

# PATTERNS OF DESYNCHRONIZATION AND RESYNCHRONIZATION IN HETEROCLINIC NETWORKS

M J FIELD

ABSTRACT. We prove results that enable the efficient and natural realization of a large class of robust heteroclinic networks in coupled identical cell systems. We also propose some general conjectures that relate a natural and large class of robust heteroclinic networks that occur in networks modelled by equations of Lotka-Volterra type, and certain networks of symmetric systems, to robust heteroclinic networks in coupled cell networks.

---

*Date:* November 16, 2016.

*2010 Mathematics Subject Classification.* Primary: 37C37, 34D06; Secondary: 34D20.

*Key words and phrases.* Heteroclinic networks, coupled cell systems, realization of heteroclinic networks, edge and face cycles.

Research supported in part by NSF Grant DMS-1265253 & Marie Curie IIF Fellowship (Project 627590).

## CONTENTS

1. Introduction	2
1.1. Heteroclinic cycles and networks	3
1.2. Robust heteroclinic networks and cycles	4
1.3. Semilinear feedback systems and coupled cell systems	5
1.4. Summary of main results	5
1.5. Related work and applications	7
2. Semilinear Feedback systems	8
2.1. Heteroclinic cycles and networks in SLF systems	10
2.2. Reduction to a simplex	11
2.3. Edge and face heteroclinic cycles and networks	12
2.4. Examples of edge and face cycles and networks	13
2.5. Lattice structure on $\mathcal{I}_k$	17
3. Coupled identical cell systems – asymmetric inputs	18
3.1. Synchrony classes and synchrony subspaces	19
4. The Synchronization Transform	26
4.1. Necessary conditions for $\mathcal{S}$ to be a lattice of synchrony subspaces	35
4.2. Summary	37
5. Transforming heteroclinic networks	37
5.1. Simple heteroclinic networks and cycles for coupled identical cell systems	38
5.2. 4 cell networks.	42
5.3. Summary of the results for 4 cell networks	44
5.4. Heteroclinic networks	47
6. Outstanding Questions and Concluding Comments	50
6.1. The Realization Conjectures	50
6.2. Transition from SLF to coupled identical cell networks	50
6.3. Symmetric inputs and sparseness of coupling	51
References	51

## 1. INTRODUCTION

This work is about the realization of robust heteroclinic networks and cycles in networks of coupled dynamical systems. The main results describe a correspondence between classes of heteroclinic networks in Lotka-Volterra (or symmetric) systems and in coupled identical cell systems. Emphasis is on structure rather than detailed construction of network vector fields realizing heteroclinic networks (for this see [26, 6]).

Unfortunately, the word *network* occurs in both ‘coupled networks of dynamical systems’ and ‘heteroclinic networks’. As an aid to the reader, we reserve the word *node* for an individual unit in a coupled dynamical network and use *vertex* for either an equilibrium on a heteroclinic network or for the vertex of the abstract graph.

**1.1. Heteroclinic cycles and networks.** The definition of a heteroclinic network varies in the literature. We start by giving a definition of a heteroclinic network appropriate for our applications and establish some mostly standard notation and terminology.

Let  $M$  be a differential manifold and  $X$  be a smooth (at least  $C^1$ ) vector field on  $M$  with associated flow  $\Phi$ . Let  $\mathbf{E} = \{\mathbf{p}_i \mid i \in I\}$  be a finite set of hyperbolic saddle points for  $X$ . If  $\mathbf{p} \in \mathbf{E}$ , let  $W^s(\mathbf{p}), W^u(\mathbf{p})$  respectively denote the (global) stable and unstable manifolds of  $\mathbf{p}$ . If  $\mathbf{p}, \mathbf{q} \in \mathbf{E}$ , a *connection* from  $\mathbf{p}$  to  $\mathbf{q}$ , denoted  $\mathbf{p} \xrightarrow{\phi} \mathbf{q}$ , will be a  $\Phi$ -trajectory  $\phi : \mathbb{R} \rightarrow M$  such that  $\phi(\mathbb{R}) \subset W^u(\mathbf{p}) \cap W^s(\mathbf{q})$ . We always assume there are no self-connections (homoclinic loops) and so  $W^s(\mathbf{p}) \cap W^u(\mathbf{p}) = \{\mathbf{p}\}$ , all  $\mathbf{p} \in \mathbf{E}$ . If  $\mathbf{C}^*$  denotes the set of all connections between equilibria in  $\mathbf{E}$ , then

$$\bigcup_{\mathbf{p}, \mathbf{q} \in \mathbf{E}, \mathbf{p} \neq \mathbf{q}} W^u(\mathbf{p}) \cap W^s(\mathbf{q}) = \bigcup_{\phi \in \mathbf{C}^*} \phi(\mathbb{R}).$$

If  $\mathbf{C} \subset \mathbf{C}^*$ , define  $\Phi(\mathbf{C}) = \bigcup_{\phi \in \mathbf{C}} \phi(\mathbb{R}) \subset M$ .

**Definition 1.1.** (Notation and assumptions as above.) A subset  $\Sigma$  of  $M$  is a *heteroclinic network*, with *equilibrium set*  $\mathbf{E}$  and *connection set*  $\mathbf{C}$ , if

- (a)  $\mathbf{C} \subset \mathbf{C}^*$ .
- (b)  $\Sigma = \mathbf{E} \cup \Phi(\mathbf{C})$ .
- (c) For all ordered pairs  $\mathbf{p}, \mathbf{q} \in \mathbf{E}$ , there is a sequence

$$(1.1) \quad \mathbf{p} \xrightarrow{\phi^{\alpha_0}} \mathbf{s}_1 \xrightarrow{\phi^{\alpha_1}} \mathbf{s}_2 \xrightarrow{\phi^{\alpha_2}} \cdots \xrightarrow{\phi^{\alpha_{k-1}}} \mathbf{s}_k \xrightarrow{\phi^{\alpha_k}} \mathbf{q}$$

of connections, where  $k = k(\mathbf{p}, \mathbf{q}) \geq 1$  if  $\mathbf{p} = \mathbf{q}$ , the  $\mathbf{s}_i$  are distinct equilibria in  $\mathbf{E} \setminus \{\mathbf{p}, \mathbf{q}\}$ , and each connection  $\phi^{\alpha_j} \in \mathbf{C}$ .

If  $\Sigma$  contains equal numbers of equilibria and connections, then  $\Sigma$  is a *heteroclinic cycle*.

*Remark 1.2.* It follows from (1.1) that  $\Sigma$  is connected and  $\Phi$ -invariant and, taking  $\mathbf{p} = \mathbf{q}$  in (1.1), that every equilibrium point  $\mathbf{p} \in \mathbf{E}$  lies on a heteroclinic cycle. It follows easily that a heteroclinic network is a union of heteroclinic cycles. ⊗

Definition 1.1 does not require that  $\Sigma$  contains *all* the connections between equilibria in  $\mathbf{E}$ . Define  $\Sigma(\mathbf{E}) \subset M$  by

$$(1.2) \quad \Sigma(\mathbf{E}) = \mathbf{E} \cup \Phi(\mathbf{C}^*) = \bigcup_{\mathbf{p}, \mathbf{q} \in \mathbf{E}} W^u(\mathbf{p}) \cap W^s(\mathbf{q}).$$

Obviously,  $\Sigma(\mathbf{E})$  is a heteroclinic network iff there is a finite subset  $\mathbf{C} \subset \mathbf{C}^*$  for which  $\mathbf{E} \cup \Phi(\mathbf{C})$  is a heteroclinic network. If  $\Sigma(\mathbf{E})$  is a heteroclinic network, then it is the *maximal* heteroclinic network with equilibrium set  $\mathbf{E}$ .

**Definition 1.3.** (Notation and assumptions as above.) The heteroclinic network  $\Sigma(\mathbf{E})$  is *clean* if

- (1)  $\Sigma(\mathbf{E})$  is a compact subset of  $M$ .
- (2)  $\Sigma(\mathbf{E}) = \bigcup_{\mathbf{p} \in \mathbf{E}} W^u(\mathbf{p})$ .

*Remark 1.4.* A relatively compact heteroclinic network  $\Sigma$  can be an asymptotically stable attractor only if  $\Sigma = \Sigma(\mathbf{E})$  and  $\Sigma(\mathbf{E})$  is clean.  $\otimes$

Let  $\Sigma$  be a heteroclinic network. If  $\dim(W^u(\mathbf{p})) = 1$ , all  $\mathbf{p} \in \mathbf{E}$ , then  $\Sigma$  is *simple*. For every connection  $\mathbf{p} \xrightarrow{\phi} \mathbf{q}$ ,  $\phi(\mathbb{R})$  will be a connected component of  $W^u(\mathbf{p}) \setminus \{\mathbf{p}\}$ . In many applications,  $M$  will have boundary (for example, be a simplex) and  $W^u(\mathbf{p}) \setminus \{\mathbf{p}\}$  may only contain a single trajectory. If this is the case for all  $\mathbf{p} \in \mathbf{E}$ , then  $\Sigma = \bigcup_{\mathbf{p} \in \mathbf{E}} W^u(\mathbf{p})$ ,  $\mathbf{C}^*$  is finite and  $\Sigma = \Sigma(\mathbf{E})$  will be a clean heteroclinic cycle.

If  $\mathbf{C}$  is not finite, then  $\Sigma$  and  $\Sigma(\mathbf{E})$  need not be closed in  $M$ . In particular, the dynamics of  $\Phi|_{\partial\Sigma(\mathbf{E})}$  may be very complex.

**1.2. Robust heteroclinic networks and cycles.** We refer to [26, §2] for a general review of heteroclinic cycles and networks and their applications. Our interest here is in *robust* heteroclinic cycles and networks. On account of the Kupka-Smale theorem [53], heteroclinic cycles and networks can only be robust, that is persist under all sufficiently small  $C^1$  perturbations of the associated vector field, if we work within a class of vector fields with additional structure. Invariably this structure is associated with the presence of invariant subspaces. Robust heteroclinic networks and cycles are well-known phenomena in models of population dynamics, ecology and game theory based on the Lotka-Volterra or replicator equations [45, 34, 35, 36, 37]. Typically, these systems are defined on a simplex, or the positive orthant  $O_k = \{x \in \mathbb{R}^k \mid x_i \geq 0, i = 1, \dots, k\}$  of  $\mathbb{R}^k$ , and have the ‘extinction’ hyperplanes  $x_i = 0$  as invariant subspaces. Heteroclinic networks and cycles also occur robustly in differential equations which are *equivariant* with respect to a compact Lie group of symmetries [55, 47, 42, 27, 43, 41, 10, 21, 25]. In

this case, robustness can occur because generic intersections of stable and unstable manifolds of equilibria in equivariant dynamics need not be transverse but can nonetheless be stable under sufficiently smooth equivariant perturbation of the underlying vector field [17, 18, 19].

If a heteroclinic network  $\Sigma$  contains a homoclinic loop then, without additional constraints,  $\Sigma$  cannot be robust by the Kupka-Smale theorem; this is the main reason we deny self-loops in definition 1.1.

**1.3. Semilinear feedback systems and coupled cell systems.** In this article we explore a relationship between robust heteroclinic networks in semilinear feedback systems and in coupled identical cell systems. Semilinear feedback systems constitute a large class of network models which includes Lotka-Volterra systems and some equivariant systems. Nodes in a semilinear feedback system can be viewed as either being in a zero state or active (non-zero). A node in a zero state will remain in a zero state for all future and past time. There is a substantial body of results on robust heteroclinic networks and cycles for semilinear feedback systems. However, most of these results concern *edge* networks. We give formal definitions in section 2 but note here that a vertex of an edge heteroclinic network corresponds to one node being active, all other nodes being null. For our purposes, we need to consider the larger class of less well-known *face* heteroclinic networks that will play a key role in our applications to coupled identical cell networks. The vertex of a heteroclinic face network will correspond to multiple active nodes. From a mathematical and conceptual point of view, the theory of robust heteroclinic networks for semilinear feedback systems is relatively elementary; in particular, it is straightforward to construct examples of clean heteroclinic networks (of edge or face type).

Coupled systems of identical cells are a natural class of models for the study of synchronization in networks [59, 31, 30]. Much less is known about heteroclinic networks, or even heteroclinic cycles, in coupled systems of identical cells. For coupled systems of identical cells, a robust heteroclinic network will typically have vertices that are clusters of synchronized nodes and connections between vertices will correspond to partial desynchronization of the clusters at the start and end of the connection. In the fully symmetric case, Ashwin *et al.* [39, 51] have examples of complex patterns of synchronization and desynchronization in phase oscillator networks that are related to heteroclinic networks. Examples of heteroclinic cycles and networks are given in [6, 5] for coupled cell networks with no global symmetry group.

**1.4. Summary of main results.** In previous work [26], it was shown that every strongly connected directed graph  $\Gamma$  with  $q$  edges and no

self-loops could be realized as the graph of a robust heteroclinic network in a coupled system of  $q + 1$  identical cells with 1-dimensional cell dynamics. This construction was limited as vertices of the heteroclinic network always corresponded to fully synchronized states and there was no obvious relation with heteroclinic networks in semilinear feedback systems. We show that this realization theorem is a special case of a general correspondence between face heteroclinic networks in semilinear feedback systems and heteroclinic networks in coupled cell systems. It turns out that for the main result theorem 4.4 of [26], a heteroclinic network corresponds to a singular edge network.

Our main results, theorems 4.5, 4.11 & 5.11, give an efficient and natural realization of a large class of robust edge and face heteroclinic networks that occur in semilinear feedback systems as robust heteroclinic networks in coupled identical cell systems. The realizations are in networks with close to the minimal number of identical cells, unlike for the realization theorem given in [26] (see also [11]). They have the attractive feature that each connection in the realization corresponds to a unique pattern of desynchronization and resynchronization along the connecting trajectory. As the formal statements require some preliminaries and background, we only give here an example to illustrate the main results: we indicate how a face heteroclinic network (realizable by a Lotka-Volterra system) is transformed into a robust heteroclinic network in a coupled identical cell system.

**Example 1.5.** Referring to figure 1(a),  $\Sigma$  denotes a heteroclinic face network defined on the simplex  $\Delta_4$  ( $\sum_{i=1}^5 x_i = 1$ , where  $x_i \geq 0$ ,  $i = 1, \dots, 5$ ). The equilibria  $\mathbf{v}_{ij}$  lie on edges  $\mathbf{F}_{ij}$  of the simplex and the connections shown in the figure lie in 2-dimensional faces  $\mathbf{F}_{ijk}$  of the simplex. For example, the connection  $\mathbf{v}_{12} \rightarrow \mathbf{v}_{24}$  lies in the face  $\mathbf{F}_{124} = \{\mathbf{x} \in \Delta_4 \mid x_3, x_5 = 0\}$  and  $\mathbf{v}_{12}$  is an interior point of the edge  $\mathbf{F}_{12} = \{\mathbf{x} \in \Delta_4 \mid x_4, x_3, x_5 = 0\}$ . The realization theorem yields the heteroclinic network  $\Sigma^T$  for an identical coupled cell system with six cells. The heteroclinic network (*not* the coupled cell system) is shown in figure 1(b). Vertices (equilibria) correspond to synchronized clusters. For example, the label (3456) for vertex  $A$  signifies that the nodes 3, 4, 5, 6 are synchronized while the label (12|346) for vertex  $B$  means that nodes 1, 2 are synchronized and nodes 3, 4, 6 are synchronized (but not to nodes 1, 2). Along the connection  $A \rightarrow B$ , nodes 3, 4, 6 are synchronized. Observe that cell 5 desynchronizes from the cluster 1, 2, 3, 4 at the start of the connection and that at the end of the connection cells 1 and 2 synchronize – but not with the cluster 3, 4, 6.

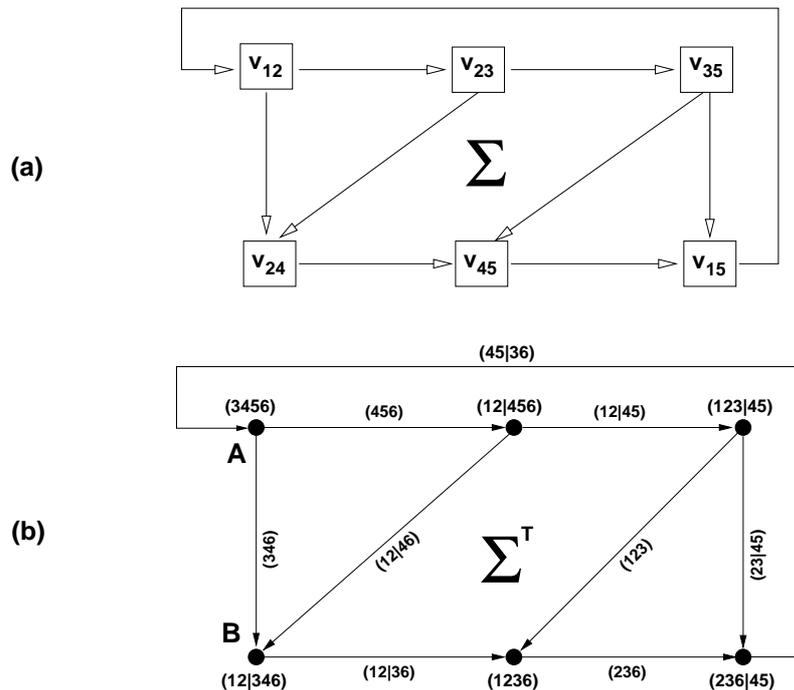


FIGURE 1. Transforming the face network  $\Sigma$  into the heteroclinic network  $\Sigma^T$ .

We discuss the heteroclinic face network  $\Sigma$  later (section 2, examples 2.10(3)) and show that the associated maximal heteroclinic network may be chosen to be clean and robust. The correspondence between  $\Sigma$  and  $\Sigma^T$ , as well as the coupled cell network that supports  $\Sigma^T$ , are discussed in example 5.10 at the end of section 5. ✱

**1.5. Related work and applications.** In [11], Ashwin and Postlethwaite consider the problem of realizing graphs as robust heteroclinic networks, though not in coupled identical cell networks. Their concept of a ‘simplex network’ realization is closely related to what we call an *edge* network (or cycle) in section 2 (see also [21]). Ashwin and Postlethwaite [12] have results on the realization of graphs as robust heteroclinic networks in a class of networks with two cell types – but not a coupled cell network in the sense of Golubitsky & Stewart [30].

There has been significant recent interest in applications of robust heteroclinic networks to neural microcircuits. It has been suggested that they provide a model to explain the function of certain neural systems [54, 50]. Specifically, heteroclinic networks and cycles model “winnerless competition” where there is a local competition between

different states but not necessarily a global winner. The models used are based on networks of generalized Lotka-Volterra equations [2, 49] and part of the motivation for the present article is to provide models based on a natural identical cell model. Our results go some way towards achieving this objective and identify some of the obstructions to obtaining realistic models based on fewer connections, more symmetry and an additive input structure [26]. We address some of these issues in more detail later and in the concluding comments. In general terms, robust heteroclinic phenomena seem particularly useful for explaining sequence generation and spatio-temporal encoding and have been found in rate-based [1] models, Hodgkin-Huxley-based models [33] and more general phase oscillator models [9] where they have been used to perform finite-state computations [8].

We conclude with a description of the contents of the work by section. In section 2 we review and develop the theory of semilinear feedback systems and describe the class of face heteroclinic cycles and networks [21]. In section 3, we describe that part of the theory of coupled identical cell systems applicable to cells with asymmetric inputs and strongly connected networks without self loops. In section 4 we introduce the idea of a *synchronization transform*. This constitutes the basis of our method of going from heteroclinic face networks for a semilinear feedback system to a heteroclinic network for a coupled cell system. We pose two conjectures concerning the scope of the synchronization transform and prove a number of associated results, notably theorems 4.5, 4.11. Section 5 is devoted to illustrative examples and includes part of the verification of the realization conjectures for identical cell networks with three or four cells. We give examples of the realization of robust heteroclinic face networks in coupled cell systems that show complex patterns of synchronization and desynchronization. In section 6, we discuss outstanding questions as well as comment on the possibility of obtaining more physically realistic models.

## 2. SEMILINEAR FEEDBACK SYSTEMS

*Semilinear Feedback systems* (*SLF* systems) are a large and transparent class of network models that naturally support many different types of robust heteroclinic network. Well-known examples include Lotka-Volterra systems and some classes of equivariant system. Parts of what we discuss here appear in [21, chapters 6 & 7] and [22] – though the terminology ‘semilinear feedback system’ was introduced later by the author (for example, [24]).

*Notational conventions.* Let  $\mathbb{N}$  denote the natural numbers (strictly positive integers),  $\mathbb{Z}_+$  the set of nonnegative integers,  $\mathbb{R}$  the real numbers, and  $\mathbb{R}_+$  the nonnegative reals. Given  $k \in \mathbb{N}$ , define  $\mathbf{k} = \{1, \dots, k\}$ , and  $\mathbf{k}^\bullet = \{0, 1, \dots, k\}$ . For clarity, we sometimes use an overline, rather than brackets; for example,  $\overline{\mathbf{k} - \mathbf{1}}$  rather than  $(\mathbf{k} - \mathbf{1})$  or  $\mathbf{k} - \mathbf{1}$ . If  $\mathbf{k}$  is an indexing set, we work mod  $k$ :  $X_{k+1} \stackrel{\text{def}}{=} X_1$ ,  $X_{1-1} \stackrel{\text{def}}{=} X_k$ .

**Definition 2.1.** An SLF system is a dynamical network modelled by differential equations

$$(2.3) \quad \dot{x}_i = f_i(x_i) + x_i F_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k), \quad i \in \mathbf{k},$$

where  $x_i \in \mathbb{R}$ ,  $F_i : \mathbb{R}^{k-1} \rightarrow \mathbb{R}$ ,  $f_i(0) = 0$ , and all maps are at least  $C^1$ .

We regard (2.3) as equations for a network  $\mathcal{N} = \{N_1, \dots, N_k\}$  consisting of  $k$  coupled nodes,  $N_1, \dots, N_k$ , where the node  $N_i$  has state  $x_i$  and phase space  $R_i$  where  $R_i$  is either  $\mathbb{R}$  or  $\mathbb{R}_+$  (we assume the same choice for all  $i \in \mathbf{k}$ ). Let  $\mathbf{R} = \prod_{i \in \mathbf{k}} R_i$  denote the *network phase space*. If the node phase space is  $\mathbb{R}_+$ , then  $\mathbf{R}$  is the positive orthant  $O_k$  of  $\mathbb{R}^k$ . Observe that since  $x_i = 0$  is flow-invariant for SLF systems, all  $i \in \mathbf{k}$ , so also is  $O_k$ .

