Hibernation prevents chaos: A logistic case study

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Abstract

We investigate how chaotic time series generated as orbits of the logistic map can be regularized by assuming continuous growth over short time intervals after fixed instants. In doing so, we obtain a Feigenbaum-like scenario which visualizes a period bisection transition away from chaos to asymptotic stability and finally ODE-like behavior. Several well-known results on population models and unimodal maps with negative Schwarzian are applied.

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1 Introduction

Suppose we are given a real-valued time series $(x_n)_{n\in\mathbb{N}_0}$ which might serve as a model for the size or biomass of a population. We assume in our scenario that this time series behaves "irregularly" in the sense that it is non-monotone or even

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forms a dense set in some compact interval, which is fulfilled, e.g., for iterates of a chaotic mapping.

On the other hand from a modeling perspective, under idealized conditions, i.e., in a lab, without influence of a fluctuating environment or other interacting species, and neglecting spatial effects, the growth of a single population usually allows a quite successful description using scalar first-order autonomous ordinary differential equations (ODEs)

$$\dot{x} = f(x)\,. \tag{1.1}$$

Their dynamical behavior, however, is trivial since all solutions are monotone yielding single point limit sets. Hence, an "irregular" sequence $(x_n)_{n\in\mathbb{N}_0}$ as above can never be obtained from the time-1-map of a solution ϕ to (1.1) via $\phi(n) = x_n$ for discrete times $n \in \mathbb{N}_0$.

This motivates the basic question of this paper: Is it possible to "regularize" the growth of a population whose size is given by the time series $(x_n)_{n\in\mathbb{N}_0}$ as follows: At the discrete times n, we isolate the population from its environment and consequently realize its continuous growth following (1.1) over a time interval of fixed length $h \in [0, 1]$. Clearly, for $h = 0$, we did not interfere, but is there a (minimal) length $h > 0$ such that the population growth is regular, e.g., monotone, convergent or asymptotically periodic? Due to the isolation of a species from its environment, we interpret the behavior over $[n, n + h]$, $n \in \mathbb{N}_0$, as *hibernation*.

Indeed, through the course of the paper, we retreat to the following simplified situation: An archetypical example of complex and chaotic behavior in form of topological transitivity, dense orbits and sensitive dependence on initial conditions is given by the dynamics of the well-known logistic map or difference equation (cf., e.g., [HK01, Ela00])

$$x_{n+1} = 4x_n(1 - x_n)\,, \tag{1.2}$$

whose right-hand side is an unimodal mapping on the compact interval $[0, 1]$. With a biological (or ecological) motivation, this problem has been studied in the survey paper [May76]. Thus, we use forward orbits generated by the logistic difference equation as our irregular sequence $(x_n)_{n\in\mathbb{N}_0}$.

On the other hand, (1.2) can be considered as forward Euler-discretization of

$$\dot{x} = 3x - 4x^2 \tag{1.3}$$

with step-size 1. As an autonomous and scalar ODE, the dynamics of (1.3) is trivial, and using separation of variables, we obtain for the solution flow

$$\phi(t,x_0) = \frac{3e^{3t}x_0}{3+4(e^{3t}-1)x_0}\,. \tag{1.4}$$

In order to associate continuous growth of (1.3) to the sequence $(x_n)_{n\in\mathbb{N}_0}$ generated by (1.2), we proceed as follows: Given an initial point x_0, we suppose that our population grows continuously following (1.3) over the interval $[0,h]$ with $h \in [0,1)$. Then we switch to discrete behavior and replace the ODE (1.3) by its forward Euler discretization with step-size $1-h$, so that the size of the population at time 1 is given by

$$x_1(h) = \phi(h,x_0) + (1-h)\phi(h,x_0)(3-4\phi(h,x_0))\,.$$

Thereafter, the growth starting at $x_1(h)$ is assumed to be continuous on the interval $[1,1+h]$, and we successively iterate this dynamical process to obtain a sequence $(x_n(h))_{n\in\mathbb{N}_0}$ depending on h. It is our goal to study the asymptotic behavior of this sequence under variation of h.

In the ensuing section, we discuss connections of our approach to dynamic equations on time scales and introduce the notion of a *time scale homotopy*. In Section 3, basic properties of the family generating by the above dynamical process are treated, and in Sections 4 and 5, we analyze both the dynamical and bifurcation situation, respectively. The proofs in Section 4 are an application of various tools for one-dimensional mappings which can be found in the textbooks [Ela00, HK01, Sed03, Wig90] to a certain extend, but also in more advanced references like [BC92, Guc79, MS93, MT88, Sin78]. To provide some further context, we refer to [Thu01] for a well-written survey on the topic of unimodal mappings.

