EXPONENTIAL MIXING FOR SMOOTH HYPERBOLIC SUSPENSION FLOWS

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Abstract. We present some simple examples of exponentially mixing hyperbolic suspension flows. We include some speculations indicating possible applications to suspension flows of algebraic Anosov systems. We conclude with some remarks about generalizations of our methods.

1. Introduction

Let M be a differential manifold and $\Lambda \subset M$ be a hyperbolic locally maximal (basic) set of a smooth (class at least C^2) flow $\Phi_t : M \rightarrow M$. We assume Λ is not a periodic orbit and that $\Phi_t|\Lambda = \phi_t$ is transitive (equivalently, Λ is connected). In this work we are interested in quantitative mixing properties of ϕ_t. Suppose then that μ is a ϕ_t-invariant equilibrium state on Λ associated to a Hölder continuous potential on Λ (μ is necessarily an ergodic probability measure on Λ; for background and more details, see [7, 24] and note that in what follows we always assume equilibrium states are associated to Hölder continuous potentials). Given $f, g \in L^2(\Lambda, \mu)$, we define the correlation function $\rho_{f,g} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\rho_{f,g}(t) = \int_\Lambda f \circ \phi_t \, g \, d\mu - \int_\Lambda f \, d\mu \int_\Lambda g \, d\mu.$$

The flow ϕ_t is μ-mixing if and only if $\rho_{f,g}(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $f, g \in L^2(\Lambda, \mu)$. We recall that if ϕ_t is μ-mixing then ϕ_t is ν-mixing with respect to all equilibrium states ν and μ-mixing is equivalent to weak-mixing or topological mixing. In future if ϕ_t is μ-mixing for some μ, we usually just say ϕ_t is mixing. Bowen [5, 6] proved that nontrivial basic sets are generically mixing; Field *et al.* [14] show that there is a C^2-open and dense set of the Axiom A flows on M for which each non-trivial basic set is mixing (indeed, rapid mixing).

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In order to quantify the rate of mixing it is necessary to assume more regularity on the functions f, g used to define the correlation function $\rho_{f,g}$. The flow ϕ_t is *exponentially μ-mixing* if, given $\alpha \in (0,1)$, there exist constants $C = C(\alpha)$, $\sigma = \sigma(\alpha) > 0$ such that for all $f, g \in C^\alpha(\Lambda)$,

$$|\rho_{f,g}(t)| \leq C\|f\|_\alpha \|g\|_\alpha e^{-\sigma t}.$$

(Typically, one proves an estimate of this type assuming stronger regularity of f, g (say C^1 or better) and then uses interpolation to get the estimate in terms of Hölder norms. See, for example, Dolgopyat [10, §1] or Liverani [21, Corollary 2.5].) We say the flow is exponentially mixing if it is exponentially μ-mixing for all equilibrium states μ.

Bowen and Ruelle [7] asked whether $\rho_{f,g}(t)$ decays at an exponential rate when f, g are restrictions of smooth functions to Λ. Subsequently, Ruelle [30] found examples of mixing Axiom A flows which did not mix exponentially. Moreover, Pollicott [26] showed that the decay rates for mixing basic sets could be arbitrarily slow. The first examples of exponential mixing (relative to volume measure) were obtained for geodesic flows on manifolds of constant negative curvature in dimensions two [9, 23, 29] and three [27]. The proofs for these results were group theoretical and unsuitable for generalization to higher dimensions or the non-constant curvature case. Following the work of Chernov [8], who proved subexponential decay of correlations for geodesic flows on surfaces of variable negative curvature, Dolgopyat [10], using methods based on thermodynamic formalism and Markov partitions, was able to establish exponential mixing for geodesic flows on surfaces of variable negative curvature as well as all transitive Anosov flows with jointly non-integrable C^1 strong stable and strong unstable foliations (arbitrary equilibrium state) as well as general results on rapid mixing (the latter results assumed strong regularity of the observables f, g [11]). More recently, Liverani [21], using a geometric approach that avoids Markov partitions, has proved exponential mixing for contact Anosov flows (Riemannian volume) [21]. Liverani's approach builds on earlier work of Blank, Keller & Liverani [4] and depends on constructing spaces of distributions on which the transfer operator acts directly. Variants of this approach have been used in a number of other works (for example, [16, 3, 2, 3]) and most recently by Tsujii [33] who proves quasi-compactness of the transfer operator for contact Anosov flows and obtains sharp estimates for decay of correlations. A number of results on exponential mixing for semi-flows have also appeared [28, 32].