*Remarks 2.2.* (1) The equation  $\dot{x}_i = f_i(x_i)$  defines the *intrinsic dynamics* of node  $N_i$ . If  $f_1 = \dots = f_k$ , nodes have *identical* intrinsic dynamics. We usually assume identical intrinsic dynamics but that is not needed for our main results. Systems of the form (2.3) often have additive input structure [26] – for example, if  $F_i$  is linear.

(2) We can require that phase spaces are  $\mathbb{R}^n$  rather than  $\mathbb{R}$  – we refer to [21] for definitions and examples. ⊛

For  $i \in \mathbf{k}$ , define  $H_i = \{\mathbf{x} \in \mathbf{R} \mid x_i = 0\}$ , and let  $\mathcal{I}_k$  denote the set of all intersections  $H_{i_1} \cap \dots \cap H_{i_p} \stackrel{\text{def}}{=} H_{i_1 \dots i_p}$ . We assume  $\mathbf{R} \in \mathcal{I}_k$  and note that  $H_{1 \dots k} = \{(0, \dots, 0)\}$ . Every  $V \in \mathcal{I}_k$  is flow-invariant for an SLF system (since this is so for  $H_i$ ,  $i \in \mathbf{k}$ ).

**Examples 2.3.** (1) A (generalized) Lotka-Volterra system is defined on the positive orthant  $O_k$  by

$$(2.4) \quad \dot{x}_i = x_i G_i(\mathbf{x}) = f_i(x_i) + x_i F_i(\mathbf{x}), \quad i \in \mathbf{k},$$

where  $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ ,  $f_i(x_i) = x_i(a_i + b_i x_i)$ ,  $F_i(\mathbf{x}) = \sum_{j \neq i} a_{ij} x_j$ , and  $a_i, b_i, a_{ij} \in \mathbb{R}$ ,  $i, j \in \mathbf{k}$ . We refer to [45, 34, 35, 36, 37] for explicit examples and applications.

(2) The cubic truncation of an equivariant system on  $\mathbb{R}^k$  with symmetry group  $G$  containing the group  $\mathbb{Z}_2^k$  of orthogonal diagonal  $k \times k$ -matrices.

For  $i \in \mathbf{k}$ ,  $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ , we have

$$f_i(x_i) = a_i x_i + b_i x_i^3, \quad F_i(\mathbf{x}) = \sum_{j \neq i} a_{ij} x_j^2.$$

We refer to [21, chapter 3] or [25, §4.5] for examples in the case when  $G$  acts absolutely irreducibly on  $\mathbb{R}^k$ . \*

*Remark 2.4.* Every  $\mathbb{Z}_2^k$ -equivariant cubic truncation restricted to  $O_k$  can be transformed into a Lotka-Volterra system under the invertible transformation  $\mathbf{x} \leftrightarrow \mathbf{x}^2 = (x_1^2, \dots, x_k^2)$  of  $O_k$ . In practice, for Lotka-Volterra systems, it is common to satisfy the constraint  $x_1 + \dots + x_k = 1$  and dynamics is naturally defined on the  $(k-1)$ -simplex  $\Delta_{k-1}$  ( $\sum_{i \in \mathbf{k}} x_i = 1$  in  $O_k$ ) using the replicator equation  $\dot{x}_i = x_i(G_i(\mathbf{x}) - \overline{G}(\mathbf{x}))$ ,  $i \in \mathbf{k}$ , where  $\overline{G}(\mathbf{x}) = \sum_{i \in \mathbf{k}} x_i G_i(\mathbf{x})$ . Analogously, in the equivariant case, given a cubic truncation, dynamics is uniquely defined on the (spherical) simplex  $\Delta_{k-1} = S^{k-1} \cap O_k$  using the phase vector field (see [25, Chapters 4,5] and section 2.1). Dynamics on the simplex and spherical simplex are topologically conjugate by the transformation  $\mathbf{x} \leftrightarrow \mathbf{x}^2$  and a time rescaling by a factor of 2. In this article we emphasise  $\mathbb{Z}_2^k$ -equivariant cubic systems rather than quadratic Lotka-Volterra systems. This is no loss of generality by the equivalences described above. Note, however, that metric/measure equivalence may fail (the transformation  $\mathbf{x} \leftrightarrow \mathbf{x}^2$  is only Hölder on  $\partial O_k$  or  $\partial \Delta_{k-1}$ ). \*

**2.1. Heteroclinic cycles and networks in SLF systems.** Heteroclinic cycles are a well-known phenomenon in Lotka-Volterra and symmetric systems. For reference, we recall a well-known example.

**Example 2.5.** Suppose that  $k = 3$  and equations are given by

$$(2.5) \quad \dot{x}_i = x_i(1 - x_i^2) - 3x_i x_{i+1}^2, \quad i \in \mathbf{3},$$

The system (2.5) has a simple asymptotically stable heteroclinic cycle  $\Sigma$  contained in a unique attractive flow invariant 2-sphere [25, Chapter 5, §2]. Each vertex of the cycle corresponds to a single active node (the other nodes will be in a zero state). We refer to this cycle as the *rock-paper-scissors*, or *RPS*, heteroclinic cycle (see [45], where the RPS cycle was first described for Lotka-Volterra equations, and [55, 21, 32] for symmetric systems). \*

Next we develop the theory of heteroclinic networks in SLF systems.

*Basic formalism and definitions.* Suppose that  $\Sigma$  is a heteroclinic network with equilibrium set  $\mathbf{E} = \{\mathbf{p}_i \mid i \in I\}$  and finite connection set  $\mathbf{C} = \{\phi^\alpha \mid \alpha \in J\}$ . The *directed graph*  $\Gamma = \Gamma(\Sigma)$  of  $\Sigma$  is defined to

have vertex set  $\mathcal{V} = \{v_i \mid i \in I\}$  and edges  $\mathcal{E} = \{e_\alpha \mid \alpha \in J\}$ , where  $e_\alpha$  is a directed edge  $v_j \rightarrow v_i$  if  $\phi^\alpha$  is a connection from  $\mathbf{p}_j$  to  $\mathbf{p}_i$  (we allow multiple edges between two vertices). Definition 1.1 implies that  $\Gamma$  is strongly connected and without self-loops.

If  $v$  is a vertex of  $\Gamma(\Sigma)$ , let  $d_v^{\text{out}}$  denote the *out-degree* of  $v$ : the number of edges in  $\mathcal{E}$  which connect  $v$  to the remaining vertices in  $\mathcal{V}$ . If  $v$  corresponds to the equilibrium  $\mathbf{p} \in \mathbf{E}$ , define  $d_{\mathbf{p}}^{\text{out}} = d_v^{\text{out}}$ .

We will mainly be interested in heteroclinic networks  $\Sigma \subset \Delta_k$  with finite connection set  $\mathbf{C}$  and

$$(2.6) \quad d_{\mathbf{p}}^{\text{out}} = \dim(W^u(\mathbf{p})), \text{ for all } \mathbf{p} \in \mathbf{E}.$$

If (2.6) holds and  $\Sigma$  is not simple, the connection set  $\mathbf{C}$  will typically be a *skeleton* or *framework* for the maximal heteroclinic network  $\Sigma(\mathbf{E})$  (1.2). This viewpoint turns out to be natural for our discussion of heteroclinic networks in SLF systems. Of course, the way in which we choose  $\mathbf{C}$  will be important. In many situations, a heteroclinic network will be robust only if the connection set  $\mathbf{C}$  is finite (and well chosen).

**2.2. Reduction to a simplex.** We consider the class of SLF systems modelled by equations of the form

$$(2.7) \quad \dot{x}_i = f_i(x_i) + x_i \left( \sum_{j \neq i} a_{ij} x_j^2 \right), \quad i \in \mathbf{k},$$

where  $f_i(x) = x - b_i x^3$ ,  $b_i > 0$ . The resulting system is  $\mathbb{Z}_2^k$ -equivariant with fundamental domain the positive orthant  $O_k$ . All our constructions will be within  $O_k$  (and so results relate directly to Lotka-Volterra systems by remark 2.4). Aside from working on  $O_k$  and using the invariance of the subspaces  $\mathcal{I}_k$ , we make no use of symmetry properties of the system. Indeed, it will generally be the case that there are no symmetries in the matrix  $A = [a_{ij}]$  of coupling coefficients and no permutation symmetries of the network. We now use a simplification exactly analogous to the reduction of a Lotka-Volterra system to the simplex (see remark 2.4).

Write (2.7) in vector form as

$$(2.8) \quad \dot{\mathbf{x}} = \mathbf{x} + \mathbf{Q}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^k,$$

where the components  $Q_i$  of the vector field  $\mathbf{Q}$  are given by  $Q_i(\mathbf{x}) = -b_i x_i^3 + x_i (\sum_{j \neq i} a_{ij} x_j^2)$ ,  $i \in \mathbf{k}$ . Note that  $\mathbf{Q} : \mathbb{R}^k \rightarrow \mathbb{R}^k$  is a homogeneous cubic polynomial map of  $\mathbb{R}^k$ . Associated to the system (2.8), we define a vector field  $\mathcal{P}_Q$  on the unit sphere  $S^{k-1}$  in  $\mathbb{R}^k$  by

$$\mathcal{P}_Q(\mathbf{u}) = \mathbf{Q}(\mathbf{u}) - (\mathbf{Q}(\mathbf{u}), \mathbf{u})\mathbf{u}, \quad \mathbf{u} \in S^{k-1},$$

where  $(\cdot, \cdot)$  denotes the standard inner product on  $\mathbb{R}^n$ . (The vector field  $\mathcal{P}_Q$  is called the *phase vector field* in [20, 25].) The dynamics of  $\mathcal{P}_Q$  encodes the nontrivial dynamics of the original system (2.8): every nonzero trajectory in  $\mathbb{R}^k$  of (2.8) is the smooth lift of a  $\mathcal{P}_Q$ -trajectory on  $S^{k-1}$ . Moreover, by the invariant sphere theorem [20, 25], if we define  $\mathbf{Q}_a(\mathbf{x}) = \mathbf{Q}(\mathbf{x}) - a\|\mathbf{x}\|^2\mathbf{x}$ ,  $a \in \mathbb{R}$ , then  $\mathcal{P}_Q = \mathcal{P}_{Q_a}$  and, for sufficiently large  $a$ ,  $\dot{\mathbf{x}} = \mathbf{x} + \mathbf{Q}_a(\mathbf{x})$  has an invariant attracting  $k-1$ -sphere  $S(a)$  such that the dynamics on  $S(a)$  is conjugate to the dynamics of  $\mathcal{P}_Q$  on  $S^{k-1}$ . In particular, every robust heteroclinic cycle or network for  $\mathcal{P}_Q$  will uniquely determine a robust heteroclinic cycle or network for  $\dot{\mathbf{x}} = \mathbf{x} + \mathbf{Q}_a(\mathbf{x})$  for sufficiently large  $a$ . Conversely, every robust heteroclinic cycle or network for  $\dot{\mathbf{x}} = \mathbf{x} + \mathbf{Q}(\mathbf{x})$  can be realized as a robust heteroclinic cycle or network for  $\mathcal{P}_Q$ .

Generally, let  $\Psi$  be a smooth ( $C^\infty$ ) flow on  $S^{k-1}$  that preserves the invariant subspace structure induced by  $\mathcal{I}_k$  (and so  $\Psi$  preserves the spherical simplex  $\Delta_{k-1}$ ). It is easy to show that there exists a smooth SLF system on  $\mathbb{R}^k$ , with flow  $\Phi$  and the origin a source (as in (2.8)), such that  $S^{k-1}$  is  $\Phi$ -invariant and attracting with  $\Phi|_{S^{k-1}}$  conjugate to  $\Psi$ . Of course,  $\Phi$  need not be given by a cubic polynomial vector field.

As a consequence of these observations, we focus on the study of robust heteroclinic phenomena for smooth vector fields on  $S^{k-1}$  that preserve the invariant subspace structure determined by  $\mathcal{I}_k$ . Generally, we restrict to the spherical simplex  $\Delta_{k-1}$ . This is no loss of generality if we have  $\mathbb{Z}_2^k$ -equivariance:  $\Delta_{k-1}$  is a fundamental domain for the  $\mathbb{Z}_2^k$ -action on  $S^{k-1}$ .

**2.3. Edge and face heteroclinic cycles and networks.** Assume  $k \geq 3$ . If  $V \in \mathcal{I}_k$  is  $r+1$ -dimensional,  $r \leq k-2$ , then  $\Delta_{k-1} \cap V$  is an  $r$ -face of  $\Delta_{k-1}$ . In particular, if  $r=0$  then  $V$  is a coordinate axis and  $\Delta_{k-1} \cap V$  is a *vertex* of  $\Delta_{k-1}$ , and if  $r=1$ ,  $\Delta_{k-1} \cap V$  is an *edge*. Clearly, the boundary of an  $r$ -face is a union of  $r-1$ -faces. In particular,  $\partial\Delta_{k-1}$  is the union of all the  $k-2$ -faces. In coordinates, the vertices of  $\Delta_{k-1}$  are given by the set  $\mathcal{V}(k) = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ , where  $\mathbf{v}_j$  is the positive unit vector along the coordinate axes  $x_j$  of  $\mathbb{R}^k$ ,  $j \in \mathbf{k}$ . Every edge is uniquely determined by two vertices, every 2-face by three vertices and so on. It is easy to verify that there are  $\binom{k}{r+1}$   $r$ -faces,  $r \leq k-2$ . If  $F = \Delta_{k-1} \cap V$  is an  $r$ -face of  $\Delta_{k-1}$ , define  $\text{Int}(F) = F \setminus \partial F$  where  $\partial F$  is the boundary of  $F$  within the sphere  $V \cap S^{k-1}$ .

**Definition 2.6** (cf. [21]). Let  $k \geq 3$ ,  $r \in \overline{\mathbf{k}-2}$ , and  $\Phi$  be a smooth flow on  $\Delta_{k-1}$  which preserves the invariant subspace structure determined by  $\mathcal{I}_k$ . Suppose that  $\Sigma$  is a heteroclinic network for  $\Phi$  with equilibrium

set  $\mathbf{E} = \{\mathbf{p}_i \mid i \in \ell\}$ , and finite connection set  $\mathbf{C} = \{\phi^i \mid i \in \mathbf{q}\}$ . We say that  $\Sigma$  is an  $r$ -face heteroclinic network if

- (1) For each  $i \in \ell$ , there exists an  $r - 1$ -face  $L_i$  of  $\Delta_{k-1}$ , such that
  - (a)  $\mathbf{p}_i \in \text{Int}(L_i)$  (if  $r = 1$ ,  $\mathbf{p}_i$  is a vertex of  $\Delta_{k-1}$ ).
  - (b) If  $i \neq j$ ,  $L_i \neq L_j$ .
- (2) For each connection  $\mathbf{p}_a \xrightarrow{\phi^i} \mathbf{p}_b \in \mathbf{C}$ ,
  - (a) there is an  $r$ -face  $F_i$  of  $\Delta_{k-1}$  such that  $\phi^i(\mathbb{R}) \subset \text{Int}(F_i)$ .
  - (b) If  $j \in \mathbf{q}$ ,  $i \neq j$ , then  $F_i \neq F_j$ .
- (3) For all  $\mathbf{p} \in \mathbf{E}$ ,  $d_{\mathbf{p}}^{\text{out}} = \dim(W^u(\mathbf{p}))$ .

*Remarks 2.7.* (1) If  $\ell = \mathbf{q}$  in definition 2.6, we say that  $\Sigma$  is an  $r$ -face heteroclinic cycle. In this case, we label so that each vertex  $\mathbf{p}_i$  is connected by  $\phi^i$  to  $\mathbf{p}_{i+1}$ .

(2) If  $r = 1$  in definition 2.6, we refer to  $\Sigma$  as an *edge network* (*edge cycle* if  $\Sigma$  is a heteroclinic cycle).

(3) If  $\dim(W^u(\mathbf{p})) = 1$ ,  $\mathbf{p} \in \mathbf{E}$ , then  $d_{\mathbf{p}}^{\text{out}} = 1$  and  $\Sigma$  will be a simple heteroclinic cycle: since  $\mathbf{p} \in \partial\Delta_{k-1}$ , there are no simple heteroclinic networks that are not heteroclinic cycles.

(4) If  $\mathbf{p}$  lies in the interior of an  $r - 1$ -face  $L$ , then  $d_{\mathbf{p}}^{\text{out}} \leq k - r - 1$  (exactly  $k - r$   $r$ -faces contain  $L$  and so  $\dim(W^u(\mathbf{p})) \leq k - r - 1$ ).  $\otimes$

**Lemma 2.8.** *If  $\Sigma$  is a  $r$ -face heteroclinic network, then  $\overline{W^u(\mathbf{p})}$  is contained in a  $d_{\mathbf{p}}^{\text{out}} + r - 1$ -face of  $\Delta_{k-1}$ .*

*Proof.* Using definition 2.6(3), the required face is spanned by the set of vertices of the faces given by definition 2.6(2b).  $\square$

*Remark 2.9.* Suppose  $\Sigma$  is an  $r$ -face heteroclinic network and  $\Sigma(\mathbf{E})$  is clean ( $\Sigma(\mathbf{E}) = \bigcup_{\mathbf{p} \in \mathbf{E}} W^u(\mathbf{p})$  and is compact). Let  $\mathbf{p} \in \mathbf{E}$  and  $F$  be the  $d_{\mathbf{p}}^{\text{out}} + r - 1$ -face of  $\Delta_{k-1}$  given by lemma 2.8. Then  $\partial W^u(\mathbf{p}) = \bigcup_{\mathbf{q} \in \mathbf{E}} W^s(\mathbf{q}) \cap \partial F$ . This condition is often robust and easy to satisfy for edge networks – unstable manifolds are open subsets of faces spanned by edge connections. On the other hand, apart from simple cycles, the description of clean  $r$ -face heteroclinic networks is less straightforward. Face heteroclinic networks may not have a clean realization and may not embed in a clean  $r$ -face network with the same equilibrium set.  $\otimes$

#### 2.4. Examples of edge and face cycles and networks.

**Examples 2.10.** (1) In figure 2(a) we show a simple edge cycle  $\Sigma^e$  on  $\Delta_3$  and in figure 2(b) an edge network  $\Sigma^n$  on  $\Delta_3$  that is the union of two edge cycles  $\Sigma_1, \Sigma_2$  which share the common edge  $\mathbf{v}_1 \rightarrow \mathbf{v}_2$ . The edge cycle  $\Sigma^e$  shown in (a) was first studied in the setting of  $\mathbb{Z}_2^4 \times \mathbb{Z}_4$ -equivariant dynamics [28] and the edge network  $\Sigma^e$  was first analysed

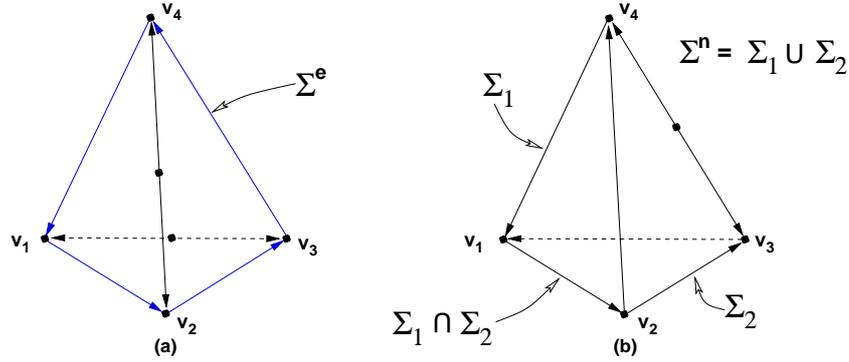


FIGURE 2. (a) Edge cycle and (b) edge network on  $\Delta_3$

by Kirk and Silber [40] & Brannath [13]. The cycle  $\Sigma^e$  is clean but the network  $\Sigma^n$  is not clean. If we change dynamics so as to remove the equilibrium on the edge joining  $\mathbf{v}_3, \mathbf{v}_4$  and instead have a connection  $\mathbf{v}_3 \rightarrow \mathbf{v}_4$ , then the resulting network, defined as the union of the unstable manifolds of the vertices of  $\Delta_3$ , is clean and robust (see [13] for this and other variations). It is straightforward to realize either cycle or network using cubic maps as in (2.7) (see [28, 11]). More generally, any edge network can be realized using cubic polynomials [11].

(2) The 2-face simple heteroclinic cycle on  $\Delta_3$  shown in figure 3

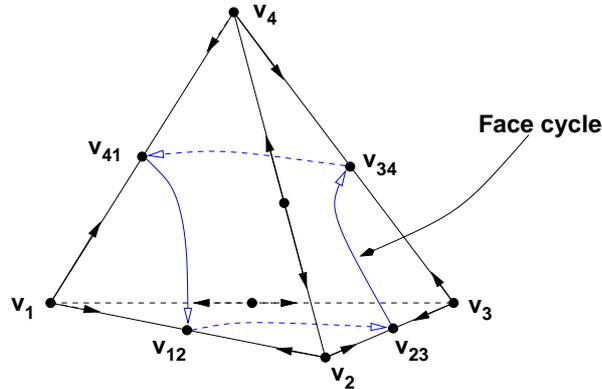


FIGURE 3. 2-face heteroclinic cycle on  $\Delta_3$

appears in [28] (we adopt the notational convention that  $\mathbf{v}_{ij}$  is an equilibrium lying in the interior of the edge  $\mathbf{F}_{12}$  joining  $\mathbf{v}_i$  to  $\mathbf{v}_j$ ). The

cycle can be realized using  $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$ -equivariant cubic polynomials. We refer to [28] for explicit polynomials realizing the cycle and remark that there will be no equilibria in the interior of any 2-faces of  $\Delta_3$ . Heteroclinic face cycles have also appeared in the ecological and game theory literature – we refer to the monograph [37, §8.3] for an example and references.