We can summarize our results: The behavior of the above dynamical process can be classified into three regimes. In the *unimodal regime* for values of h close to 0, one observes irregular dynamical behavior known from mappings including dense and periodic orbits. On the other side, in the *monotone regime* for h near 1, the behavior resembles that of scalar autonomous ODEs, in the sense that solutions are monotone or might not exist on the whole axis. Ultimately, in the intermediate regime one still has convergence to a fixed point — yet, it might not be monotone. In particular, we are able to obtain precise values for the boundaries of these three parameter regimes distinguished by characteristic dynamics.

To conclude this introduction, we remark that a somewhat inverse situation to our approach has been considered in [YM79]. Here, it is shown that Eulerdiscretizations of ODEs (1.1) are chaotic in the sense of [LY75], provided the step-size is sufficiently large and a, for instance, unimodal right-hand side f.

2 Relation to time scale dynamics

The above scenario possesses discrete as well as continuous features. Obviously, so-called dynamic equations on time scales (cf. [Hil90, BP01]) are predestinated to describe such situations of hybrid dynamics. As key observation, we note that both the discrete logistic equation (1.2), as well as the continuous ODE (1.3) are special cases of the general dynamic equation

$$x^\Delta = 3x - 4x^2 \tag{2.1}$$

for the respective time scales $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$. Its solution satisfying the initial condition $x(\tau) = \xi$ will be denoted by $\varphi(\cdot, \tau, \xi)$.

Keeping this in mind, the analysis in the paper can be rephrased as follows: We study a homotopy between the two above time scales $\mathbb{Z}$ and $\mathbb{R}$, which is given by the family of time scales

$$\mathbb{T}_h := \bigcup_{k\in\mathbb{Z}} [k, k+h] \quad \text{for all } h \in [0,1]\,,$$

and we discuss the dynamical behavior of (2.1) on this family. Here, $h \in [0,1]$ serves as a bifurcation parameter. Obviously, $\mathbb{T}_h$ exhibits both discrete and continuous behavior for $h \in (0,1)$. With this underlying time scale, the general solution φ of (2.1) satisfies the relation

$$\varphi(n+1, n, \xi) = \phi(h,\xi) + (1-h)\phi(h,\xi)[3 - 4\phi(h,\xi)] =: F_h(\xi)$$

for all $n \in \mathbb{N}_0$, and consequently $\varphi(n, 0, \xi) = F_h^n(\xi)$, where the iterates F_h^n are defined recursively by as composition

$$F_h^0(x) := x, \qquad F_h^{n+1} := F_h \circ F_h^n \quad \text{for all } n \in \mathbb{N}_0\,.$$

Hence, instead of dealing with solutions to dynamic equations as (2.1), we can restrict to the analysis of a one-dimensional discrete dynamical system $(F_h^n)_{n\in\mathbb{N}_0}$.

Nonetheless, the investigation of dynamic equations under variation of the time scale is an interesting topic extending the classical approach of generalizing ODE or difference equation results to time scales. We hope that this short article provides a small contribution and stimulates further research into this direction.

3 Basic Properties

Throughout the remaining paper, we reduce our stability and bifurcation analysis for the hybrid dynamical process described above to an investigation of the corresponding time-1-map. It is given by a one-dimensional map $F_h : [0, \infty) \to \mathbb{R}$, or equivalently the scalar autonomous difference equation

$$x_{n+1} = F_h(x_n)\,, \tag{3.1}$$

with right-hand side

$$F_h(\xi) := \phi(h,\xi) + (1-h)\phi(h,\xi)[3 - 4\phi(h,\xi)], \qquad \phi(h,\xi) = \frac{3e^{3h}\xi}{3 + 4(e^{3h}-1)\xi},$$

where $h \in [0, 1]$ serves as a parameter. Above all, we summarize some basic properties of analytic function F_h:

- First, it is easy to derive the limit relation

  $$y_\infty(h) := \lim_{x\to\infty} F_h(x) = \frac{3e^{3h}(e^{3h} + 3h - 4)}{4(e^{3h}-1)^2} \quad \text{for all } h \in (0, 1]\,;$$

  note also that $\lim_{x\to\infty} F_0(x) = -\infty$. The function $y_\infty : (0, 1] \to \mathbb{R}$ is strictly increasing with the unique zero $h_1 = \frac{1}{3}(4 - W(e^4)) \approx 0.36$ (see Figure 1 (left)). Here and in the following, $W : [-e^{-1}, \infty) \to [-1, \infty)$ denotes the *Lambert W*