There are indeed rather few examples known of smooth exponentially mixing flows and the examples that we do have are often only valid for a specific measure. Part of the difficulty in verifying exponential mixing is that existing techniques depend on establishing some regularity for the temporal distance function (see section 3.1) and, without additional geometric structure, regularity of the temporal distance function depends on the regularity of the strong stable and unstable foliations which is generally poor and even worse than that of the weak stable and unstable foliations [18, 15]. However, in the case of contact Anosov flows, stronger regularity of the temporal distance function is known (for example, Katok & Burns [19], and Liverani [21, Appendix B]).

There is, however, one other class of exponentially mixing flows obtained by Dolgopyat [12]. Motivated by work of Parry and Pollicott [25] on the stability of mixing for compact group extensions over symbolic subshifts of finite type, Dolgopyat showed that generic suspension flows over subshifts of finite type are stably exponentially mixing (for all equilibrium states). This result applies to the symbolic setting and roof functions are Lipschitz with respect to the usual d_θ-metric on symbolic shifts. As Dolgopyat points out [12, §1], this result cannot be used to give a thermodynamic formalism proof of the stability of exponential mixing for smooth Axiom A systems. Interestingly, the result of Parry and Pollicott was also influential in the paper of Field *et al.* [14], where they proved the stability of mixing and rapid mixing for Axiom A systems. A useful observation [13, 14] was that if a hyperbolic basic set Λ contained a smoothly embedded suspended subshift of finite type that was stably mixing (or rapid mixing), then Λ was forced to be stably mixing (or stably rapid mixing). The mixing case is easy to see using Bowen's well-known characterization of mixing in terms of the prime periods of the flow [5]. Alternatively, one may argue using the eigenfunction characterization of weak mixing.

In this paper we use some of the results of Dolgopyat on suspensions of symbolic subshifts of finite type to enlarge the class of known exponentially mixing flows.

Theorem 1.1. *Let Σ be a (mixing) subshift of finite type. Then we may embed Σ in a compact differential manifold M so that*

(1) *Σ is a basic set of a C^r Axiom A diffeomorphism $f : M \to M$, $2 \leq r \leq \infty$.*
(2) *There is a C^2-open and C^r-dense subset $\mathcal{U}$ of the set of C^r-roof functions on M such that for all $r \in \mathcal{U}$ the suspension flow $\phi_t : \Sigma^r \to \Sigma^r$ is exponentially mixing.*

Remark 1.2. Of course, since we can choose the embedding freely, we can require that the stable and unstable foliations are atypically regular.

The proof of theorem 1.1 is quite elementary, modulo results of Dolgopyat. We believe that some interesting issues arise even though we make strong assumptions on base dynamics. For example, we obtain exponential mixing by a *local* perturbation of the roof function which may be locally constant outside a relatively small set. To illustrate the ideas in their simplest context, we present full details only when Σ is the full 2-shift (the Smale horseshoe). The extension to the general case is routine and adds little new (but many details). We conclude with a discussion of how this type of result can yield information about mixing properties of smooth suspensions of algebraic Anosov diffeomorphisms and indicates the possibility that resonances may occur making it technically difficult to verify stable exponential mixing. We also present arguments to support our view that there may exist stably exponentially mixing suspensions of subshifts of finite type (conceivably allowing variation of the base as well as the roof function). However, we feel that the best approach to proving general results of this type is surely to use some of the more recently developed technology such as the direct transfer operator approach described above.

In more detail, we start with preliminaries on some elementary estimates together with the necessary material we need about metrics on subshifts of finite type and roof functions. Next we describe our model example and prove a result that allows us to assume roof functions depend on future coordinates — with no loss of regularity in the roof function. We conclude with generalizations and possible extensions.

2. Preliminaries

We let $|\ |_\infty$ denote the norm on $\mathbb{R}^n$ defined by $|x|_\infty = \max_{1\le i\le n} |x_i|$. We often write $|x|$ rather than $|x|_\infty$ when the meaning is clear. We denote the associated metric on $\mathbb{R}^n$ by ρ. Given a compact subset K of $\mathbb{R}^n$ and C^p map $f : K\to\mathbb{R}$, we define for $r \le p$ the C^r-semi-norm $|f|_r^K = \sup_{x\in K} \max_{|\alpha|=r} |\partial^\alpha f(x)|$ and the C^p-norm $\|f\|_p^K = \sum_{r=0}^p |f|_r^K$. We usually drop the superscript K if no confusion will result. If $\alpha \in (0, 1]$, we define the $C^{p+\alpha}$-norm $\|\ \|_{p+\alpha}$ in the usual way.