(3) Every 2-face heteroclinic network supported on  $\Delta_3$  is a heteroclinic cycle since every 2-face heteroclinic network on  $\Delta_3$  is simple (two 2-faces abut each edge of  $\Delta_3$ ). We indicate how to construct the 2-face heteroclinic network on  $\Delta_4$  used in example 1 of the introduction. The spherical simplex  $\Delta_4$  has 10 2-faces and each edge lies on 3 2-faces. We can construct a smooth vector field  $X$  on  $\Delta_4$  which has a 2-face heteroclinic network  $\Sigma$  with 9 connections between the equilibria in  $\mathbf{E} = \{\mathbf{v}_{12}, \mathbf{v}_{15}, \mathbf{v}_{23}, \mathbf{v}_{24}, \mathbf{v}_{35}, \mathbf{v}_{45}\}$  – see figure 4(a). Every vertex of  $\Gamma(\Sigma)$  has degree 3 and each connection of  $\Sigma$  will lie in a unique 2-face of  $\Delta_4$  (there is no connection in the 2-face  $\mathbf{F}_{134}$  spanned by  $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4$ ). We may choose  $X$  so that the intersections of stable and unstable manifolds of equilibria either meet in a connection lying in a 2-face or are transverse within the appropriate 3-face (one vertex will then be a sink for dynamics restricted to the 3-face). Referring to figure 4(b),  $W^u(\mathbf{v}_{23})$  will intersect  $W^s(\mathbf{v}_{35})$  transversally within the 2-face  $\mathbf{F}_{235}$ , while  $W^u(\mathbf{v}_{23})$  will intersect  $W^s(\mathbf{v}_{45})$  transversally within the 3-face  $\mathbf{F}_{2345}$ . We may choose  $X$  so that  $\Sigma(\mathbf{E})$  is robust and clean (see example 2.11 following). ✱

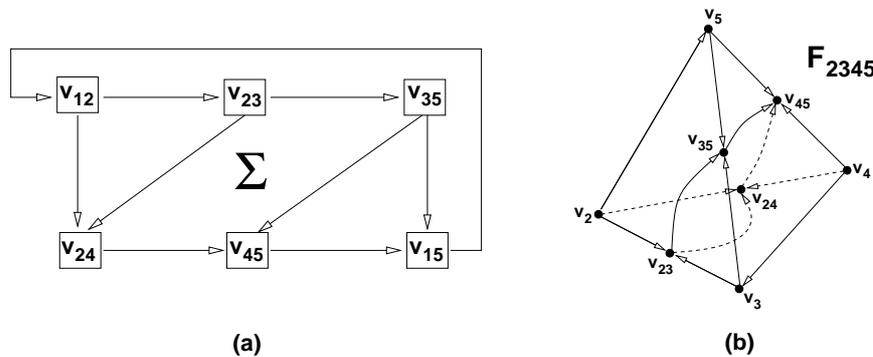


FIGURE 4. 2-face heteroclinic network on  $\Delta_4$ . (a) Robust network  $\Sigma$  with 9 connections; (b) Dynamics and equilibria on the 3-face  $\mathbf{F}_{2345}$ .

**Example 2.11** (Explicit construction using polynomials). It is obvious that we can realize 2-face cycles on  $\Delta_k$ ,  $k \geq 3$ , if we work with general

smooth vector fields of the form (2.3). Although we have not checked all the details, our expectation is every 2-face simple heteroclinic cycle can be realized on  $\Delta_k$ ,  $k \geq 3$ , using cubic polynomials of the form (2.7) (see also remark 2.13 below). It is not hard to find a cubic vector field which realizes the heteroclinic network  $\Sigma$  of examples 2.10(3). We indicate a few of the details for the reader interested in numerical experiments (we follow the notation of examples 2.10(3)). Note that we implicitly use a method, based on Bézout's theorem [25, Chapter 4, §9], to show that no 2-face  $\mathbf{F}_{ijk}$  has an equilibrium in the interior of the face (see also remark 2.13). Consider the system

$$(2.9) \quad \dot{x}_i = x_i + x_i \left( \sum_{j \neq i} \beta_{ij} x_j^2 - \alpha \|\mathbf{x}\|^2 \right), \quad i \in \mathbf{5}.$$

Assume  $\alpha \gg \max_{ij} |\beta_{ij}|$ , so that the conditions of the invariant sphere theorem [25, Chapter 5, §1] apply, and the dynamics of (2.9) is asymptotic to an invariant 4-sphere. For each  $i \in \mathbf{5}$ , (2.9) has a unique strictly positive equilibrium  $\mathbf{v}_i$  on the  $x_i$ -axis. The eigenvalue of the linearization of (2.9) at  $\mathbf{v}_i$  in direction  $\mathbf{v}_j$ ,  $i \neq j$ , is  $\beta_{ji}/\alpha$ . Noting that  $\alpha > 0$ , we assume

$$\beta_{13}, \beta_{25}, \beta_{41}, \beta_{43} < 0$$

and that all other  $\beta_{ij}$  are strictly positive. These conditions imply that there are no equilibria in the interior of the edges  $\mathbf{F}_{13}, \mathbf{F}_{25}, \mathbf{F}_{14}, \mathbf{F}_{34}$  (see figure 4(b) for the edges  $\mathbf{F}_{25}, \mathbf{F}_{34}$ ) or in the interior of any of the 2-faces (including  $\mathbf{F}_{134}$ ). Every connection of  $\Sigma$  is of the form  $\mathbf{v}_{ij} \rightarrow \mathbf{v}_{jk}$  and lies in the 2-face  $\mathbf{F}_{ijk}$  with vertices  $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$ . For example, there is a connection  $\mathbf{v}_{12} \rightarrow \mathbf{v}_{23}$  contained in  $\mathbf{F}_{123}$ . Hence the eigenvalue of linearization of (2.9) at  $\mathbf{v}_{12}$  given by the eigendirection tangent to the connection at  $\mathbf{v}_{12}$  is required to be strictly positive; that at  $\mathbf{v}_{23}$  will be strictly negative. Noting that  $\alpha \gg \max_{ij} |\beta_{ij}|$ , we find (using [25, Chapter 4, §9]) that the required eigenvalue conditions on the connection in  $\mathbf{F}_{123}$  hold if and only if

$$\frac{\beta_{31}}{\beta_{21}} + \frac{\beta_{32}}{\beta_{12}} > 1, \quad \frac{\beta_{13}}{\beta_{23}} + \frac{\beta_{12}}{\beta_{32}} < 1$$

The remaining 16 conditions for the other 8 2-faces are obtained from these inequalities by trivial permutation arguments. We claim there is a non-empty open subset of the parameters  $\beta_{ij}$  for which the inequalities hold. We start by taking

$$\beta_{13}, \beta_{31}, \beta_{25}, \beta_{52}, \beta_{14}, \beta_{41}, \beta_{34}, \beta_{43} = 0.$$

The first term in each of the 18 inequalities will then be zero. All 18 (strict) inequalities will then hold if and only if  $\beta_{21} > \beta_{51}, \beta_{15} > \beta_{45} >$

$\beta_{35}, \beta_{42} > \beta_{32} > \beta_{12}, \beta_{53} > \beta_{23}$  and  $\beta_{54} > \beta_{24}$ . Hence we can choose nonzero  $\beta_{ij}$  of the correct sign so that all inequalities hold. The vector field we have constructed has heteroclinic network  $\Sigma$  and the maximal network  $\Sigma(\mathbf{E})$  is clean.  $\ast$

**Theorem 2.12.** *Let  $k \geq 2, 0 < r \leq k - 1$ . Then  $\Delta_k$  supports (clean)  $r$ -face heteroclinic cycles. If  $r \leq k - 2$ ,  $\Delta_k$  supports (clean)  $r$ -face heteroclinic networks. In all cases, cycles and networks can be realized using smooth vector fields on  $O_{k+1}$  of form (2.3).*

*Proof.* We omit details of the routine construction of  $r$ -face heteroclinic cycles – see [21, Chapter 7] for some examples in a piecewise smooth setting. The existence of clean  $r$ -face heteroclinic networks is straightforward – see examples 2.10(3).  $\square$

*Remark 2.13.* We can require in theorem 2.12 that the nodes have identical intrinsic dynamics defined by  $f(x) = x - \alpha x^3, \alpha > 0$ . We conjecture that the conditions of the theorem can be satisfied using cubic vector fields of the form (2.7). A useful tool for the verification of the conjecture is that generically there is at most one equilibrium in the interior of each face and each equilibrium can be assumed hyperbolic. See [25, Chapter 4, §9] for the general method which depends on Bezout’s theorem and the invariant subspace structure.  $\circledast$

**2.5. Lattice structure on  $\mathcal{I}_k$ .** We describe the lattice structure on  $\mathcal{I}_k$  defined by intersection and vector space sum.

Let  $\prec$  denote the partial order on  $\mathcal{I}_k$  defined by reverse inclusion:  $V \prec W$  if  $W \subset V$ . The unique maximal and minimal elements of  $\mathcal{I}_k$  are  $\mathbf{0} = H_{1\dots k}$  and  $\mathbf{R}$  respectively. We recall some standard definitions from lattice theory (see Davey and Priestly [14] for more details). The operations of *join*  $\vee$  (least upper bound) and *meet*  $\wedge$  (greatest lower bound) are defined on  $\mathcal{I}_k$  by

$$\begin{aligned} V \vee W &= V \cap W, \\ V \wedge W &= V + W \text{ (vector space sum),} \end{aligned}$$

where  $V, W \in \mathcal{I}_k$ . It is trivial to verify that  $(\mathcal{I}_k, \vee, \wedge)$  has the structure of a (complete) lattice and that for all  $V \in \mathcal{I}_k$  we have

$$V \vee \mathbf{0} = \mathbf{0}, \quad V \wedge \mathbf{R} = \mathbf{R}.$$

When we relate SLF and coupled identical cell systems, we use a natural lattice structure on the set of synchrony subspaces. This structure will sometimes (not always) relate to the natural lattice structure on  $\mathcal{I}_k$ .

### 3. COUPLED IDENTICAL CELL SYSTEMS – ASYMMETRIC INPUTS

We refer to Stewart, Golubitsky *et al.* [59, 29, 31, 30] for general theory and background on coupled cell systems. Here we review the formalism we use for networks of coupled identical cells. We use a ‘flow-chart’ formalism, similar to that used in electrical and computer engineering, that fits well with our intended applications of constructing networks with particular properties. We give necessary definitions, establish notational conventions and refer the reader to [3, 6, 4, 23] for more details, discussion and examples. We use the term coupled cell *network* to refer to the abstract object – a directed network graph codifying the connection structure with vertices corresponding to nodes – and generally use the term coupled cell *system* when we view the coupled cell network as a system of coupled differential equations [3]. We frequently abuse notation by letting  $\mathcal{N}$  refer to both the abstract network structure as well as a realization as a coupled cell system.

Let  $\mathcal{N}$  be a coupled cell system consisting of  $m \geq 2$  identical nodes (or ‘cells’)  $N_1, \dots, N_m$  each with phase space  $M$ . Denote the state variable for node  $N_i$  by  $\mathbf{x}_i \in M$ . Let  $\mathbf{M} = M^m$  denote the network phase space. If each cell has  $p$  inputs, dynamics will be given by a system of differential equations of the form

$$(3.10) \quad \dot{\mathbf{x}}_i = f(\mathbf{x}_i; \mathbf{x}_{I_i(1)}, \dots, \mathbf{x}_{I_i(p)}), \quad i \in \mathbf{m},$$

where  $I_i : \mathbf{p} \rightarrow \mathbf{m}$ ,  $i \in \mathbf{m}$ . We refer to  $f : M \times M^p \rightarrow TM$  as the *network map* and the corresponding vector field  $\mathbf{F} : \mathbf{M} \rightarrow T\mathbf{M}$  defined by (3.10) as the *network vector field*.

Under explicitly indicated to the contrary, we assume

- (1) There are no self loops:  $i \notin I_i(\mathbf{p})$ , all  $i \in \mathbf{m}$ .
- (2) Inputs are *asymmetric*:  $f(\mathbf{x}; \mathbf{x}_1, \dots, \mathbf{x}_p)$  is not symmetric in any subset of the variables  $\mathbf{x}_1, \dots, \mathbf{x}_p$ .

*Remark 3.1.* The assumption of asymmetric inputs is a major simplification that often allows us to reduce proofs to the case where cells have a single input. From the application point of view (for example, in neuroscience) what seems to be most appropriate are asymmetric inputs that are not too far from symmetric.  $\otimes$

Associated with the coupled cell network  $\mathcal{N}$ , there is the *network graph*  $G(\mathcal{N})$ . This consists of  $m$ -vertices  $v_i$ , corresponding to the nodes  $N_i$ , and  $mp$  directed edges  $e_\alpha, v_{I_i(s)} \rightarrow v_i$ ,  $i \in \mathbf{m}$ ,  $s \in \mathbf{p}$ . We always assume that  $\mathcal{N}$  (that is,  $G(\mathcal{N})$ ) is connected. Usually,  $\mathcal{N}$  will be strongly connected (every vertex pair  $v_i, v_j$  lies on a cycle). At this level of generality, the network  $\mathcal{N}$  and graph  $G(\mathcal{N})$  represent the same structure.

Since inputs are asymmetric (condition (2)), the graph has  $p$  distinct edge types. In terms of the system, each cell has  $p$  *input types*. If  $\alpha \in \mathbf{p}$ ,  $i \in \mathbf{m}$ , let  $d_{\alpha,i}^{\text{in}}$  denote the *in-degree* of vertex  $i$  for inputs of type  $\alpha$ . Since inputs are asymmetric,  $d_{\alpha,i}^{\text{in}} = 1$  for all  $\alpha \in \mathbf{p}$ ,  $i \in \mathbf{m}$ .

**3.1. Synchrony classes and synchrony subspaces.** Let  $\mathcal{P}(m)$  denote the set of partitions of  $\mathcal{N}$ . If  $\mathcal{X} = \{X^j \mid j \in \ell\} \in \mathcal{P}(m)$ , let  $s(j)$  be the number of cells in  $X^j$ ,  $j \in \ell$ . The partition is *nontrivial* if  $\ell < m$  (at least one  $s(j)$  is strictly bigger than 1). Label cells in  $X^j$  as  $N_{i_1^j}, \dots, N_{i_{s(j)}^j}$ , where  $i_1^j < \dots < i_{s(j)}^j$ , and set  $J^j = \{i_1^j, \dots, i_{s(j)}^j\}$ . We have  $\cup_{j \in \ell} J^j = \mathbf{m}$ .

Now view  $\mathcal{N}$  as a coupled cell system. If  $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_m) \in \mathbf{M}$  denotes the state of the network, we may group states according to the partition  $\mathcal{X}$  and write  $\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^\ell)$ , where  $\mathbf{x}^j = (\mathbf{x}_{i_1^j}, \dots, \mathbf{x}_{i_{s(j)}^j}) \in M^{s(j)}$  will denote the state of the  $s(j)$  cells in  $X^j$ . Define

$$\Delta_j = \{\mathbf{x}^j \mid \mathbf{x}_{i_1^j} = \dots = \mathbf{x}_{i_{s(j)}^j}\} \subset M^{s(j)}, \quad j \in \ell,$$

and let

$$\Delta(\mathcal{X}) = \prod_{j \in \ell} \Delta_j = \{\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^\ell) \mid \mathbf{x}^j \in \Delta_j, \quad j \in \ell\}.$$

denote the corresponding *polydiagonal* subspace.

**Definition 3.2** ([6, 3], cf [59, 31, 30]). The partition  $\mathcal{X}$  is a *synchrony class* for the coupled cell network  $\mathcal{N}$  if the subspace  $\Delta(\mathcal{X})$  is dynamically invariant for every realization of  $\mathcal{N}$  as a coupled cell system. In terms of coupled cell systems, if  $\mathcal{X}$  is a synchrony class then the invariant subspace  $\Delta(\mathcal{X}) \subset \mathbf{M}$  is a *synchrony subspace* (a *polysynchronous* subspace in the terminology of [30]).

Let  $\mathcal{D}(m) = \{\Delta(\mathcal{X}) \mid \mathcal{X} \in \mathcal{P}(m)\}$  denote the set of all polydiagonal subspaces of  $\mathbf{M}$ . Clearly  $\mathcal{D}(m)$  is independent of  $M$  and  $\mathcal{D}(m) \approx \mathcal{P}(m)$ .

If  $\mathcal{X} = \{\{N_1, \dots, N_m\}\}$ , then  $\mathcal{X}$  is always a synchrony class: the *maximal synchrony class*  $\mathfrak{s}_0$ . The associated invariant space is the diagonal  $\mathbf{S}_0 \stackrel{\text{def}}{=} \Delta(\mathbf{M})$  and is referred to as the *minimal synchrony subspace*. It is the synchrony subspace of minimal dimension for every realization of a coupled cell network. If  $\mathcal{T} = \{\{N_1\}, \dots, \{N_m\}\}$  is the trivial partition of  $\mathbf{m}$ , then  $\mathcal{T}$  defines the *null synchrony class*  $\mathfrak{s}_\infty$  and  $\mathbf{M} = \Delta(\mathcal{T}) \stackrel{\text{def}}{=} \mathbf{S}_\infty$  defines the *null* or *trivial* synchrony subspace.

We give a simple and very useful criterion for synchrony subspaces (see [26], [30, §7] for greater generality).

**Proposition 3.3.** *Let  $\mathcal{N}$  be a coupled cell network as above. Suppose that  $\mathcal{X} = \{X^j \mid j \in \ell\}$  is a partition of  $\mathcal{N}$ . Then  $\mathcal{X}$  is a synchrony class iff for all  $i, j \in \mathbf{m}$  and every input type  $\alpha \in \mathbf{p}$ , either no cell in  $X^i$  receives an input of type  $\alpha$  from a cell in  $X^j$  or else every cell in  $X^i$  receives exactly one input of type  $\alpha$  from a cell in  $X^j$ .*

*Remark 3.4.* Let  $d_{\alpha,i}^{\text{in}}(j)$  denote the in-degree at node  $i$  for inputs of type  $\alpha$  originating from cells in  $X^j$ . Proposition 3.3 implies that  $\mathcal{X}$  is a synchrony class iff for every input type  $\alpha$  and all  $j, k \in \ell$ ,  $d_{\alpha,i}^{\text{in}}(j)$  is constant on  $X^k$ .  $\otimes$

As a straightforward consequence of proposition 3.3 and remark 3.4, we have a useful result that allows us to combine synchrony subspaces.

**Proposition 3.5** ([6, Theorem 3.8]). *Let  $\mathcal{N}$  be a coupled cell network with asymmetric inputs. Suppose that the partitions  $\mathcal{X} = \{X_i \mid i \in I\}$ ,  $\mathcal{Y} = \{Y_j \mid j \in J\}$  both define synchrony classes of  $\mathcal{N}$ . Then the intersection partition  $\mathcal{X} \cap \mathcal{Y} = \{X_i \cap Y_j \mid i \in I, j \in J\}$  defines a synchrony class of  $\mathcal{N}$ .*

*Remark 3.6.* Proposition 3.5 fails if cells have symmetric inputs – see [6, §7] (or later this section). We refer to Stewart [58] for the general theory when there are symmetric inputs (see also [7]).  $\otimes$

Let  $\mathcal{S} = \mathcal{S}(\mathcal{N})$  denote the set of all synchrony subspaces of the coupled cell system  $\mathcal{N}$ . We have a partial order  $\prec$  on  $\mathcal{S}$  defined by reverse inclusion.

**Proposition 3.7.**  *$\mathcal{S}$  has the natural structure of a complete lattice  $(\mathcal{S}, \vee, \wedge)$  with join and meet defined by*

- (1)  $V \vee W = V \cap W$ .
- (2)  $V \wedge W$  is the synchrony subspace defined by the intersection of the partitions defining  $V$  and  $W$ .

*The maximal element of  $(\mathcal{S}, \vee, \wedge)$  is  $\mathbf{S}_0 = \Delta(\mathbf{M})$  (the minimal synchrony subspace). The minimal element of  $\mathcal{S}$  is  $\mathbf{S}_\infty = \mathbf{M}$  (corresponding to the null synchrony class  $\mathfrak{s}_\infty$ ).*

*Proof.* Immediate from proposition 3.5.  $\square$

*Remark 3.8.* The meet and join operations are naturally defined on the set  $\mathcal{D}(m) \approx \mathcal{P}(m)$  of all polydiagonal subspaces of  $\mathbf{M}$  and so  $(\mathcal{P}(m), \vee, \wedge)$  has the structure of a complete lattice. The set  $\mathcal{S}(\mathcal{N})$  of all synchrony subspaces of a coupled identical cell system with  $m$  nodes is a sublattice of  $(\mathcal{P}(m), \vee, \wedge)$  provided that cells have asymmetric inputs [58]. In what follows we usually omit reference to the null synchrony subspace  $\mathbf{S}_\infty$ .  $\otimes$

*Notation for synchrony subspaces.* It is useful to introduce some simplified notation for synchrony subspaces. Let  $\mathcal{X} = \{X^j \mid j \in \ell\}$  be a nontrivial partition of  $\mathcal{N}$ . After relabelling we may assume that for some  $q \leq \ell$ , we have  $s(1), \dots, s(q) > 1$ ,  $s(i) = 1$ ,  $i > q$ . For  $j \in \mathbf{q}$ , we have  $\Delta_j = \{\mathbf{x}^j \mid \mathbf{x}_{i_1^j} = \dots = \mathbf{x}_{i_{s(j)}^j}\}$ . With these conventions, we write

$$\Delta(\mathcal{X}) = (i_1^1 i_2^1 \dots i_{s(1)}^1 \mid \dots \mid i_1^q \dots i_{s(q)}^q).$$

**Examples 3.9.** (1)  $\mathbf{S}_0 = \Delta(\mathbf{M}) = (12 \dots m)$ .

(2) The notation is naturally compatible with the meet and join operations. For example, if  $(125|89)$ ,  $(127|48)$  and  $(46)$  are synchrony subspaces of an identical cell network, then

- (1)  $(125|89) \vee (127|48) = (1257|489)$ .
- (2)  $(125|89) \wedge (127|48) = (12)$ .
- (3)  $(125|89) \vee (46) = (125|89|46)$ ,  $(127|48) \vee (46) = (127|468)$ .
- (4)  $(125|89) \wedge (46) = (127|48) \wedge (46) = \mathbf{S}_\infty$ . \*

Let  $(\mathcal{S}, \vee, \wedge)$  be a finite lattice with maximal element  $\mathbf{S}_0$  and minimal element  $\mathbf{S}_\infty$ . If  $\mathcal{G}$  is a nonempty subset of the finite lattice  $(\mathcal{S}, \vee, \wedge)$ , we define  $\langle \mathcal{G} \rangle$  to be the subset of  $\mathcal{S}$  generated from  $\mathcal{G}$ , using the operations  $\vee, \wedge$ , together with the minimal element  $\mathbf{S}_\infty$ .