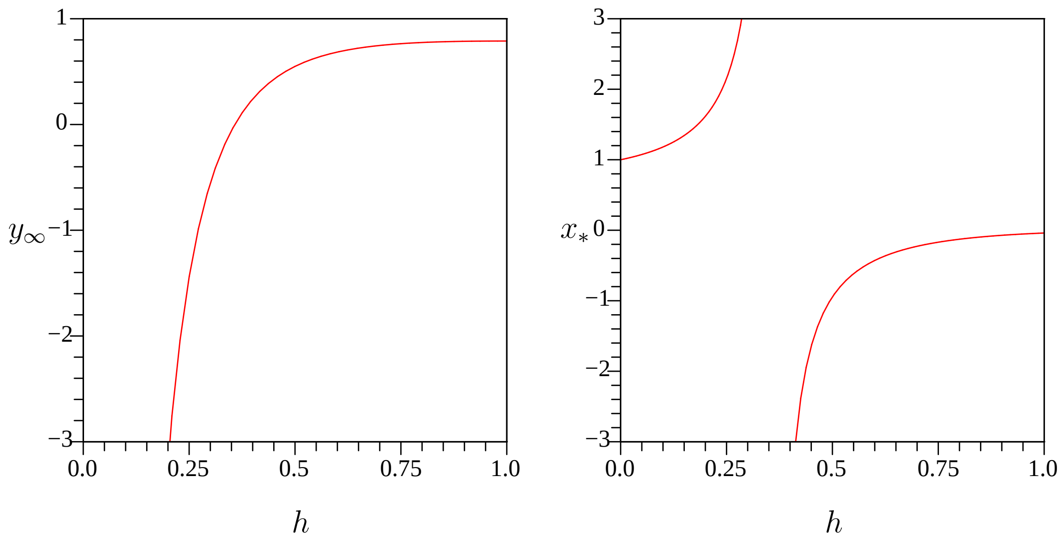

Figure 1: Graph of the functions y_∞ (left) and x_* (right)

  function, which is the inverse of the function $x \mapsto xe^x$. It is strictly increasing and for further properties, as well as applications of W, we refer to [CG+96].

- The nonlinear equation $F_h(x) = 0$ has the trivial solution $x = 0$ independently of $h \in [0,1]$, and it additionally possesses a positive solution
$$x_*(h) := \frac{3(3h-4)}{4(e^{3h}+3h-4)} \quad \text{for all } h < h_1.$$
Moreover, $x_* : [0,h_1) \to [1,\infty)$ is strictly increasing (see Figure 1 (right)).

- The function F_h attains a unique positive maximum given by
$$y_{\max}(h) := \frac{(3h-4)^2}{16(1-h)} \quad \text{for all } h \in [0,h_2)$$
at the critical point
$$x_{\max}(h) := \frac{3}{4}\frac{3h-4}{3h-4+e^{3h}(3h-2)}, \tag{3.2}$$
where $h_2 \approx 0.73$ denotes the unique solution of the transcendental equation $e^{3h} = \frac{4-3h}{3h-2}$ in $[0,1]$. The function $x_{\max} : [0,1] \to \mathbb{R}$ has a pole at h_2, which is the unique minimum of $y_{\max} : [0,1) \to [0,\infty)$ (see Figure 2). In addition, we

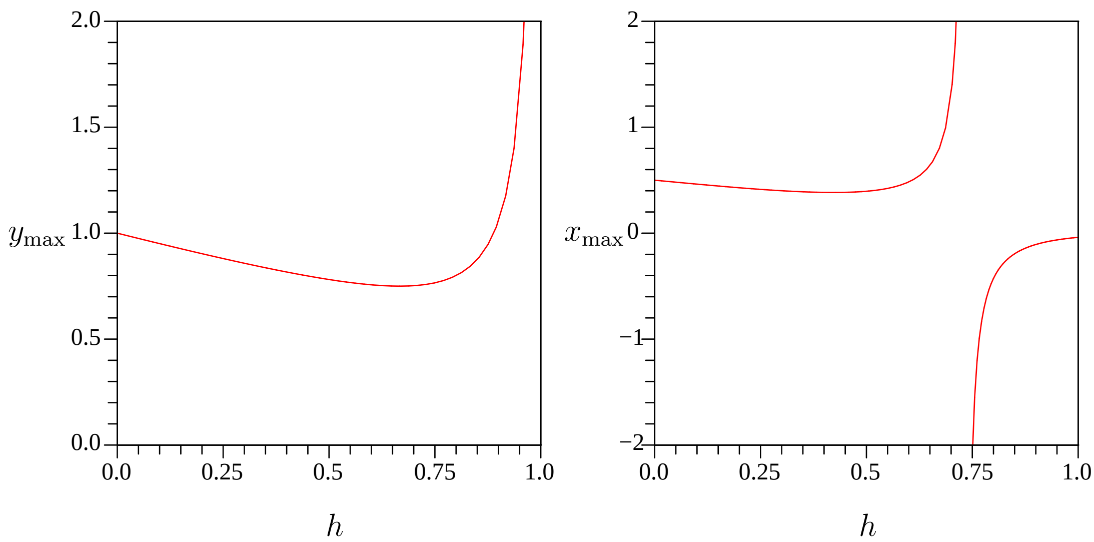

Figure 2: Graph of the functions $y_{\max}$ (left) and $x_{\max}$ (right)

compute for the second derivative
$$F_h''(x_{\max}(h)) = \frac{1}{162}\frac{\mathrm{e}^{-6h}\left(3h-4+e^{3h}(3h-2)\right)^4}{(h-1)^3}, \tag{3.3}$$
and this expression is clearly nonzero for all $h \in [0,h_2)$. Therefore, the critical point $x_{\max}(h)$ of F_h is *non-flat* (cf. [Thu01]).

