal diffeomorphism between them.*

Exercise 5.19 (Conformal maps preserve angles). *Show that conformal maps preserve angles between vectors and curves.*

Example 5.20 (Stereographic projection). *Stereographic projection from the north pole N of S^n, $\Phi\colon S^n \setminus \{N\} \to \mathbb{R}^n$, is a conformal map.*

Definition 5.21 (Locally conformally flat manifolds). *A Riemannian manifold $(\mathcal{M}, g)$ is called* locally conformally flat *if, for all $p \in \mathcal{M}$ there exists an open set $U \subset \mathcal{M}$ with $p \in U$, an open set $V \subset \mathbb{R}^n$ and a conformal diffeomorphism $f\colon U \to V$.*

Example 5.22 (The round sphere is locally conformally flat). *The round sphere (S^n, g_{Round}) is locally conformally flat. Stereographic projection is a local conformal diffeomorphism to $\mathbb{R}^n$.*

5.7 Existence of Riemannian metrics

Every smooth manifold $\mathcal{M}$ admits a Riemannian metric. One easy way to establish this fact is to appeal to the Whitney Embedding Theorem, Theorem 1.1, which guarantees the existence of an embedding $F\colon \mathcal{M} \to \mathbb{R}^N$, for some large N. The pullback of the Euclidean metric, F^*g_{Eucl}, is then a metric on $\mathcal{M}$. Such a proof is overkill, however. This section concerns a more direct proof, using a *partition of unity.*

Definition 5.23 (Partitions of unity). *Let $\mathcal{M}$ be a smooth manifold.*

- *An open cover $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ of $\mathcal{M}$ is called* locally finite *if, for every $p \in \mathcal{M}$, there exists a neighbourhood W of p such that the set*
$$\{\alpha \in \mathcal{A} \mid W \cap V_\alpha \neq \emptyset\},$$
is finite.

- *The* support *of a function $f\colon \mathcal{M} \to \mathbb{R}$is defined to be*
$$\operatorname{supp}(f) = \overline{\{p \in \mathcal{M} \mid f(p) \neq 0\}}.$$

- *Given a locally finite open cover of $\mathcal{M}$, $\{V_\alpha\}_{\alpha \in \mathcal{A}}$, a* partition of unity subordinate to $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ *is a collection $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ of smooth functions $f_\alpha\colon V_\alpha \to [0,1]$ such that* $\operatorname{supp}(f_\alpha) \subset V_\alpha$ *for all $\alpha \in \mathcal{A}$ and*
$$\sum_{\alpha \in \mathcal{A}} f_\alpha(p) = 1, \tag{9}$$
for all $p \in \mathcal{M}$.

Note that the summation (9) is finite for all $p \in \mathcal{M}$ since the cover $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ is locally finite.

Proposition 5.24 (Existence of partitions of unity). *Every smooth manifold admits a sub-atlas $\{(V_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ such that $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ is locally finite, and a partition of unity $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ subordinate to $\{V_\alpha\}_{\alpha \in \mathcal{A}}$.*

Proof. The proof is non-examinable and not given here. See, for example, [11]. ∎

The following proof follows a standard construction in Riemannian geometry of first constructing in each chart using the local homeomorphism to an open subset of $\mathbb{R}^n$ and the Euclidean metric on $\mathbb{R}^n$, and then patching the construction in each chart together using a partition of unity.

Theorem 5.25 (Existence of Riemannian metrics)**.** *Every manifold $\mathcal{M}$ admits a Riemannian metric.*

Proof. Let $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ be a locally finite open cover of $\mathcal{M}$ with charts $\phi_\alpha\colon V_\alpha \to W_\alpha \subset \mathbb{R}^n$, and let $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ be a subordinate partition of unity. For $p \in \mathcal{M}$, define

$$g_p := \sum_{\alpha\in\mathcal{A}} f_\alpha(p)\phi_\alpha^* g_{Eucl}|_p.$$

Note that the summation involves only finitely many terms for each $p \in \mathcal{M}$. Clearly g is symmetric. The positive definiteness follows from the positive definiteness of each $\phi_\alpha^* g_{Eucl}|_p$, together with the fact that $0 \le f_\alpha \le 1$ for each α and $\sum_{\alpha\in\mathcal{A}} f_\alpha(p) = 1$. $\square$

5.8 Musical isomorphisms and the inner product of tensor fields

Let $(\mathcal{M}, g)$ be a Riemannian manifold. For each $p \in \mathcal{M}$, the metric g defines a canonical isomorphism between $T_p\mathcal{M}$ and $T_p^*\mathcal{M}$.

Definition 5.26 (The flat isomorphism)**.** *Define the map "flat" $\flat\colon T_p\mathcal{M} \to T_p^*\mathcal{M}$ by*

$$X^\flat(Y) := g_p(X, Y),$$

for all $X, Y \in T_p\mathcal{M}$.

Exercise 5.27 (Flat is an isomorphism)**.** *Show that $\flat\colon T_p\mathcal{M} \to T_p^*\mathcal{M}$ is a vector space isomorphism.*

Definition 5.28 (The sharp isomorphism)**.** *Define the map "sharp" $\sharp\colon T_p^*\mathcal{M} \to T_p\mathcal{M}$ to be the inverse of the isomorphism $\flat$: $\sharp = \flat^{-1}$. For $\xi \in T_p^*\mathcal{M}$, write $\xi^\sharp$ for $\sharp(\xi)$.*

In local coordinates, if $X = X^i e_i \in T_p\mathcal{M}$ for some basis $\{e_i\}$ of $T_p\mathcal{M}$ then

$$X^\flat = X_i\theta^i,$$

where $\{\theta^i\}$ is the dual basis to $\{e_i\}$ of $T_p^*\mathcal{M}$ and

$$X_i = g_{ij}X^j,$$

and $g_{ij} = g_p(e_i, e_j)$. The operation of applying the map $\flat$ to a vector is often called "lowering an index".

Recall that $(g_{ij})_{i,j=1}^n$ is a positive definite symmetric matrix, and hence is invertible. The components of the inverse of this matrix is denoted g^{ij}, so that

$$g^{ij}g_{jk} = \delta^i_k,$$

for $i, k = 1, \ldots, n$.

If $\xi = \xi_i\theta^i \in T_p^*\mathcal{M}$, then

$$\xi^\sharp = \xi^i e_i,$$

where

$$\xi^i = g^{ij}\xi_j.$$

The operation of applying the map $\sharp$ to a covector is often called "raising an index".

The isomorphisms $\flat, \sharp$ also extend to isomorphisms

$$\flat\colon \mathfrak{X}(\mathcal{M}) \to \Gamma(T^*\mathcal{M}), \qquad \sharp\colon \Gamma(T^*\mathcal{M}) \to \mathfrak{X}(\mathcal{M}).$$

Definition 5.29 (Gradient of a function)**.** *Given a function* $f \in C^\infty(\mathcal{M})$*, define the* gradient of f, $\mathrm{grad}(f) \in \mathfrak{X}(\mathcal{M})$*, as*

$$\mathrm{grad}(f) := (df)^\sharp.$$

The operation $\sharp$ is used to extend g to an inner product on $T_p^*\mathcal{M}$. Given $\omega, \eta \in T_p^*\mathcal{M}$, define

$$g_p(\omega, \eta) := g_p(\omega^\sharp, \eta^\sharp).$$

In coordinates

$$g_p(\omega, \eta) = g^{ij}\omega_i\eta_j.$$

Similarly one can defined the norm of a covector $\omega \in T_p^*\mathcal{M}$,

$$|\omega|_g := |\omega^\sharp|_g = \sqrt{g_p(\omega^\sharp, \omega^\sharp)}.$$

Exercise 5.30 (Length of a covector in local coordinates)**.** *Let* (U, ϕ, x^i) *be a chart of* $(\mathcal{M}, g)$ *and consider* $p \in U$*. Show that, if*

$$\omega = \omega_i dx^i\big|_p \in T_p^*\mathcal{M},$$

then

$$|\omega|_g = \sqrt{g^{ij}(p)\omega_i\omega_j},$$

In particular, for any function $f\colon \mathcal{M} \to \mathbb{R}$*,*

$$|df|_g = \sqrt{g^{ij}\partial_{x^i}f\partial_{x^j}f}.$$

The operations $\flat$ and $\sharp$ are extended in the obvious way to act on higher order tensor fields. For example, if T is a $(1,2)$ tensor field, $T \in \Gamma(T_2^1\mathcal{M})$, then for each $p \in \mathcal{M}$, T_p is a map

$$T_p\colon T_p^*\mathcal{M} \times T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}.$$

Define

$$T_p^\flat\colon T_p\mathcal{M} \times T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R},$$

by

$$T_p^\flat(X, Y, Z) = T_p(X^\flat, Y, Z).$$

In coordinates, if $T = T^i{}_{jk}\frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k$, then

$$T^\flat = T_{ijk}dx^i \otimes dx^j \otimes dx^k, \qquad T_{ijk} = g_{il}T^l{}_{jk}.$$

The operations $\flat$ and $\sharp$ are defined similarly for higher order tensor fields, though one typically has to explain which argument the operation is being applied to.

The metric g can moreover be extended to act on pairs of general (r,s) tensor fields as follows. First, extend g to act on reducible (r,s) tensors by

$$g_p(X_1\otimes\ldots\otimes X_r\otimes\xi^1\otimes\ldots\otimes\xi^s, \tilde{X}_1\otimes\ldots\otimes\tilde{X}_r\otimes\tilde{\xi}^1\otimes\ldots\otimes\tilde{\xi}^s) := g_p(X_1, \tilde{X}_1)\ldots g_p(X_r, \tilde{X}_r)g_p(\xi^1, \tilde{\xi}^1)\ldots g_p(\xi^s, \tilde{\xi}^s),$$

for all $X_1, \tilde{X}_1, \ldots, X_r, \tilde{X}_r \in T_p\mathcal{M}$, $\xi^1, \tilde{\xi}^1 \ldots, \xi^s, \tilde{\xi}^s \in T_p^*\mathcal{M}$. The metric g is then extended to general (r,s) tensors by linearity.

Similarly, one defines the *length* of an (r,s) tensor T as

$$|T|_g = \sqrt{g(T,T)}.$$

Exercise 5.31 (Length of a tensor in local coordinates)**.** *Suppose, in a local chart* (U, ϕ, x^i) *of* $(\mathcal{M}, g)$*,* $T \in \Gamma(T_r^s\mathcal{M})$ *takes the form*

$$T = T^{i_1\ldots i_r}{}_{j_1\ldots j_s}\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}.$$

Show that

$$|T|_g^2 = g_{i_1k_1}\ldots g_{i_rk_r}g^{j_1l_1}\ldots g^{j_sl_s}T^{i_1\ldots i_r}{}_{j_1\ldots j_s}T^{k_1\ldots k_r}{}_{l_1\ldots l_s}.$$

6 Flows of vector fields and the Lie derivative

Recall (see Section 3) that we have a well defined way of taking derivatives of smooth functions on a manifold. We would like to have a notion of differentiation for vector fields and higher order tensor fields. There is a notion of differentiation intrinsic to a smooth manifold, know as the *Lie derivative*, introduced in this section. The Lie derivative, however, has some undesirable properties. A better notion of differentiation requires an additional structure, known as a *connection*, and is discussed in Section 7.

6.1 Derivatives of vector fields in $\mathbb{R}^n$

The existence of the canonical Cartesian coordinate system means that there is a canonical way of taking derivatives of vector fields on $\mathbb{R}^n$. It is worth taking a moment to recall this notion of differentiation and to examine the obstructions to generalising it to abstract manifolds.

In $\mathbb{R}^n$ we can, using the slight abuse of Remark 4.3, view a vector field X as a smooth function $X\colon \mathbb{R}^n \to \mathbb{R}^n$. Given a point $p \in \mathbb{R}^n$ and a direction $v \in \mathbb{R}^n$, the directional derivative of X at p in the direction v is given by

$$D_v X|_p := \lim_{t\to 0} \frac{X(p+tv) - X(p)}{t}. \tag{10}$$

Consider now a smooth manifold $\mathcal{M}$ and a vector field $X \in \mathfrak{X}(\mathcal{M})$. There are two immediate issues in generalising the definition (10) of the directional derivative of X:

1. Since now $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$, there is no definition of $p + tv$. One could replace $v \in T_p\mathcal{M}$ with a vector field $V \in \mathfrak{X}(\mathcal{M})$ and replace $p + tv$ with a curve γ with $\gamma(0) = p$ and $\dot\gamma(0) = v$. The existence of such a curve is discussed in Section 6.2.

2. Suppose one can find such a γ. Then $X_{\gamma(t)} \in T_{\gamma(t)}\mathcal{M}$ and $X_p \in T_p\mathcal{M}$ lie in two different vector spaces. In order to subtract X_p from $X_{\gamma(t)}$ one needs a way to identify these spaces. Different ways of identifying these spaces give rise to different notions of differentiation. An intrinsic way of identifying these spaces, which gives rise to the *Lie derivative*, is discussed in Section 6.3.

6.2 The flow of a vector field

Recall the local existence theorem for an ordinary differential equation in $\mathbb{R}^n$.

Theorem 6.1 (Picard/Cauchy–Lipschitz)**.** *Given an open set $U \subset \mathbb{R}^n$, a smooth vector field $V\colon U \to \mathbb{R}^n$ and a point $x_0 \in U$, there exists a unique maximal interval $(T_-, T_+) \subset \mathbb{R}$ with $T_- < 0, T_+ > 0$ and a unique solution $x\colon (T_-, T_+) \to \mathbb{R}^n$ of the initial value problem*

$$\dot x(t) = V(x(t)), \qquad x(0) = x_0.$$

The solution depends smoothly on x_0. Moreover, if $T_+ < \infty$ then, for any compact set $K \subset U$, there exists $t_K \in (0, T_+)$ such that $x(t_K, T_+) \cap K = \emptyset$.

This local existence theorem yields the following theorem on the existence of integral curves of a vector field on a manifold.

Theorem 6.2 (Existence and uniqueness of integral curves of a vector field)**.** *Given a smooth manifold $\mathcal{M}$, a vector field $V \in \mathfrak{X}(\mathcal{M})$ and a point $p \in \mathcal{M}$, there exists a unique maximal interval $(T_-, T_+) \subset \mathbb{R}$ with $T_- < 0$, $T_+ > 0$ and a unique curve $\gamma\colon (T_-, T_+) \to \mathcal{M}$ such that*

$$\dot\gamma(t) = V|_{\gamma(t)}, \qquad \gamma(0) = p,$$

for all $t \in (T_-, T_+)$. Moreover, if $T_+ < \infty$ then, for any compact set $K \subset \mathcal{M}$, there exists $t_K \in (0, T_+)$ such that $\gamma(t_K, T_+) \cap K = \emptyset$. In particular, if $\mathcal{M}$ is compact then $T_- = -\infty$, $T_+ = \infty$.

The curve γ of Theorem 6.2 is called the *integral curve* of V through p.

Definition 6.3 (Complete vector fields)**.** *Let $\mathcal{M}$ be a smooth manifold and $V \in \mathfrak{X}(\mathcal{M})$ a vector field. If, for every $p \in \mathcal{M}$, $T_- = -\infty$, $T_+ = \infty$, where $T_\pm$ are as in Theorem 6.2, then V is called* complete*.*

Exercise 6.4 (Examples of complete and incomplete vector fields)**.** *Give an example of a manifold $\mathcal{M}$ and a vector field on $\mathcal{M}$ which is not complete. Give an example of a non-compact manifold $\mathcal{M}$ and a nontrivial complete vector field on $\mathcal{M}$.*

In fact, Theorem 6.1 tells us more.

Theorem 6.5 (Flow of a vector field)**.** *Given a smooth manifold $\mathcal{M}$, a vector field $V \in \mathfrak{X}(\mathcal{M})$ and a point $p \in \mathcal{M}$, there exists $U \subset \mathcal{M}$ open with $p \in U$, $\varepsilon > 0$ and a unique smooth map*

$$\Phi\colon (-\varepsilon, \varepsilon) \times U \to \mathcal{M},$$

such that

$$\frac{\partial \Phi}{\partial s}(s,p) = V|_{\Phi(s,p)}, \qquad \Phi(0,p) = p,$$

for all $s \in (-\varepsilon,\varepsilon)$, $p \in U$. For each $s \in (-\varepsilon,\varepsilon)$, the map $\Phi_s := \Phi(s,\cdot)\colon U \to \mathcal{M}$ is a diffeomorphism onto its image. Moreover, if $s, t, s+t \in (-\varepsilon,\varepsilon)$, then $\Phi_s \circ \Phi_t = \Phi_{s+t}$.

The map Φ of Theorem 6.5 is called the *flow* of the vector field V. The curve $s \mapsto \Phi(s,p)$ is called the *integral curve*, or *flow line*, of V through p.

6.3 The Lie derivative

Given a vector field V, the Lie derivative involves using the pullback of the flow Φ_t of V to identify the tangent space $T_{\Phi_t(p)}\mathcal{M}$ with the tangent space $T_p\mathcal{M}$.

Recall that, for a given function $f\colon \mathcal{M} \to \mathcal{N}$, there is only a way of pushing forward vectors, and pulling back $(0,k)$ tensors (or, more generally, $(0,k)$ tensor *fields*). If f is a diffeomorphism then these operations can be extended using the inverse f^{-1}.

Definition 6.6 (Pullback of a vector by a diffeomorphism)**.** *If $f\colon \mathcal{M} \to \mathcal{N}$ is a diffeomorphism between smooth manifolds and $X \in T_{f(p)}\mathcal{N}$, define*

$$f^*X|_p := f^{-1}_{*f(p)}X \in T_p\mathcal{M}.$$

Definition 6.7 (Lie derivative of a vector field)**.** *Let $\mathcal{M}$ be a smooth manifold and $X, V \in \mathfrak{X}(\mathcal{M})$ be vector fields on $\mathcal{M}$. Define the* Lie derivative *of X along V to be the vector field $\mathcal{L}_V X \in \mathfrak{X}(\mathcal{M})$ defined by*

$$\mathcal{L}_V X|_p := \lim_{t\to 0} \frac{(\Phi_t^* X)_p - X_p}{t} = \lim_{t\to 0} \frac{(\Phi_{-t})_{*\Phi_t(p)}X - X_p}{t},$$

where Φ is the flow of V.

Exercise 6.8 (Lie derivative and Lie bracket)**.** *If $X, V \in \mathfrak{X}(\mathcal{M})$ are vector fields, then the Lie derivative takes the form*

$$\mathcal{L}_V X = [V, X],$$

where $[\cdot,\cdot]$ is the Lie bracket *(see Definition 3.15).*

More generally, (r,s) tensor fields can be pulled back by *diffeomorphisms.*

Definition 6.9 (Pullback of a general tensor field by a diffeomorphism)**.** *If $f\colon \mathcal{M} \to \mathcal{N}$ is a diffeomorphism between smooth manifolds and $T \in \Gamma(T^r_s\mathcal{N})$ is an (r,s) tensor field, define the* pullback of T by f*, $f^*T \in \Gamma(T^r_s\mathcal{M})$, by*

$$(f^*T)_p(\xi_1,\dots,\xi_r,X_1,\dots,X_s) := T_{f(p)}\big(((f^{-1})^*\xi_1)_{f(p)},\dots,((f^{-1})^*\xi_r)_{f(p)}, f_{*p}X_1,\dots,f_{*p}X_s\big),$$

for all $\xi_1,\dots,\xi_r \in T^*_p\mathcal{M}$ and $X_1,\dots,X_s \in T_p\mathcal{M}$.

The definition of Lie derivative can then be extended to general (r, s) tensor fields.

Definition 6.10 (Lie derivative of a tensor field)**.** *If $\mathcal{M}$ is a smooth manifold, $V \in \mathfrak{X}(\mathcal{M})$ is a vector field, and $T \in \Gamma(T^r_s\mathcal{M})$ is an (r, s) tensor field, the* Lie derivative *of T in the direction V is the (r, s) tensor field $\mathcal{L}_V T \in \Gamma(T^r_s\mathcal{M})$ defined by*

$$\mathcal{L}_V T|_p := \lim_{t\to 0} \frac{(\Phi_t^* T)_p - T_p}{t},$$

where Φ is the flow of V.

Exercise 6.11 (Properties of Lie derivative)**.** *Let $\mathcal{M}$ be a smooth manifold, $V \in \mathfrak{X}(\mathcal{M})$ be a vector field, and $T_1, T_2, T_3 \in \Gamma(T^r_s\mathcal{M})$ be (r, s) tensor fields. Show that the Lie derivative has the following properties.*

1. $\mathcal{L}_V f = Vf$ *for all* $f \in C^\infty(\mathcal{M}) = \Gamma(T^0_0\mathcal{M})$.
2. $\mathcal{L}_V (aT_1 + bT_2) = a\mathcal{L}_V T_1 + b\mathcal{L}_V T_2$ *for all* $a, b \in \mathbb{R}$.
3. $\mathcal{L}_V (T_1 \otimes T_2) = (\mathcal{L}_V T_1) \otimes T_2 + T_1 \otimes (\mathcal{L}_V T_2)$.
4. $\mathcal{L}_V (fT) = Vf \cdot T + f\mathcal{L}_V T$ *for all* $f \in C^\infty(\mathcal{M})$.
5. $\mathcal{L}_V (c_i{}^j T) = c_i{}^j(\mathcal{L}_V T)$ *for all* $1 \le j \le r$, $1 \le i \le s$.
6. *In local coordinates,*

$$\begin{aligned}(\mathcal{L}_V T)^{i_1\dots i_r}{}_{j_1\dots j_s} = {}& V(T^{i_1\dots i_r}{}_{j_1\dots j_s}) - \partial_k V^{i_1}\, T^{k i_2\dots i_r}{}_{j_1\dots j_s} - \ldots - \partial_k V^{i_r}\, T^{i_1\dots i_{r-1} k}{}_{j_1\dots j_s} \\ & + \partial_{j_1} V^k\, T^{i_1\dots i_r}{}_{k j_2\dots j_s} + \ldots + \partial_{j_s} V^k\, T^{i_1\dots i_r}{}_{j_1\dots j_{s-1} k}.\end{aligned}$$

Continuous symmetries of Riemannian manifolds are conveniently described using the Lie derivative.

Definition 6.12 (Killing vectors)**.** *On a Riemannian manifold $(\mathcal{M}, g)$, a vector field $K \in \mathfrak{X}(\mathcal{M})$ is called a* Killing vector *if*

$$\mathcal{L}_K g = 0.$$

Killing vector fields generate continuous symmetries of $(\mathcal{M}, g)$.

Remark 6.13 (Properties of Lie derivative)**.**

1. *The Lie derivative only requires a smooth manifold structure. It does not, for example, require a Riemannian metric.*
2. *The Lie derivative has the following undesirable property: if $f \in C^\infty(\mathcal{M})$, then $\mathcal{L}_{fV} T \neq f\mathcal{L}_V T$ in general (which can be easily seen using Exercise 3.17 and Exercise 6.8), i.e. $\mathcal{L}_V T|_p$ depends not just on V at $p \in \mathcal{M}$, but on V in a full neighbourhood of p.*

A better notion of derivative requires a better way of “connecting” nearby tangent spaces.

7 Affine connections and the Levi-Civita connection

Such a way of “connecting tangent spaces” requires an additional structure, called a *connection.* In Section 7.1 affine connections are introduced. The link with “connecting tangent spaces” will become apparent in Exercise 7.19. It will then be seen that a Riemannian metric on a manifold induces a canonical connection, known as the *Levi-Civita connection.*

7.1 Connections and the covariant derivative

Of primary concern to us will be *affine connections*. First, a connection on a general vector bundle is introduced.