**Definition 3.10.** (Notation and assumptions as above.) Let  $\mathcal{G}$  be a subset of the finite lattice  $(\mathcal{S}, \vee, \wedge)$  and suppose that  $\mathbf{S}_\infty \notin \mathcal{G}$ . The set  $\mathcal{G}$  is a *generating set* for  $(\mathcal{S}, \vee, \wedge)$  if  $\langle \mathcal{G} \rangle = \mathcal{S}$ . If every other generating set for  $(\mathcal{S}, \vee, \wedge)$  contains at least as many elements as  $\mathcal{G}$ , then  $\mathcal{G}$  is a *basis* for  $\mathcal{S}$ .

**Example 3.11.** The lattice  $(\mathcal{D}(m), \vee, \wedge)$  has basis  $\mathcal{B} = \{(1, j) \mid j = 2, \dots, m\} \cup \{(23 \dots m)\}$ . To see this, observe that  $(1j_1 \dots j_s) \in \langle \mathcal{B} \rangle$ ,  $2 \leq j_1 < j_2 < \dots < j_s \leq m$ , using only the  $\vee$  operation on  $\{(1, j) \mid j = 2, \dots, m\}$ . Hence,  $(j_1 \dots j_s) = (1j_1 \dots j_s) \wedge (2 \dots m) \in \langle \mathcal{B} \rangle$ ,  $2 \leq j_1 < j_2 < \dots < j_s \leq m$ . We easily obtain the remaining polydiagonal subspaces using only the  $\vee$  operation and so  $\mathcal{B}$  is a generating set. We leave it to the reader to verify that every generating set has at least  $m$  elements and so  $\mathcal{B}$  is a basis (in this case, if  $m > 2$ ,  $\mathbf{S}_\infty$  lies in the set generated from  $\mathcal{B}$  using the operations  $\vee, \wedge$ ). \*

Next we give conditions on an  $m$  cell network  $\mathcal{N}$  for  $\mathcal{S}(\mathcal{N}) = \mathcal{D}(m)$ .

**Lemma 3.12.** *Let  $\mathcal{N}$  be a coupled identical cell network with  $m \geq 3$  cells and lattice of synchrony subspaces  $\mathcal{S}$ . A necessary condition for  $\mathcal{S} = \mathcal{D}(m)$  is that  $\mathcal{N}$  has self-loops.*

*Proof.* By remark 3.4, it is enough to consider the case where cells have just one input type. Suppose  $\mathcal{S} = \mathcal{D}(m)$  and  $\mathcal{N}$  is connected.

By proposition 3.3, there exists  $j \in \mathbf{m}$  such that there is a connection  $j \rightarrow i$ , for all  $i \in \mathbf{m}$ . Hence  $N_j$  has a self-loop. If  $\mathcal{N}$  is not connected, then every connection will be a self-loop.  $\square$

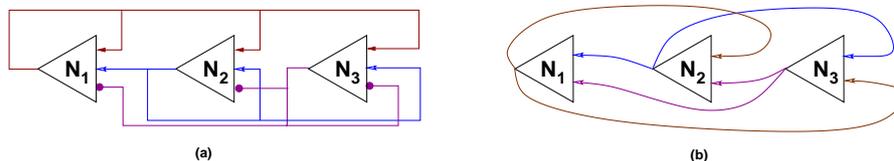


FIGURE 5. 3 cell connected networks with the maximal number of synchrony subspaces: (123), (12), (13) and (23): (a) Asymmetric inputs; (b) Symmetric inputs.

**Example 3.13.** It follows from lemma 3.12 that a strongly connected  $m$  cell network without self loops has  $\mathcal{S} \subsetneq \mathcal{D}(m)$ . For example, if  $m = 3$ , there are at most three synchrony subspaces (see [6] and also below). If we allow self-loops, then an  $m$ -cell identical cell network can be strongly connected and have  $\mathcal{S} = \mathcal{D}(m)$ , provided that cells have at least  $m$  inputs. See figure 5(a) for the case  $m = 3$ . If we allow symmetric inputs, it is easy to construct an  $m$ -cell identical cell network with  $\mathcal{S} = \mathcal{D}(m)$ , provided that cells have at least  $m - 1$  inputs. See figure 5(b) for case  $m = 3$ . From our perspective, the networks of figure 5 are not particularly interesting as neither can support robust heteroclinic cycles.  $\ast$

In [6] it was shown that up to network equivalence there were exactly two strongly connected identical three cell networks, asymmetric inputs, supporting robust heteroclinic cycles (one network had self-loops). It was shown [26] that every heteroclinic network  $\mathcal{N}$  with  $q$  connections could be realized as a robust heteroclinic network in a strongly connected identical cell network  $\mathcal{P}_{q+1}$  consisting of  $q + 1$  cells, each with  $q$  asymmetric inputs. Rather than restate the general result, we illustrate with an embedding of the RPS heteroclinic 3-cycle in  $\mathcal{P}_4$ .

**Example 3.14.** In figure 6, we show the network  $\mathcal{P}_4$  constructed in [26]. Every synchrony subspace of  $\mathcal{P}_4$  can be written as a join of generating synchrony subspaces. Assuming cells have 1-dimensional dynamics, with phase space  $\mathbb{R}$ , the heteroclinic 3 cycle  $a \rightarrow b \rightarrow c \rightarrow a$  can be realized as a robust heteroclinic cycle  $\Sigma$  in  $\mathcal{P}_4$ . The result follows from the main theorem in [26]: the equilibria  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \Sigma$  will lie on the synchrony subspace (1234); the connection  $\mathbf{a} \rightarrow \mathbf{b}$  will lie in (123); the connection  $\mathbf{b} \rightarrow \mathbf{c}$  will lie in (124); and the connection  $\mathbf{c} \rightarrow \mathbf{a}$

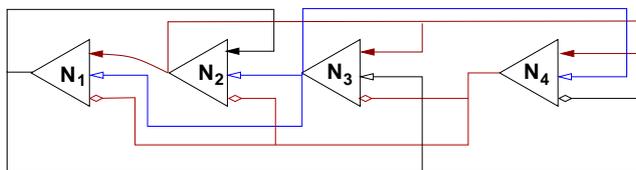


FIGURE 6. A 4 cell network with synchrony subspaces generated by (12), (13) and (14).

will lie in (134). Each equilibrium will have a 1-dimensional unstable manifold and along each connection, one of the nodes desynchronizes from the other three (synchronized) nodes. Note that  $\Sigma$  is not simple according to the strict definition we gave in section 1:  $\Sigma$  only contains one component of  $W^u(\mathbf{p}) \setminus \{\mathbf{p}\}$  for each equilibrium point  $\mathbf{p} \in \Sigma$ . Subsequently, our approach will be to work on a flow-invariant ‘fundamental domain’  $D_+ \subset \mathbb{R}^4$  for the coupled cell system so that each  $W^u(\mathbf{p}) \setminus \{\mathbf{p}\}$  has just one component in  $D_+$ . For  $\mathcal{P}_4$ , define  $D_+ = \{\mathbf{x} \in \mathbb{R}^4 \mid x_i \geq x_1, i = 2, 3, 4\}$  and note that we can construct  $\Sigma$  so that  $\Sigma \subset D_+$  and  $\Sigma$  is simple if we restrict dynamics to  $D_+$ . Henceforth, we use the term simple heteroclinic cycle (or network) in this restricted sense (we give a formal definition in section 5).  $\ast$

In example 3.14, the RPS heteroclinic cycle lying in  $\Delta_2 \subset O_3$  is realized as a heteroclinic 3 cycle in a four identical cell system. In the remaining sections, we describe a far reaching generalization of this simple result and prove that a heteroclinic  $r$ -face network  $\Sigma$  in an SLF system on  $O_k$  can be realized (in many ways) as a robust heteroclinic network in a strongly connected coupled identical cell system with  $k+1$  cells, each with  $k$  asymmetric inputs, and no self loops. Each connection will be associated to a unique pattern of desynchronization and resynchronization (at the end point equilibria); these patterns correspond to the connection structure for  $\Sigma$  viewed as an SLF system.

We conclude this section with an example illustrating some of the issues that arise if we allow symmetric inputs (see also the concluding comments at the end of the article).

**Examples 3.15.** (1) The presence of many invariant subspaces can often lead to the existence of robust heteroclinic networks. However, if cells have symmetric inputs, this intuition may fail. There are two problems: (a) ‘too many’ invariant subspaces leading to multiplicities in eigenvalues; (b) symmetry in the inputs leads to fewer free

parameters and this can make it harder to obtain specified linearizations at equilibria. For example, consider the network shown in figure 5(b) and assume 1-dimensional node dynamics. The synchrony subspaces (12), (13) and (23) all contain the line (123). Since the subspaces are all flow-invariant, this suggests that if  $\mathbf{p} \in (123)$  is an equilibrium of the network vector field  $\mathbf{F}$ , then the two eigenvalues of  $D\mathbf{F}(\mathbf{p})$  corresponding to eigendirections transverse to (123) should be equal. Computing we find that if the network vector field  $\mathbf{F}$  is given by  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $(x, y, z) \mapsto f(x; y, z)$  (symmetric in  $y, z$ ) and we set  $\alpha = \frac{\partial f}{\partial x}(\mathbf{p})$ ,  $\beta = \frac{\partial f}{\partial y}(\mathbf{p}) = \frac{\partial f}{\partial z}(\mathbf{p})$ , then  $D\mathbf{F}(\mathbf{p})$  has eigenvalues  $\alpha - \beta$  (multiplicity 2) and  $\alpha + 2\beta$ . Hence there can be no heteroclinic loops lying in two of the planes (12), (13) and (23) connecting saddle equilibria on (123). Moreover, there are only two free parameters  $\alpha, \beta$  and so it is not possible to choose three eigenvalues independently at an equilibrium on (123). Similar remarks apply to the  $(m - 1)$ -symmetric input generalization of figure 5(b) to an  $m$  cell network,  $m \geq 3$ .

(2) Notwithstanding the previous example, it is certainly possible to find robust heteroclinic cycles in identical cell networks with *some* symmetric inputs. We give an example based on [23],[6, §7.1]. Consider the 6 identical cell system  $\mathcal{N}$  with network equations

$$\begin{aligned} \dot{x}_1 &= f(x_1; x_3, x_6), \quad \dot{x}_2 = f(x_2; x_4, x_5), \quad \dot{x}_3 = f(x_3; x_5, x_1), \\ \dot{x}_4 &= f(x_4; x_5, x_1), \quad \dot{x}_5 = f(x_5; x_3, x_2), \quad \dot{x}_6 = f(x_6; x_3, x_2). \end{aligned}$$

If cells have symmetric inputs –  $f(x; y, z) = f(x; z, y)$  – there are fourteen synchrony subspaces:

$$\begin{aligned} \mathbf{S}_0 &= (123456), \quad \mathbf{S}_1 = (12|3456), \quad \mathbf{S}_2 = (34|1256), \\ \mathbf{S}_3 &= (12|34|56), \quad \mathbf{S}_4 = (34|256), \quad \mathbf{S}_5 = (34|56), \quad \mathbf{S}_6 = (34), \\ \mathbf{S}_7 &= (56), \quad \mathbf{S}_8 = (12|35|46), \quad \mathbf{S}_9 = (134|56), \quad \mathbf{S}_{10} = (1234|56), \\ \mathbf{S}_{11} &= (13|56), \quad \mathbf{S}_{12} = (25|34), \quad \mathbf{S}_{13} = (134|256). \end{aligned}$$

(For asymmetric inputs,  $\mathbf{S}_8, \dots, \mathbf{S}_{13}$  are not synchrony subspaces.) The network  $\mathcal{N}$  also has a  $\mathbb{Z}_2$ -symmetry generated by the cell permutation  $N_1 \leftrightarrow N_2$ ,  $N_3 \leftrightarrow N_5$ ,  $N_4 \leftrightarrow N_6$ . Proposition 3.5 fails if inputs are symmetric: take  $\mathcal{X}$  to be the partition defined by  $\mathbf{S}_8$  and  $\mathcal{Y}$  the partition defined by  $\mathbf{S}_3$ .

Under the assumption of symmetric inputs, we show that we can choose network dynamics determined by  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  such that there is a robust, simple (in the sense of example 3.14) and asymptotically stable heteroclinic cycle  $\Sigma$  contained in  $\mathbf{S}_6 \cup \mathbf{S}_7$  with equilibria  $\mathbf{p}, \bar{\mathbf{p}} \in \mathbf{S}_5 = \mathbf{S}_6 \cap \mathbf{S}_7$ .

Let  $\mathbf{p} = (a_1, a_2, b, b, c, c) \in \mathbf{S}_5$  and define

$$\begin{aligned} \frac{\partial f}{\partial x}(a_1; b, c) &= \alpha_1, & \frac{\partial f}{\partial y}(a_1; b, c) &= \frac{\partial f}{\partial z}(a_1; b, c) = \beta \\ \frac{\partial f}{\partial x}(a_2; b, c) &= \alpha_2, & \frac{\partial f}{\partial y}(a_2; b, c) &= \frac{\partial f}{\partial z}(a_2; b, c) = \gamma \\ \frac{\partial f}{\partial x}(b; c, a_1) &= \alpha_3, & \frac{\partial f}{\partial y}(b; c, a_1) &= \frac{\partial f}{\partial z}(b; c, a_1) = \delta \\ \frac{\partial f}{\partial x}(c; b, a_2) &= \alpha_4, & \frac{\partial f}{\partial y}(c; b, a_2) &= \frac{\partial f}{\partial z}(c; b, a_2) = \eta. \end{aligned}$$

The linearization  $J(\mathbf{p})$  of the network vector field  $\mathbf{F}$  at  $\mathbf{p}$  is given by

$$(3.11) \quad J(\mathbf{p}) = \begin{pmatrix} \alpha_1 & 0 & \beta & 0 & 0 & \beta \\ 0 & \alpha_2 & 0 & \gamma & \gamma & 0 \\ \delta & 0 & \alpha_3 & 0 & \delta & 0 \\ \delta & 0 & 0 & \alpha_3 & \delta & 0 \\ 0 & \eta & \eta & 0 & \alpha_4 & 0 \\ 0 & \eta & \eta & 0 & 0 & \alpha_4 \end{pmatrix}.$$

Noting that  $\mathbf{S}_5$  is  $J(\mathbf{p})$ -invariant, the matrix of  $J(\mathbf{p})|_{\mathbf{S}_5} = J_{BC}$  is

$$(3.12) \quad J_{BC} = \begin{pmatrix} \alpha_1 & 0 & \beta & \beta \\ 0 & \alpha_2 & \gamma & \gamma \\ \delta & 0 & \alpha_3 & \delta \\ 0 & \eta & \eta & \alpha_4 \end{pmatrix}.$$

We have similar expressions for  $J_B = J(\mathbf{p})|_{\mathbf{S}_6}$  and  $J_C = J(\mathbf{p})|_{\mathbf{S}_7}$ . For a simple cycle, all eigenvalues of  $J_{BC}(\mathbf{p})$  must have strictly negative real part and the eigenvalues corresponding to the eigenlines not contained in  $\mathbf{S}_5$  must be real, nonzero and of opposite sign. Since  $\mathbf{S}_6 \supset \mathbf{S}_5$  and both spaces are  $J(\mathbf{p})$ -invariant, we see that the eigenvalue of  $J_B$  with eigenline transverse to  $\mathbf{S}_5$  must be  $\alpha_4$  – the sum of the eigenvalues of  $J_{BC}(\mathbf{p})$  is  $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ , while the sum of the eigenvalues of  $J_B$  is  $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4$ . Similarly the eigenvalue of  $J_C$  with eigenline transverse to  $\mathbf{S}_5$  must be  $\alpha_3$ . Hence for a simple cycle with saddle at  $\mathbf{p}$  we must have  $\alpha_3\alpha_4 < 0$  and all eigenvalues of  $J_{BC}$  strictly negative. It is not hard to choose  $\alpha_1, \dots, \alpha_4, \beta, \gamma, \delta, \eta$  to achieve this. For example, if we take  $\alpha_1 = \alpha_2 = \alpha_4 = -1$ ,  $\alpha_3 = 1$ ,  $\beta = 0$ ,  $\gamma = 2$ ,  $\delta = 1$  and  $\eta = -2$ , then the characteristic equation of  $J_{BC}$  is  $\lambda^3 + \lambda^2 + 6\lambda + 1 = 0$  and so by the Routh-Hurwitz criterion all roots of the characteristic equation have strictly negative real parts. With a little more work, we may also require that the weakest contracting eigenvalue dominates the expanding eigenvalue. This is sufficient to guarantee asymptotic stability of the cycle we construct.

Our arguments show there is no obstruction to constructing a network vector field which has a hyperbolic saddle point on  $\mathbf{S}_5$  with 1-dimensional stable manifold lying in either  $\mathbf{S}_6 \setminus \mathbf{S}_5$  or  $\mathbf{S}_7 \setminus \mathbf{S}_5$ . We briefly sketch how to construct a network vector field with connecting trajectories giving a heteroclinic cycle between two hyperbolic saddle points on  $\mathbf{S}_5$ . Pick  $\mathbf{p} = (a_1, a_2, b, b, c, c)$ ,  $\bar{\mathbf{p}} = (\bar{a}_1, \bar{a}_2, \bar{b}, \bar{b}, \bar{c}, \bar{c}) \in \mathbf{S}_5$  with  $\{a_1, a_2, b, c\} \cap \{\bar{a}_1, \bar{a}_2, \bar{b}, \bar{c}\} = \emptyset$ . Choose hyperbolic linear maps  $A, \bar{A}$  at  $\mathbf{p}, \bar{\mathbf{p}}$  so that unstable eigendirections are 1-dimensional and lie in  $\mathbf{S}_6, \mathbf{S}_7$  respectively. Using  $A, \bar{A}$  we construct the network vector field on a neighbourhood of  $\mathbf{S}_5$  in  $\mathbf{M}$  with index 1 saddle points at  $\mathbf{p}, \bar{\mathbf{p}}$ . Choose connections  $\mathbf{p} \rightarrow \bar{\mathbf{p}} \subset \mathbf{S}_6, \bar{\mathbf{p}} \rightarrow \mathbf{p} \subset \mathbf{S}_7$  which match with the eigenlines transverse to  $\mathbf{S}_5$  near the equilibria  $\mathbf{p}, \bar{\mathbf{p}}$ . Using the method of [26], perturb connections so that the network vector field is well defined on the connections – regard the connections as subsets of  $\mathbf{S}_6 = \mathbf{S}_7 = \mathbb{R}^5$ . Extend the network vector field smoothly to all of  $\mathbb{R}^6$ .  $\ast$

#### 4. THE SYNCHRONIZATION TRANSFORM

We describe a general method for transforming invariant subspaces and heteroclinic networks of SLF networks to synchrony subspaces and heteroclinic networks of coupled identical cell networks. Henceforth we assume a coupled cell network  $\mathcal{N}$  satisfies

- (1) Identical cells with asymmetric inputs.
- (2) The network graph is strongly connected and without self loops.
- (3) If  $\mathcal{N}$  has  $k$  cells, each cell has at most  $k - 1$  asymmetric inputs.
- (4) Node dynamics is 1-dimensional (phase space  $\mathbb{R}$ ).

Suppose that  $k \geq 2$  and let  $\mathcal{D}(k)$  denote the lattice of all proper polydiagonal subspaces of  $\mathbf{R} = \mathbb{R}^k$  with minimal element  $\mathbf{S}_\infty = \mathbf{R}$  and maximal element (minimal synchrony subspace)  $\mathbf{S}_0 = \Delta(\mathbf{R})$ . Let  $\mathcal{L}(k)$  be the sublattice of  $\mathcal{I}_k$  generated by the subspaces  $\{H_i \mid i \in \mathbf{k}, i \neq k\}$ . The maximal element  $\mathbf{V}_0$  of  $\mathcal{L}(k)$  is the subspace  $H_{1\dots k-1}$  – the  $x_k$ -axis. The minimal element is  $\mathbf{R}$ .

**Definition 4.1.** (Notation and assumptions as above.) A *synchronization transform* of weight  $s$  is a triple  $(T, \mathcal{B}, \mathcal{S})$  where

- (1)  $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$  is a linear isomorphism: the *synchronization map*.
- (2)  $\mathcal{S}$  is a complete sublattice of  $\mathcal{D}(k)$  with maximal element  $\mathbf{S}_0$ , and minimal element  $\mathbf{S}_\infty$ .
- (3)  $\mathcal{B} = \{W_1, \dots, W_s\}$  is a subset of  $\mathcal{L}(k)$  and  $\{T(W_i) \mid i \in \mathbf{s}\}$  is a basis of  $\mathcal{S}$ .
- (4)  $T(\mathbf{V}_0) = \mathbf{S}_0 \in \mathcal{S}$ .
- (5) If  $V \in \mathcal{D}(k) \setminus \mathcal{S}$ , then  $T^{-1}(V) \notin \mathcal{L}(k)$ .

A coupled  $k$  identical cell network  $\mathcal{N}$  supports (or has) a synchronization transform if we can find a synchronization transform  $(T, \mathcal{B}, \mathcal{S})$  such that  $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$  and  $\mathcal{S} = \mathcal{S}(\mathcal{N})$ .

*Remarks 4.2.* (1) If  $(T, \mathcal{B}, \mathcal{S})$  is a synchronization transform then  $\mathbf{S}_0 \in \mathcal{S}$  by (4). In particular,  $s \geq 1$  and either  $\mathbf{V}_0 \in \mathcal{B}$  or  $\mathbf{V}_0$  lies in the sublattice generated by  $\mathcal{B}$ .

(2) Although  $\mathbf{S}_\infty \in \mathcal{S}$  (see the preamble to definition 3.10), we do not require that  $\mathbf{S}_\infty$  can be written in terms of  $T(W_1), \dots, T(W_s)$  using the  $\vee, \wedge$  operations. Of course,  $T(\mathbf{R}) = \mathbf{S}_\infty$  by (1). In what follows we generally omit reference to the minimal classes  $\mathbf{R}, \mathbf{S}_\infty$ .