Lemma 2.1. *Let $r : \mathbb{R}^n \times \mathbb{R}^m\to\mathbb{R}$ be C^2. Let $R = R_e \times R_c \subset \mathbb{R}^n \times \mathbb{R}^m$ be a compact rectangle. Fix $(x, y) \in R$. Then for all $x' \in R_e$, $y', \bar{y} \in R_c$ we have*

$$|r(x, y) - r(x, \bar{y}) + r(x', \bar{y}) - r(x', y)| \le nm\|r\|_2|x - x'||y - \bar{y}|.$$

If $r \in C^{1+\alpha}$, $\alpha \in (0, 1]$, we have the estimate

$$|r(x, y) - r(x, \bar{y}) + r(x', \bar{y}) - r(x', y)| \le nm\|r\|_{1+\alpha}|x - x'||y - \bar{y}|^\alpha.$$

Proof. Assume $n = m = 1$ — the general proof is the same though the notation is more complicated. We have

$$r(x', y) - r(x', \bar{y}) = r(x, y) - r(x, \bar{y}) + I,$$

where

$$I = \int_0^1 \left[\frac{\partial r}{\partial x}(x + s(x' - x), y) - \frac{\partial r}{\partial x}(x + s(x' - x), \bar{y}) \right] (x - x')\, ds.$$

As a corollary of the mean value theorem we have $|I| \leq C|x' - x||y - \bar{y}|$, where $C = \sup_{(u,v)\in R} |\frac{\partial^2 r}{\partial x \partial y}(u, v)| \leq \|r\|_2^R$. The argument for the case when $r \in C^{1+\alpha}$ is similar with the use of the mean value theorem replaced by an α-Hölder estimate on $\frac{\partial r}{\partial x}$. □

Lemma 2.2. *(Same assumptions as above.) Fix $(x, y) \in R$. Then for $x' \in R_e$, $y', \bar{y}, \bar{y}' \in R_c$ we have for C^2 r*

$$|r(x, y) - r(x, \bar{y}) - r(x', y') - r(x', \bar{y}')| \leq nm\|r\|_2 |x - x'||y - \bar{y}| + m\|r\|_1^R (|y - y'| + |\bar{y} - \bar{y}'|).$$

If r is $C^{1+\alpha}$, we have

$$|r(x, y) - r(x, \bar{y}) - r(x', y') - r(x', \bar{y}')| \leq nm\|r\|_{1+\alpha} |x - x'||y - \bar{y}|^\alpha + m\|r\|_1^R (|y - y'| + |\bar{y} - \bar{y}'|).$$

Proof. Write

$$r(x, y) - r(x, \bar{y}) - r(x', y') - r(x', \bar{y}') = A + B,$$

where

$$\begin{aligned} A &= r(x, y) - r(x, \bar{y}) - r(x', y) + r(x', \bar{y}) \\ B &= r(x', y) - r(x', \bar{y}) - r(x', y') + r(x', \bar{y}'). \end{aligned}$$

By lemma 2.1, if r is C^2 we have $|A| \leq nm\|r\|_2^R |x - x'||y - \bar{y}|$. The estimate on B is immediate from the mean value theorem. □

2.1. **Metrics on a subshift of finite type.** For simplicity, and with a view to our intended applications, we work only with the full shift on 2-symbols. However, everything we say generalizes straightforwardly to subshifts of finite type on n-symbols.

Let $\Sigma = \{0, 1\}^{\mathbb{Z}}$ denote the full two-sided shift on 2-symbols and denote the shift map on Σ by σ. Given $\theta \in (0, 1)$, we define the usual metric d_θ on Σ by

$$d_\theta(x, y) = \theta^N,\ x = (x_i),\ y = (y_i) \in \Sigma,$$

where N is the smallest value of $|i|$ such that $x_i \neq y_i$.

We let $|f|_\theta$ denote the d_θ Lipschitz constant of a d_θ-Lipschitz map $f : \Sigma \to \mathbb{R}$ and set $\|f\|_\theta = |f|_0 + |f|_\theta$, where $|\ |_0$ denotes the uniform norm on $C^0(\Lambda)$. We let $C_\theta(\Sigma)$ denote the Banach space of continuous d_θ-Lipschitz functions on Σ with norm $\|f\|_\theta$.