Definition 7.1 (Connections and affine connections)**.** *A* connection *on a vector bundle* $\pi\colon E \to \mathcal{M}$ *is a map*

$$\nabla\colon \mathfrak{X}(\mathcal{M}) \times \Gamma(E) \to \Gamma(E),$$

where $\nabla(V,T)$ *is denoted* $\nabla_V T$*, such that:*

- *For each* $T \in \Gamma(E)$*, the map* $V \mapsto \nabla_V T$ *is linear over* $C^\infty(\mathcal{M})$*, i.e.*

  $$\nabla_{fV+hW} T = f\nabla_V X + h\nabla_W T,$$

  for all $V, W \in \mathfrak{X}(\mathcal{M})$*,* $T \in \Gamma(E)$*,* $f, h \in C^\infty(\mathcal{M})$*.*

- *For each* $V \in \mathfrak{X}(\mathcal{M})$*, the map* $T \mapsto \nabla_V T$ *is linear over* $\mathbb{R}$*, i.e.*

  $$\nabla_V(aS + bT) = a\nabla_V S + b\nabla_V T,$$

  for all $V \in \mathfrak{X}(\mathcal{M})$*,* $S, T \in \Gamma(E)$*,* $a, b \in \mathbb{R}$*.*

- ∇ *satisfies the following product rule*

  $$\nabla_V(fT) = Vf \cdot T + f\nabla_V T,$$

  for all $V \in \mathfrak{X}(\mathcal{M})$*,* $T \in \Gamma(E)$*,* $f \in C^\infty(\mathcal{M})$*.*

An affine connection*, or* linear connection *on a smooth manifold* $\mathcal{M}$ *is a connection on the tangent bundle* $T\mathcal{M}$*.*

Given an affine connection ∇ and vector fields $V, X \in \mathfrak{X}(\mathcal{M})$, the vector field $\nabla_V X$ is called the *covariant derivative* of X in the direction V. We will see shortly that an affine connection on a manifold induces a canonical connection on each (r, s) tensor bundle.

Example 7.2 (Euclidean connection)**.** *The* canonical connection*, or* Euclidean connection*, on* $\mathbb{R}^n$ *defined by*

$$(\nabla_V X)_p := \lim_{t\to 0} \frac{X|_{p+tV_p} - X|_p}{t},$$

for all $X, V \in \mathfrak{X}(\mathbb{R}^n)$ *is the usual directional derivative.*

Remark 7.3.

1. *An affine connection* ∇ *is not a tensor field. For fixed* V *the map* $X \mapsto \nabla_V X$ *is, in particular, not linear over* $C^\infty(\mathcal{M})$*.*
2. *The notation* D *is also commonly used for connections.*

Definition 7.4 (Christoffel symbols)**.** *Given an affine connection* ∇ *on a smooth manifold* $\mathcal{M}$ *and a local frame* $\{e_i\}_{i=1}^n$ *defined on* $U \subset \mathcal{M}$*, define smooth functions* $\Gamma_{ij}^k\colon U \to \mathbb{R}$*, for* $i, j, k = 1, \dots, n$ *by*

$$\nabla_{e_i} e_j = \Gamma_{ij}^k e_k.$$

The functions $\{\Gamma_{ij}^k\}$ *are called the* Christoffel symbols *(or* connection coefficients*) of* ∇ *with respect to the frame* $\{e_i\}$*.*

Exercise 7.5 (Covariant derivative in a local frame)**.** *Given a local frame* $\{e_i\}$ *for a smooth manifold* $\mathcal{M}$ *and* $X, Y \in \mathfrak{X}(\mathcal{M})$*, show that*

$$\nabla_X Y = \big(X(Y^k) + X^i Y^j \Gamma_{ij}^k\big) e_k,$$

where $X = X^i e_i$*,* $Y = Y^i e_i$*.*

Note in particular that $\nabla_X Y|_p$ depends only on X at p, and on Y in a neighbourhood of p.

Example 7.6 (Euclidean connection in Cartesian coordinates)**.** *If $\{x^i\}$ are Cartesian coordinates on $\mathbb{R}^n$, then the canonical connection satisfies*

$$\nabla_X Y = X(Y^i)\frac{\partial}{\partial x^i} = X^j \frac{\partial Y^i}{\partial x^j}\frac{\partial}{\partial x^i},$$

for all $X, Y \in \mathfrak{X}(\mathbb{R}^n)$. The Christoffel symbols on ∇ in Cartesian coordinates therefore all identically vanish.

Exercise 7.7 (Euclidean connection in polar coordinates)**.** *Compute the Christoffel symbols of the canonical connection on $\mathbb{R}^2$ in polar coordinates.*

Proposition 7.8 (Existence of connections)**.** *Every smooth manifold admits an affine connection.*

Proof. The proof can be established following the strategy of the proof of Theorem 5.25, by pulling back the Euclidean connection in each chart and stitching each chart together using a partition of unity, and is left as an exercise. ☐

There are two alternative ways of proving Proposition 7.8:

1. Wait for the existence of the Levi-Civita connection (see Theorem 7.26) and use the existence of a Riemannian metric on any manifold, Theorem 5.25.
2. The second alternative way is the "overkill proof" of appealing to the Whitney Embedding Theorem, Theorem 1.1, and considering the induced Euclidean connection.

An affine connection ∇ on a smooth manifold $\mathcal{M}$ can be extended to act on tensor fields,

$$\nabla \colon \mathfrak{X}(\mathcal{M}) \times \Gamma(T^r_s\mathcal{M}) \to \Gamma(T^r_s\mathcal{M}).$$

Indeed, given an affine connection ∇ on $\mathcal{M}$, there is a unique connection on $T^r_s\mathcal{M}$ satisfying the following three properties.

- For any function $f \in C^\infty(\mathcal{M}) = \Gamma(T^0_0\mathcal{M})$,
$$\nabla_V f := Vf,$$
for all $V \in \mathfrak{X}(\mathcal{M})$.
- The product rule
$$\nabla_V(T_1 \otimes T_2) = (\nabla_V T_1) \otimes T_2 + T_1 \otimes (\nabla_V T_2)$$
holds for all $T_1 \in \Gamma(T^{r_1}_{s_1}\mathcal{M})$, $T_2 \in \Gamma(T^{r_2}_{s_2}\mathcal{M})$, $V \in \mathfrak{X}(\mathcal{M})$.
- ∇ commutes with contractions
$$\nabla_V\big({c_i}^j T\big) = {c_i}^j\big(\nabla_V T\big),$$
for all $V \in \mathfrak{X}(\mathcal{M})$, $T \in \Gamma(T^r_s\mathcal{M})$, $1 \le i \le s$, $1 \le j \le r$.

Explicitly, for any one form $\omega \in \Gamma(T^*\mathcal{M}) = \Gamma(T^0_1\mathcal{M})$, one computes, for any $X, Y \in \mathfrak{X}(\mathcal{M})$,

$$X(\omega(Y)) = \nabla_X\big({c_1}^1(Y \otimes \omega)\big) = {c_1}^1\big(\nabla_X(Y \otimes \omega)\big) = {c_1}^1\big(\nabla_X Y \otimes \omega + Y \otimes \nabla_X \omega\big),$$

by the first, third, and second properties above respectively, and hence

$$(\nabla_V \omega)(X) = V(\omega(X)) - \omega(\nabla_V X),$$

for all $V, X \in \mathfrak{X}(\mathcal{M})$. In components,

$$\nabla_V \omega = \big(V(\omega_i) - V^k \omega_j \Gamma^j_{ki}\big)\theta^i,$$

where Γ^j_{ki} are the Christoffel symbols of ∇ with respect to a local frame $\{e_i\}$, and $\{\theta^i\}$ is its dual frame.

Exercise 7.9 (Covariant derivative of a tensor field). *Show, in a similar way, that, for $T \in \Gamma(T^r_s\mathcal{M})$,*

$$\begin{aligned}(\nabla_X T)(\xi_1,\dots,\xi_r,Y_1,\dots,Y_s) = X\big(&T(\xi_1,\dots,\xi_r,Y_1,\dots,Y_s)\big)\\ &- T(\nabla_X\xi_1,\xi_2,\dots,\xi_r,Y_1,\dots,Y_s) - \dots - T(\xi_1,\dots,\xi_{r-1},\nabla_X\xi_r,Y_1,\dots,Y_s)\\ &- T(\xi_1,\dots,\xi_r,\nabla_X Y_1,Y_2,\dots,Y_s) - \dots - T(\xi_1,\dots,\xi_r,Y_1,\dots,Y_{s-1},\nabla_X Y_s),\end{aligned}$$

for all $Y_1,\dots,Y_s \in \mathfrak{X}(\mathcal{M})$, $\xi_1,\dots,\xi_r \in \Gamma(T^*\mathcal{M})$, and hence, in components,

$$\nabla_X T = (\nabla_X T)^{i_1\dots i_r}{}_{j_1\dots j_s} e_{i_1}\otimes\dots\otimes e_{i_r}\otimes\theta^{j_1}\otimes\dots\otimes\theta^{j_s},$$

where

$$\begin{aligned}(\nabla_X T)^{i_1\dots i_r}{}_{j_1\dots j_s} = {}& X\big(T^{i_1\dots i_r}{}_{j_1\dots j_s}\big) + X^k\,\Gamma^{i_1}_{k\,l}\,T^{l\,i_2\dots i_r}{}_{j_1\dots j_s} + \dots + X^k\,\Gamma^{i_r}_{k\,l}T^{i_1\dots i_{r-1}\,l}{}_{j_1\dots j_s}\\ &- X^k\,\Gamma^l_{k\,j_1}T^{i_1\dots i_r}{}_{l\,j_2\dots j_s} - \dots - X^k\,\Gamma^l_{k\,j_r}T^{i_1\dots i_r}{}_{j_1\dots j_{s-1}\,l}.\end{aligned}$$

Definition 7.10 (Total covariant derivative). *If ∇ is an affine connection on $\mathcal{M}$ and $T \in \Gamma(T^r_s\mathcal{M})$, define the* total covariant derivative of T*, $\nabla T \in \Gamma(T^r_{s+1}\mathcal{M})$ by*

$$(\nabla T)(\xi_1,\dots,\xi_r,Y_1,\dots,Y_s,X) := (\nabla_X T)(\xi_1,\dots,\xi_r,Y_1,\dots,Y_s).$$

Definition 7.11 (Parallel tensor fields). *A tensor field $T \in \Gamma(T^r_s\mathcal{M})$ is called* parallel *if $\nabla T \equiv 0$.*

For $k \geq 2$, one can inductively define the *k-th total covariant derivative*

$$\nabla^k\colon \Gamma(T^r_s\mathcal{M}) \to \Gamma(T^r_{s+k}\mathcal{M}),$$

by

$$\nabla^k := \nabla(\nabla^{k-1}).$$

Remark 7.12 (Second covariant derivative of a vector field). *If $X, V_1, V_2 \in \mathfrak{X}(\mathcal{M})$ then*

$$(\nabla^2 X)(V_1,V_2) \neq \nabla_{V_2}\nabla_{V_1}X,$$

in general. In fact, one computes using Exercise 7.9,

$$(\nabla^2 X)(V_1,V_2) = \nabla_{V_2}\nabla_{V_1}X - \nabla_{\nabla_{V_2}V_1}X.$$

This computation will be relevant when we discuss the Riemann curvature tensor in Section 9.

7.2 Connections along curves and parallel transport

We are typically interested not only in taking covariant derivative of vector fields on a manifold $\mathcal{M}$, but vector fields along curves γ (such as the tangent vector to γ). If a curve γ intersects itself then a given vector field along γ may not extend to a vector field in $\mathcal{M}$.

Definition 7.13 (Vector fields along curves). *Let $\gamma\colon (a,b) \to \mathcal{M}$ be a curve. A smooth map $V\colon (a,b) \to T\mathcal{M}$ is called a* vector field along γ *if $V(t) \in T_{\gamma(t)}\mathcal{M}$ (i.e. if $\pi \circ V(t) = \gamma(t)$) for all $t \in (a,b)$.*

Denote

$$\mathfrak{X}(\gamma) := \{\text{vector fields along } \gamma\}.$$

A connection ∇ defines a "connection along γ", which can be used to differentiate vector fields along γ.

If $\gamma\colon (a,b) \to \mathcal{M}$ is a curve, $t_0 \in (a,b)$, $\{e_i\}$ is a local frame around $\gamma(t_0)$, and $V \in \mathfrak{X}(\gamma)$, then we can write

$$V(t) = V^i(t)e_i|_{\gamma(t)},$$

for functions $V^i\colon (t_0-\varepsilon, t_0+\varepsilon) \to \mathbb{R}$.

Definition 7.14 (Covariant derivative along a curve). *Given a curve $\gamma\colon (a,b) \to \mathcal{M}$ and a connection ∇ on $\mathcal{M}$, the* covariant derivative along γ *is the map*

$$D_t \colon \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma),$$

defined locally by

$$D_t V|_{t_0} = (V^i)'(t_0) e_i|_{\gamma(t_0)} + V^i(t_0) \nabla_{\dot\gamma(t_0)} e_i|_{\gamma(t_0)}.$$

The covariant derivative along γ has the following properties

1. The definition of $D_t V$ is independent of the local frame $\{e_i\}$.
2. Covariant derivative along γ is linear over $\mathbb{R}$:
$$D_t(aV + bW) = aD_t V + bD_t W,$$
for all $a, b \in \mathbb{R}$, $V, W \in \mathfrak{X}(\gamma)$.
3. Covariant derivative along γ satisfies the following product rule:
$$D_t(fV) = f' \cdot V + f \cdot D_t V,$$
for all $V \in \mathfrak{X}(\gamma)$, $f \in C^\infty(a,b)$.
4. If $V \in \mathfrak{X}(\gamma)$ and there exists $\tilde V \in \mathfrak{X}(\mathcal{M})$ such that
$$\tilde V|_{\gamma(t)} = V(t),$$
for all $t \in (a,b)$, then $D_t V = \nabla_{\dot\gamma} \tilde V$.

In view of the latter property, given $V \in \mathfrak{X}(\gamma)$ we often abuse notation and write $\nabla_{\dot\gamma} V$ for $D_t V$, even when V does not extend to a vector field on $\mathcal{M}$.

Definition 7.15 (Parallel vector fields along a curve). *Given a curve γ, a vector field along γ, $V \in \mathfrak{X}(\gamma)$, is called* parallel along γ *if $D_t V \equiv 0$.*

In Section 8.1, a *geodesic* will be defined to be a curve γ whose tangent vector $\dot\gamma$ is parallel along γ.

Note that, given $p \in \mathbb{R}^n$, a vector $v \in T_p\mathbb{R}^n \cong \mathbb{R}^n$ can be uniquely extended to a parallel vector *field* on $\mathbb{R}^n$ by taking it to be constant (one easily checks that such a vector field is parallel with respect to the Euclidean connection). On a general manifold $\mathcal{M}$ there is no such procedure (indeed the notion of a vector field being "constant" is special to $\mathbb{R}^n$, relying on the canonical identification of each tangent space to $\mathbb{R}^n$ with $\mathbb{R}^n$ itself). Remarkably, there is still a way of extending a vector at a point to a parallel *vector field along a curve.*

Proposition 7.16 (Existence and uniqueness of parallel transport). *Given an affine connection ∇ on a manifold $\mathcal{M}$, a curve $\gamma\colon (a,b) \to \mathcal{M}$, $t_0 \in (a,b)$ and $v_0 \in T_{\gamma(t_0)}\mathcal{M}$, there exists a unique parallel vector field $V(t)$ along γ such that $V(t_0) = v_0$.*

Proof. The main content of the proof is in the Picard/Cauchy–Lipschitz Theorem, Theorem 6.1. Indeed, let $\{e_i\}$ be a local frame around $\gamma(t_0)$, so that

$$\dot\gamma(t) = \dot\gamma^i(t) e_i|_{\gamma(t)}, \qquad V(t) = V^i(t) e_i|_{\gamma(t)},$$

for appropriate functions $\dot\gamma^i$ and V^i. Now V satisfies

$$D_t V \equiv 0, \qquad V(t_0) = v_0,$$

if and only if

$$\Big(\frac{dV^i}{dt}(t) + \dot\gamma^j(t)V^k(t)\Gamma^i_{jk}(\gamma(t))\Big)e_i|_{\gamma(t)} = 0, \qquad V^i(t_0)e_i|_{\gamma(t_0)} = v^i_0 e_i|_{\gamma(t_0)},$$

if and only if

$$\frac{dV^i}{dt}(t) + \dot\gamma^j(t)V^k(t)\Gamma^i_{jk}(\gamma(t)) = 0, \qquad V^i(t_0) = v^i_0, \tag{11}$$

for $i = 1, \dots, n$. Note that (11) is an initial value problem for a linear system of ordinary differential equations. Theorem 6.1 guarantees there exists a unique solution. Since the system is linear, the solution exists in the entire domain of the local frame $\{e_i\}$. Indeed, the Grönwall inequality a priori implies that, for any $T \geq t_0$ such that $\gamma(t)$ lies in the domain of the local frame for all $t_0 \leq t \leq T$,

$$\sum_{i=1}^{n} \sup_{t_0\leq t\leq T} |V^i(t)| \leq \sum_{i=1}^{n} |v^i_0| \exp\Big(\sum_{i=1}^{n} \sup_{t_0\leq t\leq T} |\dot\gamma^i(t)| \sum_{i,j,k=1}^{n} \sup_{t_0\leq t\leq T} |\Gamma^i_{jk}(\gamma(t))|\Big),$$

and so there exists a compact set which $V^i(t)$ cannot leave for all $t_0 \leq t \leq T$, $i = 1, \dots, n$. If $\{e_i\}$ extends to a frame around $\gamma(t)$ for all $t \in (a, b)$ then we are done.

Otherwise, let α denote the supremum over all times $s \in [t_0, b]$ such that a unique parallel field $V(t)$ exists for all $t \in [t_0, s]$. Clearly $\alpha > t_0$ by the above. Assume $\alpha < b$. Take ε small and a local frame on $\gamma(\alpha - \varepsilon, \alpha + \varepsilon)$. By the above there exists a unique parallel vector field $\tilde V$ on $(\alpha - \varepsilon, \alpha + \varepsilon)$ such that $\tilde V(\alpha - \varepsilon/2) = V(\alpha - \varepsilon/2)$. By uniqueness V and $\tilde V$ agree on their common domain. Hence $\tilde V$ is a unique extension of V, which contradicts $\alpha < b$.

□

Definition 7.17 (Parallel transport)**.** *If $\gamma\colon (a, b) \to \mathcal{M}$ is a curve in $\mathcal{M}$ and $t_0, t \in (a, b)$, the map*

$$P_{t_0,t}\colon T_{\gamma(t_0)}\mathcal{M} \to T_{\gamma(t)}\mathcal{M},$$

defined by $P_{t_0,t}(v) := V(t)$, where $V(t)$ is the unique parallel vector field along γ with $V(t_0) = v$ (see Proposition 7.16), is called parallel transport along γ.

Remark 7.18 (Parallel transport is an isomorphism between (r, s) tensor spaces)**.** *Parallel transport $P_{t_0,t}$ is a vector space isomorphism between $T_{\gamma(t_0)}\mathcal{M}$ and $T_{\gamma(t)}\mathcal{M}$. It can moreover be extended, in the obvious way, to a vector space isomorphism*

$$P_{t_0,t}\colon T^r_s\mathcal{M}|_{\gamma(t_0)} \to T^r_s\mathcal{M}|_{\gamma(t)},$$

for all r, s.

The following exercise make the link between an affine connection and a way of "connecting nearby tangent spaces".

Exercise 7.19 ("Connecting nearby tangent spaces")**.** *If ∇ is an affine connection on $\mathcal{M}$ and $V, X \in \mathfrak{X}(\mathcal{M})$, show that*

$$\nabla_V X|_p = \lim_{t\to 0} \frac{(P^{-1}_{0,t}X - X)|_p}{t},$$

where $P_{0,t}$ denotes parallel transport along the integral curve γ of V with $\gamma(0) = p$.

Similarly for $\nabla_V T$ with $T \in \Gamma(T^r_s\mathcal{M})$.

7.3 The Levi-Civita connection

A Riemannian metric g on a manifold $\mathcal{M}$ chooses one particular connection which interacts with g in a particularly nice way.

Definition 7.20 (Compatible connections)**.** *An affine connection ∇ on a Riemannian manifold $(\mathcal{M}, g)$ is called* compatible with g *if g is parallel with respect to ∇, i.e. if $\nabla g \equiv 0$.*

Exercise 7.21 (Compatibility of a connection with a metric)**.** *Show that ∇ is compatible with g if and only if*

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$.

Note that if $\gamma \colon (a, b) \to \mathcal{M}$ is a curve and X and Y are two parallel vector fields along γ, it follows from Exercise 7.21 that the fact that ∇ is compatible with g implies that

$$\frac{d}{dt}\big(g_{\gamma(t)}(X(t), Y(t))\big) = 0,$$

i.e. if ∇ is compatible with g then lengths of vectors and angles between vectors are preserved under parallel transport. It is still possible, however, for parallel vectors along γ to "twist around γ".

Definition 7.22 (Torsion tensor field)**.** *Given a connection ∇ on $\mathcal{M}$, define the* torsion tensor *to be the map*

$$\tau \colon \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}),$$

defined by

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Exercise 7.23 (The torsion tensor field is a tensor field)**.** *Show that, if ∇ is a connection, then its torsion tensor τ is a $(1, 2)$ tensor field, $\tau \in \Gamma(T^1_2\mathcal{M})$.*

Definition 7.24 (Torsion free connections)**.** *An affine connection ∇ on a manifold $\mathcal{M}$ is called* symmetric *or* torsion free *if its torsion tensor τ identically vanishes, i.e. if*

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

for all $X, Y \in \mathfrak{X}(\mathcal{M})$.

The name "symmetric" is explained by the following exercise.

Exercise 7.25 (Symmetric Christoffel symbols)**.** *Let $\{x^i\}$ be a local coordinate system of a smooth manifold $\mathcal{M}$. Show that a connection ∇ is symmetric if and only if its Christoffel symbols with respect to the coordinate frame $\{\frac{\partial}{\partial x^i}\}$ satisfy*

$$\Gamma^k_{ij} = \Gamma^k_{ji},$$

for all $i, j, k = 1, \ldots, n$.

The following theorem is often called the fundamental theorem of Riemannian geometry.

Theorem 7.26 (Existence and uniqueness of a compatible torsion free connection)**.** *Given a Riemannian manifold $(\mathcal{M}, g)$, there exists a unique affine connection ∇ on $\mathcal{M}$ which is torsion free and compatible with g. This affine connection ∇ moreover satisfies*

$$2g(\nabla_X Y, Z) = X\big(g(Y, Z)\big) + Y\big(g(X, Z)\big) - Z\big(g(X, Y)\big) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \quad (12)$$

Proof. The proof proceeds by showing that any such connection must satisfy (12), and then checking that (12) indeed defines a torsion free compatible connection. Indeed, suppose ∇ is such a connection. Then, since ∇ is compatible with g,

$$g(\nabla_X Y, Z) = X\big(g(Y,Z)\big) - g(Y, \nabla_X Z),$$

(see Exercise 7.21) and, since ∇ is torsion free,

$$g(\nabla_X Y, Z) = X\big(g(Y,Z)\big) - g(Y, [X,Z]) - g(Y, \nabla_Z X). \tag{13}$$

Similarly, interchanging the roles of X, Y and Z,

$$g(\nabla_Y Z, X) = Y\big(g(X,Z)\big) + g(Z, [X,Y]) - g(Z, \nabla_X Y), \tag{14}$$

and

$$g(\nabla_Z X, Y) = Z\big(g(X,Y)\big) - g(X, [Y,Z]) - g(X, \nabla_Y Z). \tag{15}$$

Adding (13) to (14) and subtracting (15) then yields (12). The uniqueness of such a connection is then evident. For existence, it is left as an exercise to check that (12) indeed defines a torsion free compatible connection. E.g., to check that (12) defines a connection, let $\{e_i\}$ be a local orthonormal frame, set

$$\nabla_X Y = \sum_{k=1}^{n} g(\nabla_X Y, e_k) e_k,$$

and replace $g(\nabla_X Y, e_k)$ using (12). Check that this ∇ satisfies the properties of Definition 7.1, and then check that ∇ is torsion free and compatible with g. $\square$

Definition 7.27 (The Levi-Civita connection of a Riemannian manifold)**.** *Given a Riemannian manifold $(\mathcal{M}, g)$, the connection ∇ of Theorem 7.26 is called the* Levi-Civita connection *of $(\mathcal{M}, g)$.*

In a given chart (U, ϕ) of a Riemannian manifold $(\mathcal{M}, g)$ with local coordinates $\{x^i\}$, (12) gives an expression for the Christoffel symbols Γ^i_{jk} of ∇ with respect to the coordinate frame $\{\frac{\partial}{\partial x^i}\}$ in terms of the components of g. Indeed, setting $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ and $Z = \frac{\partial}{\partial x^k}$, (12) gives

$$2 g_{kl} \Gamma^l_{ij} = \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k},$$

from which it follows that

$$\Gamma^k_{ij} = \frac{g^{kl}}{2}\Big(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\Big). \tag{16}$$

The Levi-Civita connection is a geometric object.