(3) It is neither required, nor true, that  $T$  induces a lattice isomorphism between  $\mathcal{S}$  and the sublattice of  $\mathcal{L}(k)$  generated by  $\mathcal{B}$ .  $\otimes$

**Examples 4.3** (Networks supporting synchronization transforms).

(1) Suppose  $k = 3$ . Up to permutation of coordinates, there are three nonempty subsets (sublattices here) of  $\mathcal{L}(3)$  containing  $\mathbf{V}_0$ :

$$\mathcal{B}_0 = \{\mathbf{V}_0, H_1, H_2\} = \mathcal{L}(3), \quad \mathcal{B}_1 = \{\mathbf{V}_0, H_1\}, \quad \mathcal{B}_\infty = \{\mathbf{V}_0\}.$$

If  $\alpha \in \{0, 1, \infty\}$  and  $(T_\alpha, \mathcal{B}_\alpha, \mathcal{S}_\alpha)$  is a synchronization transform, then (up to permutation of  $\{1, 2, 3\}$ ) we have

- (1)  $\mathcal{S}_0 = \{(123), (12), (13)\}$  ( $s = 2$ ).
- (2)  $\mathcal{S}_1 = \{(123), (12)\}$  ( $s = 2$ ).
- (3)  $\mathcal{S}_\infty = \{(123)\}$  ( $s = 1$ ).

The synchronization map  $T_0$  is given by a linear isomorphism satisfying  $T_0(H_1) = (12)$ ,  $T_0(H_2) = (13)$  and  $T_0(\mathbf{V}_0) = (123)$ . In matrix form,

$$T_0 = \begin{pmatrix} a & b & 1 \\ a_2 & b & 1 \\ a & b_3 & 1 \end{pmatrix},$$

where  $(a_2 - a)(b_3 - b) \neq 0$ . For example, we may take  $a = b = 0$ ,  $a_2 b_3 \neq 0$ . Finding explicit synchronization maps for the other two cases is a simple computation. Note that  $(12) \wedge (13) = \mathbf{S}_\infty$  for  $\mathcal{S}_0$  but that  $\mathbf{S}_\infty$  cannot be expressed in terms of basis elements for  $\mathcal{S}_1, \mathcal{S}_\infty$ . In all cases,  $T_\alpha$  will induce a lattice isomorphism  $T_\alpha^* : \mathcal{B}_\alpha \rightarrow \mathcal{S}_\alpha$ .

We have already given a 3 identical cell network that realizes  $\mathcal{S}_0$  – see figure 5(b) and assume asymmetric inputs. In figure 7 we show identical cell networks that realize  $\mathcal{S}_1$  and  $\mathcal{S}_\infty$ . Note that for (a) we need *two* input types and at least one cell will have a duplicated input – in this case  $N_3$ . For (b) we only need *one* input – shown by the unbroken line in figure 7(b). The second set of inputs (broken line in the figure) can be filled using any configuration (denying self loops).



FIGURE 7. Three node identical cell system with nontrivial synchrony subspaces (a)  $(12)$ ,  $(123)$ , and (b)  $(123)$ .

(2) Take  $k = 5$ ,  $s = 3$ ,  $\mathcal{B} = \{H_{12}, H_{34}, H_{124}\}$  and require  $T(H_{12}) = (12|34)$ ,  $T(H_{34}) = (15|23)$  and  $T(H_{124}) = (12|345)$ . We have

- (1)  $\mathbf{V}_0 = H_{12} \cap H_{34}$ .
- (2)  $\mathcal{S} = \{(12|34), (15|23), (12|345), (12345)\}$ .

We may realize a synchronization map with the matrix

$$T = [t_{ij}] = \begin{pmatrix} a & b & c & d & 1 \\ a_2 & b_2 & c & d & 1 \\ a_2 & b_2 & c_3 & d_3 & 1 \\ a_4 & b_4 & c_3 & d_3 & 1 \\ a & b & c_3 & d_5 & 1 \end{pmatrix}.$$

where  $(c - c_3)(d_5 - d_3)$ ,  $[(a_2 - a)(b_4 - b_2) - (b_2 - b)(a_4 - a_2)] \neq 0$  are required for  $T$  to be nonsingular and

$$a_2 - a_4, a - a_2, a_4 - a_2, b - b_2, b_2 - b_4, b - b_4, d - d_5, d - d_3 \neq 0$$

are needed to ensure  $\mathcal{S} = \{(12|34), (15|23), (12|345), (12345)\}$  and (3c) of definition 4.1 are satisfied. The coupled cell system shown in figure 8 has exactly these synchrony subspaces.

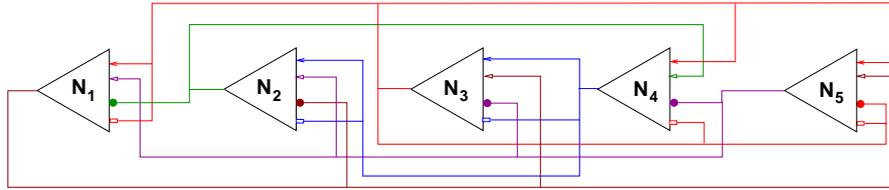


FIGURE 8. A five node identical cell system with nontrivial synchrony subspaces:  $(12|34)$ ,  $(15|23)$ ,  $(12|345)$ ,  $(12345)$ .

Note that  $T$  does not induce a lattice isomorphism between  $\mathcal{S}$  and the sublattice of  $\mathcal{L}(k)$  generated by  $\mathcal{B}$  under the join and meet operations on  $\mathcal{I}_k$ . Indeed,  $H_{45} \wedge H_{235} = H_5$  and  $T(H_5) \notin \mathcal{S}$ .

(3) If  $k > 2$ , it is not possible to choose a synchronization transform  $(T, \mathcal{B}, \mathcal{S})$  such that  $\mathcal{S} = \mathcal{D}(k)$  since the maximal number of elements in  $\mathcal{S}$  is  $2^{k-1}$  (the cardinality of  $\mathcal{L}(k)$ ). Since we deny self loops, and  $k > 2$ , there are no strongly connected coupled cell networks for which every polydiagonal subspace is a synchrony subspace (lemma 3.12).

(4) If  $k = 4$  and we take the sublattice  $\mathcal{S}$  of  $\mathcal{D}(4)$  generated by (123), (124), (12|34), then there is no synchronization transform. It is straightforward to verify that there is no strongly connected network  $\mathcal{N}$ , *without self loops*, for which  $\mathcal{S}(\mathcal{N}) = \mathcal{S}$  (start by assuming the contrary and observe that the resulting network must contain the unique three cell two input network of figure 7(a)).  $\ast$

*Remark 4.4.* We conjecture that if  $(T, \mathcal{B}, \mathcal{S})$  is a synchronization transform of weight  $s$ ,  $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$ , then (a)  $\mathcal{S}$  contains at most  $2^{k-1}$  synchrony classes, (b)  $s \leq k - 1$ . Note that (a) is immediate if  $T$  induces a lattice isomorphism between the lattice generated by  $\mathcal{B}$  and  $\mathcal{S}$ .  $\circledast$

**Theorem 4.5.** *Let  $k \geq 2$ .*

- (1) *If  $\mathcal{B} = \{\mathbf{V}_0\}$ , there exists a strongly connected coupled identical  $k$  cell network  $\mathcal{N}$  without self loops for which  $\mathcal{S}(\mathcal{N}) = \{(12 \dots k)\}$ . It suffices that cells have  $s \leq 2$  input types.*
- (2) *Suppose that  $(a_i, b_i) \in \mathcal{D}(k)$ ,  $i \in \overline{k-1}$ . Let  $\mathcal{S} \subset \mathcal{D}(k)$  be the sublattice generated by  $\{(a_i, b_i) \mid i \in \overline{k-1}\}$  and suppose that  $\vee_{i \in \overline{k-1}} (a_i, b_i) = \mathbf{S}_0$ . Then there exists a strongly connected coupled identical  $k$  cell network  $\mathcal{N}$  without self loops for which  $\mathcal{S}(\mathcal{N}) = \mathcal{S}$ . Each cell will have  $k - 1$  input types.*

*In both cases, there are synchronization transforms  $(T, \mathcal{B}, \mathcal{S})$  for which  $\mathcal{S}$  is generated by  $\{T(W_i) \mid W_i \in \mathcal{B}\}$ .*

*Proof.* (1) Let  $\mathcal{N}_1$  be the  $k$  identical cell network with one input type and connections  $N_2 \rightarrow N_1, \dots, N_k \rightarrow N_{k-1}, N_1 \rightarrow N_k$ . If  $k$  is prime, the only synchrony subspace will be  $(12 \dots k)$ . If  $k$  is not prime, then it is straightforward exercise, based on proposition 3.3, to show that every synchrony subspace of  $\mathcal{N}_1$  is of the form

$$\mathbf{S}_{PQ} = (1Q+1 \dots (P-1)Q+1 | 2 \dots (P-1)Q+2 | \dots | Q \ 2Q \dots PQ),$$

where  $k = PQ$  is a proper factorization of  $k$  (note that  $S_{PQ} \neq S_{QP}$  unless  $P = Q$ ). Choose a second input type and connections  $N_{j+1} \rightarrow N_j$ ,  $j > 1$  and  $N_3 \rightarrow N_1$  so as to define the network  $\mathcal{N}_2$  with two input types. The only synchrony subspace of  $\mathcal{N}_2$  is  $(12 \dots k)$ .

(2) We proceed by induction on  $k$ . The result is true trivially if  $k = 2$  and follows by examples 4.3(1) if  $k = 3$ . Suppose the result

is proved for  $k < n$ . We prove for  $k = n$ . By a simple counting argument, there exists  $i \in \mathbf{n}$  such that either  $a_i$  or  $b_i$  occurs just once in  $\{a_1, b_1, a_2, \dots, a_{n-1}, b_{n-1}\}$ . Relabelling, suppose that  $i = n - 1$ ,  $b_{n-1} = n$  occurs just once and  $a_{n-1} = 1$ . Then  $\{(a_i, b_i) \mid i \in \overline{\mathbf{n} - 2}\}$  satisfies the hypotheses of the proposition for  $k = n - 1$  and so there exists a strongly connected coupled identical  $n - 1$  cell network  $\mathcal{N}_{n-1}$  without self loops for which  $\mathcal{S}(\mathcal{N}_{n-1})$  is generated by  $(a_1, b_1), \dots, (a_{n-2}, b_{n-2})$ . Change cells in  $\mathcal{N}_{n-1}$  by adding one new input type and form the  $n$  cell network  $\mathcal{N}$  by adding one new cell  $N_n$  to  $\mathcal{N}_{n-1}$ , and

- (1) connecting  $N_n$  to the  $n - 1$ -input of cells  $N_1, \dots, N_{n-1}$ ,
- (2) connecting  $N_1$  to the  $n - 1$ -input of cell  $N_n$ ,
- (3) connecting  $N_j$  to the  $j - 1$ -input of cell  $N_n$ ,  $j = 2, \dots, n - 1$ .

It is straightforward to check that  $\mathcal{S}(\mathcal{N})$  is equal to the sublattice generated by  $\{(a_i, b_i) \mid i \in \overline{\mathbf{n} - 1}\}$ .

Finally, it is easy to construct synchronization transforms  $(T, \mathcal{B}, \mathcal{S})$  for which  $\mathcal{S}$  is generated by  $\{T(W_i) \mid W_i \in \mathcal{B}\}$ . In case (2)  $T$  will map  $x_j = 0$  to  $x_{a_j} = x_{b_j}$ ,  $j \in \overline{\mathbf{k} - 1}$ .  $\square$

*Remarks 4.6.* (1) The proof of theorem 4.5 gives an inductive construction of the coupled cell network realizing the given synchrony. It is not hard to show that, up to relabelling of cells and input types, the networks given in part (2) of the theorem are unique. In particular, the networks cannot be constructed with fewer than  $k - 1$  input types and there are no multiple connections between cells: each cell receives exactly one input from each of the remaining cells.

(2) If we take generating set  $\{(1, j) \mid j = 2, \dots, k\}$ , then the network given by the proposition will be the network  $\mathcal{P}_k$  of [26]. Even though the sublattices of theorem 4.5(2) are always isomorphic to the sublattice generated by  $\{(1, j) \mid j = 2, \dots, k\}$ , the associated networks are generally *not* linearly or dynamically equivalent [16, 3] (allowing for permutation of cells).  $\otimes$

Suppose that  $\mathcal{N} = \{N_1, \dots, N_k\}$  is an identical cell network and that each cell has  $p \geq 1$  asymmetric inputs. We construct a new identical cell network  $\widehat{\mathcal{N}} = \{\widehat{N}_1, \dots, \widehat{N}_{k+1}\}$  such that each cell  $\widehat{N}_j$  has  $p + 1$  asymmetric inputs and  $\mathcal{N}$  is naturally embedded in  $\widehat{\mathcal{N}}$ . To do this, we start with the network  $\mathcal{N}$  and add one input type to each cell  $N_j$  to obtain a new cell  $\widehat{N}_j$  with  $p + 1$  input types. Add a new cell  $\widehat{N}_{k+1}$ . It remains to fill the inputs of type  $p + 1$  for  $\widehat{N}_1, \dots, \widehat{N}_k$  and the inputs of type  $1, \dots, p + 1$  of  $\widehat{N}_{k+1}$ . The output of  $\widehat{N}_{k+1}$  will go to the input of type  $p + 1$  for each of the cells  $\widehat{N}_1, \dots, \widehat{N}_k$ . We fill all the inputs of  $\widehat{N}_{k+1}$  with the output of  $\widehat{N}_1$ . This defines the identical cell network  $\widehat{\mathcal{N}}$ .

**Lemma 4.7.** *(Notation and assumptions as above.) Let  $\mathcal{N}$  be a strongly connected identical cell network without self loops and with cells having  $p \geq 1$  asymmetric inputs.*

- (1)  $\widehat{\mathcal{N}}$  is a strongly connected identical cell network without self loops.
- (2)  $\mathcal{S}(\widehat{\mathcal{N}}) = \mathcal{S}(\mathcal{N}) \cup \{\mathbf{S}_0\}$ , where  $\mathbf{S}_0$  is the maximal element of  $\mathcal{S}(\widehat{\mathcal{N}})$ .
- (3)  $\widehat{\mathcal{N}}$  has a synchronization transform  $(\widehat{T}, \widehat{\mathcal{B}}, \mathcal{S}(\widehat{\mathcal{N}}))$  of weight  $s+1$  if and only if  $\mathcal{N}$  has a synchronization transform  $(T, \mathcal{B}, \mathcal{S}(\mathcal{N}))$  of weight  $s$ .

*Proof.* The first statement is trivial. Concerning (2), it suffices to show that the only synchrony subspace for  $\widehat{\mathcal{N}}$  containing  $N_{k+1}$  is  $(1 \dots k+1)$ . Let  $(A_1 | \dots | A_s)$  be a synchrony subspace for  $\widehat{\mathcal{N}}$  containing  $N_{k+1}$  as a synchronized node. Without loss of generality, we may suppose that  $k+1 \in A_1$ . Since there are connections  $\widehat{N}_{k+1} \rightarrow \widehat{N}_j$ ,  $j \in \mathbf{k}$ , and  $\widehat{N}_1 \rightarrow \widehat{N}_{k+1}$ , all to inputs of type  $p+1$ , we must have  $1 \in A_1$ . Hence for all  $i \in \mathbf{k}$ ,  $j \in \mathbf{k} \cap A_1$ ,  $\ell \in \mathbf{p}$ , if there is a connection from  $\widehat{N}_i$  to the  $\ell$ -input of  $\widehat{N}_j$ , then  $i \in A_1$  (since there is a connection from  $\widehat{N}_1$  to the  $\ell$ -input of  $\widehat{N}_{k+1}$ ). It follows by the strong connectivity of  $\widehat{\mathcal{N}}$  that  $A_1 = (1 \dots k+1)$ , proving (2). Finally, (3) is a routine computation.  $\square$

**Examples 4.8.** (1) Let  $k = 4$ . Up to permutation of nodes, theorem 4.5 gives three strongly connected identical cell networks with synchrony subspaces generated by (a) (1234), (b) (12), (13), (14), and (c) (12), (23), (34). Using the results for  $k = 3$ , lemma 4.7 yields three additional 4 cell identical network with synchrony subspaces generated by (d) (12), (123), (1234), (e) (123), (1234), (f) (12), (23), (1234). However, this list is far from complete – see the concluding remarks 4.2 at the end of the section and the tables at the end of section 5.

(2) For  $k \geq 2$ , there is a coupled identical  $k$  cell network with synchrony subspaces (12), (123), (1234),  $\dots$ ,  $(12 \dots k)$ . This follows by applying lemma 4.7 a total of  $k - 2$  times to the 2 identical cell strongly connected network with one input type. All networks constructed support a synchronization transform.  $\ast$

### Realization Conjecture, part I

Let  $k \geq 2$  and  $\mathcal{S}$  be a complete sublattice of  $\mathcal{D}(k)$ . If there exists a synchronization transform  $(T, \mathcal{B}, \mathcal{S})$ , then there is a strongly connected coupled identical  $k$  cell network  $\mathcal{N}$ , without self loops and with lattice of synchrony subspaces  $\mathcal{S}(\mathcal{N}) = \mathcal{S}$ . Cells of  $\mathcal{N}$  have (at most)  $k - 1$  asymmetric inputs.

*Remarks 4.9.* (1) The conjecture is not interesting if we allow self loops (see example 3.13(1)) or do not require strong connectivity (see examples 4.3(3)). In particular, the converse to the conjecture – that a synchronization transform is necessary – is obviously false if we do not require strong connectivity and no self-loops. The converse may also fail if the network equations have symmetry which acts non trivially on  $\mathcal{S}$  (see example 4.13).

(2) There may be more than one identical cell network with specified lattice of synchrony subspaces (see example 4.13).  $\otimes$

When the Realization Conjecture holds, it is often true that the coupled cell network  $\mathcal{N}$  can be chosen so that

- (A) Generically, the linearizations of network vector fields at equilibria on synchrony subspaces have no multiple eigenvalues.
- (B) If  $\mathbf{U}_0, \mathbf{U}_1, \dots, \mathbf{U}_\ell$  are distinct synchrony subspaces satisfying  $\mathbf{U}_i \cap \mathbf{U}_j = \mathbf{U}_0$  for all  $i \neq j \in \ell$ ,  $m_0 \in \{0, \dots, \dim(\mathbf{U}_0)\}$ ,  $m_i \in \{0, \dots, \dim(\mathbf{U}_i) - \dim(\mathbf{U}_0)\}$ ,  $i \in \ell$ , and  $m \in \{\sum_{i \in \ell} m_i, \dots, k\}$ , then for every  $\mathbf{p} \in \mathbf{U}_0$ , there exists a network vector field  $F$ , with hyperbolic equilibrium at  $\mathbf{p}$ , such that the index of  $F|_{\mathbf{U}_0}$  at  $\mathbf{p}$  is  $m_0$ , the index of  $F|_{\mathbf{U}_i}$  at  $\mathbf{p}$  is  $m_0 + m_i$ ,  $i \in \ell$ , and the index of  $F$  at  $\mathbf{p}$  is  $m$ .

*Remarks 4.10.* (1) In order to verify (A), it suffices to show that equilibria on the minimal synchrony subspace  $\mathbf{S}_0$  generically do not have multiple eigenvalues. More generally, if generically there are no multiple eigenvalues for equilibria on a synchrony subspace  $\mathbf{U} \in \mathcal{S}$ , the same will be true for all  $\mathbf{T} \in \mathcal{S}$  such that  $\mathbf{T} \supset \mathbf{U}$ . If there is a synchronization transform, we have found no examples where the presence of multiple eigenvalues impacts the existence of robust heteroclinic networks or cycles (see also the last paragraph of example 4.13).

(2) Condition (B) implies that the synchrony subspace structure does not constrain the indices of network vector fields at equilibria. We note that if (B) holds when  $\mathbf{U}_0 = \mathbf{S}_0$ , then (B) holds generally.  $\otimes$

**Theorem 4.11.** *Let  $k \geq 2$  and  $\mathcal{S}$  be a complete sublattice of  $\mathcal{D}(k)$ . Part I of the Realization Conjecture is true if any of the following conditions hold:*

- (a)  $\mathcal{S} = \{(12 \dots k)\}$ .
- (b)  $\mathcal{S}$  has generators  $\{(a_i, b_i) \mid i \in \overline{k-1}\}$  where  $\vee_{i \in \overline{k-1}} (a_i, b_i) = (12 \dots k)$ .
- (c) We can write  $\mathcal{S}$  as the disjoint union  $\mathcal{S}' \cup \{(12 \dots k)\}$ , where  $\mathcal{S}'$  is a complete sublattice of  $\mathcal{D}(k-1)$ , and part I of the Realization Conjecture holds for  $\mathcal{S}'$ .

(d)  $k \leq 4$ .

Condition (A) holds in cases (a,b) and condition (B) holds in full generality in case (b).

*Proof.* The statement about part I of the Realization Conjecture follow for (a,b) from theorem 4.5, and for (c) from lemma 4.7. In particular, in cases (a,b,c) there is a synchronization transform. Statement (d) follows for  $k = 3$  by examples 4.3(1), and for  $k = 4$  on a case-by-case basis (see tables 1–3 of section 5). Note that for (d) we require as part of the hypotheses of the Realization Conjecture that there is a synchronization transform. For the statements about condition (A), use the comments preceding the statement of the theorem. For verification of (B) for (b), use induction on  $k$  (the case  $k = 2$  is trivial).

*Remarks 4.12.* (1) Condition (B) does not generally hold in case (a). For example, if  $k = 3$ , the real parts of the two eigenvalues corresponding to directions transverse to  $\mathbf{S}_0$  have the same sign.

(2) For the validity of (A,B) in case (d), see the tables of section 5.

(3) For (c), we conjecture that if condition (A) holds for the network  $\mathcal{N}'$  realizing  $\mathcal{S}'$ , then the same is true for  $\mathcal{S}$ . However, condition (B) may or may not hold. For example, if  $k = 4$  and  $\mathcal{S}'$  is the 3 cell network of figure 7(a), conditions (B) holds for the network given by lemma 4.7. On the other hand if we take the coupled cell system  $\mathcal{N}'$  given by

$$\dot{x}_1 = f(x_1; x_2, x_2) \quad \dot{x}_2 = f(x_2; x_1, x_3) \quad \dot{x}_3 = f(x_3; x_2, x_2),$$

then  $\mathcal{S}(\mathcal{N}') = \mathcal{S}'$  but (B) fails for the network given by lemma 4.7.  $\otimes$

### Realization Conjecture, part II

Let  $k \geq 2$  and  $\mathcal{N}$  be a  $k$  identical cell network with lattice of synchrony subspaces  $\mathcal{S}$ . Let  $G \subset S_k$  denote the group of permutation symmetries of  $\mathcal{N}$ . A synchronization transform  $(T, \mathcal{B}, \mathcal{S})$  exists if and only if  $G$  acts as the identity on  $\mathcal{S}$ . In particular, if  $G_{\mathcal{S}}$  is the subgroup of  $S_k$  preserving  $\mathcal{S}$  then a sufficient condition for a synchronization transform is that  $G_{\mathcal{S}}$  acts as the identity on  $\mathcal{S}$ .