We similarly define d_θ on the one sided shift Σ^+ and the corresponding norm $\|f\|_\theta$ and Banach space $C_\theta(\Sigma^+)$.

For our purposes, it is worth remarking a slight variant of the metric d_θ. Specifically, we define the metric d^1_θ on Σ by

$$d^1_\theta(x, y) = (N + 1)\theta^N,$$

where N is defined by $d_\theta(x, y) = \theta^N$. Let $C^1_\theta(\Sigma)$ and $C^1_\theta(\Sigma^+)$ denote the corresponding function spaces with norm $\|f\|^1_\theta = |f|_0 + |f|^1_\theta$.

Remarks 2.3. (1) If $\theta' > \theta$, then there exists $C > 1$ such that $d_{\theta'}(x, y) \geq Cd^1_\theta(x, y)$ for all $x, y \in \Sigma$.
(2) If $f \in C^1_\theta(\Sigma) \cap C_\theta(\Sigma)$, then f is the $\|\ \|^1_\theta$-limit of a sequence of locally constant functions.
(3) All the well known and standard theory for transfer operators generalizes easily to the metric d^1_θ. For example, the basic transfer operator inequality $|L^n_f w|_\theta \leq C|w|_0 + \theta^n|w|_\theta$ ([24, Proposition 2.1]) becomes $|L^n_f w|^1_\theta \leq C|w|_0 + (n+1)\theta^n|f|^1_\theta$ which more than suffices to prove standard results such as the existence of equilibrium states for d^1_θ-Lipschitz potentials.

2.2. **Roof functions and the suspension flow.** Let M be a compact differential manifold. For $r \geq 1$, let $R^r(M)$ denote the set of C^r strictly positive functions $r : M \to \mathbb{R}$. Elements of $R^r(M)$ are called *roof* functions. Let $f : M \to M$ be a C^r diffeomorphism of M and $r \in R^r(M)$. Let $M^{r,f}$ be the quotient manifold defined by $M \times \mathbb{R}/\sim$ where $(x, t) \sim (f(x), t+r(x))$, $(t, x) \in M \times \mathbb{R}$. The suspension flow $\Phi^{f,r}_t$ on $M^{r,f}$ is the C^r-flow induced on $M^{r,f}$ by the flow $(x, s) \mapsto (x, s + t)$ on $M \times \mathbb{R}$. See figure 1.

3. Model example

We realize the Smale Horseshoe as a hyperbolic invariant set Λ of a smooth diffeomorphism f of $\mathbb{R}^2$. Referring to figure 2, we assume that f is affine linear restricted to the vertical rectangles J_0 and J_1 and that the linear part of $f|J_0$ is the map $(x, y) \mapsto (\lambda x, \mu y)$, and the linear part of $f|J_1$ is the map $(x, y) \mapsto (\lambda x, -\mu y)$, where $0 < \mu < 1 < \lambda$. (It does not affect the issue if we take both restrictions to be $(x, y) \mapsto (\lambda x, \mu y)$ — indeed, this is the form in which the shift will appear when we look at suspensions over the 'cat map'. We define the horizontal rectangles