Exercise 7.28 (The Levi-Civita connection is a geometric object)**.** *Suppose $\varphi\colon (\mathcal{M}, g) \to (\tilde{\mathcal{M}}, \tilde{g})$ is a local isometry. Show that $\varphi_*(\nabla_X Y) = \tilde{\nabla}_{\varphi_* X} \varphi_* Y$ for all $X, Y \in \mathfrak{X}(\mathcal{M})$, where ∇ and $\tilde{\nabla}$ are the Levi-Civita connections of $(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$ respectively.*

7.4 Divergence

The Levi-Civita connection on a Riemannian manifold $(\mathcal{M}, g)$ defines the divergence operator.

If V is a vector space, recall the contraction map ${c_1}^1\colon V \otimes V^* \to \mathbb{R}$. This map is also called *trace*, and is denoted $\mathrm{tr} := {c_1}^1$.

Definition 7.29 (Divergence)**.** *Let $(\mathcal{M}, g)$ be a Riemannian manifold with Levi-Civita connection ∇. Define the* divergence *of $X \in \mathfrak{X}(\mathcal{M})$ by*

$$\mathrm{div} X := \mathrm{tr}\big(\nabla X\big).$$

If $\xi \in \Gamma(T^*\mathcal{M})$ is a one form, define

$$\mathrm{div}\xi := \mathrm{div}\xi^\sharp.$$

In local coordinates, if $X = X^i \frac{\partial}{\partial x^i}$, then

$$\mathrm{div} X = \nabla_i X^i.$$

(Recall that $\nabla_i X^j := \nabla X(dx^j, \frac{\partial}{\partial x^i}) \neq \partial_{x^i}(X^j)$ in general.) If $\xi = \xi_i dx^i$, then

$$\mathrm{div}\xi = g^{ij}\nabla_i \xi_j.$$

Exercise 7.30 (Divergence in local coordinates)**.** *In a local coordinate system $\{x^i\}$ for $(\mathcal{M}, g)$, show that, for any vector field $X \in \mathfrak{X}(\mathcal{M})$,*

$$\mathrm{div} X = \frac{1}{\sqrt{\det g}} \partial_{x^i}(\sqrt{\det g} X^i),$$

where

$$\det g = \det\big(g_{ij}\big).$$

Unless specified otherwise, we adopt the (not completely standard) convention that the divergence of $T \in \Gamma(T^r_0\mathcal{M})$, for some $r \geq 1$, is defined to be $\mathrm{div}T \in \Gamma(T^{r-1}_0\mathcal{M})$ by taking the divergence with respect to the *first* argument,

$$(\mathrm{div}T)(\xi_1, \ldots, \xi_{r-1}) := \mathrm{tr}\big(\nabla T(\cdot, \xi_1, \ldots, \xi_{r-1})\big).$$

In components, if $T = T^{i_1 \ldots i_r} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}}$ then $\mathrm{div}T = (\mathrm{div}T)^{i_1 \ldots i_{r-1}} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_{r-1}}}$, where

$$(\mathrm{div}T)^{i_1 \ldots i_{r-1}} = \nabla_j T^{j i_1 \ldots i_{r-1}}.$$

Similarly, unless specified otherwise, if $T \in \Gamma(T^0_s\mathcal{M})$ for some $s \geq 1$, define $\mathrm{div}T \in \Gamma(T^0_{s-1}\mathcal{M})$ by raising the index of the *first* argument and taking the divergence,

$$\mathrm{div}T(X_1, \ldots, X_{s-1}) := \mathrm{tr}\big(\nabla T(\cdot, X_1, \ldots, X_{s-1})^\sharp\big).$$

In components, if $T = T_{i_1 \ldots i_r} dx^{i_1} \otimes \ldots \otimes dx^{i_r}$ then $\mathrm{div}T = (\mathrm{div}T)_{i_1 \ldots i_{r-1}} dx^{i_1} \otimes \ldots \otimes dx^{i_{r-1}}$, where

$$(\mathrm{div}T)_{i_1 \ldots i_{r-1}} = g^{jk}\nabla_j T_{k i_1 \ldots i_{r-1}}.$$

To take the divergence of a general $T \in \Gamma(T^r_s\mathcal{M})$, one has to specify which argument the divergence is being taken with respect to.

Definition 7.31 (Laplace–Beltrami operator)**.** *Given a Riemannian manifold $(\mathcal{M}, g)$, define the* Laplace–Beltrami operator $\Delta_g \colon C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ *(or more generally $\Delta_g \colon \Gamma(T^r_s\mathcal{M}) \to \Gamma(T^r_s\mathcal{M})$) by*

$$\Delta_g f := \mathrm{div}\nabla f,$$

for all $f \in C^\infty(\mathcal{M})$.

One has the following expression for the Laplace–Beltrami operator.

Exercise 7.32 (Laplace–Beltrami operator in local coordinates)**.** *In a local coordinate system $\{x^i\}$ for $(\mathcal{M}, g)$, show that the Laplace–Beltrami operator takes the form*

$$\Delta_g f = \frac{1}{\sqrt{\det g}} \partial_{x^i}\big(g^{ij}\sqrt{\det g}\partial_{x^j} f\big),$$

where

$$\det g = \det\big(g_{ij}\big).$$

8 Geodesics and Riemannian distance

Each Riemmanian manifold has a notion of *geodesic*, or "curve which is as straight as possible". Moreover, there is a notion of Riemannian distance, with the distance between two points defined to be the infimum of the length over all curves which join these two points. This notion of distance gives each Riemannian manifold a metric space structure. When this metric space is complete, this infimum is always attained by a geodesic.

From now on it will be assumed that $(\mathcal{M}, g)$ is a Riemmanian manifold with Levi-Civita connection ∇.

8.1 Geodesics

A geodesic in $(\mathcal{M}, g)$ should be viewed as a curve which is "as straight as possible" and thus is a generalisation of a straight line in $\mathbb{R}^n$. Note that straight lines in $\mathbb{R}^n$ have the property that they minimise the distance between any two points, and moreover that, when traversed at constant speed, they are the unique curves γ with "zero acceleration" in the sense that $\ddot{\gamma} \equiv 0$. One could define geodesics in $(\mathcal{M}, g)$ to be curves which locally minimise length, but it turns out to be much simpler to take the latter property of straight lines in $\mathbb{R}^n$ as a definition and deduce this local length minimising property.

Note that, if $\gamma\colon (a,b) \to \mathcal{M}$, then $\dot{\gamma} \in \mathfrak{X}(\gamma)$.

Definition 8.1 (Geodesics)**.** *A curve $\gamma\colon (a,b) \to \mathcal{M}$ is called a* geodesic *if $D_t\dot{\gamma}(t) = 0$ for all $t \in (a,b)$, i.e. if $\dot{\gamma}$ is parallel along γ.*

Recall the definition 7.14 of covariant derivative along γ and hence, given a local coordinate system $\{x^i\}$, γ is a geodesic if and only if

$$\Big(\ddot{\gamma}^i(t) + \dot{\gamma}^j(t)\dot{\gamma}^k(t)\Gamma^i_{jk}(\gamma(t))\Big)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)} = 0,$$

if and only if

$$\ddot{\gamma}^i(t) + \dot{\gamma}^j(t)\dot{\gamma}^k(t)\Gamma^i_{jk}(\gamma(t)) = 0, \tag{17}$$

for $i = 1, \dots, n$. The equations (17) are called the *geodesic equations*.

The following theorem guarantees that, for a given initial position and velocity, there locally exists a unique geodesic.

Theorem 8.2 (Local existence and uniqueness of geodesics with given initial point and velocity)**.** *Given $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$, there exists a unique maximal interval $(T_-, T_+) \subset \mathbb{R}$, with $T_- < 0$ and $T_+ > 0$, and a unique geodesic $\gamma\colon (T_-, T_+) \to \mathcal{M}$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.*

Proof. Let (U, ϕ) be a chart with $p \in U$, and let $\{x^i\}$ be the associated local coordinates. The vector v can then be written $v = v^i \frac{\partial}{\partial x^i}|_p$. By Theorem 6.1[4] there exists $\varepsilon > 0$ and $\gamma^1, \dots, \gamma^n\colon (-\varepsilon, \varepsilon) \to \mathcal{M}$ satisfying the system of nonlinear ordinary differential equations (17) together with the initial conditions

$$\gamma^i(0) = \phi(p)^i, \qquad \dot{\gamma}^i(0) = v^i.$$

The maximality part of the statement follows from the maximality part of Theorem 6.1, being careful that γ may leave the domain of the chart (U, ϕ) (as in the proof of Proposition 7.16). □

Alternative proof of Theorem 8.2 (which is really just a more geometric take on the previous proof). Recall that a chart (U, ϕ), with local coordinates $x^1, \dots, x^n$, for $\mathcal{M}$ defines a chart $(\pi^{-1}(U), \Phi)$ for $T\mathcal{M}$, where $\Phi\colon \pi^{-1}(U) \to \mathbb{R}^{2n}$ is defined by

$$\Phi\Big(p, v^i \frac{\partial}{\partial x^i}\Big|_p\Big) = (x^1(p), \dots, x^n(p), v^1, \dots, v^n).$$

[4] To bring the system (17) to the correct form to directly apply Theorem 6.1, one has to perform the well known trick of converting a second order ordinary differential equation on $\mathbb{R}^n$ to a first order ordinary differential equation on $\mathbb{R}^{2n}$. See also the "alternative proof" below.

The associated local coordinates $\{x^i\}$ for $\mathcal{M}$ are thus extended to a local coordinate system $\{x^i, v^i\}$ for $T\mathcal{M}$.

The second order geodesic equations (17) can be rewritten in first order form

$$\dot{\gamma}^i(t) = v^i(t), \qquad \dot{v}^i(t) = -v^j(t)v^k(t)\Gamma^i_{jk}(\gamma(t)).$$

Solutions of this system are exactly the integral curves of the vector field

$$G = v^i \partial_{x^i} - v^j v^k \Gamma^i_{jk} \partial_{v^i}.$$

Note that $G \in \mathfrak{X}(T\mathcal{M}) = \Gamma(TT\mathcal{M})$ is a vector field not on $\mathcal{M}$ but on the tangent bundle $T\mathcal{M}$. The existence and uniqueness then follows from Theorem 6.2 (with $T\mathcal{M}$ in place of $\mathcal{M}$). $\square$

Definition 8.3 (Geodesic spray and geodesic flow)**.** *Given a Riemannian manifold $(\mathcal{M}, g)$, the vector field $G \in \mathfrak{X}(T\mathcal{M}) = \Gamma(TT\mathcal{M})$ defined locally by*

$$G = v^i \partial_{x^i} - v^j v^k \Gamma^i_{jk} \partial_{v^i},$$

is called the geodesic spray *of $\mathcal{M}$. The flow of G is called the* geodesic flow *of $\mathcal{M}$.*

The maximal geodesic of Theorem 8.2 is often denoted $\gamma_{p,v}$ or sometimes just γ_v.

Example 8.4 (Examples of geodesics)**.**

1. *Consider $(\mathbb{R}^n, g_{Eucl})$. In Cartesian coordinates $g_{ij} = \delta_{ij}$ and so $\Gamma^i_{jk} \equiv 0$ for all $i, j, k = 1, \ldots, n$. The geodesic equations then take the form*
$$\ddot{\gamma}^i(t) = 0,$$
for $i = 1, \ldots, n$. All solutions take the form,
$$\gamma(t) = a + tb,$$
for some $a, b \in \mathbb{R}^n$, i.e. all geodesics in $\mathbb{R}^n$ are straight lines.

2. *Consider now (S^2, g_{Round}). Recall that, in spherical polar coordinates,*
$$g_{Round} = d\theta^2 + \sin^2\theta \, d\phi^2.$$
Using the expression (16) *one computes*
$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta},$$
and
$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\phi} = \Gamma^\phi_{\phi\phi} = \Gamma^\phi_{\theta\theta} = 0.$$
The geodesic equations for $\gamma(t) = (\theta(t), \phi(t))$ then take the form
$$\ddot{\theta}(t) - \big(\dot{\phi}(t)\big)^2 \sin\theta(t)\cos\theta(t) = 0, \qquad \ddot{\phi}(t) + 2\dot{\theta}(t)\dot{\phi}(t)\frac{\cos\theta(t)}{\sin\theta(t)} = 0.$$
One sees, for example, that $\theta(t) = t$, $\phi(t) = \phi_0$ is a solution for any $\phi_0 \in (0, 2\pi)$. The image of $(\theta(t), \phi(t))$ is a "great circle" i.e. the intersection of S^2 with a plane through the origin in $\mathbb{R}^3$. In fact, the image of any geodesic in (S^2, g_{Round}) is a great circle.

Definition 8.5 (Geodesic completeness)**.** *A Riemannian manifold $(\mathcal{M}, g)$ is called* geodesically complete *if, for all $(p, v) \in T\mathcal{M}$, the maximal geodesic $\gamma_{p,v}$ is defined for all $t \in \mathbb{R}$, i.e. if for all $(p, v) \in T\mathcal{M}$, the $T_\pm$ of Theorem 8.2 satisfy $T_\pm = \pm\infty$.*

Example 8.6 (Examples of geodesic completeness)**.** *The Riemannian manifold $(\mathbb{R}^n, g_{Eucl})$ is geodesically complete. The Riemannian manifold $(\mathbb{R}^n \smallsetminus \{0\}, g_{Eucl})$ is not.*

Exercise 8.7 (Compact manifolds are geodesically complete)**.** *Show that every compact Riemannian manifold is geodesically complete.*

8.2 The exponential map and normal coordinates

The *exponential map* on the tangent bundle of a Riemannian manifold is defined by following the geodesic corresponding to a given element of $T\mathcal{M}$ for time 1.

Definition 8.8 (The exponential map)**.** *Let $(\mathcal{M}, g)$ be a Riemannian manifold and, given $p \in \mathcal{M}$, let $\mathcal{E}_p \subset T_p\mathcal{M}$ be the set of vectors $v \in T_p\mathcal{M}$ such that $\gamma_{p,v}$ is defined on an interval containing $[0,1]$. Define the* exponential map at p, $\exp_p \colon \mathcal{E}_p \to \mathcal{M}$, *by*

$$\exp_p(v) := \gamma_{p,v}(1).$$

Set $\mathcal{E} = \bigcup_{p\in\mathcal{M}} \{p\} \times \mathcal{E}_p \subset T\mathcal{M}$ and define the exponential map, $\exp\colon \mathcal{E} \to \mathcal{M}$, *by*

$$\exp(p, v) = \exp_p(v) = \gamma_{p,v}(1).$$

Note that $\exp(p, v) = \pi \circ \Phi_G(p, v)$, where $\pi\colon T\mathcal{M} \to \mathcal{M}$ is the natural projection and Φ_G is the geodesic flow (see Definition 8.3). In particular, exp is smooth.

Lemma 8.9 (Rescaling geodesic directions)**.** *Given $(p, v) \in T\mathcal{M}$ and $\lambda \in \mathbb{R}$, $\gamma_{p,\lambda v}(t) = \gamma_{p,v}(\lambda t)$ whenever either side is defined.*

Proof. Suppose λ is such that $\gamma_{p,v}(\lambda t)$ is defined. Set $\tilde{\gamma}(t) = \gamma_{p,v}(\lambda t)$. Clearly $\tilde{\gamma}(0) = p$ and, in local coordinates,

$$\tilde{\gamma}'(0)^i = \frac{d}{dt}\Big|_{t=0} \gamma^i_{p,v}(\lambda t) = \lambda \gamma'_{p,v}(0)^i = \lambda v^i,$$

so that $\tilde{\gamma}'(0) = \lambda v$. Since $\tilde{\gamma}$ is also a geodesic (check that it solves the geodesic equation in local coordinates) it follows by uniqueness that $\tilde{\gamma} = \gamma_{p,\lambda v}$, i.e.

$$\gamma_{p,\lambda v}(t) = \gamma_{p,v}(\lambda t).$$

$\square$

In particular, Lemma 8.9 implies that

$$\exp(p, \lambda v) = \gamma_{p,v}(\lambda),$$

whenever either side is defined and, if $\exp(p, v)$ is defined then so is $\exp(p, \lambda v)$ for all $0 \le \lambda \le 1$. Hence $\mathcal{E}_p$ is *star shaped* for all $p \in \mathcal{M}$.

Lemma 8.10 (The exponential map is a local diffeomorphism around the origin)**.** *For each $p \in \mathcal{M}$ there exists an open neighbourhood $U \subset T_p\mathcal{M}$ with $0 \in U$ and an open neighbourhood $V \subset \mathcal{M}$ with $p \in V$ such that*

$$\exp_p \colon U \to V,$$

is a diffeomorphism.

Proof. The proof follows from the Inverse Function Theorem after establishing that the differential of the exponential map at the origin, $(\exp_p)_{*0}$ is invertible. Since $T_p\mathcal{M}$ is a vector space there is a canonical identification between $T_0T_p\mathcal{M}$ and $T_p\mathcal{M}$ and so, abusing notation slightly, we write,

$$(\exp_p)_{*0} \colon T_0T_p\mathcal{M} = T_p\mathcal{M} \to T_p\mathcal{M}.$$

We will show that $(\exp_p)_{*0}$ is the identity map.

Recall that, if $F\colon \mathcal{M} \to \mathcal{N}$ is a smooth map between manifolds, $p \in \mathcal{M}$ and $\beta\colon (-\varepsilon, \varepsilon) \to \mathcal{M}$ satisfies $\beta(0) = p$, $\beta'(0) = v \in T_p\mathcal{M}$, then $F_{*p}v = (F \circ \beta)'(0)$.

Given $v \in T_p\mathcal{M}$, it follows that

$$(\exp_p)_{*0}v = \frac{d\exp_p(tv)}{dt}\Big|_{t=0} = \frac{d\gamma_{p,v}(t)}{dt}\Big|_{t=0} = v,$$

by Lemma 8.9, i.e. $(\exp_p)_{*0} = Id_{T_p\mathcal{M}}$. $\square$

Definition 8.11 (Normal neighbourhoods)**.** *Such a neighbourhood V of p, as in Lemma 8.10, is called a* normal neighbourhood *of p.*

Lemma 8.10 in particular implies that, for all $p \in \mathcal{M}$, there exists $\varepsilon > 0$ such that $\exp_p \colon B(0,\varepsilon) \subset T_p\mathcal{M} \to \mathcal{M}$ is a diffeomorphism onto its image. The image $\exp_p(B(0,\varepsilon)) \subset \mathcal{M}$ is called a *geodesic ball*.

For each point $p \in \mathcal{M}$, the exponential map defines a particularly nice system of local coordinates on a normal neighbourhood of p, known as *normal coordinates*.

Definition 8.12 (Normal coordinates)**.** *Consider $p \in \mathcal{M}$ and an orthonormal basis $\{e_i\}$ of $T_p\mathcal{M}$, define the isomorphism*

$$E \colon \mathbb{R}^n \to T_p\mathcal{M}, \qquad E(x^1, \ldots, x^n) = x^i e_i.$$

If V is a normal neighbourhood of p, define a chart

$$\phi \colon V \to W \subset \mathbb{R}^n, \qquad \phi := E^{-1} \circ (\exp_p)^{-1}.$$

The local coordinates $\{x^i\}$ associated to this chart (V, ϕ) are called normal coordinates *centred at p.*

Normal coordinates centred at p have the following nice properties.

Exercise 8.13 (Properties of normal coordinates)**.** *Given $p \in \mathcal{M}$, show the following properties of normal coordinates centred at p:*

1. *The coordinates of p are $(0, \ldots, 0)$.*
2. *Given $v = v^i e_i \in T_p\mathcal{M}$, the associated geodesic satisfies $\gamma_{p,v}(t) = (tv^1, \ldots, tv^n)$.*
3. *At the point p the components of the metric take the form*
$$g_{ij}(p) = \delta_{ij}.$$
4. *At the point p,*
$$\Gamma^k_{ij}(p) = 0, \qquad \frac{\partial g_{ij}}{\partial x^k}(p) = 0,$$
for all $i, j, k = 1, \ldots, n$.

8.3 Riemannian distance

A Riemannian metric g on $\mathcal{M}$ defines a metric (in the metric spaces sense) $d \colon \mathcal{M} \times \mathcal{M} \to [0, \infty)$ called Riemannian distance.

Definition 8.14 (Smooth curves on closed intervals)**.** *A map from a closed interval $c \colon [a,b] \to \mathcal{M}$ is called a* smooth curve *if $c|_{(a,b)} \colon (a,b) \to \mathcal{M}$ is a smooth curve in the usual sense, c is continuous, and all derivatives of c with respect to any chart containing $c(a)$ extends continuously to $c(a)$ and all derivatives of c with respect to any chart containing $c(b)$ extends continuously to $c(b)$.*

Riemannian distance is most conveniently defined not using *smooth* curves, but rather *piecewise smooth* curves.

Definition 8.15 (Piecewise smooth curves)**.** *A continuous map $c \colon [a,b] \to \mathcal{M}$ is called a* piecewise smooth curve *if there exists a finite subdivision $a = t_0 < t_1 < \ldots < t_k = b$ of $[a,b]$ such that $c|_{[t_i,t_{i+1}]}$ is a smooth curve for all $i = 0, 1, \ldots, k-1$.*

If $c \colon [a,b] \to \mathcal{M}$ is such a piecewise smooth curve, define the length of c by

$$L(c) := \sum_{i=0}^{k-1} L\big(c|_{(t_i,t_{i+1})}\big) = \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} |c'(t)|_g dt.$$

Such a curve c is said to join *$c(a)$ and $c(b)$.*

Given $p, q \in \mathcal{M}$, define the set

$$C_{p,q} = \{c\colon [a,b] \to \mathcal{M} \text{ piecewise smooth } \mid c(a) = p, c(b) = q\}.$$

Recall that we always assume $\mathcal{M}$ is connected, and hence path connected, and so $C_{p,q} \neq \emptyset$ for all $p, q \in \mathcal{M}$.

Definition 8.16 (Riemannian distance)**.** *The map $d\colon \mathcal{M} \times \mathcal{M} \to [0, \infty)$ defined by*

$$d(p,q) := \inf\{L(c) \mid c \in C_{p,q}\},$$

is called the Riemannian distance *of $(\mathcal{M}, g)$.*

A curve γ which joins p and q is called *length minimising* if $L(\gamma) = d(p,q)$.

Proposition 8.17 (Riemannian manifolds are metric spaces)**.** *With d defined as above, $(\mathcal{M}, d)$ is a metric space.*

The proof uses the following exercise.

Exercise 8.18 (Lengths of curves are invariant under reparameterisation)**.** *Show that lengths of curves are invariant under reparameterisation.*

Proof of Proposition 8.17. It follows from the definition of $L(c)$ that $d(p,q) = d(q,p) \geq 0$ and $d(p,p) = 0$ for all $p, q \in \mathcal{M}$.