**Example 4.13** (Scope and converse to the realization conjectures). Suppose that  $k = 4$ ,  $s = 3$ ,  $\mathcal{B} = \{H_{12}, H_{13}, H_{23}\}$ , and we require  $T(H_{12}) = (12|34)$ ,  $T(H_{13}) = (13|24)$  and  $T(H_{23}) = (14|23)$ . We have  $\mathcal{S} = \{(12|34), (13|24), (14|23), (1234)\}$ . We can realize a synchronization map with the matrix

$$T = [t_{ij}] = \begin{pmatrix} a & b & c & 1 \\ a_2 & b_2 & c & 1 \\ a_2 & b & c_3 & 1 \\ a & b_2 & c_3 & 1 \end{pmatrix}.$$

where  $a \neq a_2, b \neq b_2, c \neq c_3$ . The coupled cell system shown in figure 9(a) has synchrony subspaces  $\mathcal{S}$ . A new feature is that  $\mathcal{S}$ , and the

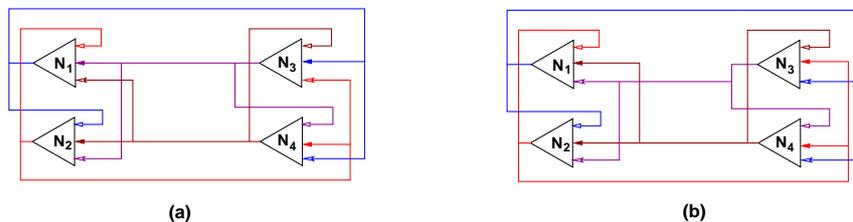


FIGURE 9. Four identical cell networks with nontrivial synchrony subspaces (a)  $(12|34), (13|24), (14|23), (1234)$ , (b)  $(12), (34), (13|24), (12|34), (1234)$

network realizing these synchrony subspaces, have a nontrivial symmetry group – the Klein four-group  $K$  – generated by the involutions  $(12)(34), (13)(24), (14)(23)$ . However,  $\mathcal{S}$  is *pointwise fixed* by  $K$  and there is no action induced on  $\mathcal{B}$  – this is compatible with part II of the Realization Conjecture.

If we take  $\mathcal{S} = \{(12), (34), (13|24), (12|34), (1234)\}$ , generated by  $(12), (34), (13|24)$ , then it is *not* possible to find a synchronization transform even though there is a coupled identical 4 cell network  $\mathcal{N}$  that realizes these synchrony subspaces (see figure 9(b)). Observe that the symmetry group of  $\mathcal{S}$  (or  $\mathcal{N}$ ) contains the involution  $\sigma = (24)(13)$ . Although  $\sigma$  preserves  $\mathcal{S}$  it does not act trivially –  $\sigma((12)) = (34)$ . As a consequence, if there is a synchronization map  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ , there is a nontrivial  $\mathbb{Z}_2$  action induced on  $\mathcal{B} \subset \mathcal{L}(4)$ . This gives an extra constraint on  $T$  which cannot be satisfied and we cannot define generators for  $\mathcal{S}$  in terms of elements of  $\mathcal{L}(4)$ .

In a related direction, suppose

$$\mathcal{S} = \{(12), (12|34), (13|24), (14|23), (1234)\}.$$

A basis for  $\mathcal{S}$  contains *four* elements. The symmetry group of  $\mathcal{S}$  contains the involution  $\sigma = (12)$  and acts nontrivially on  $\mathcal{S}$ . It is easy to verify directly that we cannot find  $\mathcal{B} = \{W_1, W_2, W_3, W_4\}$  and a synchronization map  $T$  such that  $\{T(W_i) \mid i \in \mathbf{4}\}$  generate  $\mathcal{S}$  – the conditions force  $T$  to be singular. A case by case analysis of the one input connection structures that support  $(12|34), (13|24), (14|23)$  and  $(12)$  shows that  $(34)$  will always be a synchrony subspace and so it is not possible to add a second input type that allows  $(12)$  and denies  $(34)$  (self loops do not help here). Although we can find a 4 identical cell network with synchrony subspaces  $\mathcal{S}'$  generated by  $(12), (34), (13|24), (14|23)$ ,

the resulting network is not strongly connected if we deny self loops. The involution (13)(24) is a symmetry of  $\mathcal{S}'$  that does not act trivially on  $\mathcal{S}'$  – this is consistent with part II of the Realization Conjecture.

Finally suppose that  $\mathcal{S}$  is generated by (12), (34), (1234). The involution  $\sigma = (13)(24)$  is a nontrivial symmetry of  $\mathcal{S}$ . However, it is not possible to find a coupled cell network  $\mathcal{N}$  with  $\mathcal{S}(\mathcal{N}) = \mathcal{S}$  such that  $\sigma$  is a symmetry of  $\mathcal{N}$ . Here there exist coupled cell networks  $\mathcal{N}$  with  $\mathcal{S}(\mathcal{N}) = \mathcal{S}$ , and which all have associated synchronization transforms, but  $\sigma$  is never a symmetry of  $\mathcal{N}$ . For example, we may take the network equations

$$\begin{aligned} \dot{x}_1 &= f(x_1; x_2, x_2, x_3), & \dot{x}_2 &= f(x_2; x_1, x_1, x_3), \\ \dot{x}_3 &= f(x_3; x_4, x_1, x_2), & \dot{x}_4 &= f(x_4; x_3, x_1, x_2). \end{aligned}$$

It is easy to check that conditions (A,B) are satisfied on (1234). On the other hand, if we take the network equations

$$\begin{aligned} \dot{x}_1 &= f(x_1; x_2, x_4, x_3), & \dot{x}_2 &= f(x_2; x_1, x_4, x_3), \\ \dot{x}_3 &= f(x_3; x_4, x_2, x_2), & \dot{x}_4 &= f(x_4; x_3, x_2, x_2), \end{aligned}$$

then conditions (A,B) fail on (1234) but hold on  $(12|34) \setminus (1234)$ .  $\ast$

**4.1. Necessary conditions for  $\mathcal{S}$  to be a lattice of synchrony subspaces.** Fix  $k \geq 4$  and let  $\mathcal{S}$  be a complete sublattice of  $\mathcal{D}(k)$ . We are interested in giving structural conditions on  $\mathcal{S}$  that imply  $\mathcal{S}$  cannot be the lattice of synchrony subspaces of a strongly connected  $k$  identical cell network without self loops.

Suppose that  $A$  is a subset of  $\mathbf{k}$ . Let  $|A|$  denote the number of elements in  $A$ . In what follows we assume  $3 \leq |A| < k$  and set  $|A| = p$ . Relabelling, suppose that  $A = \{1, \dots, p\}$ . If  $\mathfrak{s} = (A_1 | \dots | A_r) \in \mathcal{S}$ , define  $\text{supp}(\mathfrak{s}) = \cup_{i \in \mathbf{r}} A_i$  and set  $\mathcal{S}_A = \{\mathfrak{s} \in \mathcal{S} \mid \text{supp}(\mathfrak{s}) \subset A\}$ .

**Definition 4.14.** (Notation and assumptions as above.) The set  $\mathcal{S}_A$  is a *synchrony substructure* of  $\mathcal{S}$  if

- (1)  $(1 \dots p) \in \mathcal{S}_A$ .
- (2)  $\mathcal{S}_A$  is a sublattice of  $\mathcal{S}$ .

The synchrony substructure  $\mathcal{S}_A$  is *indecomposable* if given  $\mathfrak{s}, \mathfrak{t} \in \mathcal{S}_A$  there is a chain  $\mathfrak{s} = \mathfrak{s}_1, \dots, \mathfrak{s}_q = \mathfrak{t}$  of synchrony subspaces lying in  $\mathcal{S}_A$  such that  $\text{supp}(\mathfrak{s}_i) \cap \text{supp}(\mathfrak{s}_{i+1}) \neq \emptyset$ ,  $i \in \overline{\mathbf{q} - 1}$ .

*Remark 4.15.* If  $\mathcal{S}_A$  is a synchrony substructure, then  $\mathcal{S}_A$  naturally embeds as a sublattice of  $\mathcal{D}(p)$  with maximal element  $(1 \dots p)$ .

**Lemma 4.16.** (Notation and assumptions as above.) Let  $\mathcal{N}$  be an identical  $k$  cell network with lattice of synchrony subspaces  $\mathcal{S}(\mathcal{N}) = \mathcal{S}$ .

Assume cells have  $q$  asymmetric inputs. Suppose that  $A \subset \mathbf{k}$  and  $\mathcal{S}_A$  is a synchrony substructure of  $\mathcal{S}$ . For each input type  $j \in \mathbf{q}$  one of the following conditions holds

- (1) There exists  $\ell \in \mathbf{k} \setminus A$  such that for all  $i \in A$ , there is a connection  $N_\ell \rightarrow N_i$  to input  $j$  of  $N_i$ .
- (2) For all  $i \in A$ , there exists  $\ell = \ell(i) \in A$ , such that there is a connection  $N_\ell \rightarrow N_i$  to input  $j$  of  $N_i$ .

In particular,  $\mathcal{S}_A$  determines an identical cell network  $\mathcal{N}_A$ , with cells  $\{N_i \mid i \in A\}$  defined by deleting all inputs to, and outputs from, cells  $N_i \in \mathcal{N}_A$  which connect to cells  $N_k$ ,  $k \notin A$ .

*Proof.* Relabelling cells, we may assume  $A = \mathbf{p}$ . Since  $(1 \dots p) \in \mathcal{S}_A$ , the result is immediate from lemma 3.3.  $\square$

**Proposition 4.17.** Let  $k \geq 4$  and  $\mathcal{S}$  be a complete sublattice of  $\mathcal{D}(k)$ .

- (1) A necessary condition for there to exist a strongly connected  $k$  identical cell network  $\mathcal{N}$ , without self loops, with  $\mathcal{S}(\mathcal{N}) = \mathcal{S}$ , is that for every indecomposable synchrony substructure  $\mathcal{S}_A$ , with  $|A| = p$ , there exists a  $p$  identical cell network  $\mathcal{N}_A$ , without self loops, and with  $\mathcal{S}(\mathcal{N}_A) = \mathcal{S}_A$ .
- (2) A necessary condition for there to exist a strongly connected  $k$  identical cell network  $\mathcal{N}$ , without self loops, with a synchronization transform  $(T, \mathcal{B}, \mathcal{S})$ , is that for every indecomposable synchrony substructure  $\mathcal{S}_A$ , with  $|A| = p$ , there exists a synchronization transform  $(T_A, \mathcal{B}_A, \mathcal{S}_A)$ .

*Proof.* Both statements follows easily from lemma 4.16; we omit the routine details.  $\square$

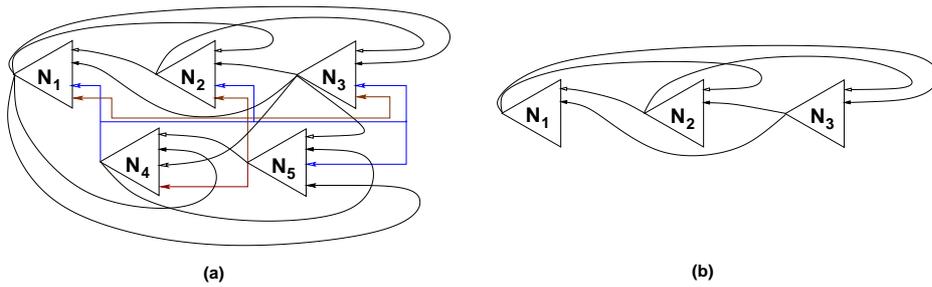


FIGURE 10. A 5 identical cell network with synchrony substructure  $\mathcal{S}_3$ .

**Example 4.18.** In figure 10(a) we show a 5 cell network  $\mathcal{N}$  with  $\mathcal{S}(\mathcal{N}) \supset \{(12), (13), (123)\}$ . Taking  $A = \mathbf{3}$ , it is easy to see that  $\mathcal{S}_A$

is a synchrony substructure. The associated network  $\mathcal{N}_A$  is shown in figure 10(b). \*

**Examples 4.19.** (1) Suppose that  $k = 6$  and  $\mathcal{S}$  is generated by

$$\{(12), (34), (13|24), (14|23), (1256)\}.$$

It follows from example 4.13 and proposition 4.16 that it is not possible to find a strongly connected 6 identical cell network  $\mathcal{N}$ , without self loops, such that  $\mathcal{S}(\mathcal{N}) = \mathcal{S}$ .

(2) Again, using example 4.13 and proposition 4.16, it is not possible to find a synchronization transform for the 6 cell network with synchrony subspaces generated by  $\{(12), (34), (13|24), (2356)\}$ . Note that these synchrony spaces cannot be realized by a strongly connected 6 cell network without self loops. \*

**4.2. Summary.** The realization conjectures suggest a way to identify coupled identical cell networks which have synchrony subspaces closely related to invariant subspaces of SLF systems. In examples 4.8(1), example 4.13 we have identified eight coupled cell networks where the realization conjectures apply. Parts 1 and 2 (where applicable) of the conjectures also hold when  $\mathcal{S}$  has generating sets (i)  $\{(12), (13|24)\}$ , (j)  $\{(12), (134)\}$ , (k)  $\{(12), (12|34), (1234)\}$ , (l)  $\{(12|34), (13|24)\}$ , and (m)  $\{(12), (34), (1234)\}$ . We discuss some of these examples further in the next section where our focus is on investigating what this relationship between coupled cell networks and SLF systems implies about heteroclinic cycles and networks in coupled cell networks.

## 5. TRANSFORMING HETEROCLINIC NETWORKS

In this section we investigate ways of proving the existence of heteroclinic cycles and networks in coupled identical cell systems using known heteroclinic cycles and networks in SLF systems. Specifically, suppose that  $(T, \mathcal{B}, \mathcal{S})$  is a synchronization transform, with synchronization map  $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$ , and that there is a strongly connected  $k$  identical cell network, without self loops, which has lattice of synchrony subspaces  $\mathcal{S}(\mathcal{N}) = \mathcal{S}$  – that is, part I of the Realization Conjecture holds. We show how we can use the synchronization map to transform robust heteroclinic cycles and networks for SLF systems, with  $k$  nodes, into corresponding results on robust heteroclinic cycles and networks for a coupled  $k$  identical cell network  $\mathcal{N}$  with  $\mathcal{S}(\mathcal{N}) = \mathcal{S}$ . Our focus will be on examples that illustrate the approach; we present general conjectures in the next section.

**5.1. Simple heteroclinic networks and cycles for coupled identical cell systems.** As we indicated in example 3.14, heteroclinic cycles in coupled identical cell systems will generally not be simple (as defined in section 1) unless we restrict domains (just as we do for equivariant maps and generalized Lotka-Volterra systems). We give a definition to cover the case of greatest interest.

Let  $(T, \mathcal{B}, \mathcal{S})$  be a synchronization transform with  $\mathcal{B} = \{H_i \mid i < k\}$ ,  $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$  and  $\mathcal{S}$  generated by  $\{T(H_i) = (a_i, b_i) \mid i < k\}$ , as in the statement of theorem 4.5(2). We have  $T(H_i) = \{\mathbf{y} \in \mathbb{R}^k \mid y_{a_i} = y_{b_i}\}$ . Let  $\mathcal{N}$  denote the associated coupled cell network given by theorem 4.5(2).

Define the domain  $D_+(T) = D_+ \subset \mathbb{R}^k$  by

$$D_+ = \{T(\mathbf{x}) \mid x_i \geq 0, i < k\}.$$

Observe that  $D_+$  is flow-invariant for coupled cell dynamics on  $\mathbb{R}^k$ ,  $\partial D_+ = D_+ \cap \bigcup_{i < k} T(H_i)$ ,  $D_+ \supset \Delta(\mathbb{R}^k)$ , and there are  $2^{k-1}$  different choices for  $D_+$  depending on the signs of the matrix entries for  $T$ . Up to homeomorphism,  $D_+ = \mathbb{R} \times O_{k-1}$ . A heteroclinic cycle  $\Sigma$  will be a simple heteroclinic cycle for network dynamics on  $\mathcal{N}$ , if we can choose  $T$  so that  $\Sigma \subset D_+$  and  $\Sigma$  is simple in the sense of section 1 for dynamics restricted to  $D_+$ .

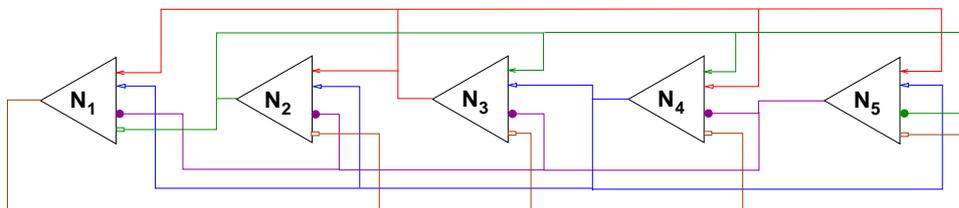


FIGURE 11. A 5 node identical cell network  $\mathcal{N}$  with synchrony subspaces (12), (23), (34), (25), (123), (125), (234), (235), (12|34), (25|34), (1234), (1235), (2345), (125|34), and (12345).

**Example 5.1** (A five cell system). The coupled cell network of figure 11 has synchrony  $\mathcal{S}(\mathcal{N})$  generated by  $\{(12), (23), (34), (25)\}$  (and so theorem 4.5(2) applies). Assuming 1-dimensional node dynamics, the equations for the system of figure 11 are

$$\begin{aligned} \dot{x}_1 &= f(x_1; x_3, x_4, x_5, x_2), & \dot{x}_2 &= f(x_2; x_3, x_4, x_5, x_1), \\ \dot{x}_3 &= f(x_3; x_2, x_4, x_5, x_1), & \dot{x}_4 &= f(x_4; x_2, x_3, x_5, x_1), \\ & & \dot{x}_5 &= f(x_5; x_3, x_4, x_2, x_1). \end{aligned}$$

We claim that we can choose  $f$  so that the identical cell system  $\mathcal{N}$  has a robust simple heteroclinic cycle  $\Sigma$

$$\cdots \rightarrow (235) \xrightarrow{23} (123) \xrightarrow{12} (12|34) \xrightarrow{34} (25|34) \xrightarrow{25} (235) \rightarrow \cdots$$

Here vertices and connections are labelled by synchrony type. Observe that one cell desynchronizes along each connection of  $\Sigma$  and there is a resynchronization at the end point of the connection (to a different cluster of synchronized cells from those at the initial point).

It follows from theorem 4.5(2) that there is an associated synchronization transform  $(T, \mathcal{B}, \mathcal{S})$  where

$$H_1 \mapsto (12), \quad H_2 \mapsto (34), \quad H_3 \mapsto (25), \quad H_4 \mapsto (23).$$

We use the synchronization map  $T$  for a formal derivation of  $\Sigma$  from the simple heteroclinic 2-face cycle  $\Sigma_F \subset \Delta_4$

$$\cdots \rightarrow \mathbf{v}_{12} \rightarrow \mathbf{v}_{23} \rightarrow \mathbf{v}_{34} \rightarrow \mathbf{v}_{14} \rightarrow \mathbf{v}_{12} \rightarrow \cdots$$

Observe that  $\Sigma_F$  is supported on the 3-face  $\mathbf{F}_{1234} \approx \Delta_3$  of  $\Delta_4$ .

We have  $T(\mathbf{v}_{12}) = (u, V, V, w, V) \in (235)$ ,  $T(\mathbf{v}_{23}) \in (123)$ ,  $T(\mathbf{v}_{34}) \in (12|34)$ , and  $T(\mathbf{v}_{25}) \in (25|34)$ . Each connection for the face cycle  $\Sigma_F$  will be mapped to the corresponding connection for  $\Sigma$ .

We assumed above that the heteroclinic face cycle  $\Sigma_F$  was a subset of  $H_5$  – although this makes it easier to visualize the face cycle, the assumption is unnecessary and a little misleading. If we assume that the  $x_5$ -coordinates of all of the vertices of the cycle are non-zero, then  $\Sigma_F$  will be a 3-face heteroclinic cycle lying in  $\Delta_4$ . Since the synchrony map is a linear isomorphism it preserves the dimension of invariant subspaces. For example, the vertex  $\mathbf{v}_{125} \in \text{Int}(\mathbf{F}_{125})$ , and  $T$  embeds  $\text{Int}(\mathbf{F}_{125})$  into the 3-dimensional synchrony subspace space  $(235) = T(H_{34})$ . The connection  $\mathbf{v}_{125} \rightarrow \mathbf{v}_{235}$  lies in the 3-face  $\mathbf{F}_{1235}$  and the interior of this face gets embedded by  $T$  in the 4-dimensional synchrony space  $(23) = T(H_4)$ . If we take  $x_5 = 0$ , then  $\mathbf{v}_{12}$  lies on the edge  $\mathbf{F}_{12}$ . The interior of  $\mathbf{F}_{12}$  still gets mapped into  $(235)$  by  $T$ .

What is essential is that the number of zero coordinates of a vertex on the face cycle, not counting the  $x_5$ -coordinate, is the same for all vertices on the cycle. In our example, there are two zero coordinates, not counting the  $x_5$ -coordinate. We remark that when we have connections between fully synchronous equilibria, there is no choice: we always take equilibria on the  $x_5$ -axis  $\mathbf{V}_0$ .

Along similar lines, we can realize

$$\cdots \rightarrow (2345) \xrightarrow{25|34} (125|34) \xrightarrow{12|34} (1234) \xrightarrow{123} (1235) \xrightarrow{235} (2345) \rightarrow \cdots$$

as a robust heteroclinic cycle  $\Sigma_e$ . Using the synchronization map  $T$ ,  $\Sigma_e$  can be derived from the heteroclinic edge cycle (cf. figure 2(a))

$$\cdots \rightarrow \mathbf{v}_1 \rightarrow \mathbf{v}_4 \rightarrow \mathbf{v}_3 \rightarrow \mathbf{v}_2 \rightarrow \mathbf{v}_1 \rightarrow \cdots$$

Equivalently,  $\Sigma_e$  may be derived from the 2-face cycle

$$\cdots \rightarrow \mathbf{v}_{15} \rightarrow \mathbf{v}_{45} \rightarrow \mathbf{v}_{35} \rightarrow \mathbf{v}_{25} \rightarrow \mathbf{v}_{15} \rightarrow \cdots$$

Having identified two potential heteroclinic cycles via the above mechanism, it is easily verified by direct computation (using the methods of [26]) that network dynamics can be chosen so that either cycle is realized as a robust heteroclinic cycle in the coupled cell network architecture  $\mathcal{N}$ .