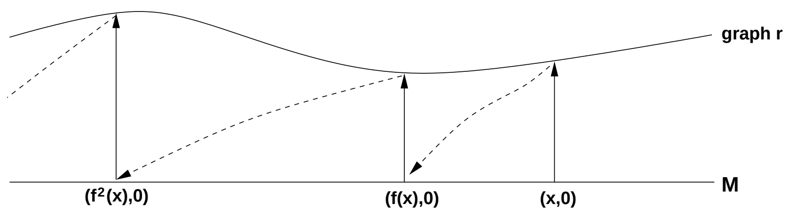

FIGURE 1. The suspension flow

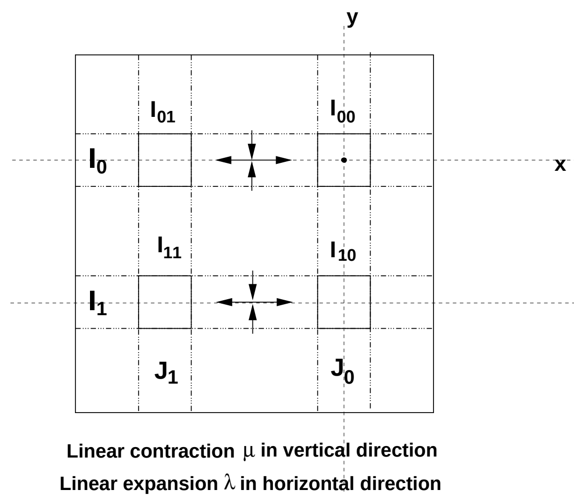

FIGURE 2. Smale Horseshoe

$I_i = f(J_i)$, $i = 1, 2$. If we define $I_{ij} = I_i \cap J_j$, $i, j = 1, 2$, then $\Lambda \subset \cup_{i,j} I_{ij}$ (see figure 2). If $X \in \Lambda$, we let $W^s(X)$ and $W^u(X)$ respectively denote the stable and unstable manifolds of X. If $X \in I$, where $I = I_0$ or I_1, we let $W^s_{\text{loc}}(X)$ denote the connected component of $W^s(X) \cap I$ containing X. That is, $W^s_{\text{loc}}(X)$ will be the vertical line segment through X. We similarly define $W^u_{\text{loc}}(X)$ to be the horizontal line segment through X. We let $\mathbf{W}^u(X) = W^u(X) \cap \Lambda$ denote the unstable set of X and similarly define $\mathbf{W}^s(X)$, $X \in \Lambda$.

The rectangles I_0, I_1 give the usual coding of Λ as the full shift on 2-symbols. In particular, if $X \in I_{ij} \cap \Lambda$ has coding (x_n), then $x_0 = i$, $x_1 = j$.

Henceforth we assume $\lambda\mu \leq 1$ (else replace f by f^{-1}), define $\theta = \lambda^{-1}$ and take the metric d_θ on Σ. After linear rescaling and replacing $|x - x'|$ by $\max\{1, |x - x'|\}$, the natural itinerary map $\iota : \Lambda \rightarrow \Sigma$ restricts to an isometry between the unstable sets $\mathbf{W}^u(X)$ and $\mathbf{W}^u(\iota(X))$ for all $X \in \Lambda$. On the stable sets $\iota : \mathbf{W}^s(X) \rightarrow \mathbf{W}^s(\iota(X))$ will be α-Hölder, where $\alpha = -\log(\lambda)/\log(\mu)$. Set $\iota^{-1} = \chi : \Sigma \rightarrow \Lambda$. We have that $\chi : \mathbf{W}^s(\iota(X)) \rightarrow \mathbf{W}^s(X)$ is Lipschitz and so $\chi : \Sigma \rightarrow \Lambda$ is Lipschitz: $\rho(\chi(X), \chi(Y)) \leq d_\theta(X, Y)$ for all $X, Y \in \Sigma$.

Remark 3.1. If $\lambda\mu = 1$, then $\iota : \Lambda \rightarrow \Sigma$ is an isometry.

Let $C^0(\Lambda)$ denote the space of continuous real valued functions on Λ and $\| \|_0$ denote the uniform norm on $C^0(\Lambda)$. Let $p \geq 1$. A function $f : \Lambda \rightarrow \mathbb{R}$ is C^p if f extends to a C^p function defined on some open neighbourhood of Λ. Let $C^p(\Lambda)$ denote the space of all C^p functions on Λ. If $f \in C^p(\Lambda)$, we define $\|f\|_p$ to be $\|F\|_p^\Lambda$, where F is a C^p extension of f to some open neighbourhood of Λ. For $p \geq 0$, $(C^p(\Lambda), \| \|_p)$ is a Banach space.

We have a natural mapping $\chi^\star : C^0(\Lambda) \rightarrow C^0(\Sigma)$ defined by $\chi^\star(f) = f \circ \chi$. As an immediate consequence of the mean value theorem and our choice of norm on $\mathbb{R}^2$ we have

Lemma 3.2. *For* $p \geq 1$, $\chi^\star : C^p(\Lambda) \rightarrow C_\theta(\Sigma)$ *and*

$$|\chi^\star(f)|_\theta \leq 2\|f\|_1, \ \|\chi^\star(f)\|_\theta \leq \|f\|_0 + 2\|f\|_1, \ f \in C^1(\Lambda).$$

Referring to figure 3, take horizontal lines $\ell \subset I_0$ and $m \subset I_1$ so that ℓ, m are components of an unstable manifold of a periodic point of f — in this case, we can take the unstable manifold of the unique fixed point for f in I_{00}. We define a smooth map $\Pi : I_0 \cup I_I \rightarrow \ell \cup m$ by projecting I_0 onto ℓ and I_1 onto m along stable directions, Given $X = (x, y) \in \Lambda$, we set $\pi(x, y) = (x, \bar{y})$, $\pi(X) = \bar{X}$ (see figure 3).