Further simple calculations culminate in the subsequent theorem, whose elementary proof is omitted (see Figure 3).

Theorem 3.1 (zeros of F_h)**.** *The function $F_h : [0, \infty) \to \mathbb{R}$ has the following properties:*

1. *For $h = 0$, the function F_0 has a unique positive zero $x_*(0) = 1$ and achieves its unique maximum $y_{\max}(0) = 1$ at $x_{\max}(0) = \frac{1}{2}$.*
2. *For $h \in (0, h_1)$, the function F_h has a unique positive zero $x_*(h) > 1$, achieves its unique maximum $y_{\max}(h) < 1$ at $x_{\max}(h) \in (0, x_*(h))$ and converges to the value $y_\infty(h) < 0$ as $x \to \infty$.*
3. *For $h = h_1$, the function F_h has no positive zero, achieves its unique maximum $y_{\max}(h_1)$ at $x_{\max}(h_1)$ and converges to 0 as $x \to \infty$.*
4. *For $h \in (h_1, h_2)$, the function F_h has no positive zero, achieves its unique maximum $y_{\max}(h) \in (\frac{3}{4}, 1)$ at $x_{\max}(h) \leq \frac{1}{2}$ and converges to $y_\infty(h) > 0$ as $x \to \infty$.*
5. *For $h \in [h_2, 1]$, the function F_h is strictly increasing and converges to $y_\infty(h) > 0$ as $x \to \infty$.*

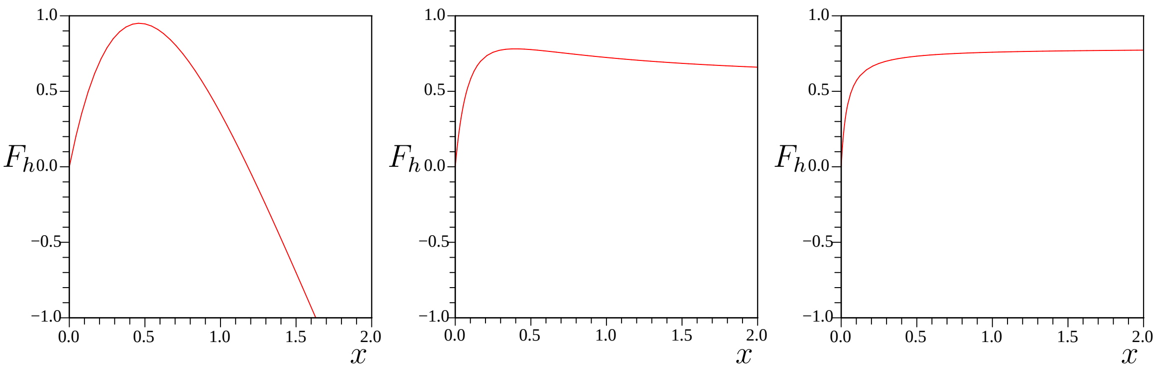

Figure 3: Representative graph of the function F_h for $h \in [0, h_1)$ (left), $h \in [h_1, h_2)$ (middle) and $h \in [h_2, 1]$ (right)

In the subsequent sections, we have to restrict our map $F_h : [0, \infty) \to \mathbb{R}$ to certain intervals $I_h \subseteq [0, \infty)$ which are *forward invariant*, i.e., $F_h(I_h) \subseteq I_h$. Hence, for such an interval I_h, also the restriction $F_h : I_h \to I_h$ is well-defined.

The above Theorem 3.1 implies that, if $h \in [0, h_1)$, then we have

$$F_h([0, x_*(h)]) \subseteq [0, 1] \subseteq [0, x_*(h)],$$

and this means that the compact interval $[0, x_*(h)]$ is forward invariant. Moreover, for $h \in [h_1, 1]$, the unbounded interval $[0, \infty)$ is invariant, and we define

$$I_h := \begin{cases} [0, x_*(h)] & \text{for } h \in [0, h_1)\,, \\ [0, \infty) & \text{for } h \in [h_1, 1]\,. \end{cases}$$

Obviously, the unit interval $[0, 1]$ is also invariant for $h \in [0, h_1)$, but we have chosen I_h as above, since then F_h is unimodal for $h \in [0, h_1)$. Recall that the definition of a unimodal function is given as follows (cf. also, e.g., [MS93, p. 92]):

Definition 3.2 (unimodal function)**.** A function $f : I \to I$ on an interval $I \subseteq \mathbb{R}$ is called *unimodal*, if it is piecewise monotone with a unique maximum at an interior point and $f(\partial I) \subseteq \partial I$.