Consider $p, q, r \in \mathcal{M}$. If $\gamma_1\colon [a,b] \to \mathcal{M}$ satisfies $\gamma_1(a) = p$, $\gamma_1(b) = q$ and $\gamma_2\colon [c,d] \to \mathcal{M}$ satisfies $\gamma_2(c) = q$, $\gamma_2(d) = r$, define $\gamma\colon [0,2] \to \mathcal{M}$, by

$$\gamma(s) = \begin{cases} \gamma_1\big((b-a)s + a\big) & \text{if } s \in [0,1], \\ \gamma_2\big((d-c)(s-1) + c\big) & \text{if } s \in (1,2]. \end{cases}$$

By Exercise 8.18

$$L(\gamma) = L(\gamma_1) + L(\gamma_2).$$

Taking the infimum over all such γ_1, γ_2 gives the triangle inequality

$$d(p,r) \leq d(p,q) + d(q,r).$$

It remains to show that $d(p,q) > 0$ for all $p \neq q$. Consider $p \in \mathcal{M}$, $\varepsilon > 0$ small and normal coordinates on the geodesic ball $V_\varepsilon := \exp_p(B(0,\varepsilon))$ centred at p. If ε is sufficiently small there exist $C, c > 0$ such that

$$c\sqrt{\delta_{ij}v^i v^j} \leq |v|_{g(r)} \leq C\sqrt{\delta_{ij}v^i v^j},$$

for all $r \in V_\varepsilon$, $v \in T_r\mathcal{M}$. (Indeed $|v|_{g(p)} = \sqrt{\delta_{ij}v^i v^j}$ for all $v \in T_p\mathcal{M}$ by Exercise 8.13. The claim then follows by continuity.) Take ε sufficiently small so that $q \notin V_\varepsilon$ and take any curve γ joining p and q. Let t_0 be the first time such that $\gamma(t_0) \in \partial V_\varepsilon$. Then

$$L(\gamma) \geq L(\gamma|_{[a,t_0]}) = \int_a^{t_0} |\gamma'(t)|_g dt \geq c\int_a^{t_0} \sqrt{\delta_{ij}\gamma'(t)^i\gamma'(t)^j} dt \geq c d_{Eucl}\big(\phi(\gamma(t_0)), \phi(\gamma(a))\big) = c\varepsilon.$$

Taking the infimum over all such curves γ gives

$$d(p,q) \geq c\varepsilon > 0.$$

$\square$

8.4 Variations of curves

The main result of this section is the statement that the geodesic equation is the *Euler–Lagrange equation* of the length functional, a statement known as the *first variation formula*. See Proposition 8.23. It follows that every length minimising curve is a geodesic.

Definition 8.19 (Admissible curves and admissible vector fields). *An* admissible curve *is a continuous piecewise smooth curve $\gamma\colon [a,b] \to \mathcal{M}$ such that $\dot\gamma$ is nonvanishing and left and right limits of $\dot\gamma$ exist (in every chart) everywhere.*

If γ is an admissible curve, a vector field $V \in \mathfrak{X}(\gamma)$ is called admissible *if it is continuous and piecewise smooth.*

Such an admissible vector field V is called proper *if $V(a) = V(b) = 0$.*

Definition 8.20 (Variation of an admissible curve). *A* variation *of an admissible curve $\gamma\colon [a,b] \to \mathcal{M}$ is a continuous map*

$$A\colon (-\varepsilon,\varepsilon) \times [a,b] \to \mathcal{M},$$

such that

1. *$A(0,t) = \gamma(t)$ for all $t \in [a,b]$.*
2. *There exists a subdivision $a = t_0 < t_1 < \ldots < t_k = b$ such that A is smooth on each $(-\varepsilon,\varepsilon) \times (t_i, t_{i+1})$.*
3. *For each $s \in (-\varepsilon,\varepsilon)$ the curve $t \mapsto A_s(t) := A(s,t)$ is an admissible curve.*

A variation A of γ is called proper *if $A(s,a) = \gamma(a)$, $A(s,b) = \gamma(b)$ for all $s \in (-\varepsilon,\varepsilon)$.*

If A is a variation of an admissible curve γ then the vector fields along γ,

$$\partial_t A(0,t) := \frac{d}{dt} A(s,t)|_{s=0}, \qquad \partial_s A(0,t) := \frac{d}{ds} A(s,t)|_{s=0},$$

are admissible vector fields along γ.

Exercise 8.21 (Every admissible vector field arises from a variation). *Show that every admissible vector field V along γ arises as $V = \partial_s A(0,t)$ for some variation A of γ.*

If V is a vector field on the image of a variation A, let D_t denote the covariant derivative along the curves $t \mapsto A_s(t) := A(s,t)$, and let D_s denote the covariant derivative along the curves $s \mapsto A^t(s) := A(s,t)$.

Lemma 8.22 (Covariant derivatives of a variation commute). *If $A\colon (-\varepsilon,\varepsilon) \times [a,b] \to \mathcal{M}$ is a variation of an admissible curve $\gamma\colon [a,b] \to \mathcal{M}$, then*

$$D_s \partial_t A = D_t \partial_s A,$$

on any rectangle $(-\varepsilon,\varepsilon) \times (t_i, t_{i+1})$ on which A is smooth.

Proof. Consider local coordinates $\{x^i\}$ around a point $A(s_0,t_0)$ at which A is smooth. Write A in coordinates as $A(s,t) = (A^1(s,t), \ldots, A^n(s,t))$, so that

$$\partial_t A = \frac{\partial A^i}{\partial t} \partial_{x^i}, \qquad \partial_s A = \frac{\partial A^i}{\partial s} \partial_{x^i},$$

and

$$D_s \partial_t A = \frac{\partial^2 A^i}{\partial s \partial t} \partial_{x^i} + \frac{\partial A^i}{\partial t} \nabla_{\frac{\partial A}{\partial s}} \partial_{x^i} = \frac{\partial^2 A^i}{\partial s \partial t} \partial_{x^i} + \frac{\partial A^i}{\partial t} \frac{\partial A^j}{\partial s} \Gamma_{ij}^k \partial_{x^k}.$$

Similarly,

$$D_t \partial_s A = \frac{\partial^2 A^i}{\partial t \partial s} \partial_{x^i} + \frac{\partial A^i}{\partial s} \frac{\partial A^j}{\partial t} \Gamma_{ij}^k \partial_{x^k}.$$

The proof follows from the fact that $\Gamma_{ij}^k = \Gamma_{ji}^k$ with respect to any coordinate frame since ∇ is torsion free. $\square$

Proposition 8.23 (First variation formula)**.** *If $\gamma\colon [a,b] \to \mathcal{M}$ is an admissible curve, smooth on (t_i, t_{i+1}) for $i = 0, \dots, k$, parameterised by unit speed (i.e. $|\dot\gamma(t)|_g = 1$ for all t), if A is a proper variation of γ then*

$$\frac{d}{ds}\Big|_{s=0} L(A_s) = -\int_a^b g_{\gamma(t)}(V, D_t\dot\gamma)dt - \sum_{i=1}^{k-1} g_{\gamma(t_i)}\big(V|_{t_i}, \dot\gamma(t_i^+) - \dot\gamma(t_i^-)\big),$$

where $V(t) = \partial_s A(0,t)$ and $\dot\gamma(t_i^\pm)$ denote the left and right limits of $\dot\gamma$ at t_i,

$$\dot\gamma(t_i^-) = \lim_{t\uparrow t_i} \dot\gamma(t), \qquad \dot\gamma(t_i^+) = \lim_{t\downarrow t_i} \dot\gamma(t),$$

(defined with respect to appropriate local coordinate charts) i.e. the geodesic equation is the Euler–Lagrange equation *of the length functional.*

Proof. First note that

$$\begin{aligned}\partial_s\big(g(\partial_t A(s,t), \partial_t A(s,t))\big) &= 2g\big(D_s\partial_t A(s,t), \partial_t A(s,t)\big) = 2g\big(D_t\partial_s A(s,t), \partial_t A(s,t)\big)\\ &= 2\partial_t\big(g\big(\partial_s A(s,t), \partial_t A(s,t)\big)\big) - 2g\big(\partial_s A(s,t), D_t\partial_t A(s,t)\big),\end{aligned}$$

by Lemma 8.22 and the fact that ∇ is compatible with g. Hence,

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0} L(A_s) &= \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} \partial_s\big(\sqrt{g(\partial_t A(s,t), \partial_t A(s,t))}\big)dt\Big|_{s=0}\\ &= \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} \partial_t\big(g(\partial_s A(s,t), \partial_t A(s,t))\big) - g(\partial_s A(s,t), D_t\partial_t A(s,t))dt\Big|_{s=0}\\ &= \sum_{i=0}^{k-1}\Big(g_{\gamma(t_{i+1})}\big(V|_{t_{i+1}}, \dot\gamma(t_{i+1}^-)\big) - g_{\gamma(t_i)}\big(V|_{t_i}, \dot\gamma(t_i^+)\big) - \int_{t_i}^{t_{i+1}} g_{\gamma(t)}(V, D_t\dot\gamma(t))dt\Big),\end{aligned}$$

since γ is parameterised by unit speed. The proof follows from the fact that $V|_a = V|_b = 0$ since A is a proper variation. □

The following proposition in particular implies that every length minimising curve is a geodesic.

Proposition 8.24 (Geodesics and critical points of the length functional)**.** *A unit speed admissible curve γ is a critical point of the length functional L if and only if it is a geodesic.*

Proof. If γ is a geodesic then the first variation formula implies that $\frac{d}{ds}|_{s=0}L(A_s) = 0$ for every variation A of γ, i.e. γ is a critical point of L.

Suppose now that γ is a unit speed admissible curve which is a critical point of L. For each $i = 0, \dots, k-1$, let φ be a smooth bump function such that $\varphi(t) = 0$ for $t \le t_i$, $t \ge t_{i+1}$ and $\varphi(t) > 0$ for $t \in (t_i, t_{i+1})$. Setting $V(t) = \varphi(t)D_t\dot\gamma(t)$ in the first variation formula (using Exercise 8.21) gives

$$\int_{t_i}^{t_{i+1}} \varphi(t)|D_t\dot\gamma(t)|_g^2 dt = 0.$$

It follows that $D_t\dot\gamma(t) = 0$ for all $t \in (t_i, t_{i+1})$ for each $i = 0, \dots, k-1$, and hence γ is a geodesic on each interval on which it is smooth. To see that γ is actually a geodesic, for each i take V such that

$$V(t_i) = \dot\gamma(t_i^+) - \dot\gamma(t_i^-), \qquad V(t_j) = 0 \text{ for } i \ne j.$$

The first variation formula then implies that

$$0 = |\dot\gamma(t_i^+) - \dot\gamma(t_i^-)|_g^2,$$

for each i, i.e. $\dot\gamma(t_i^+) = \dot\gamma(t_i^-)$ for each i and so γ is a geodesic.

□

8.5 Geodesics are locally length minimising

In the previous section it was shown that all length minimising curves are geodesics. The converse is not true in general (consider great circles on the sphere). In this section a partial converse is shown: geodesics are *locally* length minimising.

The main content is in the following.

Proposition 8.25 (Gauss Lemma)**.** *Given $p \in \mathcal{M}$ and normal coordinates $\{x^i\}$ centred at p on $V_\varepsilon = \exp_p(B(0,\varepsilon))$, the vector field $\frac{x^i}{r}\partial_{x^i}$ on $V_\varepsilon \setminus \{p\}$ is orthogonal to the geodesic spheres $\exp_p(\partial B(0,\delta))$ for all $0 < \delta < \varepsilon$. Here $r(x) = \sqrt{(x^1)^2 + \ldots + (x^n)^2}$.*

Exercise 8.26 (Scaling vector has unit length)**.**

1. *Show that if $\gamma\colon (a,b) \to \mathcal{M}$ is a geodesic, then $|\dot\gamma(t)|_g$ is conserved along γ.*
2. *Show that, on $V_\varepsilon \setminus \{p\}$, the vector field $\frac{x^i}{r}\partial_{x^i}$ has unit length*

$$\Big|\frac{x^i}{r}\partial_{x^i}\Big|_g = 1.$$

Hint: use the fact that $\frac{x^i}{r}\partial_{x^i} = \dot\gamma(1)$, where $\gamma(t) = \exp(p, t\frac{x^i}{r}e_i)$ for some orthonormal basis $\{e_i\}$ for $T_p\mathcal{M}$.

Proof of Proposition 8.25. Consider $q \in \exp_p(\partial B(0,\delta))$ and $X \in T_q\mathcal{M}$ such that X is tangential to $\exp_p(\partial B(0,\delta))$. There exists $v \in T_p\mathcal{M}$ such that $q = \exp(p,v)$ and $w \in T_vT_p\mathcal{M} = T_p\mathcal{M}$ such that $X = (\exp_p)_{*v}w$. Note that $V = x^i(q)e_i$ where $(x^1(q),\ldots,x^n(q))$ are the coordinates of q. Set $\gamma(t) = \exp(p,tv)$. Then $\gamma(0) = p$, $\gamma(1) = q$ and

$$\dot\gamma(1) = v^i \frac{\partial}{\partial x^i}\Big|_{\gamma(1)} = x^i(q)\frac{\partial}{\partial x^i}\Big|_q = \delta\frac{x^i}{r}\frac{\partial}{\partial x^i}\Big|_q,$$

by the definition of normal coordinates. The proposition will follow from the fact that

$$g_q(X, \dot\gamma(1)) = 0.$$

Let $\sigma\colon (-\varepsilon,\varepsilon) \to \partial B(0,\delta) \subset T_p\mathcal{M}$ be a curve such that $\sigma(0) = v$, $\dot\sigma(0) = w$. Define the smooth variation

$$A(s,t) = \exp(p, t\sigma(s)).$$

Note that

$$\begin{aligned}\partial_s A(0,0) &= \frac{d}{ds}\Big|_{s=0}\exp(p,0) = 0,\\ \partial_s A(0,1) &= \frac{d}{ds}\Big|_{s=0}\exp(p,\sigma(s)) = (\exp_p)_{*v}w = X,\\ \partial_t A(0,1) &= \frac{d}{dt}\Big|_{t=1}\exp(p,tv) = \dot\gamma(1),\end{aligned}$$

so it suffices to show that

$$\frac{d}{dt}\Big(g(\partial_s A(s,t), \partial_t A(s,t))\Big)\Big|_{s=0} = 0.$$

Now,

$$\begin{aligned}\frac{d}{dt}\Big(g(\partial_s A(s,t), \partial_t A(s,t))\Big) &= g(D_t\partial_s A(s,t), \partial_t A(s,t)) + g(\partial_s A(s,t), D_t\partial_t A(s,t))\\ &= g(D_s\partial_t A(s,t), \partial_t A(s,t))\\ &= \frac{1}{2}\frac{d}{ds}\Big(g(\partial_t A(s,t), \partial_t A(s,t))\Big) = 0,\end{aligned}$$

by Lemma 8.22, the fact that the curves $t \mapsto A(s,t)$ are geodesics and so $|\partial_t A(s,t)|_g = |\partial_t A(s,0)|_g = |\sigma(s)|_g = \delta$ for all s, by Exercise 8.26. ☐

Given $p \in \mathcal{M}$ and q in a normal neighbourhood of p, $q \in V_\varepsilon = \exp_p(B(0,\varepsilon))$ with $q \neq p$, there exists $0 < \delta < \varepsilon$ and $v \in T_p\mathcal{M}$ with $|v|_g = 1$ such that $q \in \exp_p(\partial B(0,\delta))$ and $q = \exp(p, \delta v)$.

Proposition 8.27 (Geodesics are locally length minimising)**.** *The geodesic* $\gamma\colon [0,\delta] \to \mathcal{M}$ *defined by* $\gamma(t) = \exp(p, tv)$ *is the unique (up to reparameterisation) length minimising curve joining* p *and* q *in* $\mathcal{M}$*.*

Proof. Note that $|\dot\gamma(t)|_g = 1$ for all $t \in [0,\delta]$, hence $L(\gamma) = \delta$. Let $\sigma\colon [a,b] \to \mathcal{M}$ be another curve joining p and q, so that $\sigma(a) = p$, $\sigma(b) = q$. Assume that σ is parameterised by arc length, so that $|\dot\sigma(t)|_g = 1$ for all $t \in [a,b]$. Suppose moreover that $\sigma(t) \neq p$ for all $t > a$. Define

$$b_0 = \min\{t \in [a,b] \mid \sigma(t) \in \exp_p(\partial B(0,\delta))\}.$$

For any $t \in (a, b_0]$ the tangent vector to σ can be written

$$\dot\sigma(t) = \alpha(t)\frac{x^i}{r}\partial_{x^i} + X(t),$$

for some $\alpha(t)$ and some vector field $X(t)$ which is tangent to the geodesic sphere through $\sigma(t)$, $\exp_p(\partial B(0, r(\sigma(t))))$. Note that $X(t)r = 0$ for all t since $X(t)$ is tangential to the level sets of r, and

$$\frac{x^i}{r}\partial_{x^i} r = \sum_{i=1}^{n} \frac{x^i x^i}{r^2} = 1,$$

and so $\dot\sigma(t)r = \alpha(t)$. By the Gauss Lemma, Proposition 8.25,

$$g\Big(\frac{x^i}{r}\partial_{x^i}, X(t)\Big) = 0,$$

for all t, and so

$$|\dot\sigma(t)|_g^2 = (\alpha(t))^2 + |X(t)|_g^2 \geq (\alpha(t))^2,$$

by Exercise 8.26. Hence,

$$\begin{aligned} L(\sigma) \geq L(\sigma|_{(a,b_0)}) = \int_a^{b_0} |\dot\sigma(t)|_g dt \geq \int_a^{b_0} \alpha(t) dt = \int_a^{b_0} \dot\sigma(t) r dt \\ = \int_a^{b_0} \frac{dr(\sigma(t))}{dt} dt = r(\sigma(b_0)) - r(\sigma(a)) = \delta = L(\gamma). \end{aligned}$$

Suppose now that $L(\sigma) = L(\gamma)$. Then, by the above, $L(\sigma) = L(\sigma|_{(a,b_0)})$ so it must be the case that $b_0 = b$ and

$$\int_a^b |\dot\sigma(t)|_g dt = \int_a^b \alpha(t) dt.$$

It must then be the case that $X(t) = 0$ for all t and, since $|\dot\sigma(t)|_g = 1$, it must be the case that $\alpha(t) = 1$ for all t, and so

$$\dot\sigma(t) = \frac{x^i}{r}\partial_{x^i}.$$

Since it is also the case that $\dot\gamma(t) = \frac{x^i}{r}\partial_{x^i}$ it follows that σ and γ are both integral curves of the same vector field passing through p, and therefore must be equal. $\square$

8.6 Completeness

Recall that a Riemannian manifold comes with a notion of geodesic completeness (see Definition 8.5). Since $\mathcal{M}$ together with the Riemannian distance function d is a metric space, there is also a notion of completeness of $(\mathcal{M}, g)$ as a metric space. (Recall that a metric space (M, d) is complete if every Cauchy sequence converges to a limit in M.) These two notions of completeness coincide.

Theorem 8.28 (Hopf–Rinow)**.** *Let $(\mathcal{M}, g)$ be a (connected) Riemannian manifold. The following are equivalent.*

1. *$(\mathcal{M}, d)$ is a complete metric space.*
2. *$(\mathcal{M}, g)$ is geodesically complete.*
3. *There exists $p \in \mathcal{M}$ such that $\mathcal{E}_p = T_p\mathcal{M}$.*
4. *The closed bounded subsets of $\mathcal{M}$ are compact.*

Moreover, any one of the above conditions imply that, for any points $p, q \in \mathcal{M}$, there exists a minimising geodesic joining p and q.

The proof of the Hopf–Rinow Theorem is non-examinable.

Definition 8.29 (Complete Riemannian manifolds)**.** *A Riemannian manifold $(\mathcal{M}, g)$ is called* complete *if either $(\mathcal{M}, g)$ is geodesically complete or, equivalently, if $(\mathcal{M}, d)$ is a complete metric space.*

9 Curvature

One simple way to show that two Riemannian manifolds are not locally isometric is to show that they have different *curvature*, which is the subject of this section. (Note that we still have not ruled out the possibility that every Riemannian manifold is locally isometric to Euclidean space.)

9.1 The Riemann curvature tensor

In order to motivate the definition, recall that, in Euclidean space, every vector $Z_p \in T_p\mathbb{R}^n$ can be extended to a globally parallel vector field, i.e. to a vector field $Z \in \mathfrak{X}(\mathbb{R}^n)$ such that $\nabla^{Eucl} Z \equiv 0$. Indeed, one simply takes $Z = Z_p^i \partial_{x^i}$, where $(x^1, \ldots, x^n)$ are Cartesian coordinates on $\mathbb{R}^n$. This property, in a sense, is a local characterisation of Euclidean space and is taken as the basis for the definition of curvature.

On a Riemannian manifold $(\mathcal{M}, g)$, what is the obstruction to extending $Z_p \in T_p\mathcal{M}$ to a parallel vector field? For simplicity, suppose $n = 2$. Consider local coordinates (x, y) around $p \in \mathcal{M}$. Take $Z_p \in T_p\mathcal{M}$ and extend to Z along $\{y = 0\}$ by parallel transport

$$\nabla_{\partial_x} Z|_{\{y=0\}} = 0.$$

Now extend along $\{x = \mathit{const}\}$ by parallel transport

$$\nabla_{\partial_y} Z = 0.$$

Is it the case that Z is parallel? I.e. is it the case that $\nabla_{\partial_x} Z|_{\{y=c\}} = 0$ for $c \neq 0$? It would be the case (by uniqueness of parallel transport) if $\nabla_{\partial_y}\nabla_{\partial_x} Z = 0$, so what is the obstruction to this? If it were the case that covariant derivatives commute (as they do in Euclidean space) then it would be the case that

$$\nabla_{\partial_y}\nabla_{\partial_x} Z = \nabla_{\partial_x}\nabla_{\partial_y} Z = 0.$$

Curvature is therefore defined to be the failure of covariant derivatives to commute.

Definition 9.1 (Riemann curvature tensor)**.** *Given a Riemannian manifold* $(\mathcal{M}, g)$*, define the* Riemann curvature tensor *of* $(\mathcal{M}, g)$ *to be the map* $R\colon \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ *defined by*

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for $X, Y, Z \in \mathfrak{X}(\mathcal{M})$*, where* ∇ *is the Levi-Civita connection of* g*.*

We also refer to R simply as the *curvature tensor*.