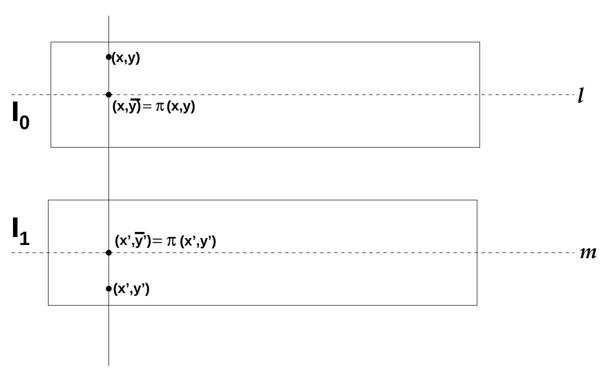

Figure 3. Projection along stable manifolds

Proposition 3.3. *Given $r \in C^2(\Lambda)$, define $h : \Lambda \to \mathbb{R}$ by*

$$h(X) = \sum_{n=0}^{\infty} r(f^n(X)) - r(f^n(\bar{X})),\ X \in \Lambda.$$

(a) *The series for $h(X)$ converges uniformly on Λ and we have $\|h\|_0 \leq \|r\|_1 (1-\mu)^{-1}$.*
(b) *If $\lambda\mu < 1$, then $\chi^\star(h) \in C_\theta(\Sigma)$.*
(c) *If $\lambda\mu = 1$, then $\chi^\star(h) \in C_\theta^1(\Sigma)$.*
(d) *For $\lambda\mu \leq 1$, $\chi^\star(h) \circ \sigma - \chi^\star(h) \in C_\theta(\Sigma)$.*

If we let $\Theta : C^2(\Lambda) \to C_\theta(\Sigma)$ denote the linear map defined by $\Theta(r) = \chi^\star(h) \circ \sigma - \chi^\star(h)$, then Θ is continuous with respect to the C^2-topology on $C^2(\Lambda)$ and $\| \ \|_\theta$-topology on $C_\theta(\Sigma)$.

Proof. (a) If $X \in \Lambda$, then $f^n(X), f^n(\bar{X})$ lie in the same rectangle $I_{i(n)}$ for all $n \geq 0$. The result is immediate from the Mean Value theorem and linearity of f on I_0, I_1.
(b) Suppose $X, X' \in \Lambda$ satisfy $d_\theta(\iota X, \iota X') = \theta^N$. Note that this condition implies that $|X - X'|_\infty \leq \theta^N$ with equality if X, X' lie on the same unstable manifold and $N > 0$. In particular, $f^n(X), f^n(X')$ lie in the same rectangle $I_{i(n)}$ for $0 \leq n < N$. Since $f^n(X), f^n(\bar{X})$ lie in the same rectangle for all $n \geq 0$, we have

$$\begin{aligned} |r(f^n(X)) - r(f^n(\bar{X}))| &\leq \|r\|_1 \mu^n |y - \bar{y}|, \\ &\leq \|r\|_1 \theta^n |y - \bar{y}|. \end{aligned}$$

The same estimate holds for $|r(f^n(X')) - r(f^n(\bar{X}'))|$ with $|y - \bar{y}|$ replaced by $|y' - \bar{y}'|$. Consequently, we may choose $C > 0$ (equal to the height of a rectangle) so that for all $X, X' \in \Lambda$ with $|X - X'|_\infty \leq \theta^N$, we have
(3.1)
$$|r(f^n(X)) - r(f^n(\bar{X}))|, |r(f^n(X)) - r(f^n(\bar{X}))| \leq C\|r\|_1 \theta^n,\ n \geq N.$$

Hence
(3.2)
$$|\sum_{n=N}^{\infty} r(f^n(X)) - r(f^n(\bar{X})) - (r(f^n(X')) - r(f^n(\bar{X}')))| \leq C\|r\|_1 \theta^N,$$

where $C = C(\theta) > 0$. Next we estimate $T_n = |r(f^n(X)) - r(f^n(\bar{X})) - r(f^n(X')) + r(f^n(\bar{X}'))|$, for $0 \leq n < N$. For this we use lemma 2.2 to obtain

$$\begin{aligned} T_n &\leq \|r\|_2 \lambda^n \mu^n |x - x'||y - \bar{y}| + \mu^n \|r\|_1 (|y - y'| + |\bar{y} - \bar{y}'|), \\ &\leq (\lambda\mu)^n |y - \bar{y}| \|r\|_2 |x - x'| + \mu^n \|r\|_1 |y - y'|, \end{aligned}$$

where we note that the y-components of $f^n(\bar{X})$ and $f^n(\bar{X}')$ are equal for $0 \leq n < N$. It follows that

$$|\sum_{n=0}^{N-1} r(f^n(X)) - r(f^n(\bar{X})) - (r(f^n(X')) - r(f^n(\bar{X}')))| \leq C(\|r\|_2, \lambda\mu, |y - \bar{y}|)\theta^N.$$