Corollary 3.3. *For $h \in [0, h_1)$, the mapping $F_h : I_h \to I_h$ is unimodal.*

Proof. This follows from Theorem 3.1. □

As easily seen, the function $F_h : I_h \to I_h$ has two fixed points, and generically their stability can be determined using the derivative F_h'. Nevertheless, in the critical nonhyperbolic case where F_h' has absolute value 1, we make use of the *Schwarzian derivative* of F_h defined by

$$SF_h(x) := \frac{F_h'''(x)}{F_h'(x)} - \frac{3}{2}\left(\frac{F_h''(x)}{F_h'(x)}\right)^2 .$$

The Schwarzian derivative of the function F_h turns out to be negative, i.e., F_h is a so-called *S-unimodal mapping*, which is defined to be a unimodal mapping with negative Schwarzian derivative.

Lemma 3.4. *For all $h \in [0, h_1)$ and $x \in [0, x_*(h))$, we have $SF_h(x) < 0$.*

Proof. The condition $SF_h(x) < 0$ is equivalent to $3F_h''(x)^2 - 2F_h'(x)F_h'''(x) > 0$, which in turn allows the representation

$$\begin{aligned}
&3F_h''(x)^2 - 2F_h'(x)F_h'''(x) \\
=&\frac{1296(e^{3h})^2}{\left(3 + 4xe^{3h} - 4x\right)^8} \cdot \big[- 12 + 16x + 9h + 21e^{3h} - 12hx - 8xe^{3h} - 8x(e^{3h})^2 \\
&+ 12h(e^{3h})^2x - 18he^{3h}\big] \cdot \big[- 108 + 96x + 81h + 252e^{3h} - 72hx - 72xe^{3h} \\
&- 96x(e^{3h})^2 + 144h(e^{3h})^2x - 216he^{3h} \\
&- 32x^2e^{3h} + 64x^2 - 48hx^2 - 32x^2(e^{3h})^2 + 48hx^2\left(e^{3h}\right)^2\big]
\end{aligned}$$

$$=\frac{1296(e^{3h})^2}{\big(3+4xe^{3h}-4x\big)^8}f_1(x,h)f_2(x,h)$$

with the factors

$$\begin{aligned}f_1(x,h) =& -12+9h+21e^{3h}-18he^{3h}+4[4-3h-2e^{3h}-2(e^{3h})^2+3h(e^{3h})^2]x,\\ f_2(x,h) =& -108+81h+252e^{3h}-216he^{3h}\\ &+24[4-3h-3e^{3h}-4(e^{3h})^2+6h(e^{3h})^2]x\\ &+16[4-2e^{3h}-3h-2(e^{3h})^2+4h(e^{3h})^2]x^2\,.\end{aligned}$$

One can check that f_1 and f_2 are positive functions for the prescribed values of x and the parameter h, which implies the assertion. □

The behavior of the functions F_h allows the following dichotomy: For parameters $h \in [0, h_1)$, it is a unimodal mapping on the compact interval $I_h = [0, x_*(h)]$, and we speak of the *unimodal regime*. However, for $h \in [h_1, 1]$, the functions $F_h : [0, \infty) \to [0, \infty)$ fall into the class of population models, and the parameter range $[h_1, 1]$ is denoted as *population model regime*. The notion of a population model itself is due to [Cul81]:

Definition 3.5 (population model)**.** A function $f : [0, \infty) \to [0, \infty)$ is called *population model*, if $f(0) = 0$, there exists a unique positive fixed point x^* of f with

$$f(x) > x \quad \text{for all } x \in (0, x^*), \quad f(x) < x \quad \text{for all } x \in (x^*, \infty)$$

and if f has a maximum $x_{\max} \in (0, x^*)$, then f decreases monotonously on (x^*, ∞).

Corollary 3.6. *For $h \in [h_1, 1]$, the mapping $F_h : I_h \to I_h$ is a population model.*

Proof. This again results from Theorem 3.1. □

The population model regime has a subset $[h_2, 1]$ (with h_2 defined as above) for which F_h is strictly increasing from 0 to $y_\infty(h) < \infty$; in this situation, we speak of the *monotone regime*. The above Figure 3 illustrates three representative graphs of the function F_h in these regimes.

4 Dynamics

For the reader's convenience, before studying dynamical properties of the autonomous difference equation (3.1), we summarize various notions from the theory of scalar C^1-mappings $f : I \to I$ defined on intervals $I \subseteq \mathbb{R}$.