Summarizing, we propose a method for constructing robust heteroclinic cycles and networks for coupled identical cell networks. If part I of the Realization Conjecture is true, and we can satisfy conditions (A,B) where required, then, modulo geometric obstructions to making connections, the method will always work to yield robust heteroclinic cycles and networks.  $\ast$

*Remark 5.2.* If we assume that the first and second inputs of cells are symmetric in the previous example, then it can be shown that it is still possible to construct the robust simple cycle  $\Sigma$ . However, because of multiple eigenvalues, it is no longer possible to construct a robust simple cycle  $\Sigma_e$ . Essentially, synchrony transforms of face heteroclinic cycles and networks work better than edge cycles if we want to allow for some symmetry in the input structure to cells.  $\otimes$

**Example 5.3** (A four cell SLF system). Next we consider two ways of deriving a heteroclinic cycle between synchronized states from the RPS heteroclinic 3 cycle  $\Sigma$  shown in figure 12.

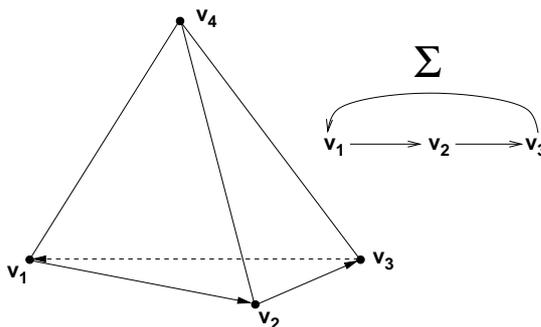


FIGURE 12. An SLF system with heteroclinic 3 cycle.

We start by realizing  $\Sigma$  in the coupled cell network of figure 6 – the coupled cell network  $\mathcal{P}_4$  of [26], see example 3.14. This network has lattice of synchrony subspaces  $\mathcal{S}$  generated by (12), (13), and (14) and so satisfies the conditions of theorem 4.5(2). Let  $(T, \mathcal{B}, \mathcal{S})$  be the associated synchronization transform. The synchronization map  $T$  maps hyperplanes as follows

$$H_1 \mapsto (12), \quad H_2 \mapsto (13), \quad H_3 = 0 \mapsto (14).$$

The heteroclinic cycle  $\Sigma$  transforms to the heteroclinic cycle (134)  $\xrightarrow{13}$  (123)  $\xrightarrow{12}$  (124)  $\xrightarrow{14}$  (134) which can be realized as a robust simple heteroclinic cycle in the network  $\mathcal{P}_4$ . Note that this realization is different from that given by [26, Theorem 4.4] – see section 5.2. We sketch some of the straightforward computations. With node phase space  $\mathbb{R}$ , network dynamics on  $\mathcal{P}_4$  is given by the system

$$\begin{aligned} \dot{x}_1 &= f(x_1; x_2, x_3, x_4), & \dot{x}_2 &= f(x_2; x_1, x_3, x_4), \\ \dot{x}_3 &= f(x_3; x_2, x_1, x_4), & \dot{x}_4 &= f(x_4; x_2, x_3, x_1). \end{aligned}$$

If  $\mathbf{t} = (a, a, a, b) \in (123)$  is an equilibrium of the network vector field  $\mathbf{F}$ , then the eigenvalues of the linearization  $D\mathbf{F}(\mathbf{t})$  are

$$\begin{aligned} \lambda_{1,4} &= \frac{1}{2} \left( (\bar{\alpha}_1 + \sigma) \pm \sqrt{(\bar{\alpha}_1 + \sigma)^2 - 4(\bar{\alpha}_1\sigma - \alpha_4\bar{\sigma})} \right) \\ \lambda_2 &= \alpha_1 - \alpha_2, \\ \lambda_3 &= \alpha_1 - \alpha_3, \end{aligned}$$

where  $\bar{\mathbf{t}} = (b, a, a, a)$ , then  $\alpha_i = \frac{\partial f}{\partial x_i}(\mathbf{t})$ ,  $\bar{\alpha}_i = \frac{\partial f}{\partial x_i}(\bar{\mathbf{t}})$ ,  $i \in \mathbf{4}$ ,  $\sigma = \sum_{i \in \mathbf{3}} \alpha_i$ ,  $\bar{\sigma} = \sum_{i \in \mathbf{3}} \bar{\alpha}_i$  and we have set  $\mathbf{t} = (b, a, a, a)$ . Consequently, we can choose  $\mathbf{F}$  near  $\mathbf{t}$  so that  $\mathbf{t}$  is of index 1 with  $W^u(\mathbf{t}) \subset (13)$ . Similar computations hold for equilibria in (134) and (124). It follows there are no local obstructions to realizing the cycle in  $\mathcal{P}_4$ . Using the method of [26, §4.5], it is easily shown there are no geometric obstructions to realizing  $\Sigma$  as a robust (simple) heteroclinic cycle in  $\mathcal{P}_4$ .

Next we indicate how we can realize  $\Sigma$  in the coupled cell network shown in figure 13. Theorem 4.5(2) again applies and there is an associated synchronization transform  $(T, \mathcal{B}, \mathcal{S})$ . The synchronization map  $T$  maps hyperplanes as follows

$$H_1 \mapsto (12), \quad H_2 \mapsto (34), \quad H_3 \mapsto (13).$$

Using  $T$ ,  $\Sigma$  transforms to the heteroclinic cycle

$$\cdots \rightarrow (134) \xrightarrow{34} (12|34) \xrightarrow{12} (123) \xrightarrow{134} (134) \rightarrow \cdots$$

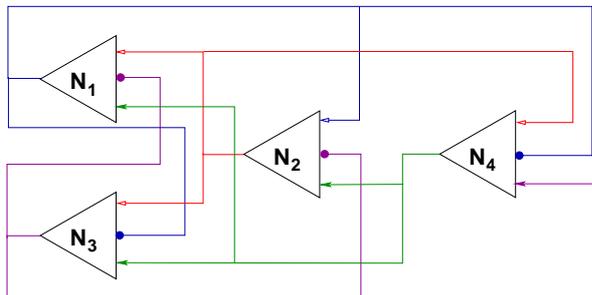


FIGURE 13. A 4 identical cell network with synchrony subspaces generated by (12), (13), (34).

The heteroclinic cycle may be realized as a robust heteroclinic network in the identical cell network of figure 13 which has equations

$$\begin{aligned}\dot{x}_1 &= f(x_1; x_2, x_3, x_4), & \dot{x}_2 &= f(x_2; x_1, x_3, x_4), \\ \dot{x}_3 &= f(x_3; x_2, x_1, x_4), & \dot{x}_4 &= f(x_4; x_2, x_1, x_3).\end{aligned}$$

Explicit realization follows the method of [26, §4.5]. \*

The previous examples all concern strongly connected identical cell networks which have the maximal number of synchrony subspaces. We next look at examples with smaller lattices of synchrony subspaces. For simplicity, we restrict to 4 cell networks.

## 5.2. 4 cell networks.

**Example 5.4.** The identical cell network  $\mathcal{N}$  with network equations

$$\begin{aligned}\dot{x}_1 &= f(x_1; x_2, x_3, x_4), & \dot{x}_2 &= f(x_2; x_1, x_3, x_4), \\ \dot{x}_3 &= f(x_3; x_2, x_3, x_4), & \dot{x}_4 &= f(x_4; x_2, x_1, x_3).\end{aligned}$$

has lattice of synchrony subspaces  $\mathcal{S} = \{(12), (134), (1234)\}$ .

The network is associated to the synchronization transform  $(T, \mathcal{B}, \mathcal{S})$ , where  $T$  maps hyperplanes according to

$$H_1 \mapsto (12), \quad H_{23} \mapsto (134).$$

The nontrivial synchrony subspaces have different dimensions and so we do not expect to see heteroclinic cycles that are transforms of edge or face cycles. Indeed, for this example, the only possible *robust* heteroclinic cycles for  $\mathcal{N}$  must have equilibria on  $(1234) = \mathbf{S}_0$  and so we expect to use the synchrony transform  $T$  to transform a heteroclinic cycle with equilibria on  $\mathbf{V}_0$  to a robust heteroclinic cycle for  $\mathcal{N}$  with equilibria on  $\mathbf{S}_0$ . A heteroclinic cycle with equilibria on  $\mathbf{V}_0$  clearly

cannot be a face or edge heteroclinic cycle. Instead, the equilibria correspond to different active states of the *same* node.

It is not hard to check that an SLF system in  $\mathbb{R}^4$  can support a robust heteroclinic cycle with equilibria on  $\mathbf{V}_0$ . Connections in  $H_1$  (respectively,  $H_{23}$ ) will then be transformed into connections in (12) (respectively, (134)). Given  $\mathbf{p}, \mathbf{q} \in \mathbf{S}_0$ , we search for a network map  $f$  such that the coupled cell system  $\mathcal{N}$  supports a robust heteroclinic cycle  $\Sigma$  with equilibria  $\mathbf{p}, \mathbf{q} \in \mathbf{S}_0$ , and connections  $\mathbf{p} \xrightarrow{12} \mathbf{q}$  and  $\mathbf{q} \xrightarrow{134} \mathbf{p}$ . Observe that *two* nodes desynchronize along the connection  $\mathbf{p} \rightarrow \mathbf{q}$ .

Assuming node phase space  $\mathbb{R}$ , the eigenvalues  $\lambda_i(\mathbf{s})$  of  $D\mathbf{F}(\mathbf{s})$  at an equilibrium  $\mathbf{s} \in \mathbf{S}_0$  of the network vector field  $\mathbf{F}$  are

$$\begin{aligned} \lambda_1(\mathbf{s}) &= \sum_{i \in \mathbf{4}} \alpha_i, \quad \lambda_2(\mathbf{s}) = \alpha_1 - \alpha_2, \\ \lambda_{3,4}(\mathbf{s}) &= \alpha_1 - \frac{1}{2} \left( (\alpha_3 + \alpha_4) \pm i\sqrt{3}(\alpha_3 - \alpha_4) \right), \end{aligned}$$

where  $\alpha_i = \frac{\partial f}{\partial x_i}(\mathbf{s})$ ,  $i \in \mathbf{4}$ . The eigenvalue  $\lambda_1$  corresponds to dynamics on  $\mathbf{S}_0$ ; the eigenvalues  $\lambda_1, \lambda_2$  to dynamics on (134); and the eigenvalues  $\lambda_1, \lambda_3, \lambda_4$  to dynamics on (12). Since  $\lambda_3, \lambda_4$  are complex conjugate, we cannot find a simple heteroclinic cycle with equilibria at  $\mathbf{p}, \mathbf{q}$ .

One possibility is to require  $\lambda_1(\mathbf{p}), \lambda_2(\mathbf{p}) < 0$ ,  $\text{Re}(\lambda_{3,4}(\mathbf{p})) > 0 > \text{Re}(\lambda_{3,4}(\mathbf{q}))$ ,  $\lambda_1(\mathbf{q})$  and  $W^u(\mathbf{p}) \setminus \{\mathbf{p}\} \subset W^s(\mathbf{q}) \subset (12)$ . If, in addition, we require that  $\lambda_2(\mathbf{q}) > 0$ , and there is a connection from  $\mathbf{q}$  to  $\mathbf{p}$  in (134), we obtain a robust heteroclinic cycle with equilibria at  $\mathbf{p}, \mathbf{q}$  (there will be a continuum of connections from  $\mathbf{p}$  to  $\mathbf{q}$ ).  $\ast$

**Example 5.5.** The network of figure 14 is associated to the synchronization transform  $(T, \mathcal{B}, \mathcal{S})$ , where  $T$  maps hyperplanes according to

$$H_{12} \mapsto (12|34), \quad H_{23} \mapsto (13|24).$$

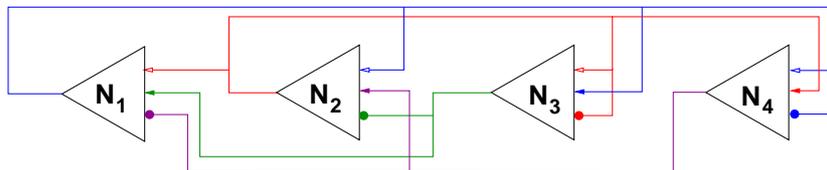


FIGURE 14. A 4 identical cell network with synchrony subspaces  $\mathcal{S} = \{(12|34), (13|24), (1234)\}$ .

The equations for the network are

$$\begin{aligned}\dot{x}_1 &= f(x_1; x_2, x_3, x_4), \quad \dot{x}_2 = f(x_2; x_1, x_4, x_3), \\ \dot{x}_3 &= f(x_3; x_2, x_1, x_2), \quad \dot{x}_4 = f(x_4; x_1, x_2, x_1).\end{aligned}$$

It is straightforward to verify that the network supports a heteroclinic cycle with vertices  $\mathbf{p}, \mathbf{q} \in \mathbf{S}_0$  and connections  $\mathbf{p} \xrightarrow{12|34} \mathbf{q}$  and  $\mathbf{q} \xrightarrow{13|24} \mathbf{p}$ . This cycle is derived from a cycle for an SLF system on  $\mathbb{R}^4$  (but not from a face heteroclinic cycle). The eigenvalues of the linearization of the network vector field at an equilibrium  $\mathbf{s} \in \mathbf{S}_0$  are

$$\lambda_1 = \sum_{i \in \mathbf{4}} \alpha_i, \quad \lambda_2 = \alpha_1 - \alpha_3 - \alpha_4, \quad \lambda_3 = \alpha_1 - \alpha_3 - \alpha_4, \quad \lambda_4 = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4,$$

where  $\alpha_i = \frac{\partial f}{\partial x_i}(\mathbf{s})$ ,  $i \in \mathbf{4}$ . The eigenvalue  $\lambda_1$  corresponds to dynamics on  $\mathbf{S}_0$ ; the eigenvalues  $\lambda_1, \lambda_2$  to dynamics on  $(12|34)$ ; and the eigenvalues  $\lambda_1, \lambda_3$  to dynamics on  $(13|24)$ .

Similarly, we predict and find that the 4 cell identical system of figure 9(a), with synchrony subspaces  $(12|34)$ ,  $(13|24)$ ,  $(14|23)$ , and  $(1234)$ , supports a robust heteroclinic cycle  $\Sigma$  connecting three equilibria  $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbf{S}_0$  and with three connections, each lying in one of the synchrony subspaces  $(12|34)$ ,  $(13|24)$ ,  $(14|23)$ . We remark the example gives an interesting variation on the standard representation of the RPS heteroclinic 3 cycle.

The 4-cell network of figure 9(b), which does not have a synchronization transform, does not support robust heteroclinic cycles with equilibria on  $\mathbf{S}_0$ . The network does support a robust heteroclinic cycle with equilibria on  $(12|34)$  and connections in  $(12)$  and  $(34)$ .  $\ast$

**5.3. Summary of the results for 4 cell networks.** Assume networks are strongly connected, without self loops and that cells have asymmetric inputs. By simple heteroclinic cycle we mean here that unstable manifolds of equilibria are one-dimensional. In every case, a simple heteroclinic cycle will extend to a heteroclinic network which is simple in the sense of section 1 and has the same set of equilibria.

In table 1, we list the 4 identical cell networks which do not support robust heteroclinic cycles. These networks all have synchronization transforms and satisfy parts I and II of the realization conjecture as well as conditions (A,B).

TABLE 1. 4 cell networks which do not support robust heteroclinic cycles

	Network equations	Generators of $\mathcal{S}(\mathcal{N})$	Multiple eigenvalues
$\mathcal{U}_1$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_3, x_1, x_4)$ $\dot{x}_3 = f(x_3; x_4, x_4, x_2)$ $\dot{x}_4 = f(x_4; x_1, x_3, x_1)$	(1234)	No
$\mathcal{U}_2$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_4, x_2, x_4)$ $\dot{x}_4 = f(x_4; x_3, x_1, x_2)$	(12), (1234)	No
$\mathcal{U}_3$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_3, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_1, x_2, x_4)$ $\dot{x}_4 = f(x_4; x_1, x_1, x_1)$	(123), (1234)	No
$\mathcal{U}_4$	$\dot{x}_1 = f(x_1; x_2, x_4, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_4, x_1, x_2)$ $\dot{x}_4 = f(x_4; x_3, x_2, x_2)$	(12 34), (1234)	No
$\mathcal{U}_5$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_2, x_4)$ $\dot{x}_4 = f(x_4; x_3, x_1, x_3)$	(12), (123), (1234)	No

In table 2, we list the 4 cell networks that support robust heteroclinic cycles.

*Remarks 5.6.* (1) The network  $\mathcal{V}_1$  does not support robust simple heteroclinic cycles and does not satisfy condition (B). We refer to example 5.4. The networks  $\mathcal{V}_2$ ,  $\mathcal{V}_4$  support both simple and not simple robust heteroclinic cycles. All three networks show the phenomenon of two cells desynchronizing along one of the connections.

(2) All networks have a synchronization transform and satisfy the realization conjectures. All networks satisfy condition (A); other than  $\mathcal{V}_1$ , all networks satisfy condition (B).  $\otimes$

TABLE 2. 4 cell networks supporting robust heteroclinic cycles.

	Network equations	Generators of $S(\mathcal{N})$	Multiple eigenvalues	Robust heteroclinic cycles	Simple heteroclinic cycles
$\mathcal{V}_1$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_2, x_4)$ $\dot{x}_4 = f(x_4; x_3, x_1, x_3)$	(12), (134)	No	Yes, but see remarks	No: complex eigenvalues
$\mathcal{V}_2$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_4, x_1, x_2)$ $\dot{x}_4 = f(x_4; x_3, x_3, x_2)$	(12), (13 24)	No	Yes	Yes, but complex eigenvalues possible
$\mathcal{V}_3$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_4, x_3)$ $\dot{x}_3 = f(x_3; x_2, x_1, x_2)$ $\dot{x}_4 = f(x_4; x_1, x_2, x_1)$	(12 34), (13 24)	No	Yes	Yes
$\mathcal{V}_4$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_4, x_1, x_2)$ $\dot{x}_4 = f(x_4; x_3, x_2, x_2)$	(12), (12 34), (1234)	No	Yes	Yes, but complex eigenvalues possible
$\mathcal{V}_5$	$\dot{x}_1 = f(x_1; x_2, x_2, x_3)$ $\dot{x}_2 = f(x_2; x_1, x_1, x_3)$ $\dot{x}_3 = f(x_3; x_4, x_1, x_2)$ $\dot{x}_4 = f(x_4; x_3, x_1, x_2)$	(12), (34), (1234)	No	Yes	Yes
$\mathcal{V}_6$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_1, x_4)$ $\dot{x}_4 = f(x_4; x_1, x_1, x_1)$	(12), (13), (1234)	No	Yes	Yes
$\mathcal{V}_7$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_2, x_4)$ $\dot{x}_4 = f(x_4; x_1, x_2, x_3)$	(12), (123), (12 34)	No	Yes	Yes
$\mathcal{V}_8$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_4, x_3)$ $\dot{x}_3 = f(x_3; x_4, x_2, x_1)$ $\dot{x}_4 = f(x_4; x_3, x_1, x_2)$	(12 34), (13 24), (14 23)	No	Yes	Yes
$\mathcal{V}_9$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_2, x_4)$ $\dot{x}_4 = f(x_4; x_1, x_3, x_1)$	(123), (124)	No	Yes	Yes
$\mathcal{V}_{10}$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_1, x_4)$ $\dot{x}_4 = f(x_4; x_2, x_3, x_1)$	(12), (13), (14)	No	Yes	Yes
$\mathcal{V}_{11}$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_1, x_4)$ $\dot{x}_4 = f(x_4; x_2, x_1, x_3)$	(12), (13), (34)	No	Yes	Yes

Finally, in table 3, we list the 4 cell networks that support robust heteroclinic cycles but do not satisfy condition (A).

TABLE 3. 4 cell networks which support robust heteroclinic cycles but do not satisfy condition (A).

	Network equations	Generators of $\mathcal{S}(\mathcal{N})$	Multiple eigenvalues	Robust heteroclinic cycles	Simple heteroclinic cycles
$\mathcal{W}_1$	$\dot{x}_1 = f(x_1; x_2, x_4, x_3)$ $\dot{x}_2 = f(x_2; x_1, x_4, x_3)$ $\dot{x}_3 = f(x_3; x_4, x_2, x_2)$ $\dot{x}_4 = f(x_4; x_3, x_2, x_2)$	(12), (34), (1234)	Yes: (1234)	Yes: (12 34)	Yes
$\mathcal{W}_2$	$\dot{x}_1 = f(x_1; x_2, x_4, x_3)$ $\dot{x}_2 = f(x_2; x_1, x_4, x_3)$ $\dot{x}_3 = f(x_3; x_4, x_2, x_1)$ $\dot{x}_4 = f(x_4; x_3, x_2, x_1)$	(12), (34), (13 24)	Yes: (1234)	Yes: (12 34)	Yes

*Remarks 5.7.* (1)  $\mathcal{W}_1$  has a synchronization transform. Although there is no robust heteroclinic cycle joining equilibria on (1234), the network does support robust heteroclinic cycles joining equilibria on (12|34). Note that network  $\mathcal{V}_5$  has same synchrony classes as  $\mathcal{W}_1$  and does support robust heteroclinic cycles joining equilibria on (1234).

(2)  $\mathcal{W}_2$  does not support a synchronization transform. Although there is no robust heteroclinic cycle joining equilibria on (1234),  $\mathcal{W}_2$  does support robust heteroclinic cycles joining equilibria on (12|34).  $\otimes$

We have identified 17 inequivalent networks which are associated to a synchronization transform and for which the realization conjectures hold. There is a network  $\mathcal{W}_2$  – figure 9(b) – which is not associated to a synchronization transform and is consistent with part II of the realization conjecture. Of the 17 networks which are associated to a synchronization transform, 12 support robust heteroclinic cycles which in every case are related to heteroclinic cycles in an SLF system.

**5.4. Heteroclinic networks.** We start with two examples of robust heteroclinic network which can be realized in a coupled cell network which is the synchronization transform of an SLF face network. We end with a general theorem.