Combining this estimate with (3.2), we have shown that $|h(X)-h(X')| \leq C\theta^N$ if $d_\theta(\iota X, \iota X') = \theta^N$. Since this holds for all $N \geq 1$, we have shown that $\chi^\star(h) \in C_\theta(\Sigma)$.
(c) The proof is similar to that of (b) except that the estimate for T_n is now

$$T_n \leq \|r\|_2 |x - x'| + \mu^n \|r\|_1 |y - y'|,$$

and this leads to the term $N\theta^N$ in the estimate for $|h(X) - h(X')|$.
(d) The proof is similar to that of (b). We can estimate relative to $|\ |_\theta$ as most of the low order terms in the sums for $h(f(X)) - h(X)$, $h(f(X')) - h(X')$ cancel.
The final statement follows from the proofs of statements (b,d). □

Corollary 3.4. *Assume that $\lambda\mu \leq 1$. There is a continuous linear map $C^2(\Lambda) \rightarrow C_\theta(\Sigma^+)$, $r \mapsto \tilde{r}$, such that $\chi^\star(r)$ is cohomologous to $\tilde{r}$.*

Proof. We define

$$\tilde{r}(x) = \chi^\star(r)(x) + \Theta(r)(x), \ x \in \Sigma.$$

It is trivial to check that $\tilde{r}$ depends only on future coordinates and by (d) of proposition 3.3, $\tilde{r} \in C_\theta(\Sigma)$ even if $\lambda\mu = 1$. (cf the proof of proposition 1.2 [24]).

Remarks 3.5. (1) We emphasize that while $\tilde{r} \in C_\theta(\Sigma^+)$ and $\tilde{r} = \chi^\star(r) + \chi^\star(h) \circ \sigma - \chi^\star(h))$, $\chi^\star(h)$ does not generally lie in $C_\theta(\Sigma)$ if $\lambda\mu = 1$.
(2) If r is a roof function, $\tilde{r}$ need not be a roof function but it is *eventually positive* in the sense defined by Dolgopyat [12, §1]. That is all that is needed (and assumed) in Dolgopyat's analysis. In particular, the suspension space $\Lambda^{\tilde{r}}$ is compact Hausdorff and the suspension flow is defined.

Examples 3.6. (1) Suppose that $r(x, y) = 1 + ax$ on I_{00} and $r \equiv 1$ on I_{01}, I_1. Since r is constant on $W^s_{\mathrm{loc}}(X)$ for all $X \in \Lambda$, $\tilde{r} = \chi^\star(r)$.
(2) Suppose that $r(x, y) = 1 + axy$ on I_{00}, $a \neq 0$, and $r \equiv 1$ on I_{01}, I_1. It is easy to check directly that if $\lambda\mu = 1$ then (a) $\chi^\star(h) \in C^1_\theta(\Sigma) \setminus C_\theta(\Sigma)$, (b) $\tilde{r} \in C_\theta(\Sigma^+)$.

3.1. **Temporal distance function.** Suppose that the points $\omega_1, \cdots, \omega_4$ lie in the same rectangle, I_0 or I_1, and that $\omega_1, \cdots, \omega_4$ define the vertices of a rectangle given by the stable and unstable foliation of Λ — see figure 4.

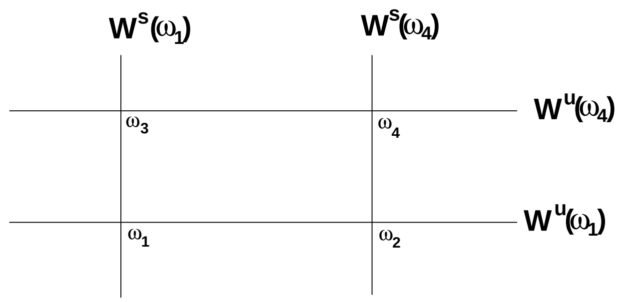

FIGURE 4. Temporal distance function

Given a roof function $r \in C^1(\Lambda)$, define

$$\phi(\omega_1, \cdots, \omega_4) = \sum_{n=-\infty}^{\infty} r(f^n(\omega_1)) - r(f^n(\omega_2)) - r(f^n(\omega_3)) + r(f^n(\omega_4)).$$

Lemma 3.7.

$$|\phi(\omega_1, \cdots, \omega_4)\| \leq C\|r\|_1|\omega_1 - \omega_4|.$$

where $C = C(\lambda, \mu)$.