Exercise 9.2 (Failure of covariant derivatives to commute)**.** *Show that*

$$R(X,Y)Z = \nabla^2 Z(Y,X) - \nabla^2 Z(X,Y),$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$*.*

Remark 9.3 (Properties of the Riemann curvature tensor)**.**

1. *The definition of the curvature of a Riemannian manifold involves only the Levi-Civita connection* ∇*, and not the metric* g *directly. One can therefore define similarly the curvature of any connection on a vector bundle.*

2. *The curvature tensor* R *is anti-symmetric in the first two arguments, i.e.*

$$R(X,Y)Z = -R(Y,X)Z,$$

   for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$*.*

3. *There is no agreed upon sign convention for the Riemann curvature tensor. Many books define the Riemann curvature tensor to be the negative of the above* R*.*

Proposition 9.4 (The Riemann curvature tensor is a tensor field)**.** *For any* $X, Y, Z \in \mathfrak{X}(\mathcal{M})$*,* $R(X,Y)Z|_p$ *depends only on* $X|_p$*,* $Y|_p$*,* $Z|_p$*, i.e.* R *is a* $(1,3)$ *tensor field* $R \in \Gamma(T^1_3\mathcal{M})$*.*

Proof. Given $f \in C^\infty(\mathcal{M})$, it suffices to show that,

$$R(fX,Y)Z = R(X,fY)Z = R(X,Y)fZ = fR(X,Y)Z.$$

It follows from Exercise 9.2 that $R(fX,Y)Z = R(X,fY)Z = fR(X,Y)Z$. Now

$$\nabla_X \nabla_Y (fZ) = \nabla_X((Yf)Z + f\nabla_Y Z) = XYf \cdot Z + Yf \cdot \nabla_X Z + Xf \cdot \nabla_Y Z + f \cdot \nabla_X \nabla_Y Z,$$

and similarly

$$\nabla_Y \nabla_X (fZ) = YXf \cdot Z + Xf \cdot \nabla_Y Z + Yf \cdot \nabla_X Z + f \cdot \nabla_Y \nabla_X Z,$$

and

$$\nabla_{[X,Y]}(fZ) = [X,Y]fZ + f\nabla_{[X,Y]}Z,$$

from which it follows that $R(X,Y)fZ = fR(X,Y)Z$. $\square$

Define the $(0,4)$ Riemann curvature tensor of a Riemannian manifold $(\mathcal{M}, g)$, by

$$Rm(X,Y,Z,W) = g(R(X,Y)Z,W),$$

for all $X, Y, Z, W \in \mathfrak{X}(\mathcal{M})$, i.e. $Rm = R^\flat$, so that $Rm \in \Gamma(T^0_4\mathcal{M})$. We often still write R for Rm. In local coordinates

$$R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}, \qquad Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

where $R_{ijkl} = g_{lm} R_{ijk}{}^m$.

Exercise 9.5 (Riemann curvature tensor in local coordinates). *Show that*

$$R_{ijk}{}^{l} = \partial_{x^i}\Gamma^l_{jk} - \partial_{x^j}\Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}.$$

Exercise 9.6 (Expansion for metric in normal coordinates). *Let $\{x^i\}$ be normal coordinates centred at $p \in \mathcal{M}$. Show that*

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{iklj}(p)x^k x^l + \mathcal{O}(|x|^3),$$

around $p = (0, \dots, 0)$. (Hint: attempt this problem after reading Section 11.)

It seems a priori that it requires n^4 functions to locally describe R. In fact, R possesses many symmetries.

Proposition 9.7 (Algebraic symmetries of the curvature tensor). *For all $X, Y, Z, W \in \mathfrak{X}(\mathcal{M})$,*

(i) $Rm(X,Y,Z,W) = -Rm(Y,X,Z,W)$,

(ii) $Rm(X,Y,Z,W) = -Rm(X,Y,W,Z)$,

(iii) $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$ *(first Bianchi identity),*

(iv) $Rm(X,Y,Z,W) = Rm(Z,W,X,Y)$.

Proof. *(i)* is immediate from the definition.

For *(ii)*, it follows from the fact that ∇ is compatible with g that

$$XY(g(W,Z)) = g(\nabla_X\nabla_Y W, Z) + g(\nabla_Y W, \nabla_X Z) + g(\nabla_X W, \nabla_Y Z) + g(W, \nabla_X\nabla_Y Z), \tag{18}$$

and similarly

$$YX(g(W,Z)) = g(\nabla_Y\nabla_X W, Z) + g(\nabla_X W, \nabla_Y Z) + g(\nabla_Y W, \nabla_X Z) + g(W, \nabla_Y\nabla_X Z), \tag{19}$$

Subtracting (19) from (18) gives

$$[X,Y](g(W,Z)) = g(\nabla_X\nabla_Y W - \nabla_Y\nabla_X W, Z) + g(W, \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z),$$

and *(ii)* then follows from the fact that

$$[X,Y](g(W,Z)) = g(\nabla_{[X,Y]}W, Z) + g(W, \nabla_{[X,Y]}Z).$$

(iii) follows from a direct computation, using the torsion free property of ∇.

Finally, for *(iv)*, note that *(iii)* implies that

$$\begin{aligned}
Rm(X,Y,Z,W) + Rm(Z,X,Y,W) + Rm(Y,Z,X,W) &= 0,\\
Rm(Y,Z,W,X) + Rm(W,Y,Z,X) + Rm(Z,W,Y,X) &= 0,\\
Rm(Z,W,X,Y) + Rm(X,Z,W,Y) + Rm(W,X,Z,Y) &= 0,\\
Rm(W,X,Y,Z) + Rm(Y,W,X,Z) + Rm(X,Y,W,Z) &= 0.
\end{aligned}$$

Summing, and using *(i)* and *(ii)*, gives

$$2Rm(Z,X,Y,W) - 2Rm(Y,W,Z,X) = 0.$$

□

Accounting for the symmetries of Proposition 9.7, R can in fact be described locally by $\frac{1}{12}n^2(n^2-1)$ functions.

The total covariant derivative ∇R possesses an additional symmetry.

Proposition 9.8 (Second Bianchi identity)**.** *For all $V, W, X, Y, Z \in \mathfrak{X}(\mathcal{M})$,*

$$\nabla_X Rm(Y,Z,V,W) + \nabla_Z Rm(X,Y,V,W) + \nabla_Y Rm(Z,X,V,W) = 0.$$

Exercise 9.9. *Prove Proposition 9.8. Hint: use normal coordinates.*

The Riemann curvature tensor is invariant under local isometries.

Exercise 9.10 (The Riemann curvature tensor is a geometric object)**.** *Suppose $\varphi\colon (\mathcal{M}, g) \to (\tilde{\mathcal{M}}, \tilde{g})$ is a local isometry. Show that $\varphi_*(R(X,Y)Z) = \tilde{R}(\varphi_* X, \varphi_* Y)\varphi_* Z$ for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$.*

Vanishing of the Riemann curvature tensor is a local characterisation of Euclidean space.

Theorem 9.11 (Riemann curvature characterises flatness)**.** *A Riemannian manifold is flat if and only if its curvature tensor vanishes identically.*

Proof. A flat Riemannian manifold clearly has vanishing curvature tensor, so consider the converse. Compare the proof with the motivating discussion at the beginning of the section.

We will use the following non-examinable fact: *If $\{e_i\}$ is a local frame around $p \in \mathcal{M}$ such that $[e_i, e_j] = 0$ for all $i, j = 1, \dots, n$, then there exists local coordinates $\{y^i\}$ around p such that $e_i = \partial_{y^i}$ for $i = 1, \dots, n$.* Using this fact it suffices, given $p \in \mathcal{M}$, to find an orthonormal frame $\{e_i\}$ around p such that $[e_i, e_j] = 0$ for all i, j, for then

$$g_{ij} = g(\partial_{y^i}, \partial_{y^j}) = g(e_i, e_j) = \delta_{ij},$$

and so the map $q \mapsto (y^1(q), \dots, y^n(q))$ is a local isometry to Euclidean space.

Let $\{x^i\}$ be normal coordinates around p and set $e_i|_p = \frac{\partial}{\partial x^i}|_p$. Extend each e_i along $\{x^2 = \ldots = x^n = 0\}$ by parallel transport,

$$\nabla_{\partial_{x^1}} e_i|_{\{x^2 = \ldots = x^n = 0\}} = 0.$$

Now extend along $\{x^3 = \ldots = x^n = 0\}$ by parallel transport,

$$\nabla_{\partial_{x^2}} e_i|_{\{x^3 = \ldots = x^n = 0\}} = 0,$$

etc. Let $M_k = \{x^{k+1} = \ldots = x^n = 0\}$. We will inductively show that, for each $k = 1, \dots, n$,

$$\nabla_{\partial_{x^j}} e_i|_{M_k} = 0, \qquad \text{for all } j = 1, \dots, k. \tag{20}$$

Indeed, for $k = 1$ (20) holds by definition. Assume then that (20) holds for some $1 \leq k \leq n-1$. Then

$$\nabla_{\partial_{x^{k+1}}} e_i|_{M_{k+1}} = 0,$$

by definition, and it remains to check that

$$\nabla_{\partial_{x^j}} e_i|_{M_{k+1}} = 0,$$

for $j = 1, \dots, k$. Since R vanishes, it follows that

$$\nabla_{\partial_{x^{k+1}}} \nabla_{\partial_{x^j}} e_i = \nabla_{\partial_{x^j}} \nabla_{\partial_{x^{k+1}}} e_i,$$

and so

$$\nabla_{\partial_{x^{k+1}}} \nabla_{\partial_{x^j}} e_i|_{M_k} = 0.$$

By the inductive hypothesis $\nabla_{\partial_{x^j}} e_i = 0$ on M_k and so it follows that $\nabla_{\partial_{x^j}} e_i = 0$ on M_{k+1} by uniqueness of parallel transport.

Now $\nabla e_i = 0$ for each $i = 1, \dots, n$ and so

$$[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i = 0,$$

and the proof follows. $\square$

The Riemann curvature tensor R can be extended to a map $R\colon \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \times \Gamma(T^r_s\mathcal{M}) \to \Gamma(T^r_s\mathcal{M})$,

$$R(X,Y)T := \nabla_X\nabla_Y T - \nabla_Y\nabla_X T - \nabla_{[X,Y]}T,$$

for all $r, s \geq 0$.

Exercise 9.12 (Properties of the Riemann curvature tensor)**.** *Show that the Riemann curvature tensor satisfies the following properties, for all $X, Y \in \mathfrak{X}(\mathcal{M})$, $T \in \Gamma(T^{r_1}_{s_1}\mathcal{M})$, $S \in \Gamma(T^{r_2}_{s_2}\mathcal{M})$.*

1. $R(X,Y)(T \otimes S) = \big(R(X,Y)T\big) \otimes S + T \otimes \big(R(X,Y)S\big)$.
2. $R(X,Y)f = 0$ *for all* $f \in C^\infty(\mathcal{M}) = \Gamma(T^0_0\mathcal{M})$.
3. $R(X,Y)(c(T)) = c\big(R(X,Y)T\big)$ *for any contraction* c.

9.2 Sectional curvature

Since the Riemann curvature tensor is a $(1,3)$ tensor field, it is difficult to make statements of the form "this manifold has positive curvature". Such statements are most easily made in terms of *sectional curvature*.

Definition 9.13 (Sectional curvature)**.** *Consider a Riemannian manifold $(\mathcal{M}, g)$, $p \in \mathcal{M}$ and a 2 dimensional subspace (a 2-plane) $\Pi_p \subset T_p\mathcal{M}$. If U, V is a basis of Π_p, the* sectional curvature *of Π_p is defined by*

$$\kappa(\Pi_p) := \frac{Rm_p(U,V,V,U)}{g_p(U,U)g_p(V,V) - \big(g_p(U,V)\big)^2}.$$

If $X, Y \in T_p\mathcal{M}$ are linearly independent then we often write

$$\kappa(X,Y) := \kappa(\mathrm{span}\{X,Y\}).$$

Remark 9.14 (Properties of sectional curvature)**.**

1. *If the basis $\{U, V\}$ is orthonormal, then*

$$\kappa_p(\Pi_p) = Rm_p(U,V,V,U).$$

2. *The sectional curvature $\kappa(\Pi_p)$ does note depend on the choice of basis for Π_p.*
3. *Recall the Gauss curvature of a surface. Given a 2-plane $\Pi_p \subset T_p\mathcal{M}$, define the local sub-surface of $\mathcal{M}$ by $S := \exp_p(\Pi_p \subset W)$, where $W \subset T_p\mathcal{M}$ is a open neighbourhood of 0 such that such that $\exp_p|_W$ is a diffeomorphism onto its image. Then $T_pS = \Pi_p$ and $\kappa(\Pi_p)$ is the Gauss curvature of S at p.*

Clearly the Riemann curvature tensor determines all sectional curvatures. The converse is also true. The following proposition guarantees that the Riemann curvature tensor can be recovered from all of the sectional curvatures.

Proposition 9.15 (Sectional curvature determines the Riemann curvature tensor)**.** *Suppose two functions $R_1, R_2\colon T_p\mathcal{M} \times T_p\mathcal{M} \times T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ satisfy the algebraic symmetries of the Riemann curvature tensor Rm and*

$$\frac{R_1(X,Y,Y,X)}{|X|_g^2|Y|_g^2 - \big(g_p(X,Y)\big)^2} = \frac{R_2(X,Y,Y,X)}{|X|_g^2|Y|_g^2 - \big(g_p(X,Y)\big)^2},$$

for all $X, Y \in T_p\mathcal{M}$. Then $R_1 = R_2$.

Proof. Setting $R_3 = R_1 - R_2$, it follows that $R_3(X, Y, Y, X) = 0$ for all $X, Y \in T_p\mathcal{M}$ and it suffices to show that

$$R_3(X, Y, Z, W) = 0,$$

for all $X, Y, Z, W \in T_p\mathcal{M}$. Now, for any $X, Y, Z \in T_p\mathcal{M}$,

$$\begin{aligned} 0 = R_3(X+Y, Z, Z, X+Y) = R_3(X, Z, Z, Y) + R_3(Y, Z, Z, X) + R_3(X, Z, Z, Y) + R_3(Y, Z, Z, Y) \\ = 2R_3(X, Z, Z, Y). \end{aligned}$$

Hence, for all $X, Y, Z, W \in T_p\mathcal{M}$,

$$0 = R_3(X, Z+W, Z+W, Y) = R_3(X, Z, W, Y) + R_3(X, W, Z, Y).$$

The first Bianchi identity then implies

$$0 = R_3(X, Y, Z, W) + R_3(Z, X, Y, W) + R_3(Y, Z, X, W) = 3R_3(X, Y, Z, W).$$

□

The following exercise gives a geometric interpretation to sectional curvature.

Exercise 9.16 (Geometric interpretation of sectional curvature). *Given $p \in \mathcal{M}$ and a 2-plane $\Pi_p \subset T_p\mathcal{M}$, for $r > 0$ sufficiently small let $C_r^0 \subset \Pi_p$ be the circle of radius r centred at the origin,*

$$C_r^0 = \{v \in \Pi_p \mid |v|_g = r\},$$

and let $C_r := \exp_p(C_r^0)$ be its image under the exponential map. Show that

$$\lim_{r \to 0} \frac{2\pi r - L(C_r)}{r^3} = \frac{\pi}{3}\kappa(\Pi_p),$$

where $L(C_r)$ is the length of C_r.

Sectional curvature therefore measures the infinitesimal deviation of $L(C_r)$ with the length of a circle of radius r in Euclidean space.

Definition 9.17 (Constant curvature manifolds). *A Riemannian manifold $(\mathcal{M}, g)$ is called a* constant curvature *manifold if there exists $c \in \mathbb{R}$ such that $\kappa(\Pi_p) = c$ for all 2-planes $\Pi_p \subset T_p\mathcal{M}$ for all $p \in \mathcal{M}$.*

Proposition 9.18 (Riemann curvature tensor of constant curvature manifolds). *Given $p \in \mathcal{M}$, if $c \in \mathbb{R}$ then $\kappa(\Pi_p) = c$ for all 2-planes $\Pi_p \subset T_p\mathcal{M}$ if and only if*

$$R_p(X, Y)Z = c\big(g_p(Y, Z)X - g_p(X, Z)Y\big), \tag{21}$$

for all $X, Y, Z \in T_p\mathcal{M}$.

Proof. If (21) holds then it is straightforward to check that $\kappa(\Pi_p) = c$ for all 2-planes $\Pi_p \subset T_p\mathcal{M}$. Suppose then that $\kappa(\Pi_p) = c$ for all 2-planes $\Pi_p \subset T_p\mathcal{M}$. Define

$$S(X, Y, Z, W) = c\big(g_p(Y, Z)g_p(X, W) - g_p(X, Z)g_p(Y, W)\big),$$

and note that S has the same symmetries as the Riemann curvature tensor Rm. For any 2-plane $\Pi_p = \operatorname{span}\{U, V\}$,

$$\frac{S(U, V, V, U)}{|U|_g^2|V|_g^2 - (g_p(U, V))^2} = c = \kappa(\Pi_p),$$

and so $S = Rm_p$ by Proposition 9.15. □

It follows that $(\mathcal{M}, g)$ is a constant curvature manifold if and only if

$$R(X, Y)Z = c\big(g(Y, Z)X - g(X, Z)Y\big),$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$.

9.3 Model spaces of constant curvature

Euclidean space $(\mathbb{R}^n, g_{Eucl})$ has constant curvature 0.

Exercise 9.19 (The round sphere is a constant curvature manifold)**.** *Show that the unit round sphere (S^n, g_{Round}) has constant curvature 1.*

Recall the Poincaré ball model of hyperbolic space

$$B^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid (x^1)^2 + \ldots + (x^n)^2 < 1\}, \qquad g_{B^n} = \frac{4}{(1-|x|^2)^2}\big((dx^1)^2 + \ldots + (dx^n)^2\big).$$

Exercise 9.20 (Poincaré ball and Poincaré half space models of hyperbolic space are isometric)**.** *Show that (B^n, g_{B^n}) is isometric to the Poincaré half space model of hyperbolic space*

$$H^n\{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n > 0\}, \qquad g_{H^n} = \frac{1}{(x^n)^2}\big((dx^1)^2 + \ldots + (dx^n)^2\big).$$

Hint: define $f\colon B^n \to H^n$ by

$$f(x^1, \ldots, x^n) = \frac{1}{(x^1)^2 + \ldots + (x^{n-1})^2 + (x^n - 1)^2}(2x^1, \ldots, 2x^{n-1}, 1 - |x|^2).$$

Check that

$$f^{-1}(x^1, \ldots, x^n) = \frac{1}{(x^1)^2 + \ldots + (x^{n-1})^2 + (x^n + 1)^2}(2x^1, \ldots, 2x^{n-1}, |x|^2 - 1),$$

so that f is a diffeomorphism, and then check that $f^* g_{H^n} = g_{B^n}$.

Exercise 9.21 (Hyperbolic space is a constant curvature manifold)**.** *Show that (B^n, g_{B_n}) (and hence also (H^n, g_{H^n})) has constant curvature -1.*

9.4 Ricci curvature and scalar curvature

The Riemann curvature tensor is a complicated object and carries a lot of information. The Ricci curvature and scalar curvature are simpler, but less definitive, measures of curvature.

If V is a vector space, recall the contraction map ${c_1}^1\colon V \otimes V^* \to \mathbb{R}$. This map is also called *trace*, and is denoted $\mathrm{tr} := {c_1}^1$. Recall that $R \in \Gamma(T^1_3\mathcal{M})$ and so, for $p \in \mathcal{M}$ and $X, Y \in T_p\mathcal{M}$, one can view $R_p(\cdot, X)Y$ as a map $R_p(\cdot, X)Y\colon T^*_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ given by $R_p(\cdot, X)Y(\xi, W) = \xi(R_p(W, X)Y)$.

Definition 9.22 (Ricci curvature tensor)**.** *The* Ricci curvature tensor *of a Riemannian manifold $(\mathcal{M}, g)$ is the $(0,2)$ tensor field $Ric \in \Gamma(T^0_2\mathcal{M})$ defined by*

$$Ric(X, Y) := \mathrm{tr}\big(R(\cdot, X)Y\big).$$

We often write $Ric(g)$ for Ric when it is necessary to emphasise the dependence on g.

Remark 9.23 (Properties of Ricci curvature)**.**

1. *In local coordinates the Ricci curvature tensor takes the form,*

$$Ric = Ric_{ij}\, dx^i \otimes dx^j,$$

   where $Ric_{ij} = {R_{kij}}^k = g^{kl} R_{kijl}$.

2. *Due to the symmetries of R, taking the trace over any other two arguments gives either $-Ric$ or 0.*

3. *If $\{e_i\}$ is an orthonormal basis for $T_p\mathcal{M}$, then*

$$Ric(X, Y) = \sum_{i=1}^{n} Rm(e_i, X, Y, e_i).$$

4. *The Ricci curvature tensor is symmetric,*

$$Ric(X, Y) = Ric(Y, X),$$

for all $X, Y \in \mathfrak{X}(\mathcal{M})$.

5. *The Ricci curvature tensor is described locally by $\frac{1}{2}n(n+1)$ functions (compare with the $\frac{1}{12}n^2(n^2-1)$ functions required to locally describe the Riemann curvature tensor).*

6. *The Ricci curvature tensor features in several famous geometric equations such as the Einstein equations of general relativity, and the Ricci flow, which was famously used to resolve the Poincaré conjecture.*

7. *Since Ric is symmetric and bilinear, it is completely determined by the associated quadratic form $X \mapsto Ric(X, X)$.*

8. *Given $V \in T_p\mathcal{M}$ with $|V|_g = 1$, if V is extended to an orthonormal basis $\{V = e_1, e_2, \ldots, e_n\}$ then*

$$Ric(V, V) = \sum_{k=1}^{n} Rm(e_k, e_1, e_1, e_k) = \sum_{k=2}^{n} \kappa(V, e_k).$$

The following exercise gives a geometric interpretation to Ricci curvature.

Exercise 9.24 (Geometric interpretation of Ricci curvature)**.** *Let $\{x^i\}$ be normal coordinates centred at $p \in \mathcal{M}$. Show that*

$$\sqrt{\det g_{ij}(x)} = 1 - \frac{1}{6} Ric_{kl}(p) x^k x^l + \mathcal{O}(|x|^3),$$

around $p = (0, \ldots, 0)$. (Hint: use Exercise 9.6.)

It follows from Exercise 9.24 that, if γ is a unit speed geodesic such that $\gamma(0) = p$, then

$$\lim_{t \to 0} \frac{\sqrt{\det g_{ij}(\gamma(t))} - 1}{t^2} = -\frac{1}{6} Ric(\dot{\gamma}(0), \dot{\gamma}(0)),$$

i.e., recalling Exercise 5.9, if $v \in T_p\mathcal{M}$ is a unit vector then $Ric(v, v)$ measures the infinitesimal deviation between the volume of a small wedge in the direction $\gamma_{p,v}$ with the volume of the corresponding wedge in Euclidean space.

If one views the Riemann curvature tensor as a "twisted hessian" of the metric g, then it is appropriate to view the Ricci curvature tensor as some sort of Laplacian of g. Recall that, appropriately viewed, the Laplacian of a function contains much of the information of the whole hessian of that function.

Exercise 9.25 (Ricci curvature in harmonic coordinates)**.** *Recall the Laplace–Beltrami operator (see Definition 7.31). Show that, in local coordinates x^i which satisfy $\Delta_g(x^i) = 0$, the Ricci curvature takes the form*

$$Ric(g)_{ij} = -\frac{1}{2} \Delta_g(g_{ij}) + N_{ij}(g, \partial g),$$

for appropriate functions N_{ij}.

Definition 9.26 (Scalar curvature)**.** *The* scalar curvature *of a Riemannian manifold $(\mathcal{M}, g)$ is the function $S \colon \mathcal{M} \to \mathbb{R}$ defined by*

$$S = \operatorname{tr}(Ric^\sharp).$$

Again, we often write $S(g)$ for S to emphasise the dependence on g.

Remark 9.27 (Properties of scalar curvature)**.**

1. *In local coordinates*
$$S = g^{ij} Ric_{ij} = g^{ij} R_{kij}{}^{k}.$$

2. *If $p \in \mathcal{M}$ and $\{e_i\}$ is an orthonormal basis of $T_p\mathcal{M}$, then*
$$S = \sum_{i=1}^{n} Ric(e_i, e_i) = \sum_{i,j=1}^{n} Rm(e_j, e_i, e_i, e_j) = \sum_{j \neq i} \kappa(e_i, e_j).$$

The following exercise gives a geometric interpretation of scalar curvature. Since it involves volume, the exercise is non-examinable.