**Example 5.8.** In figure 15 we show a robust 3-face heteroclinic network  $\Sigma_N$  supported in  $\Delta_5 \subset \mathbb{R}^6$  (alternatively, view as a 2-face heteroclinic network supported in  $F_{12345} \subset \Delta_5$ ). Referring to figure 15,  $\mathbf{v}_{346}$  is an interior equilibrium point on the 2-face  $\mathbf{F}_{346}$ . Each connection between equilibria lies in a 3-face of  $\Delta_5$ . For example,  $\mathbf{v}_{346} \rightarrow \mathbf{v}_{246}$  lies

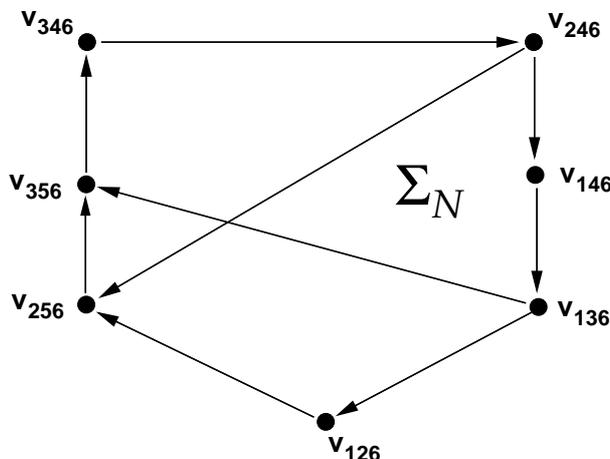


FIGURE 15. Schematic for 3-face heteroclinic network  $\Sigma_N$  supported in  $\Delta_5$ .

in the 3-face  $\mathbf{F}_{2346}$ . The index of the equilibria  $\mathbf{v}_{246}$  and  $\mathbf{v}_{136}$  is 2, all other equilibria have index 1.

Applying theorem 4.5(2), there is a six identical cell network  $\mathcal{N}$  with lattice  $\mathcal{S}$  of synchrony subspaces generated by  $a_1 = (14)$ ,  $a_2 = (13)$ ,  $a_3 = (15)$ ,  $a_4 = (25)$ , and  $a_5 = (46)$ . We have a synchronization transform  $(T, \mathcal{B}, \mathcal{S})$  where  $T$  maps  $\mathcal{B} = \{H_j \mid j \in \mathbf{5}\}$  to  $\mathcal{S}$  according to  $H_j \mapsto a_j$ ,  $j \in \mathbf{5}$ . Applying  $T$  to the heteroclinic network  $\Sigma_N$  ( $T(\mathbf{v}_{346}) \in a_1 \vee a_2 \vee a_5 = (1346)$ , etc), we obtain the predicted heteroclinic network  $\Sigma$  for the architecture  $\mathcal{N}$ . We show the result in figure 16 (nodes and connections are labelled by synchrony type).

Explicit equations for the network  $\mathcal{N}$  are given by

$$\begin{aligned} \dot{x}_1 &= f(x_1; x_4, x_3, x_5, x_2, x_6), & \dot{x}_2 &= f(x_2; x_4, x_3, x_1, x_5, x_6) \\ \dot{x}_3 &= f(x_3; x_4, x_1, x_5, x_2, x_6), & \dot{x}_4 &= f(x_4; x_1, x_3, x_5, x_2, x_6) \\ \dot{x}_5 &= f(x_5; x_4, x_3, x_1, x_2, x_6), & \dot{x}_6 &= f(x_6; x_1, x_3, x_5, x_2, x_4) \end{aligned}$$

It is straightforward, using the methods of [26], to show that  $f$  can be chosen so that  $\Sigma$  is a robust heteroclinic network in the architecture  $\mathcal{N}$  with 1-dimensional cell dynamics. In particular, all equilibria will be hyperbolic saddles with the correct index, and connections will lie in the synchrony subspaces indicated in figure 16.  $\ast$

*Remark 5.9.* The heteroclinic network  $\Sigma$  has 7 equilibria and 9 connections and no two connections lie in the same synchrony subspace. The realization is more efficient than that given in [26, Theorem 1.1] which realizes the heteroclinic network in an identical 10 cell network.  $\circledast$

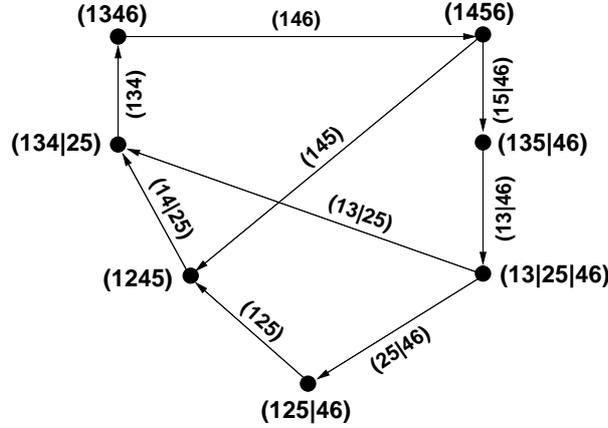


FIGURE 16. The robust heteroclinic network  $\Sigma$  realized in the coupled cell network  $\mathcal{N}$  with synchrony classes generated by (14), (13), (15), (25), (46).

**Example 5.10.** We conclude with example 1.5 from the introduction (see figure 1). We have previously discussed the 2-face cycle  $\Sigma$  of figure 1(a) in examples 2.10(3). We use the coupled 6 identical coupled cell  $\mathcal{N}$  system with synchrony generated by (12), (23), (36), (45), (46) and given by theorem 4.5(2). The associated network vector field is

$$\begin{aligned} \dot{x}_1 &= f(x_1; x_2, x_3, x_6, x_5, x_4), & \dot{x}_2 &= f(x_2; x_1, x_3, x_6, x_5, x_4) \\ \dot{x}_3 &= f(x_3; x_1, x_2, x_6, x_5, x_4), & \dot{x}_4 &= f(x_4; x_1, x_2, x_3, x_5, x_6) \\ \dot{x}_5 &= f(x_5; x_1, x_2, x_3, x_4, x_6), & \dot{x}_6 &= f(x_6; x_1, x_2, x_3, x_5, x_4) \end{aligned}$$

The heteroclinic network  $\Sigma^T$  of figure 1(b) is derived from  $\Sigma$  (viewed as a subset of  $\mathbf{F}_{12345} \subset \mathbf{\Delta}_5$ ) using the synchronization transform exactly as in the previous example. Although  $\Sigma$  is clean we do not know whether it is possible to obtain a clean realization of  $\Sigma^T$  in  $\mathcal{N}$ .  $\ast$

Let  $(a_i, b_i) \in \mathcal{D}(k+1)$ ,  $i \in \mathbf{k}$ . Let  $\mathcal{S} \subset \mathcal{D}(k+1)$  be the sublattice generated by  $\{(a_i, b_i) \mid i \in \mathbf{k}\}$  and suppose that  $\vee_{i \in \mathbf{k}} (a_i, b_i) = \mathbf{S}_0$ . Let  $\mathcal{N}$  be the coupled identical  $k+1$  cell network given by theorem 4.5(2) and  $(T, \mathcal{B}, \mathcal{S})$  be a synchronization transform, where  $T: \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{k+1}$ .

**Theorem 5.11.** (Notation and assumptions as above). Suppose that  $\Sigma$  is a robust  $r$ -face heteroclinic network in  $\mathbb{R}^k$ ,  $r \geq 1$ . Then  $\Sigma$  may be realized as a robust heteroclinic network  $\Sigma^T$  in  $\mathcal{N}$ . Vertices of  $\Sigma^T$  will lie in  $r+1$ -dimensional synchrony subspaces and connections will lie in  $r+2$ -dimensional synchrony subspaces. Synchrony of vertices and

connections is given by the synchronization map  $T$  applied to vertices and connections of  $\Sigma$ .

*Proof.* The identification of the network  $\Sigma^T$  follows the approach of the previous two examples – embed  $\Sigma$  as an  $r + 1$ -face network in  $\mathbb{R}^{k+1}$ . Local analysis near the vertices of the network  $\Sigma^T$  is straightforward and along the lines presented in [26, §§4.1,4.2]. Similarly, the proof of existence of connections depends on the transversality and extension arguments given in [26, §4.5].  $\square$

## 6. OUTSTANDING QUESTIONS AND CONCLUDING COMMENTS

### 6.1. The Realization Conjectures.

- (1) Part I of the conjectures proposes that if there is a synchronizations transform  $(T, \mathcal{B}, \mathcal{S})$ , then there is a coupled identical cell network  $\mathcal{N}$  with  $\mathcal{S}(\mathcal{N}) = \mathcal{S}$ . Is it possible to find an algorithm that gives  $\mathcal{N}$  (that is, the architecture) in terms of  $(T, \mathcal{B}, \mathcal{S})$ ?
- (2) Is it possible to find strong sufficient conditions on a coupled identical cell network  $\mathcal{N}$  that imply there are no restrictions on index or multiplicity of linearization of network vector fields at equilibria? (Conditions (A,B) of section 4.)

Regarding (2), symmetries of  $\mathcal{N}$  and  $\mathcal{S}(\mathcal{N})$  seem to play a role. However, conditions such as requiring symmetries of  $\mathcal{N}$  to act trivially on  $\mathcal{S}$  or all symmetries of  $\mathcal{S}$  to act trivially on  $\mathcal{S}$  are either false (see  $\mathcal{W}_2$ ) or too weak.

### 6.2. Transition from SLF to coupled identical cell networks.

It is generally straightforward to construct explicit vector fields that realize heteroclinic cycles and networks in an SLF system [21, 11, 12]. Often (perhaps always) cubic vector fields suffice. On the other hand it seems difficult to construct explicit vector fields that realize heteroclinic cycles and networks in coupled identical cell systems. Can these difficulties be overcome by working with heterogeneous networks with two cell types or by allowing the node (or coupling) dynamics to be 2-dimensional? We refer to the recent work of Ashwin & Postlethwaite [12] for the use of two cell types and to the comments on scalar signalling in [3, 26] for restrictions on coupling. We are inclined to the view that there should be a natural way of transforming from a coupled identical cell system of the type we construct to systems with two or more distinct cell types for which it is easy to construct explicit and natural vector fields realizing heteroclinic cycles and networks<sup>1</sup>.

<sup>1</sup>However, this might require thresholds and vector fields may only be piecewise smooth or pulse coupled [48].

In short, we would like to think of the identical cell network, with one dimensional node dynamics, as a *minimal model* for coupled cell networks supporting robust heteroclinic phenomena (just as the Kuramoto phase oscillator model can be viewed as a minimal model for describing the phase dynamics of a weakly coupled system of nonlinear oscillators). The question then is how to unfold the minimal model to obtain more physically realistic models with, for example, additive input structure [26, §2, §5] (a crucial assumption in the reduction of weakly coupled networks of nonlinear oscillators to phase oscillator models [44, 38]).

If a heteroclinic network is clean, can we always require the same of the network when realized in an coupled identical cell system? In particular, how straightforward is it to construct asymptotically stable heteroclinic attractors in coupled cell systems?

**6.3. Symmetric inputs and sparseness of coupling.** Suppose that an  $r$ -face heteroclinic network is realized in a coupled identical cell system using one of the architectures given by theorem 4.5(2). Under what conditions can we ‘symmetrize’ some of the inputs to the cells without breaking the robust structure – for example, by introducing multiple eigenvalues. Is it the case that the larger  $r$  is the more inputs we can symmetrize and can this be quantified?

In biological and technological networks, coupling is typically sparse and far from “all-to-all”. In examples 3.15(2) we gave a simple six identical cell example, with two symmetric inputs, which supported a robust heteroclinic cycle. Is it possible to find families of identical cell networks which have relatively few inputs, compared to the total number of cells, such that (a) the networks support robust heteroclinic networks, and (b) robustness persists if we allow for (approximately) symmetric inputs? We would expect that a positive answer to this question would yield potentially realistic networks for which there were robust heteroclinic networks connecting relatively small clusters of synchronized cells.

### Acknowledgements

I appreciate many helpful comments made by the referees.

### REFERENCES

- [1] V S Afraimovich, V P Zhigulin, and M I Rabinovich. ‘On the origin of reproducible sequential activity in neural circuits’, *Chaos* **14**(4) (2004), 1123–1129.
- [2] V Afraimovich, I Tristan, R Huerta, and M I Rabinovich. ‘Winnerless competition principle and prediction of the transient dynamics in a LotkaVolterra model’, *Chaos* **18**(4) (2008).

- [3] N Agarwal and M Field. ‘Dynamical equivalence of networks of coupled dynamical systems I: Asymmetric inputs’, *Nonlinearity* **23** (2010), 1245–1268.
- [4] N Agarwal and M Field. ‘Dynamical equivalence of networks of coupled dynamical systems II: general case’, *Nonlinearity* **23** (2010), 1269–1289.
- [5] M A D Aguiar and A Dias. ‘Heteroclinic network dynamics on joining coupled cell networks’, preprint (2015).
- [6] M A D Aguiar, P Ashwin, A Dias, and M Field. ‘Dynamics of coupled cell networks: synchrony, heteroclinic cycles and inflation’, *J. Nonlinear Science* **21**(2) (2011), 271–323.
- [7] M A D Aguiar and A P S Dias. ‘The Lattice of Synchrony Subspaces of a Coupled Cell Network: Characterization and Computation Algorithm’, *J. Nonlinear Science* **24** (2014), 949–996.
- [8] P Ashwin and J Borresen. ‘Discrete computation using a perturbed heteroclinic network’, *Phys. Rev. E* **70** (2004), 026203.
- [9] P Ashwin, G Orosz, J Wordsworth, and S Townley. ‘Reliable switching between cluster states for globally coupled phase oscillators’, *SIAM Journal on Applied Dynamical Systems* **6** (2007), 728–758.
- [10] P Ashwin and M Field. ‘Heteroclinic networks in coupled cell systems’, *Arch. Rational Mech. Anal.* **148** (1999), 107–143.
- [11] P Ashwin and C M Postlethwaite. ‘On designing heteroclinic networks from graphs’, *Physica D* **265** (2013), 26–39.
- [12] P Ashwin and C Postlethwaite. ‘Designing heteroclinic and excitable networks in phase space using two populations of coupled cells’, *J. Nonlinear Science* **26**(2) (2016), 345–364.
- [13] W Brannath. ‘Heteroclinic networks on the tetrahedron’, *Nonlinearity* **7** (1994), 1367–1384.
- [14] B A Davey and H A Priestley. *Introduction to Lattices and Order* (Cambridge University Press, Cambridge, 1990).
- [15] A P S Dias, B Dionne, and I Stewart. ‘Heteroclinic Cycles and Wreath Product Symmetries’, *Dynamics and Stability of Systems* **15** (2000), 353–385.
- [16] A P S Dias and I Stewart. ‘Linear equivalence and ODE-equivalence for coupled cell networks’, *Nonlinearity* **18** (2005), 1003–1020.
- [17] M J Field. ‘Equivariant dynamical systems’, *Bull. Amer. Math. Soc.*, (1970), 1314–1318.
- [18] M J Field. ‘Transversality in  $G$ -manifolds’, *Trans. Amer. Math. Soc.* **231**(2) (1977), 429–450.
- [19] M J Field. ‘Equivariant dynamical systems’, *Trans. Amer. Math. Soc.* **259**(1) (1980), 185–205.
- [20] M J Field. ‘Equivariant Bifurcation Theory and Symmetry Breaking’, *J. Dynamics and Diff. Eqns.* **1**(4) (1989), 369–421.
- [21] M J Field. *Dynamics, Bifurcation and Symmetry*, Pitman Research Notes in Mathematics, **356**, 1996.
- [22] M J Field. ‘Heteroclinic cycles in symmetrically coupled systems’, *Proc. IMA workshop on Pattern Formation in Continuous and Coupled Systems*, May 11–18, 1998 (eds Golubitsky, Luss, Strogatz), IMA volumes no 115, Springer-Verlag, 1999, 49–64.
- [23] M J Field. ‘Combinatorial Dynamics’, *Dynamical Systems* **19** (2004), 217–243.

- [24] M J Field. Unpublished notes and talks given in 2005 on Heteroclinic cycles and coupled cell systems (talks accessible at URL: [math.rice.edu/~mjf8/research/networks/SLF](http://math.rice.edu/~mjf8/research/networks/SLF)).
- [25] M J Field. *Dynamics and Symmetry* (Imperial College Press Advanced Texts in Mathematics — Vol. 3, 2007.)
- [26] M J Field. ‘Heteroclinic Networks in Homogeneous and Heterogeneous Identical Cell Systems’, *J. Nonlinear Science* **25**(3) (2015), 779–813.
- [27] M J Field and R W Richardson. ‘Symmetry breaking and branching patterns in equivariant bifurcation theory II’, *Arch. Rational Mech. and Anal.* **120** (1992), 147–190.
- [28] M J Field and J Swift. ‘Static bifurcation to limit cycles and heteroclinic cycles’, *Nonlinearity* **4**(4) (1991), 1001–1043.
- [29] M Golubitsky, M Pivato, and I Stewart. ‘Interior symmetry and local bifurcation in coupled cell networks’. *Dynamical Systems* **19** (2004), 389–407.
- [30] M Golubitsky and I Stewart. ‘Nonlinear dynamics of networks: the groupoid formalism’. *Bull. Amer. Math. Soc.* **43** (2006), 305–364.
- [31] M Golubitsky, I Stewart, and A Török. ‘Patterns of Synchrony in Coupled Cell Networks with Multiple Arrows’, *SIAM J. Appl. Dynam. Sys.* **4**(1) (2005), 78–100.
- [32] J Guckenheimer and P Holmes. ‘Structurally stable heteroclinic cycles’, *Math. Proc. Camb. Phil. Soc.* **103** (1988), 189–192.
- [33] D Hansel, G Mato, C Meunier. ‘Phase dynamics for weakly coupled Hodgkin-Huxley neurons’, *Europhys Letts* **23** (1993), 367–372.
- [34] J Hofbauer. ‘Heteroclinic cycles on the simplex’, *Proc. Int. Conf. Nonlinear Oscillations, Janos Bolyai Math. Soc. Budapest*, 1987.
- [35] J Hofbauer. ‘Heteroclinic cycles in ecological differential equations’, *Tatra Mountains Math. Publ.* **4** (1994), 105–116.
- [36] J Hofbauer and K Sigmund. *The Theory of Evolution and Dynamical Systems*, Cambridge University Press, Cambridge, 1988.
- [37] J Hofbauer and K Sigmund. *Evolutionary Games and Population Dynamics*, Cambridge University Press, Cambridge, 1998.
- [38] F C Hoppensteadt and E M Izhikevich. *Weakly Connected Neural Networks* (Springer, Applied Mathematical Science **126**, Springer-Verlag, 1997).
- [39] A Ismail and P Ashwin. ‘Multi-cluster dynamics in coupled phase oscillator networks’, *Dynamical Systems—An International Journal* **30**(1) (2015), 122–135.
- [40] V Kirk and M Silber. ‘A competition between heteroclinic cycles’, *Nonlinearity* **7**(6) (1994), 1605–1622.
- [41] M Krupa. ‘Robust heteroclinic cycles’, *J. Nonlinear Science* **7** (1997), 129–176.
- [42] M Krupa and I Melbourne. ‘Asymptotic stability of heteroclinic cycles in systems with symmetry’, *Ergod. Th. & Dynam. Sys.* **15** (1995), 121–147.
- [43] M Krupa and I Melbourne. ‘Asymptotic stability of heteroclinic cycles in systems with symmetry, II’, *Proc. Roy. Soc. Edinburgh A* **134A** (2004), 1177–1197.
- [44] Y Kuramoto. *Chemical oscillations, Waves and Turbulence* (Dover publications, New York, 2003; reprint with corrections of 1984 edition, Springer-Verlag, New York).
- [45] R M May and W J Leonard. ‘Nonlinear aspects of competition between three species’, *SIAM J. Appl. Math.*, **29** (1975), 243–253.

- [46] I Melbourne. ‘An example of a nonasymptotically stable attractor’. *Nonlinearity* **4** (1991), 835–844.
- [47] I Melbourne, P Chossat, and M Golubitsky. ‘Heteroclinic cycles involving periodic solutions in mode interactions with  $O(2)$  symmetry’, *Proc. Roy. Soc. Edinburgh* **113A** (1989), 315–345.
- [48] R E Mirolo and S H Strogatz. ‘Synchronization of pulse-coupled biological oscillators’, *SIAM J. Appl. Math.* **50**(6) (1990), 1645–1662.
- [49] M K Muezzinoglu, I Tristan, R Huerta, V S Afraimovich, and M I Rabinovich. ‘Transients versus attractors in complex networks’, *Int. Jnr. of Bifn. and Chaos* **20**(6) (2010), 1653–1675.
- [50] T Nowotny and M Rabinovich. ‘Dynamical Origin of Independent Spiking and Bursting Activity in Neural Microcircuits’, *Phys. Rev. Letts* **98** (2007), 128106.
- [51] G Orosz, J Moehlis, and P Ashwin. ‘Designing the dynamics of globally coupled oscillators’, *Progress of Theoretical Physics*, **122**(3) (2009), 611–630.
- [52] A Rodrigues and M J Field. ‘Attractors in Complex Networks’, in preparation.
- [53] M M Peixoto. ‘On an approximation theorem of Kupka and Smale’, *J. Diff. Eqn.* **3** (1966), 214–227.
- [54] M Rabinovich, R Huerta, and G Laurent. ‘Transient Dynamics for Neural Processing’, *Science* **321** (2008), 48–50.
- [55] G L dos Reis. ‘Structural stability of equivariant vector fields on two manifolds’, *Trans. Amer. Math. Soc.*, **283** (1984), 633–643.
- [56] L P Shilnikov, A L Shilnikov, D V Turaev, and L O Chua. *Methods of Qualitative Theory in Nonlinear Dynamics, Part 1* (World Scientific, Singapore, 1998).
- [57] A L Shilnikov, D V Turaev, L O Chua, and L P Shilnikov. *Methods of Qualitative Theory in Nonlinear Dynamics, Part 2* (World Scientific, Singapore, 2002).
- [58] I Stewart. ‘The lattice of balanced equivalence relations of a coupled cell network’, *Math. Proc. Cambridge Philos. Soc.* **143**(1) (2007), 165–183.
- [59] I Stewart, M Golubitsky, and M Pivato. ‘Symmetry groupoids and patterns of synchrony in coupled cell networks’, *SIAM J. Appl. Dynam. Sys.* **2** (4) (2003), 606–646.

MICHAEL FIELD, DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, SOUTH KENSINGTON CAMPUS, LONDON SW7 2AZ, UK

*E-mail address:* mikefield@gmail.com, Michael.Field@imperial.ac.uk