A subset $\Omega \subseteq I$ is called *invariant*, if $f(\Omega) = \Omega$. An example of an invariant set is the *ω-limit set* of a point $x \in I$ given by

$$\omega_f(x) := \bigcap_{n\in\mathbb{N}} \overline{\{f^k(x) \in I : k > n\}}.$$

For an invariant set Ω, we can define its *domain* (or *basin*) *of attraction*

$$A_f(\Omega) := \{x \in I : \omega_f(x) \subseteq \Omega\}.$$

With given $p \in \mathbb{N}$, a *p-periodic point* $x^* \in I$ of f is a fixed point of the p-th iterate f^p; the period p is called *minimal*, provided $f^n(x^*) \neq x^*$ for all $n \in \{1, \dots, p-1\}$. Moreover, $x^* \in I$ is said to be (cf., for instance, [HK01, Ela00, Sed03])

- *stable*, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$f^{pn}((x^* - \delta, x^* + \delta) \cap I) \subseteq (x^* - \epsilon, x^* + \epsilon) \quad \text{for all } n \in \mathbb{N}_0,$$

- *unstable*, if x^* is not stable,
- *attracting*, if the interior of $A_f(\{f^n(x^*)\}_{n=0}^{p-1})$ is nonempty,
- *nearly attracting*, if $|(f^p)'(x^*)| \leq 1$,
- *asymptotically stable*, if x^* is stable and attracting.

It is shown in [Sed03, p. 31, Corollary 2.1.3] that the notions of asymptotic stability and attractivity are equivalent — this is characteristic for the scalar situation.

Finally, a forward invariant set $\Omega \subseteq I$ is said to be a *metric attractor*, if the sets $A_f(\Omega)$ and $A_f(\Omega) \setminus A_f(\Omega')$ have positive Lebesgue measure for every forward invariant set Ω', which is strictly contained in Ω (cf. [Mil85]).

Theorem 4.1 (fixed points of F_h). *The function $F_h : [0, \infty) \to \mathbb{R}$ has exactly two fixed points. While the trivial point 0 is unstable for $h \in [0, 1]$, the fixed point $x^* = \frac{3}{4}$ fulfills:*

1. *For $h \in [0, h^*)$ the fixed point x^* is unstable,*
2. *for $h \in [h^*, 1]$ the fixed point x^* is asymptotically stable,*

where $h^* = \frac{1}{3}(2 - W(e^2)) \approx 0.15$ is in the unimodal regime.

Proof. The function F_h has exactly the three fixed points 0, $\frac{3}{4}$ and $3\frac{1-4e^{3h}+3he^{3h}}{4(e^{3h}-1)^2}$, where the latter one is negative. We set $x^* := \frac{3}{4}$ and compute

$$F_h'(0) = (4-3h)e^{3h}, \qquad\qquad F_h'(x^*) = (3h-2)e^{-3h}.$$

Taking into account that $|F_h'(x^*)| < 1$ for $h \in (h^*, 1]$, as well as $|F_h'(x^*)| > 1$ for $h \in [0, h^*)$, this implies the assertions for all $h \neq h^*$ (note that $F_{h^*}'(x^*) = -1$). Furthermore, due to Lemma 3.4 and $h^* < h_1$, the Schwarzian derivative $SF_{h^*}(x^*) < 0$ yields that the nontrivial fixed point x^* is asymptotically stable for $h = h^*$ (see [Ela00, Theorem 1.5, p. 24] or [MS93, Theorem 6.1, p. 155]). □

The results we obtained in Section 3 for the mapping F_h consecutively enable us to employ the whole arsenal of tools for monotone, unimodal or S-unimodal maps on intervals. Above all, unimodal functions with negative Schwarzian have various nice properties, and e.g., Singer's Theorem (see, e.g., [Sin78,MS93,Ela00]) leads to the next assertion.

Theorem 4.2. *In the* ***unimodal regime*** $h \in [0, h_1)$, *equation* (3.1) *has*

1. *at most one attracting periodic orbit, and*
2. *at most three periodic orbits which are nearly attractive.*

Proof. ad 1.: We apply [MS93, Corollary to Theorem 6.1, p. 156]. This is possible, since the trivial fixed point 0 is repelling (note that $F_h'(0) = e^{3h}(4-3h) > 1$ for every $h \in [0, 1]$) and, by Lemma 3.4, the map F_h is unimodal with a negative Schwarzian derivative.
ad 2.: This follows from [MT88, p. 511]. □

Another property of unimodal maps with negative Schwarzian is that such mappings have no wandering intervals. An interval $J \subset I_h$ is said to be a *wandering interval* for the function F_h if the intervals $\{J, F_h(J), F_h(F_h(J)), \dots\}$ are pairwise disjoint and J is not contained in the basin of an attracting periodic orbit.