Exercise 9.28 (*Geometric interpretation of scalar curvature)**.** *Given $p \in \mathcal{M}$, show that*
$$\lim_{r \to 0} \frac{\mathrm{Vol}(B(p,r)) - \mathrm{Vol}_{Eucl}(B_{Eucl}(0,r))}{r^{n+2}} = -\frac{\omega_{n-1}}{6n(n+2)} S(p),$$
where $\mathrm{Vol}(B(p,r))$ *is the Riemannian volume of the geodesic ball $B(p,r) \subset \mathcal{M}$,* $\mathrm{Vol}_{Eucl}(B_{Eucl}(0,r))$ *is the Euclidean volume of $B_{Eucl}(0,r) = \{x \in \mathbb{R}^n \mid |x| < r\}$, $n = \dim\mathcal{M}$, and $\omega_{n-1} = \int_{S^{n-1}} d\sigma_{S^{n-1}}$ is the Euclidean volume of S^{n-1}.*

Exercise 9.28 in particular implies that scalar curvature $S(p)$ is an infinitesimal measure of the deviation of the volume of a geodesic ball centred at p with the volume of a Euclidean ball.

Definition 9.29 (Einstein manifolds)**.** *A Riemannian manifold $(\mathcal{M}, g)$ is called an* Einstein manifold *if there exists a constant $\lambda \in \mathbb{R}$ such that*
$$Ric(g) = \lambda g.$$

Exercise 9.30 (Curvature and Einstein manifolds)**.** *Show that the scalar curvature of an Einstein manifold is constant. Show that, if $(\mathcal{M}^n, g)$ has constant curvature c, then $(\mathcal{M}^n, g)$ is an Einstein manifold with constant $\lambda = (n-1)c$.*

Remark 9.31 (Riemann curvature tensor and Ricci and scalar curvatures in low dimensions)**.**

1. *In dimension* 2 *the Riemann curvature tensor is determined by the scalar curvature and g,*
$$\kappa(T_p\mathcal{M}) = \frac{1}{2} S(p).$$

2. *In dimension* 3*, the Riemann curvature tensor is determined by the Ricci curvature and g,*
$$R_{ijkl} = Ric_{il} g_{jk} - Ric_{ik} g_{jl} + Ric_{jk} g_{il} - Ric_{jl} g_{ik} - \frac{1}{2} S (g_{il} g_{jk} - g_{ik} g_{jl}).$$

3. *A* 3 *dimensional Riemannian manifold is an Einstein manifold if and only if it has constant curvature.*

10 Submanifolds

If $\overline{\mathcal{M}} \subset \mathcal{M}$ is a submanifold of a Riemannian manifold $(\mathcal{M}, g)$, many of the geometric properties of $\overline{\mathcal{M}}$ with the induced metric $\overline{g}$ can be related to the geometric properties of $(\mathcal{M}, g)$.

10.1 Second fundamental form

Let $\overline{\mathcal{M}} \subset \mathcal{M}$ be a submanifold of a Riemannian manifold $(\mathcal{M}, g)$ and let $\overline{g}$ denote the induced metric. Given $p \in \overline{\mathcal{M}}$, the tangent space to $\mathcal{M}$ at p can be decomposed as

$$T_p\mathcal{M} = T_p\overline{\mathcal{M}} \oplus N_p\overline{\mathcal{M}},$$

where $T_p\overline{\mathcal{M}}$ is the tangent space to $\overline{\mathcal{M}}$ at p and $N_p\overline{\mathcal{M}} = (T_p\overline{\mathcal{M}})^\perp$ is the orthogonal complement of $T_p\overline{\mathcal{M}} \subset T_p\mathcal{M}$ with respect to g, called the *normal space*. The disjoint union

$$N\overline{\mathcal{M}} = \bigcup_{p\in\overline{\mathcal{M}}} \{p\} \times N_p\overline{\mathcal{M}},$$

together with the natural projection map $\pi\colon N\overline{\mathcal{M}} \to \overline{\mathcal{M}}$, $\pi(p, n) = p$ is a vector bundle over $\overline{\mathcal{M}}$, called the *normal bundle*.

For all $p \in \overline{\mathcal{M}}$, any $v \in T_p\mathcal{M}$ can be uniquely decomposed as

$$v = v^\top + v^\perp,$$

where $v^\top \in T_p\overline{\mathcal{M}}$, $v^\perp \in N_p\overline{\mathcal{M}}$. Similarly, any vector field $V \in \mathfrak{X}(\mathcal{M})$ when restricted to $\overline{\mathcal{M}}$ can be uniquely decomposed

$$V|_{\overline{\mathcal{M}}} = V^\top + V^\perp,$$

with $V_p^\top \in T_p\overline{\mathcal{M}}$, $V_p^\perp \in N_p\overline{\mathcal{M}}$ for all $p \in \overline{\mathcal{M}}$.

Definition 10.1 (Second fundamental form)**.** *Define the* second fundamental form *of $\overline{\mathcal{M}}$ to be the map $\Pi\colon \mathfrak{X}(\overline{\mathcal{M}}) \times \mathfrak{X}(\overline{\mathcal{M}}) \to \Gamma(N\overline{\mathcal{M}})$ given by*

$$\Pi(X, Y) := (\nabla_X Y)^\perp.$$

Proposition 10.2 (Properties of second fundamental form)**.** *The second fundamental form Π is symmetric and bilinear over $C^\infty(\overline{\mathcal{M}})$.*

Proof. Consider vector fields $X, Y \in \mathfrak{X}(\overline{\mathcal{M}})$ and recall that $[X, Y] \in \mathfrak{X}(\overline{\mathcal{M}})$. Now

$$\Pi(X, Y) - \Pi(Y, X) = \big(\nabla_X Y - \nabla_Y X\big)^\perp = \big([X, Y]\big)^\perp = 0,$$

since ∇ is torsion free, i.e. Π is symmetric. Clearly Π is linear over $C^\infty(\overline{\mathcal{M}})$ in the first argument. The $C^\infty(\mathcal{M})$ linearity over the second argument follows by the fact that it is symmetric. ☐

In the case that $\overline{\mathcal{M}}$ is an oriented *hypersurface* in $\mathcal{M}$, i.e. $\dim\overline{\mathcal{M}} = \dim\mathcal{M} - 1$, we often write

$$\Pi(X, Y) = k(X, Y)n,$$

where n is the oriented unit normal to $\overline{\mathcal{M}}$. By a slight abuse, we also refer to k as the second fundamental form. It follows from Proposition 10.2 that k is a symmetric $(0, 2)$ tensor field, $k \in \Gamma(T^0_2\overline{\mathcal{M}})$. Since k is symmetric and bilinear, it is determined by the associated quadratic form $X \mapsto k(X, X)$.

Exercise 10.3 (Second fundamental form of a hypersurface)**.** *Show that, for all $X, Y \in \mathfrak{X}(\overline{\mathcal{M}})$,*

$$k(X, Y) = -g(\nabla_X n, Y).$$

Proposition 10.4 (Covariant derivative of submanifold)**.** *If $\overline{\mathcal{M}}$ is a submanifold of $(\mathcal{M}, g)$ then, for any vector fields $X, Y \in \mathfrak{X}(\overline{\mathcal{M}})$,*

$$\nabla_X Y = \overline{\nabla}_X Y + \Pi(X, Y),$$

where $\overline{\nabla}$ is the Levi-Civita connection of the induced metric $\overline{g}$.

Proof. The proof follows from checking that the map $\nabla^\top \colon \mathfrak{X}(\overline{\mathcal{M}}) \times \mathfrak{X}(\overline{\mathcal{M}}) \to \mathfrak{X}(\overline{\mathcal{M}})$, defined by $\nabla^\top_X Y := (\nabla_X Y)^\top$, is a connection on $\overline{\mathcal{M}}$ which is symmetric and compatible with the induced metric $\overline{g}$ (**Exercise**). □

The second fundamental form should be viewed as a measure of *extrinsic* curvature: it describes how the submanifold curves within the ambient manifold.

Exercise 10.5 (Second fundamental form measures failure of $(\overline{\mathcal{M}}, \overline{g})$ geodesics to be $(\mathcal{M}, g)$ geodesics)**.** *Let $\overline{\mathcal{M}}$ be a submanifold of $(\mathcal{M}, g)$ with the induced metric $\overline{g}$. Show that any geodesic $\gamma \colon [a, b] \to \overline{\mathcal{M}}$ is also a geodesic in $(\mathcal{M}, g)$ if and only if the second fundamental form Π vanishes identically.*

If γ is a geodesic in $(\overline{\mathcal{M}}, \overline{g})$, then $k(\dot\gamma, \dot\gamma)$ measures the failure of γ to be a geodesic in $(\mathcal{M}, g)$.

Exercise 10.6 (Second fundamental form of cylinder in $\mathbb{R}^3$)**.** *Show that the cylinder*

$$C := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^3,$$

with the induced Euclidean metric is flat. Consider the coordinate chart $U = \{(x, y, z) \in C \mid y > 0\}$ and $\phi \colon U \to \mathbb{R}^2$ defined by $\phi(x, y, z) = (x, z)$. Show that, in the associated (x, z) local coordinate system, the second fundamental form of C is given by

$$k = \frac{1}{x^2 - 1} dx^2.$$

When $\overline{\mathcal{M}}$ is an oriented hypersurface in $\mathcal{M}$, one can consider the trace of the second fundamental form k. The trace of the second fundamental form measures the change in volume as the hypersurface is deformed in the normal direction.

Exercise 10.7 (Mean curvature measures change in volume under deformations in normal direction)**.** *Let $\overline{\mathcal{M}}$ be an oriented compact hypersurface of a compact Riemannian manifold $(\mathcal{M}, g)$. Consider the oriented unit normal vector field $n \in \Gamma(N\overline{\mathcal{M}})$ and define*

$$\overline{\mathcal{M}}_t := \{\Phi_t(p) \mid p \in \overline{\mathcal{M}}\} \subset \mathcal{M},$$

where $\Phi_t \colon \overline{\mathcal{M}} \to \mathcal{M}$ is defined by $\Phi_t(p) := \exp_p(tn(p))$. For sufficiently small t, $\overline{\mathcal{M}}_t$ is a hypersurface of $\mathcal{M}$. Show that, in any local coordinate system,

$$\frac{d}{dt}\Big|_{t=0} \sqrt{\det\big((\overline{g}_t)_{ij}\big)}(p) = -\mathrm{tr}k(p) \sqrt{\det\big((\overline{g}_t)_{ij}\big)}(p),$$

where $\overline{g}_t$ is the pullback by Φ_t of the induced metric on $\overline{\mathcal{M}}_t$.

The trace of the second fundamental form of a hypersurface, $\mathrm{tr}k$, is called the *mean curvature*.

10.2 The Gauss equation

The curvature of a submanifold can be related to the curvature of the ambient manifold using the second fundamental form.

Theorem 10.8 (The Gauss equation)**.** *Let $\overline{\mathcal{M}}$ is a submanifold of $(\mathcal{M}, g)$ with the induced metric $\overline{g}$. The Riemann curvature tensor of $(\overline{\mathcal{M}}, \overline{g})$ satisfies*

$$\overline{Rm}(X, Y, Z, W) = Rm(X, Y, Z, W) + g(\Pi(X, W), \Pi(Y, Z)) - g(\Pi(X, Z), \Pi(Y, W)),$$

for all $X, Y, Z, W \in \mathfrak{X}(\overline{\mathcal{M}})$.

Proof. Proposition 10.4 implies that

$$\nabla_X \nabla_Y Z = \nabla_X \big(\overline{\nabla}_Y Z + \Pi(Y,Z)\big) = \overline{\nabla}_X \overline{\nabla}_Y Z + \Pi(X, \overline{\nabla}_Y Z) + \nabla_X \big(\Pi(Y,Z)\big),$$

where $\overline{\nabla}$ is the Levi-Civita connection of the induced metric $\overline{g}$. Since $g(W, \Pi(A,B)) = 0$ for all $A, B \in \mathfrak{X}(\overline{\mathcal{M}})$, it follows that

$$g(\Pi(X, \overline{\nabla}_Y Z), W) = 0, \qquad g\big(\nabla_X \big(\Pi(Y,Z)\big), W\big) = -g\big(\Pi(Y,Z), \nabla_X W\big) = -g\big(\Pi(Y,Z), \Pi(X,W)\big).$$

Hence

$$g(\nabla_X \nabla_Y Z, W) = g(\overline{\nabla}_X \overline{\nabla}_Y Z, W) - g\big(\Pi(Y,Z), \Pi(X,W)\big),$$

and similarly

$$g(\nabla_Y \nabla_X Z, W) = g(\overline{\nabla}_Y \overline{\nabla}_X Z, W) - g\big(\Pi(X,Z), \Pi(Y,W)\big),$$

and

$$g(\nabla_{[X,Y]} Z, W) = g\big(\overline{\nabla}_{[X,Y]} Z + \Pi([X,Y], Z), W\big) = g(\overline{\nabla}_{[X,Y]} Z, W).$$

The proof follows. □

If $(\overline{\mathcal{M}}, \overline{g})$ is an embedded surface in Euclidean space, Proposition 10.8 in particular implies that

$$\overline{Rm}(X,Y,Z,W) = k(X,W)k(Y,Z) - k(X,Z)k(W,Y).$$

This is the celebrated *Theorema Egregium* ("Remarkable Theorem") of Gauss.

11 Jacobi fields

Jacobi fields are vector fields along a geodesic which measure the effect of curvature on neighbouring geodesics.

11.1 Definition and basic properties

The definition of a Jacobi field can immediately be given.

Definition 11.1 (Jacobi fields and the Jacobi equation)**.** *If $\gamma\colon [a,b] \to \mathcal{M}$ is a geodesic, a vector field J along γ, $J \in \mathfrak{X}(\gamma)$, is called a* Jacobi field *if it satisfies the* Jacobi equation*,*

$$D_t^2 J = R(\dot{\gamma}, J)\dot{\gamma}. \tag{22}$$

Equation (22) *is called the* Jacobi equation*.*

We now proceed to uncover the link between Jacobi fields and properties of neighbouring geodesics.

Definition 11.2 (Variation through geodesics)**.** *Let $\gamma\colon [a,b] \to \mathcal{M}$ be a geodesic. A* variation of γ through geodesics *is a smooth map $A\colon (-\varepsilon, \varepsilon) \times [a,b] \to \mathcal{M}$ such that $A(0,t) = \gamma(t)$ for all $t \in [a,b]$ and, for each $s \in (-\varepsilon, \varepsilon)$, the curve $t \mapsto A(s,t)$ is a geodesic.*

Recall that a variation of γ gives rise to a *variational vector field* along γ,

$$\frac{\partial A(s,t)}{\partial s}\Big|_{s=0} \in \mathfrak{X}(\gamma).$$

Theorem 11.3 (Variations through geodesics and Jacobi fields)**.** *If γ is a geodesic and A is a variation of γ through geodesics, then $J(t) := \frac{\partial A(s,t)}{\partial s}|_{s=0} \in \mathfrak{X}(\gamma)$ is a Jacobi field, i.e. J satisfies the Jacobi equation* (22)*.*

Proof. It suffices to show that, for any vector field $V \in \mathfrak{X}(A(s,\cdot))$ for all s,

$$D_s D_t V - D_t D_s V = R(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t})V. \tag{23}$$

Indeed, if (23) holds, then, since $t \mapsto A(s,t)$ is a geodesic for all $s \in (-\varepsilon, \varepsilon)$,

$$0 = D_s D_t \frac{\partial A}{\partial t} = D_t D_s \frac{\partial A}{\partial t} + (\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t})\frac{\partial A}{\partial t},$$

by Lemma 8.22. Setting $s = 0$ gives (22).

To see that (23) holds, consider local coordinates $\{x^i\}$ and write $V(s,t) = V^i(s,t)\partial_{x^i}$. Then

$$D_s D_t V = \frac{\partial^2 V^i}{\partial s \partial t}\partial_{x^i} + \frac{\partial V^i}{\partial t} D_s \partial_{x^i} + \frac{\partial V^i}{\partial s} D_t \partial_{x^i} + V^i D_s D_t \partial_{x^i},$$

and similarly,

$$D_t D_s V = \frac{\partial^2 V^i}{\partial t \partial s}\partial_{x^i} + \frac{\partial V^i}{\partial s} D_t \partial_{x^i} + \frac{\partial V^i}{\partial t} D_s \partial_{x^i} + V^i D_t D_s \partial_{x^i},$$

and so

$$D_s D_t V - D_t D_s V = V^i \big(D_s D_t \partial_{x^i} - D_t D_s \partial_{x^i}\big).$$

Now

$$D_s D_t \partial_{x^i} = \frac{\partial^2 A^j}{\partial s \partial t} \nabla_{\partial_{x^j}} \partial_{x^i} + \frac{\partial A^j}{\partial t}\frac{\partial A^k}{\partial s} \nabla_{\partial_{x^k}} \nabla_{\partial_{x^j}} \partial_{x^i},$$

and

$$D_t D_s \partial_{x^i} = \frac{\partial^2 A^j}{\partial t \partial s} \nabla_{\partial_{x^j}} \partial_{x^i} + \frac{\partial A^j}{\partial s}\frac{\partial A^k}{\partial t} \nabla_{\partial_{x^k}} \nabla_{\partial_{x^j}} \partial_{x^i},$$

and so

$$D_s D_t \partial_{x^i} - D_t D_s \partial_{x^i} = R\Big(\frac{\partial A}{\partial s}, \frac{\partial A}{\partial t}\Big)\partial_{x^i},$$

and (23) follows. □

The Jacobi equation (22) should be viewed as a second order ordinary differential equation.

Proposition 11.4 (Initial value problem for Jacobi equation)**.** *If $\gamma\colon [a,b] \to \mathcal{M}$ is a geodesic then, given $X, Y \in T_{\gamma(a)}\mathcal{M}$, there exists a Jacobi field J along γ such that*

$$J(a) = X, \qquad D_t J(a) = Y.$$

Proof. Let $\{e_i|_p\}$ be an orthonormal basis of $T_p\mathcal{M}$. Extend to an orthonormal frame $\{e_i(t)\}$ along γ by parallel transport $D_t e_i = 0$ (recall that parallel transport preserves lengths and angles). Writing $J(t) = J^i(t)e_i$, the Jacobi equation becomes

$$\ddot{J}^i(t) = R_{jkl}{}^i(\gamma(t))\dot{\gamma}^j(t)J^k(t)\dot{\gamma}^l(t),$$

for $i = 1, \ldots, n$. The proof reduces to an initial value problem for this linear system of ODE with initial conditions

$$J^i(a) = X^i, \qquad \dot{J}^i(a) = Y^i,$$

for which existence and uniqueness follows from Theorem 6.1. (As in the proof of Proposition 7.16, since the equation is linear, the solution $J^i(t)$ lives for all $t \in [a,b]$, for all $i = 1, \ldots, n$.) □

Exercise 11.5 (Variations through geodesics and Jacobi fields)**.** *Show that every Jacobi field J along a geodesic $\gamma\colon [a,b] \to \mathcal{M}$ arises as the variational vector field of some variation of γ through geodesics. Hint: let $\sigma\colon (-\varepsilon, \varepsilon) \to \mathcal{M}$ be a curve in $\mathcal{M}$ such that $\sigma(0) = \gamma(a)$, $\dot{\sigma}(0) = J(a)$ and let $W \in \mathfrak{X}(\sigma)$ be a vector field such that $W(0) = \dot{\gamma}(a)$, $D_s W(0) = D_t J(a)$, then define $A(s,t) = \exp(\sigma(s), tW(s))$.*

Given a geodesic γ, define the space of Jacobi fields

$$\mathcal{J}(\gamma) = \{\text{Jacobi fields along } \gamma\}.$$

Since the Jacobi equation (22) is linear, it follows that $\mathcal{J}(\gamma)$ is a vector space. It follows from Proposition 11.4 that $\mathcal{J}(\gamma)$ is $2n$ dimensional, where $n = \dim\mathcal{M}$.

Along any geodesic $\gamma\colon [a, b] \to \mathcal{M}$ there are two trivial Jacobi fields $\dot\gamma(t)$ and $t\dot\gamma(t)$. These two Jacobi fields arise from the variations $A(s, t) = \gamma(s + t)$ and $A(s, t) = \gamma(e^s t)$ which reparameterise γ. These two Jacobi fields span a two dimensional subspace of $\mathcal{J}(\gamma)$, and so the space of nontrivial Jacobi fields along γ has dimension $2n - 2$.

Exercise 11.6 (Normal Jacobi fields)**.** *Let $\gamma\colon [a, b] \to \mathcal{M}$ be a geodesic. Suppose $X, Y \in T_{\gamma(a)}\mathcal{M}$ are normal to γ, i.e. $g(X, \dot\gamma) = g(Y, \dot\gamma) = 0$. Show that the Jacobi field J arising from Proposition 11.4 is normal to γ, i.e. $g(J(t), \dot\gamma(t)) = 0$ for all $t \in [a, b]$.*

The nontrivial Jacobi fields along γ can be determined from the trivial Jacobi fields and those normal to γ.

The Jacobi fields of constant curvature manifolds which vanish initially can be computed explicitly.

Proposition 11.7 (Jacobi fields of constant curvature manifolds)**.** *Suppose $(\mathcal{M}, g)$ has constant curvature c and let $\gamma\colon [a, b] \to \mathcal{M}$ be a unit speed geodesic. If $J(t)$ is a normal Jacobi field along γ which vanishes initially, $J(a) = 0$, then*

$$J(t) = \begin{cases} \frac{\sin(t\sqrt{c})}{\sqrt{c}} W(t) & \text{if } c > 0, \\ tW(t) & \text{if } c = 0, \\ \frac{\sinh(t\sqrt{-c})}{\sqrt{-c}} W(t) & \text{if } c < 0, \end{cases}$$

where $W(t)$ is parallel along γ, $D_t W(t) = 0$.

Proof. Recall that $(\mathcal{M}, g)$ has constant curvature c if and only if

$$R(X, Y)Z = c\big(g(Y, Z)X - g(X, Z)Y\big).$$

Hence

$$D_t^2 J = c\big(g(\dot\gamma, J)\dot\gamma - g(\dot\gamma, \dot\gamma)J\big),$$

and so

$$D_t^2 J + cJ = 0.$$

Suppose $J(t) = f(t)W(t)$, where W is normal to γ and parallel along γ. Then

$$\big(f''(t) + cf(t)\big)W(t) = 0,$$

and so, if J is nontrivial,

$$f''(t) + cf(t) = 0.$$

The unique solution of this ordinary differential equation with $f(0) = 0$ is a constant multiple of

$$f(t) = \begin{cases} \frac{\sin(t\sqrt{c})}{\sqrt{c}} & \text{if } c > 0, \\ t & \text{if } c = 0, \\ \frac{\sinh(t\sqrt{-c})}{\sqrt{-c}} & \text{if } c < 0. \end{cases}$$

The proof then follows from the fact that the dimension of the space of normal Jacobi fields which vanish initially is $n - 1$. $\square$

11.2 The second variation formula, the index form, and conjugate points

In this section we are concerned with the question of when geodesics fail to be length minimising with respect to "nearby" curves.

If A is a variation of a geodesic $\gamma\colon [a,b] \to \mathcal{M}$ then, if γ is length minimising amongst all nearby curves, it must be the case that

$$\frac{d^2}{ds^2}\Big|_{s=0} L(A(s,\cdot)) \geq 0,$$

i.e. γ is a local minimum of the length functional.