Theorem 4.3. *In the* ***unimodal regime*** $h \in [0, h_1)$, *equation* (3.1) *has no wandering intervals.*

Proof. We apply [Guc79]. Since F_h is unimodal with negative Schwarzian derivative (cf. Lemma 3.4), we only have to check that $x_{\max}(h)$ is non-flat. This, nevertheless, is a consequence of (3.3). □

Due to a result from [BL91], unimodal maps with negative Schwarzian derivative and a non-flat critical point come in only a few different flavors:

Theorem 4.4. *In the* ***unimodal regime*** $h \in [0, h_1)$, *equation* (3.1) *has a unique metric attractor* $\Omega_h \subseteq I_h$ *such that* $\Omega_h = \omega_{F_h}(x)$ *for Lebesgue almost all* $x \in I_h$. *The attractor is of one of the following types:*

- *an attracting periodic orbit,*
- *a Cantor set of measure zero, or*
- *a finite union of intervals with a dense orbit;*

in the first two cases, one has $\Omega_h = \omega_{F_h}(x_{\max}(h))$.

Proof. See [BL91]. □

In the population model regime, the dynamical behavior of (3.1) is much simpler. We obtain the following global attraction result for the nontrivial fixed point:

Theorem 4.5. *In the* ***population model regime*** $h \in [h_1, 1]$, *one has* $A_{F_h}(\frac{3}{4}) = (0, \infty)$.

Proof. We aim to use various results and distinguish several cases:

- In case $h \geq h_2$, the function F_h is strictly increasing by Theorem 3.1, and consequently, there exists no maximum in $(0, \frac{3}{4})$. Thus, we can apply [Cul81, Theorem 1(a)] yielding the assertion.
- In case $\frac{2}{3} < h \leq h_2$, the point $x_{\max}(h)$, where F_h achieves its unique maximum, satisfies $\frac{3}{4} < x_{\max}(h)$. This can be deduced from the explicit form of the function $x_{\max}$ given in (3.2). Therefore, F_h has no maximum in $(0, \frac{3}{4})$, and again, [Cul81, Theorem 1(a)] yields the assertion.
- In the remaining case $h_1 \leq h \leq \frac{2}{3}$, we argue with [Cul88, Theorem 2] and indicate that (3.1) has no 2-cycle of minimal period 2. Indeed, the corresponding fixed point equation $F_h^2(x) = x$ is explicitly solvable, yielding three nontrivial solution branches $x_1(h)$, $x_2(h)$ and $x_3(h)$, where $x_3(h) = \frac{3}{4}$ is the uniquely determined fixed point of F_h. Due to their algebraic complexity, we do not state the expressions for x_1 and x_2, but we remark that these two solutions are negative for $h > h^*$. Consequently, we have no positive 2-periodic cycle with minimal period 2 for $h \geq h_1 > h^*$, which finishes the proof of this theorem.

□

A further simplification of the dynamical behavior occurs in the monotone regime $[h_2, 1]$. Here, the function F_h is strictly increasing and converges towards the limit $y_\infty(h) \geq \frac{3}{4}$ as $x \to \infty$. Therefore, $F_h : [0, \infty) \to [0, y_\infty(h))$ is bijective, and the restriction $F_h|_{[0,3/4]}$ is a self-mapping. This implies that F_h allows no points of period $p > 1$, and (3.1) has the behavior of a scalar autonomous ODE.

Theorem 4.6. *In the **monotone regime** $h \in [h_2, 1]$, equation* (3.1) *satisfies:*

1. *For initial values $x_0 \in \{0, \frac{3}{4}\}$, there exists a unique constant solution $(x_n)_{n\in\mathbb{Z}}$,*
2. *for initial values $x_0 \in (0, \frac{3}{4})$, there exists a unique strictly increasing complete solution $(x_n)_{n\in\mathbb{Z}}$ connecting the fixed points 0 and $\frac{3}{4}$,*
3. *for initial values $x_0 > \frac{3}{4}$, the forward solution $(x_n)_{n\in\mathbb{N}_0}$ is strictly decreasing to $\frac{3}{4}$.*

Proof. From Theorem 3.1, we know that the mapping F_h is strictly increasing. Then [HK01, p. 82, Lemma 3.14] guarantees that forward solutions to (3.1) are monotone sequences. Together with the considerations preceding this theorem, we obtain the assertion. □

5 Bifurcation

In the proof of the above Theorem 4.5, we observed that F_h admits 2-periodic cycles for parameters $h \in [0, h^*)$. In addition, Theorem 4.1 says that the asymptotic stability of the fixed point $\frac{3}{4}$ of F_h is lost at $h = h^*$. These observations suggest that this loss of stability inside the unimodal regime is associated with a flip (or period doubling) bifurcation.