Proposition 11.8 (Second variation formula)**.** *Let $\gamma\colon [a,b] \to \mathcal{M}$ be a unit speed geodesic and let $A\colon (-\varepsilon,\varepsilon)\times [a,b] \to \mathcal{M}$ be a proper variation of γ. Then*

$$\frac{d^2}{ds^2}\Big|_{s=0} L(A(s,\cdot)) = \int_a^b |D_t V^\perp(t)|_g^2 - Rm(V^\perp(t),\dot\gamma(t),\dot\gamma(t),V^\perp(t))dt,$$

where $V^\perp$ is the normal component of $V = \frac{\partial A}{\partial s}|_{s=0}$:

$$V^\perp = V - g(V,\dot\gamma)\dot\gamma.$$

Proof. Assume, for simplicity, that A is smooth. In the proof of the first variation formula, Proposition 8.23, we showed that

$$\frac{d}{ds} L(A(s,\cdot)) = \int_a^b \frac{g(D_t\partial_s A, \partial_t A)}{\sqrt{g(\partial_t A,\partial_t A)}}dt.$$

Now

$$\begin{aligned}\frac{d}{ds}\Big(\frac{g(D_t\partial_s A,\partial_t A)}{\sqrt{g(\partial_t A,\partial_t A)}}\Big) &= \frac{g(D_sD_t\partial_s A,\partial_t A) + g(D_t\partial_s A, D_s\partial_t A)}{|\partial_t A|} - \frac{g(D_t\partial_s A,\partial_t A)g(D_s\partial_t A,\partial_t A)}{|\partial_t A|^3}\\ &= \frac{g(D_tD_s\partial_s A + R(\partial_s A,\partial_t A)\partial_s A,\partial_t A) + |D_t\partial_s A|^2}{|\partial_t A|} - \frac{g(D_t\partial_s A,\partial_t A)^2}{|\partial_t A|^3},\end{aligned}$$

by Lemma 8.22 and equation (23). Hence

$$\frac{d^2}{ds^2}\Big|_{s=0} L(A(s,\cdot)) = \int_a^b g(D_tD_s\partial_s A|_{s=0},\dot\gamma) - Rm(V(t),\dot\gamma(t),\dot\gamma(t),V(t)) + |D_t V|_g^2 - g(D_t V,\dot\gamma)^2 dt.$$

Clearly,

$$Rm(V(t),\dot\gamma(t),\dot\gamma(t),V(t)) = Rm(V(t)^\perp,\dot\gamma(t),\dot\gamma(t),V(t)^\perp),$$

and

$$D_t V = D_t V^\perp + g(D_t V,\dot\gamma)\dot\gamma,$$

so that

$$|D_t V|_g^2 = |D_t V^\perp|_g^2 + g(D_t V,\dot\gamma)^2.$$

The proof follows from the fact that

$$g(D_tD_s\partial_s A,\dot\gamma) = \frac{d}{dt}\big(g(D_s\partial_s A,\dot\gamma)\big),$$

and so

$$\int_a^b g(D_tD_s\partial_s A,\dot\gamma)dt = 0,$$

since A is a proper variation. ☐

Definition 11.9 (Index form)**.** *Let $\gamma\colon [a,b] \to \mathcal{M}$ be a geodesic. For vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$ which are normal to γ, the symmetric bilinear map defined by*

$$I(X,Y) := \int_a^b g(D_tX, D_tY) - Rm(X, \dot{\gamma}, \dot{\gamma}, Y)dt,$$

is called the index form *of γ.*

The second variation formula, Proposition 11.8, implies that, for any variation A of γ,

$$\frac{d^2}{ds^2}\Big|_{s=0} L(A(s,\cdot)) < 0,$$

if and only if

$$I(V,V) < 0, \qquad V = \frac{\partial A}{\partial s}\Big|_{s=0}.$$

If $X, Y \in \mathfrak{X}(\gamma)$ are admissible vector fields then

$$I(X,Y) = -\int_a^b g(D_t^2X - R(\dot{\gamma}, X)\dot{\gamma}, Y)dt - \sum_{i=0}^{k} g(D_tX(t_i^+) - D_tX(t_i^-), Y(t_i)),$$

where $t_i \in (a,b)$, for $i = 0, \dots, k$ are the points at which X is not smooth.

Definition 11.10 (Conjugate points)**.** *If $p \in \mathcal{M}$ and $\gamma\colon [a,b] \to \mathcal{M}$ is a geodesic such that $\gamma(a) = p$, a point $q = \gamma(t_*)$ for some $t_* \in (a,b]$ is called a* conjugate point *to p along γ if there exists a nontrivial Jacobi field J along γ such that $J(a) = J(b) = 0$.*

Exercise 11.11 (Normal Jacobi fields)**.** *Let J be a Jacobi field along $\gamma\colon [a,b] \to \mathcal{M}$ such that $J(a) = J(b) = 0$. Show that J is normal to γ.*

The following theorem implies that geodesics are not length minimising past conjugate points.

Theorem 11.12 (Geodesics are not length minimising past conjugate points)**.** *If $\gamma\colon [a,b] \to \mathcal{M}$ is a unit speed geodesic and there exists $c \in (a,b)$ such that $\gamma(c)$ is a conjugate point to $\gamma(a)$, then there exists a proper normal vector field $V \in \mathfrak{X}(\gamma)$ such that $I(V,V) < 0$.*

Proof. There exists a nontrivial Jacobi field J along γ such that $J(a) = J(c) = 0$. Define

$$X(t) := \begin{cases} J(t) & \text{if } t \in [a,c], \\ 0 & \text{if } t \in (c,b]. \end{cases}$$

It follows from Exercise 11.11 that X is a normal proper vector field along γ, which is smooth everywhere except at c. Let $Y \in \mathfrak{X}(\gamma)$ be such that $Y(c) = -D_tX(c^-)$. Note that $Y \neq 0$ as J is nontrivial. Now, given $\varepsilon > 0$, set $V = X + \varepsilon Y$. Then, since $I(X,X) = 0$ and $I(X,Y) = -|D_tX(c^-)|^2$, it follows that

$$I(V,V) = -2\varepsilon|D_tX(c^-)|^2 + \varepsilon^2 I(Y,Y) < 0,$$

if ε is sufficiently small. □

12 Classical theorems in Riemannian geometry

This section concerns two classical theorems in Riemannian geometry, the Bonnet–Myers Theorem and the Cartan–Hadamard Theorem, which both involve deducing topological properties of Riemannian manifolds based on some geometric assumptions satisfied by the Riemannian metric.

12.1 The Bonnet–Myers Theorem

The Bonnet–Myers Theorem guarantees that, if the Ricci curvature of a Riemannian manifold obeys an appropriate lower bound, then the manifold is compact and satisfies a *diameter* bound.

Definition 12.1 (Diameter of a Riemannian manifold)**.** *If $(\mathcal{M}, g)$ is a Riemannian manifold, the* diameter *of $(\mathcal{M}, g)$ is defined by*

$$\operatorname{diam}(\mathcal{M}) := \sup\{d_g(p,q) \mid p, q \in \mathcal{M}\},$$

where d_g is the Riemannian distance.

Theorem 12.2 (Bonnet–Myers)**.** *Let $(\mathcal{M}^n, g)$ be a complete (and connected) Riemannian manifold and suppose $\rho > 0$ is such that*

$$Ric(X,X) \geq \frac{n-1}{\rho^2} g(X,X),$$

for all $X \in \mathfrak{X}(\mathcal{M})$. Then $\mathcal{M}$ is compact and $\operatorname{diam}(\mathcal{M}) \leq \pi\rho$.

Remark 12.3 (Remarks on Bonnet–Myers Theorem)**.**

1. *Think of the Bonnet–Myers Theorem as a "weak topological assumption" (completeness) plus a "curvature assumption" (the lower bound on the Ricci curvature) guaranteeing a "strong topological statement" (the diameter bound) and "topological information" (compactness).*

2. *The round sphere (S^n, g_{Round}) satisfies $Ric = (n-1)g$ and $\operatorname{diam}(S^n) = \pi$, so the theorem is sharp is this sense.*

3. *In fact, the Bonnet–Myers Theorem involves "comparing with (S^n, g_{Round})". The assumption involves comparing the Ricci curvature of $(\mathcal{M}, g)$ with that of (S^n, g_{Round}). The $V_i(t)$ considered in the proof are exactly the Jacobi fields of the round sphere which vanish at $t = 0$ (see Proposition 11.7).*

4. *The lower bound on the Ricci curvature cannot be relaxed to $Ric(X,X) > 0$ for all $X \in \mathfrak{X}(\mathcal{M})$ as the the paraboloid*

   $$\{(x,y,z) \in \mathbb{R}^3 \mid z = x^2 + y^2\},$$

   *with the induced Euclidean metric satisfies $Ric(X,X) > 0$ (**Exercise**) but is complete and not compact. The theorem is then also sharp in this sense.*

Proof of Bonnet–Myers Theorem. Suppose that the diameter bound fails. Then, there exists $p, q \in \mathcal{M}$ and a length minimising unit speed geodesic $\gamma\colon [0, L] \to \mathcal{M}$ such that $\gamma(0) = p$, $\gamma(L) = q$, and $L(\gamma) > \rho\pi$. Then, for every proper vector field $V \in \mathfrak{X}(\gamma)$,

$$I(V,V) \geq 0. \tag{24}$$

We will construct a V which contradicts this fact.

Extend $e_1 := \dot\gamma(0)$ to an orthonormal basis $\{e_k\}_{k=1}^n$ for $T_p\mathcal{M}$ and extend to an orthonormal frame along γ by parallel transport. Set

$$V_i(t) := \sin\left(\frac{\pi t}{L}\right) e_i(t),$$

for $i = 1, \dots, n$, where $L = L(\gamma)$. Then

$$\begin{aligned}\sum_{i=2}^n I(V_i, V_i) = \sum_{i=2}^n \int_0^L -g(D_t^2 V_i - R(\dot\gamma, V_i)\dot\gamma, V_i)dt &= \sum_{i=2}^n \int_0^L \frac{\pi^2}{L^2}\sin^2\left(\frac{\pi t}{L}\right) - \sin^2\left(\frac{\pi t}{L}\right) Rm(e_i, \dot\gamma, \dot\gamma, e_i)dt \\ &\leq \int_0^L (n-1)\sin^2\left(\frac{\pi t}{L}\right)\left(\frac{\pi^2}{L^2} - \frac{1}{\rho^2}\right)dt < 0,\end{aligned}$$

and so there exists $i = 2, \dots, n$ such that $I(V_i, V_i) < 0$, which contradicts (24). Hence $\operatorname{diam}(\mathcal{M}) \leq \rho\pi$. Given $p \in \mathcal{M}$, it follows that $\exp_p\colon \overline{B(0, 2\rho)} \to \mathcal{M}$ is a surjection, hence $\mathcal{M}$ is compact. ∎

12.2 The Cartan–Hadamard Theorem

The Cartan–Hadamard Theorem guarantees that a *simply connected* Riemannian manifold with non-positive curvature is diffeomorphic to $\mathbb{R}^n$.

Definition 12.4 (Simply connected topological space)**.** *A topological space $\mathcal{M}$ is called* simply connected *if, for all pairs of curves $\gamma_1, \gamma_2 \colon [0,1] \to \mathcal{M}$ such that $\gamma_1(0) = \gamma_1(1) = \gamma_2(0) = \gamma_2(1)$, there exists a continuous map $A \colon [0,1]^2 \to \mathcal{M}$ such that $A(0,t) = \gamma_1(t)$ and $A(1,t) = \gamma_2(t)$ for all $t \in [0,1]$.*

Theorem 12.5 (Cartan–Hadamard)**.** *If $(\mathcal{M}^n, g)$ is a complete simply connected Riemannian manifold of non-positive curvature (i.e. $\kappa(\Pi_p) \leq 0$ for all 2-planes $\Pi_p \subset T_p\mathcal{M}$ for all $p \in \mathcal{M}$), then $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^n$. More precisely, for all $p \in \mathcal{M}$ the exponential map $\exp_p \colon T_p\mathcal{M} \to \mathcal{M}$ is a diffeomorphism.*

The proof of the Cartan–Hadamard Theorem relies on the following.

Lemma 12.6 (Conjugate points and the exponential map)**.** *Given $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$, the exponential map $\exp_p \colon T_p\mathcal{M} \to \mathcal{M}$ is a local diffeomorphism around v if and only if $q = \exp_p(v)$ is not conjugate to p along the geodesic $t \mapsto \exp_p(tv) =: \gamma(t)$.*

Proof. By the inverse function theorem $\exp_p$ is a local diffeomorphism near v if and only if $(\exp_p)_{*v} \colon T_v T_p\mathcal{M} = T_p\mathcal{M} \to T_q\mathcal{M}$ is an isomorphism if and only if $(\exp_p)_{*v}$ is injective. Consider the variation of geodesics $A \colon (-\varepsilon, \varepsilon) \times [0,1] \to \mathcal{M}$ defined by $A(s,t) = \exp_p(t(v + sw))$. The variational vector field $J(t) = \partial_s A(0,t)$ is a Jacobi field along γ satisfying $J(0) = 0$, $J(1) = (\exp_p)_{*v} w$. Since the dimension of the space of Jacobi fields with $J(0) = 0$ is n, all such Jacobi fields with $J(0) = 0$ take this form for some $w \in T_p\mathcal{M}$. Hence $(\exp_p)_{*v}$ fails to be injective if and only if there exists $w \neq 0$ such that $(\exp_p)_{*v} w = 0$ if and only if there exists a nontrivial Jacobi field J satisfying $J(0) = J(1) = 0$. □

Proof of the Cartan–Hadamard Theorem. We will only show that, for all $p \in \mathcal{M}$, the exponential map $\exp_p \colon T_p\mathcal{M} \to \mathcal{M}$ is a local diffeomorphism. This step contains most of the content of the proof.

Consider $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$, and let J be a Jacobi field along $t \mapsto \exp_p(tv)$ such that $J(0) = 0$. Then

$$\frac{d^2}{dt^2}(g(J,J)) = -2Rm(J, \dot\gamma, \dot\gamma, J) + 2|D_t J|^2 = -2\kappa(J, \dot\gamma)\big(|J|^2|\dot\gamma|^2 - (g(\dot\gamma, J))^2\big) + 2|D_t J|^2 \geq 0,$$

by the Cauchy–Schwarz inequality. It follows that

$$\frac{d}{dt}(g(J,J)) \geq 0,$$

i.e. $|J(t)|_g^2$ is non-decreasing. Since $J(0) = 0$, if $J(t_*) = 0$ for some $t_* > 0$ then it must be the case that $J \equiv 0$. Hence there are no conjugate points to p. Lemma 12.6 then implies that $\exp_p \colon T_p\mathcal{M} \to \mathcal{M}$ is a local diffeomorphism. □

13 *Lorentzian geometry and Penrose's incompleteness theorem

Recall that, for $n \geq 1$, a semi Riemannian metric g on an $n+1$ dimensional manifold $\mathcal{M}$ is called a *Lorentzian metric* if it has signature $(-, +, \ldots, +)$, i.e. if, for each $p \in \mathcal{M}$, there exists $e_0, e_1, \ldots, e_n \in T_p\mathcal{M}$ such that

$$g_p(e_0, e_0) = -1, \qquad g_p(e_i, e_0) = 0 \text{ for } i = 1, \ldots, n, \qquad g_p(e_i, e_j) = \delta_{ij}, \text{ for } i, j = 1, \ldots, n.$$

A pair $(\mathcal{M}, g)$ is called a *Lorentzian manifold.* We call such a manifold $n+1$ dimensional to emphasise the Lorentzian signature of the metric. Much (but certainly not all) of the theory of Riemannian manifolds carries over to Lorentzian manifolds, for example a Lorentzian metric on a manifold defines a unique Levi-Civita connection, which in turn defines parallel transport and a Riemann curvature tensor.

Lorentzian manifolds are of particular interest in general relativity – a theory of gravity postulated by Einstein in 1915, which concerns Lorentzian manifolds satisfying the *Einstein equations*. In the absence of matter, the *vacuum* Einstein equations take the remarkably simple form

$$Ric(g) = 0. \tag{25}$$

Appropriately viewed, (25) are a system of nonlinear hyperbolic partial differential equations for g.

The goal of this section is to sketch a proof of the *Penrose Incompleteness Theorem* – a seminal result in Lorentzian geometry, which profoundly shaped the way we view black holes and general relativity, and was accordingly celebrated with the award of the 2020 Nobel prize in physics to Roger Penrose. The theorem can, in a sense, be viewed as the Lorentzian analogue of the Bonnet–Myers Theorem and is thus a nice application of many of the ideas developed in this course. Before stating the theorem there are several notions in Lorentzian geometry which must first be introduced.

For the rest of this section it will be assumed that $(\mathcal{M}, g)$ is a $3+1$ dimensional Lorentzian manifold. The prototype of such a space is *Minkowski space*, namely $\mathbb{R}^{3+1}$ equipped with the *Minkowski metric*

$$g = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

13.1 Timelike, null, and spacelike vectors, curves, and submanifolds

The failure of the metric g to be positive definite means that a Lorentzian manifold does not retain the metric space structure of a Riemannian manifold. A Lorentzian metric, however, retains some convexity properties when restricted to certain directions.

Definition 13.1 (Timelike, spacelike, and null vectors)**.** *Given $p \in \mathcal{M}$, a vector $v \in T_p\mathcal{M}$ is called* timelike *if $g_p(v,v) < 0$,* spacelike *if $g_p(v,v) > 0$, or* null *if $g_p(v,v) = 0$. A vector $v \in T_p\mathcal{M}$ is called* causal *if it is either timelike or null.*

Definition 13.2 (Timelike, spacelike, and null curves)**.** *A curve $\gamma\colon (a,b) \to \mathcal{M}$ is called* timelike *if $g(\dot\gamma(t), \dot\gamma(t)) < 0$ for all $t \in (a,b)$,* spacelike *if $g(\dot\gamma(t), \dot\gamma(t)) > 0$ for all $t \in (a,b)$, or* null *if if $g(\dot\gamma(t), \dot\gamma(t)) = 0$ for all $t \in (a,b)$.*

A symmetric bilinear map h on a vector space V is called *degenerate* if there exists $X \in V$ with $X \neq 0$ such that $h(X,Y) = 0$ for all $Y \in V$.

Definition 13.3 (Timelike, spacelike, and null submanifolds)**.** *A submanifold $\mathcal{N} \subset \mathcal{M}$ is called* timelike *if the induced metric $g|_{\mathcal{N}}$ is Lorentzian,* spacelike *if the induced metric $g|_{\mathcal{N}}$ is Riemannain, and* null *if the induced metric $g|_{\mathcal{N}}$ is degenerate.*

13.2 Time orientations and causal structure

One new feature of Lorentzian geometry, not present in Riemannian geometry, is the notion of *causality*.

Definition 13.4 (Time orientations)**.** *A* time orientation *on a Lorentzian manifold $(\mathcal{M}, g)$ is a global (nowhere vanishing) timelike vector field T. A causal vector v in a time oriented Lorentzian manifold $(\mathcal{M}, g, T)$ is called* future directed *if $g(v,T) < 0$, and* past directed *if $g(v,T) > 0$.*

From now on, Lorentzian manifolds will be assumed to be time oriented.

Definition 13.5 (Causal and chronological futures)**.** *Given an Lorentzian manifold $(\mathcal{M}, g)$ and a set $S \subset \mathcal{M}$, the* causal future *of S is the set*

$$J^+(S) := \{p \in \mathcal{M} \mid \textit{there exist a causal curve } \gamma\colon [a,b] \to \mathcal{M} \textit{ with } \gamma(a) \in S, \gamma(b) = p\}.$$

The chronological future *of S is the set*

$$I^+(S) := \{p \in \mathcal{M} \mid \textit{there exist a timelike curve } \gamma\colon [a,b] \to \mathcal{M} \textit{ with } \gamma(a) \in S, \gamma(b) = p\}.$$

There are similar definitions of causal past and chronological past but, as we are progressive, let us not concern ourselves with these here.

13.3 Global hyperbolicity

Recall that a curve $\gamma\colon (a,b) \to \mathcal{M}$ in a manifold $\mathcal{M}$ is called *inextendible* if it is not the restriction of a curve $\tilde{\gamma}$ defined on a strictly larger domain than (a,b).

Definition 13.6 (Cauchy hypersurfaces and global hyperbolicity)**.** *Let $(\mathcal{M},g)$ be a Lorentzian manifold. A spacelike hypersurface $\mathcal{H} \subset \mathcal{M}$ is called a* Cauchy hypersurface *if every inextendible causal curve in $\mathcal{M}$ intersects $\mathcal{H}$ exactly once. A Lorentzian manifold $(\mathcal{M},g)$ is called* globally hyperbolic *if it admits a Cauchy hypersurface.*

When g is a Lorentzian metric, the associated Laplace–Beltrami operator (see Definition 7.31) is a *hyperbolic* operator and hence is denoted $\Box_g$ (and the notation Δ_g is reserved for when g is Riemannian and hence the associated Laplace–Beltrami operator is of *elliptic* type). For example, when g is the Minkowski metric, the associated Laplace–Beltrami operator is the standard wave operator $-\partial_t^2 + \Delta_x$.

The significance of globally hyperbolic manifolds is that solutions of hyperbolic equations, whose symbol is related to that of the wave equation

$$\Box_g \psi = 0,$$

such as the wave equation itself, or the vacuum Einstein equations (25) (recall Exercise 9.25), are uniquely determined by their Cauchy data on a Cauchy hypersurface.

13.4 Closed trapped 2-surfaces and black holes

If $S \subset \mathcal{M}$ is a spacelike 2-surface then, for each $p \in S$, there are two associated future directed null normal directions, spanned by future directed null vectors L_p and $\underline{L}_p$ normalised so that $g(L_p, \underline{L}_p) = -2$.[5] If S is a closed surface then we can designate the vector field $\underline{L}$ on S to be *incoming* and the vector field L on S to be *outgoing*. To each null normal L and $\underline{L}$ one can associate a second fundamental form χ and $\underline{\chi}$ respectively,

$$\chi(X,Y) = g(\nabla_X L, Y), \qquad \underline{\chi}(X,Y) = g(\nabla_X \underline{L}, Y),$$

for all $X, Y \in T_pS$.[6]

Definition 13.7 (Closed trapped surfaces)**.** *A closed spacelike 2-surface S is called a* closed trapped surface *if the traces of the two associated null second fundamental forms are negative,*

$$\mathrm{tr}\chi < 0, \qquad \mathrm{tr}\underline{\chi} < 0,$$

on all of S.

In view of Exercise 10.7, one should view a closed trapped surface as a closed surface whose area decreases when deformed in each of the two null directions. Contrast with the standard spacelike 2-spheres in Minkowski space.

The notion of a closed trapped surface is closely tied to the notion of a *black hole*. The definition of a black hole is more difficult to give than that of a closed trapped surface due to the global aspect of the definition. Informally, points are said to be in a black hole region if their causal future does not ever meet "distant observers". In order to make the definition more precise one needs to make this notion of "distant observers" more precise.

Definition 13.8 (Informal definition of a black hole)**.** *A Lorentzian manifold $(\mathcal{M},g)$ is said to contain a* black hole region *if there is an appropriate asymptotic boundary, called future null infinity and denoted $\mathcal{I}^+$, representing "distant observers", and a point $p \in \mathcal{M}$ such that*

$$J^+(p) \cap \mathcal{I}^+ = \emptyset.$$

[5] Note that these vectors are not unique, as any positive multiples of L_p and $\underline{L}_p$ also describe these directions. Since these vectors are null, and hence have zero length, there is no way to fix them by, for example, insisting that they each have unit length. Any rescaling $L_p \mapsto aL_p$, $\underline{L}_p \mapsto a^{-1}\underline{L}_p$ will preserve the condition that $g(L_p, \underline{L}_p) = -2$, but the choice of L_p and $\underline{L}_p$ are otherwise unique.

[6] Note the difference in sign convention from that of Section 10.

The set of all such p is called the black hole region, denoted $\mathcal{B}$.

Again, note the global aspect of Definition 13.8. In contrast, compare with the definition of a closed trapped surface, which is purely local. One can easily show that, if a Lorentzian manifold $(\mathcal{M}, g)$ contains a closed trapped surface and obeys an appropriate Ricci curvature assumption (for example, if $(\mathcal{M}, g)$ solves the vacuum Einstein equations (25)), then, if $(\mathcal{M}, g)$ admits an appropriate notion of future null infinity $\mathcal{I}^+$, $(\mathcal{M}, g)$ contains a black hole region (in the sense of Definition 13.8) which contains the closed trapped surface.