Theorem 5.1 (flip bifurcation)**.** *At $h^* = \frac{1}{3}(2 - W(e^2))$, the fixed point $\frac{3}{4}$ undergoes a subcritical flip bifurcation.*

Proof. We check the corresponding conditions stated in, e.g., [Wig90, p. 373] or [Sed03]. They basically require to detect a bifurcation of pitchfork type for the second iterate mapping $G(h, x) := F_h(F_h(x))$. We obtain $F'_{h^*}(\frac{3}{4}) = -1$,

$$\frac{\partial G}{\partial h}(h^*, \tfrac{3}{4}) = 0\,, \qquad \frac{\partial^2 G}{\partial^2 x}(h^*, \tfrac{3}{4}) = 0\,,$$

$$\frac{\partial^2 G}{\partial x \partial h}(h^*, \tfrac{3}{4}) = -\frac{W(e^2)+1}{W(e^2)} \neq 0\,, \qquad \frac{\partial^3 G}{\partial^3 x}(h^*, \tfrac{3}{4}) = -\frac{64}{3}\left(\frac{W(e^2)+1}{W(e^2)^2}\right)^2 \neq 0\,,$$

which implies the claimed subcritical flip bifurcation. □

Apart from this flip bifurcation, there is a series of further subcritical period doubling bifurcations for $h < h^*$. In addition, we obtain a similar Feigenbaum diagram as the well-known one for the logistic map $x \mapsto hx(1-x)$, $h \in [0,4]$ (see Figure 4). Finally, the appearance of a window with 3-periodic points for F_h is illustrated in Figure 5. There is numerical evidence that these 3-periodic points exist in the unimodal regime for $h \in [0, 0.02257]$. Thus, the celebrated Sharkovskii theorem (see, for example, [MS93, Ela00, HK01, Sed03]) guarantees the existence of points with arbitrary period and F_h is chaotic in the sense of [BC92] with corresponding dynamical consequences, like the existence of a homoclinic point or positive topological entropy. Moreover, [Kie98, p. 154, Satz 4.4.2] ensures that F_h is also topologically transitive, has dense periodic orbits and sensitive dependence on initial conditions, i.e., is chaotic in the sense of [Dev89]. This, in turn, furthermore implies chaos in the sense of [LY75] (cf. [Kie98, p. 159, 4.4.4 Satz]).

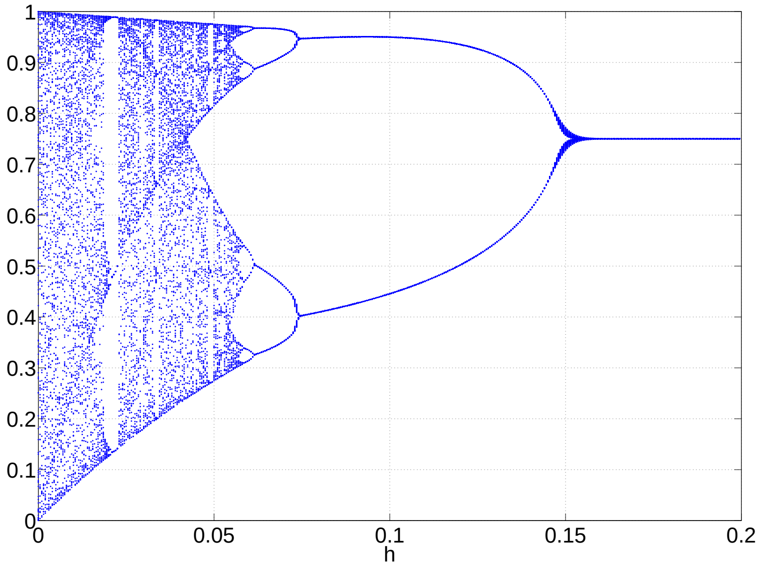

Figure 4: Feigenbaum diagram for F_h

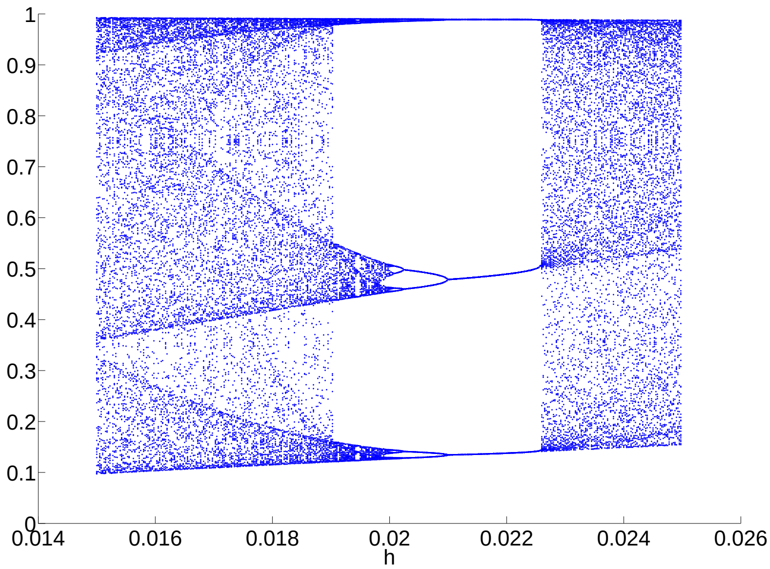

Figure 5: Window in the Feigenbaum diagram for F_h with a 3-periodic point

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