13.5 The Penrose Incompleteness Theorem

The Penrose theorem can now be stated.

Theorem 13.9 (Penrose, 1965)**.** *Let $(\mathcal{M}, g)$ be a $3+1$ dimensional globally hyperbolic Lorentzian manifold, with a non-compact Cauchy hypersurface, whose Ricci curvature satsifies*

$$Ric(V, V) \geq 0, \tag{26}$$

for all null vectors V. Suppose moreover that $(\mathcal{M}, g)$ contains a closed trapped surface, in the sense of Definition 13.7. Then $(\mathcal{M}, g)$ is future geodesically incomplete.

Note that the assumption (26) is trivially satisfied by solutions of the vacuum Einstein equations (25).

Theorem 13.9 is often called a *singularity* theorem. Note however that the conclusion of the theorem involves geodesic incompleteness (hence the name "incompleteness theorem") rather than any singular behaviour (which one usually associates with something becoming infinite). One of the most remarkable aspects of Theorem 13.9 is that it applies in different cases where the nature of the incompleteness is very different. Two primary examples are discussed in Section 13.6 below. The fact that one can make such a soft conclusion as geodesic incompleteness of solutions of a highly nonlinear system of partial differential equations such as (25) without in fact understanding the nature of that incompleteness is another remarkable aspect of Theorem 13.9.

Theorem 13.9 can be viewed as a Lorentzian analogue of the Bonnet–Myers Theorem, Theorem 12.2. Indeed, following [5], the Bonnet–Myers Theorem can be phrased

$$\begin{array}{c} \text{Completeness,} \\ \text{Lower bound on Ricci curvature,} \end{array} \Rightarrow \text{Compactness.}$$

The Penrose Theorem, stated in contrapositive form, can similarly be phrased,

$$\begin{array}{c} \text{Completeness,} \\ \text{Lower bound on Ricci curvature,} \\ \text{Existence of closed trapped surface,} \end{array} \Rightarrow \text{Compactness of every Cauchy hypersurface.}$$

13.6 The Schwarzschild and Kerr families of black holes

In order to properly understand the significance of Theorem 13.9 it is helpful to view it in the light of some examples of solutions of the vacuum Einstein equations (25).

The most famous family of solutions of (25) is the one parameter *Schwarzschild family.* Discovered in 1915, very shortly after Einstein arrived at the final form of the field equations of general relativity (which reduce to (25) in the absence of matter), the Schwarzschild metrics take the most familiar form, for fixed $M > 0$,

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{27}$$

Compare with the Minkowski metric which, in polar coordinates on $\mathbb{R}^3$, takes the form

$$g = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

There was much confusion surrounding the metric (27) for many years after its discovery, in particular due to its apparently singular nature at $r = 0$ and $r = 2M$. Nowadays, we understand that the (t, r, θ, ϕ) coordinate chart in which (27) is written covers only (appropriate subsets of) the regions $r > 2M$ and $0 < r < 2M$ of the *maximally extended Schwarzschild solution*

$$g_M = -\frac{8M^3}{r} e^{-\frac{r}{2M}} dU dV + r^2 \mathring{\gamma}, \tag{28}$$

defined on the manifold $\mathcal{M} = W \times S^2$, where $W = \{(U, V) \subset \mathbb{R}^2 \mid UV < 1\}$ and $\mathring{\gamma}$ is the unit round metric on S^2. In the expression (28), the function $r = r(U, V)$ is defined implicitly by the relation

$$\left(\frac{r}{2M} - 1\right) \exp\left(\frac{r}{2M}\right) = -UV.$$

The Lorentzian manifold $(\mathcal{M}, g_M)$ is inextendible as a suitably regular Lorentzian manifold, is globally hyperbolic with non-compact Cauchy hypersurface $\{U + V = 0\}$, and each spacelike 2-sphere defined by $U = U_0$, $V = V_0$ is a closed trapped surface if $U_0 > 0$ and $V_0 > 0$. The Penrose Theorem thus applies to $(\mathcal{M}, g_M)$. The nature of the incompleteness is a *curvature singularity* at $r = 0$ (or equivalently at $UV = 1$), as can be seen by observing that $|Rm|_{g_M} \sim \frac{M}{r^3}$ as $r \to 0$. This singular behaviour, in an otherwise perfectly reasonably solution of (25), disturbed people at first. A common opinion was that this singular behaviour was a pathology, resulting from the high degrees of symmetry possessed by the Schwarzschild solution. Given that any small perturbation of the Schwarzschild solution will also possess a closed trapped surface (noting that the conditions $\mathrm{tr}\chi < 0$, $\mathrm{tr}\underline{\chi} < 0$ are open conditions), the Penrose Theorem guarantees that, on the contrary, this apparently pathological behaviour, provided one is willing to weaken "singularity" to "geodesic incompleteness", is a generic feature of solutions of (25).

What could the nature of the geodesic incompleteness predicted by the Penrose Theorem be if not a singularity? The answer to this question is best addressed by way of another example, in the form of the *Kerr family.* For given parameters $|a| \leq M$, the Kerr metric takes the most familiar form

$$g_{a,M} = -\frac{\Delta}{\varrho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\varrho^2}{\Delta}dr^2 + \varrho^2 d\theta^2 + \frac{\sin^2\theta}{\varrho^2}(adt - (r^2 + a^2)d\phi)^2,$$

where

$$\Delta = r^2 - 2Mr + a^2, \qquad \varrho^2 = r^2 + a^2\cos^2\theta.$$

Note that when $a = 0$ this expression reduces to (27). As with the Schwarzschild metric in the form (27), the Kerr metric in the above form arises most naturally in a coordinate chart of a maximally extended globally hyperbolic Lorentzian Kerr manifold. In contrast to the case of Schwarzschild, however, the maximally extended globally hyperbolic Kerr manifold is smoothly extendible as a Lorentzian manifold, provided $a \neq 0$.

The geodesic incompleteness guaranteed by Theorem 13.9 thus has nothing to do in Kerr with any singular behaviour, but rather a breakdown of global hyperbolicity. In view of the discussion in Section 13.3, this means that the incompleteness in Kerr involves a breakdown in deterministic properties of equations such as the vacuum Einstein equations (25). One may view this breakdown as being even worse for a theory than the presence of the type of singular behaviour exhibited by the Schwarzschild solution.

It is conjectured that the situation in Kerr with $a \neq 0$ is not typical, and that the incompleteness guaranteed by the Penrose Theorem is *generically* a result of some form of singularity (such as the singular nature of the incompleteness in Schwarzschild). This conjecture, first formulated by Penrose, goes by the name of the *Strong Cosmic Censorship Conjecture*, and constitutes one of the most important open problems in classical general relativity.

13.7 Sketch proof of the Penrose Theorem

This sketch proof follows [4].

Let S be a closed trapped surface, in the sense of Definition 13.7. Recall the future directed null vector fields L and $\underline{L}$ on S. At each $p \in S$, there is a unique geodesic maximal geodesic $\gamma_p \colon (T_-(p), T_+(p)) \to \mathcal{M}$ such that $\gamma_p(0) = p$, $\dot\gamma_p(0) = L_p$. Similarly, there is a unique geodesic maximal geodesic $\underline{\gamma}_p \colon (\underline{T}_-(p), \underline{T}_+(p)) \to \mathcal{M}$ such that $\underline{\gamma}_p(0) = p$, $\dot{\underline{\gamma}}_p(0) = L_p$. For fixed $s \geq 0$, define the sets

$$S_s = \{\gamma_p(s) \mid p \in S\}, \qquad \underline{S}_s = \{\underline{\gamma}_p(s) \mid p \in S\}.$$

Define also the sets

$$C(s) = \{\gamma_p(s') \mid p \in S, 0 \leq s' \leq s\}, \qquad \underline{C}(s) = \{\underline{\gamma}_p(s') \mid p \in S, 0 \leq s' \leq s\},$$

and

$$C = \{\gamma_p(s) \mid p \in S, 0 \leq s \leq T_+(p)\}, \qquad \underline{C} = \{\underline{\gamma}_p(s) \mid p \in S, 0 \leq s \leq \underline{T}_+(p)\},$$

Lemma 13.10 (The boundary of the causal future of a spacelike 2-sphere)**.** *If s is sufficiently small then S_s and $\underline{S}_s$ are spacelike 2-spheres and the geodesic congruences $C(s)$ and $\underline{C}(s)$ are null hypersurfaces (with boundary). Moreover,*

$$\partial J^+(S) \subset C \cup \underline{C}. \tag{29}$$

Note that the reverse inclusion of (29) is not true in general. Consider, for example, a standard 2-sphere in Minkowski space.

The curves γ_p and $\underline{\gamma}_p$ are called the *null generators* of C and $\underline{C}$ respectively. If s is such that $C(s)$ and $\underline{C}(s)$ are smooth null hypersurfaces, then the vector fields L and $\underline{L}$ extend to vector fields along $C(s)$ and $\underline{C}(s)$ respectively by

$$L_{\gamma_p(s)} = \dot\gamma_p(s), \qquad \underline{L}_{\underline{\gamma}_p(s)} = \dot{\underline{\gamma}}_p(s),$$

and therefore χ and $\underline{\chi}$ extend to $(0,2)$ tensor fields on C and $\underline{C}$ respectively. Note that

$$\nabla_L L = 0 \text{ on } C, \qquad \nabla_{\underline{L}} \underline{L} = 0 \text{ on } \underline{C}.$$

Moreover there exists a unique extension of $\underline{L}$ to a future directed null vector field $\underline{L}$ on $C(s)$, normal to the spheres S_s, satisfying $g(L, \underline{L}) = -2$, and a unique extension of L to a future directed null vector field L on $\underline{C}(s)$, normal to the spheres $\underline{S}_s$, satisfying $g(L, \underline{L}) = -2$.

The proof of Theorem 13.9 is based on understanding behaviour of certain Jacobi fields along the geodesics γ_p and $\underline{\gamma}_p$. As opposed to conjugate points, which, for example, feature in the proof of Theorem 12.5, it is slightly more convenient to introduce a related notion of *focal points*.[7]

Definition 13.11 (Focal points)**.** *Given $p \in S$, a point $q = \gamma_p(s_*)$ for some $s_* \in (0, T_+(p))$ is called a* focal point *to p if there exists a nontrivial (non-identically vanishing) normal vector field $J \in \mathfrak{X}(\gamma_p)$ along γ_p such that $[L(s), J(s)] = 0$ for all $s \in [0, T_+(p))$ and $J(s_*) = 0$. Similarly, a point $q = \underline{\gamma}_p(s_*)$ for some $s_* \in (0, \underline{T}_+(p))$ is called a* focal point *to p if there exists a nontrivial normal vector field $\underline{J} \in \mathfrak{X}(\underline{\gamma}_p)$ along $\underline{\gamma}_p$ such that $[\underline{L}(s), \underline{J}(s)] = 0$ for all $s \in [0, \underline{T}_+(p))$ and $\underline{J}(s_*) = 0$.*

Note that such a J is a Jacobi field along γ_p. Indeed, since $\nabla_L J - \nabla_J L = [L, J] = 0$,

$$\nabla_L \nabla_L J = \nabla_L \nabla_J L = R(L, J)L,$$

since $\nabla_L L = 0$. Similarly for $\underline{J}$.

Compare the following lemma to Theorem 11.12.

Lemma 13.12 (Null generators beyond focal points)**.** *Consider $p \in S$ and let $s_* > 0$ be such that $\gamma_p(s_*)$ is a focal point to p along γ_p. Then $\gamma_p(s_*) \in I^+(S)$. Similarly, if $s_* > 0$ is such that $\underline{\gamma}_p(s_*)$ is a focal point to p along $\underline{\gamma}_p$. Then $\underline{\gamma}_p(s_*) \in I^+(S)$.*

[7]The distinction between conjugate points and focal points made here is non-standard. The terms are often used interchangeably to mean either of the two notions introduced here.

Proof. See Proposition 2 of [4]. $\square$

As a consequence of Lemma 13.12, it in particular follows that, if $s_* > 0$ is such that γ_p and $\underline{\gamma}_p$ each contain a focal point before time s_*, for all $p \in S$, then (29) can be improved to

$$\partial J^+(S) \subset C(s_*) \cup \underline{C}(s_*). \tag{30}$$

Let $\not{g}$ denote the induced (Riemannian) metric on the spacelike 2-spheres S_s and $\underline{S}_s$, and define

$$\hat{\chi} = \chi - \frac{1}{2}\mathrm{tr}\chi \not{g}, \qquad \underline{\hat{\chi}} = \underline{\chi} - \frac{1}{2}\mathrm{tr}\underline{\chi} \not{g},$$

to be the trace free parts of the second fundamental forms χ and $\underline{\chi}$ respectively.

Lemma 13.13 (Null mean curvatures satisfy Raychaudhuri equaions)**.** *The quantities* $\mathrm{tr}\chi$ *and* $\mathrm{tr}\underline{\chi}$ *satisfy the* Raychaudhuri equations

$$L(\mathrm{tr}\chi) = -\frac{1}{2}(\mathrm{tr}\chi)^2 - |\hat{\chi}|^2_{\not{g}} - Ric(L,L), \qquad \underline{L}(\mathrm{tr}\underline{\chi}) = -\frac{1}{2}(\mathrm{tr}\underline{\chi})^2 - |\underline{\hat{\chi}}|^2_{\not{g}} - Ric(\underline{L},\underline{L}).$$

Proof. Consider some $p \in S$ and a local frame e_1, e_2 for S around p. For convenience, extend e_1, e_2 along γ_p by solving $[L, e_A] = 0$ for $A = 1, 2$. One then computes

$$L(\chi(e_A, e_B)) = g(\nabla_L \nabla_{e_A} L, e_B) + g(\nabla_{e_A} L, \nabla_L e_B) = g(\nabla_{e_A} \nabla_L L, e_B) + g(\nabla_{e_A} L, \nabla_L e_B) + Rm(L, e_A, L, e_B),$$

and so, since $\nabla_L L = 0$ and $\nabla_{e_A} L = \chi_A{}^C e_C$,

$$\begin{aligned}&(\nabla_L \chi)(e_A, e_B) = L(\chi(e_A, e_B)) - \chi(\nabla_L e_A, e_B) - \chi(e_A, \nabla_L e_B)\\ &= g(\nabla_{e_A} L, \nabla_L e_B) - Rm(e_A, L, L, e_B) - g(\nabla_{\nabla_L e_A} L, e_B) - g(\nabla_{e_A} L, \nabla_L e_B) = -\chi_A{}^C \chi_{CB} - Rm(e_A, L, L, e_B).\end{aligned}$$

The first equation then follows by using the fact that g is compatible with ∇ and so

$$L(\mathrm{tr}\chi) = g^{AB}(\nabla_L \chi)(e_A, e_B) = -\chi^{AB}\chi_{AB} - g^{AB} Rm(e_A, L, L, e_B) = -\frac{1}{2}(\mathrm{tr}\chi)^2 - |\hat{\chi}|^2_{\not{g}} - Ric(L, L).$$

The second equation follows similarly. $\square$

Lemma 13.13 and the curvature assumption in Theorem 13.9 imply that

$$L(\mathrm{tr}\chi) \leq -\frac{1}{2}(\mathrm{tr}\chi)^2. \tag{31}$$

In particular, $\mathrm{tr}\chi$ satisfies the remarkable monotonicity property

$$L(\mathrm{tr}\chi) \leq 0.$$

Recall that one views the assumption that $\mathrm{tr}\chi < 0$ on S as meaning that the area of S can only decrease when *infinitesimally* deformed in the direction of L. This monotonicity property, together with the fact that S is a closed trapped surface, in particular implies that the area must continue to decrease as one deforms further in the L direction.

The main analytic content of the proof of Theorem 13.9 is contained in the following proposition, which is an easy application of the quantitative form (31) of the monotonicity property of $\mathrm{tr}\chi$.

Proposition 13.14 (Occurrence of focal points)**.** *Under the assumptions of Theorem 13.9, if, for all* $p \in S$, $T_+(p)$ *and* $\underline{T}_+(p)$ *are sufficiently large then, for all* $p \in S$, *the null generators* γ_p *and* $\underline{\gamma}_p$ *both contain a focal point to* p.

Proof. Consider first the null generators γ_p. Since S is compact, there exists $k > 0$ such that

$$\sup_{p \in S} \mathrm{tr}\chi(p) \leq -k.$$

The inequality (31) implies that

$$L(-(\mathrm{tr}\chi)^{-1}) \leq -\frac{1}{2},$$

and so, integrating along the integral curves of L,

$$-(\mathrm{tr}\chi(\gamma_p(s)))^{-1} \leq \frac{1}{k} - \frac{s}{2},$$

i.e.,

$$\mathrm{tr}\chi(\gamma_p(s)) \leq \left(\frac{s}{2} - \frac{1}{k}\right)^{-1}, \tag{32}$$

for all $s \geq 0$. It follows that $\mathrm{tr}\chi(\gamma_p(s)) \to -\infty$ before time $s = \frac{2}{k}$. The inequality (32) forms the main content of the proof. Some extra work is required to check that this indeed leads to the existence of a focal point.

Consider some $p \in S$. Let e_1, e_2 be an orthonormal frame for T_pS and extend to an orthonormal frame for $T_{\gamma_p(s)}S_s$ for all s by solving

$$\nabla_L e_A = -\frac{1}{2} g(\nabla_{e_A} L, \underline{L})L, \qquad A = 1, 2.$$

This choice ensures that $L(g(e_A, K)) = 0$ for $K = e_B, L, \underline{L}$, so that e_1, e_2 is an orthonormal frame for each $T_{\gamma_p(s)}S_s$. Consider now an arbitrary vector $v \in T_pS$. Define a vector field J along γ_p by solving $[L, J] = 0$, $J(0) = v$. One can check that

$$L(g(J, L)) = L(g(J, \underline{L})) = 0,$$

and so $J(s) \in T_{\gamma_p(s)}S_s$ for all s. By the linearity of the equation $[L, J] = 0$ in J over $\mathbb{R}$ we can write $J^A(s) = M^A{}_B(s)v^B$ for some matrix $M(s)$ (independent of v). Clearly $M(0) = Id$ and so $\det M(0) = 1$ and $\det M(s) \neq 0$ for small s. If there exists s such that $\det M(s) = 0$ then there exists $v \in T_p\mathcal{M}$ with $v \neq 0$ such that $J(s) = 0$, i.e. to find a focal point it suffices to find a zero of $\det M$. Now

$$L(J^A) = L(g(J, e_A)) = g(\nabla_L J, e_A) + g(J, \nabla_L e_A) = g(\nabla_J L, e_A) - g(\nabla_{e_A} L, \underline{L})g(L, e_A) = \chi(J, e_A),$$

and so

$$L(M^A{}_B(s))v^B = \chi(J(s), e_A) = \chi(e_A, e_C)M^C{}_B v^B.$$

Since v was arbitrary, it follows that

$$L(M^A{}_B(s)) = \chi(e_A, e_C)M^C{}_B.$$

Using the well known identity

$$\frac{d \det M}{ds} = \det M \,\mathrm{tr}\Big(M^{-1}\frac{dM}{ds}\Big),$$

it follows that

$$\frac{d \log \det M}{ds} = \mathrm{tr}\chi.$$

The inequality (32), together with the fact that $\log \det M(0) = 0$, then implies that

$$\det M(s) \leq \left(\frac{s}{2} - \frac{1}{k}\right)^2,$$

i.e. there exists a focal point along γ_p by time $s = \frac{2}{k}$.

The proof for the null generators $\underline{\gamma}_p$ is identical, using now the Raychaudhuri equation for $\mathrm{tr}\underline{\chi}$. $\square$

A set $K \subset \mathcal{M}$ is called a *future set* if $p \in K$ implies that $I^+(p) \subset K$. A set $A \subset \mathcal{M}$ is called *achronal* if there is no pair $p, q \in A$ such that $q \in I^+(q)$.

Lemma 13.15 (Topological boundary of a future set is achronal)**.** *If $K \subset \mathcal{M}$ is a future set, then its topological boundary $\partial K \subset \mathcal{M}$ is a closed achronal three dimensional embedded Lipschitz submanifold (without boundary).*

Proof. See Proposition 6.3.1 of [8]. □

The remaining ingredients of the proof of Theorem 13.9 consist of the following topological facts.

Proposition 13.16 (Topological facts)**.**

1. *If A is a compact topological space, B is a Hausdorff topological space and $h \colon A \to B$ is a continuous bijection, then h is a homeomorphism.*
2. *If $\mathcal{M}$ is a (connected) topological manifold and $\mathcal{N} \subset \mathcal{M}$ is a compact submanifold (without boundary) such that* $\mathrm{dim}\mathcal{N} = \mathrm{dim}\mathcal{M}$*, then $\mathcal{M}$ is compact.*

Proof. See Section 5.5 of [1]. □

The proof of Theorem 13.9 can now be completed.

Proof of Theorem 13.9. Suppose, for the sake of contradiction, that $(\mathcal{M}, g)$ is future geodesically complete. The null generators γ_p and $\underline{\gamma}_p$ are therefore future complete (i.e. $T_+(p) = \underline{T}_+(p) = \infty$ for all $p \in S$). By Proposition 13.14 there therefore exists $s_* > 0$ such that, for all $p \in S$, the null generators γ_p and $\underline{\gamma}_p$ contain a focal point to p by time s_*. Lemma 13.12 then implies (see (30)) that

$$\partial J^+(S) \subset C(s_*) \cup \underline{C}(s_*).$$

Since $C(s_*) \cup \underline{C}(s_*)$ is compact and $\partial J^+(S)$ is closed (the topological boundary of a set always being closed), it follows that $\partial J^+(S)$ is compact.

Note that $J^+(S)$ is a future set, and hence $\partial J^+(S)$ is achronal by Lemma 13.15. Since $(\mathcal{M}, g)$ is time oriented there exists a globally timelike vector field T. By assumption $(\mathcal{M}, g)$ admits a non-compact Cauchy hypersurface $\mathcal{H}$. For each $q \in \partial J^+(S)$ the integral curve of T through q intersects $\mathcal{H}$ exactly once and, since $\partial J^+(S)$ is achronal, does not intersect $\partial J^+(S)$ again. The map $F \colon \partial J^+(S) \to \mathcal{H}$, where $F(q)$ is defined to be the unique point where the integral curve of T intersects $\mathcal{H}$, is therefore a continuous injection. By Proposition 13.16, $\partial J^+(S)$ is homeomorphic to $F(\partial J^+(S))$. Since $\partial J^+(S)$ is a compact three dimensional submanifold of $\mathcal{M}$ (see Lemma 13.15) it follows that $F(\partial J^+(S))$ is a compact three dimensional topological submanifold of the Cauchy hypersurface $\mathcal{H}$. Proposition 13.16 then implies that $\mathcal{H}$ is compact, which contradicts the assumption that $\mathcal{H}$ is non-compact. Hence $(\mathcal{M}, g)$ is geodesically incomplete. □

References

[1] Aretakis, S., *Lecture notes on general relativity, Columbia University.*

[2] Berger, M., *A panoramic view of Riemannian geometry.*

[3] Boothby, W., *An introduction to differentiable manifolds and Riemannian geometry.*

[4] Christodoulou, D., *Mathematical problems in general relativity II.*

[5] Dafermos, M., *Part III Differential geometry lecture notes.*

[6] Do Carmo, M., *Riemannian geometry.*

[7] Gallot, S., Hulin, D., Lafontaine, J., *Riemannian geometry.*

[8] Hawking, S., Ellis, G., *The large scale structure of space-time.*

[9] Jost, J., *Riemannian geometry and geometric analysis.*

[10] Lee, J., *Riemannian manifolds.*

[11] Lee, J., *Introduction to smooth manifolds.*

[12] O'Neill, B., *Semi–Riemannian geometry with applications to relativity.*

[13] Petersen, P., *Riemannian geometry.*

[14] Spivak, M., *A comprehensive introduction to differential geometry, volumes 1–5.*