Proof. A straightforward application of the mean value theorem together with $|\omega_1 - \omega_4| \geq |\omega_1 - \omega_2|, |\omega_3 - \omega_4|, |\omega_1 - \omega_3|, |\omega_2 - \omega_4|$. □

Proposition 3.8. . *Let* $r \in R^2(\Lambda)$. *There exists* $C = C(\theta) > 0$ *such that if* $\lambda\mu \leq 1$ *then*

$$|\phi(\omega_1, \cdots, \omega_4)\| \leq C\|r\|_2 d_\theta(\iota(\omega_1), \iota(\omega_2)).$$

Proof. By corollary 3.4, we may replace r by $\tilde{r} \in C_\theta(\Sigma^+)$ without changing the value of ϕ. The result is immediate by proposition 3.3 since

$$\phi(\omega_1, \cdots, \omega_4) = \sum_{n=-\infty}^{-1} \tilde{r}(\sigma^n(\tilde{\omega}_1)) - \tilde{r}(\sigma^n(\tilde{\omega}_2)) - \tilde{r}(\sigma^n(\tilde{\omega}_3)) + \tilde{r}(\sigma^n(\tilde{\omega}_4)),$$

where we have set $\iota(\omega_i) = \tilde{\omega}_i$, $1 \leq i \leq 4$ (cf [12, Proposition 2.1]). □

Next we give Dolgopyat's definition of strong non-integrability adapted to our context.

Definition 3.9 (cf Dolgopyat [12, §2])**.** The roof function $r \in R^1(\Lambda)$ is *strongly non-integrable* if there exist $\alpha, \beta \in \Lambda$, $\delta > 0$ and an open neighbourhood U of α such that

(a) α, β lie in the same rectangle (I_0 or I_1) and $\beta \in W^s_{\text{loc}}(\alpha)$.
(b) For every $\omega_1 \in U \cap \Lambda$ with $\omega_1 \in W^u_{\text{loc}}(\alpha)$, and $N \geq 1$ there exist $\omega_2, \omega_3, \omega_4$ with $\omega_2 \in W^u_{\text{loc}}(\alpha)$, $\omega_3, \omega_4 \in W^u_{\text{loc}}(\beta)$, $\omega_3 \in W^s_{\text{loc}}(\omega_1)$, $\omega_4 \in W^s_{\text{loc}}(\omega_2)$ such that $d_\theta(\iota(\omega_1), \iota(\omega_2)) = \theta^N$ and
$$|\phi(\omega_1, \cdots, \omega_4)| \geq \delta d_\theta(\iota(\omega_1), \iota(\omega_2)).$$

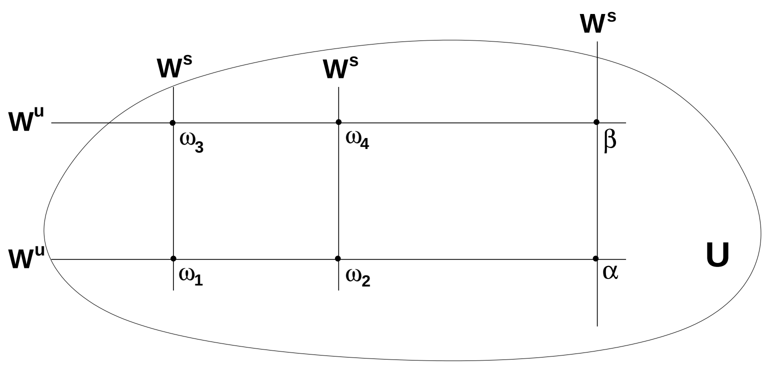

FIGURE 5. Strong non-integrability

Although definition 3.9 seems quite restrictive, it implies estimates on the temporal distance function for general points $\omega_1 \in U$ (cf Dolgopyat [12, §2]).

Lemma 3.10. *Suppose that $r \in R^1(\Lambda)$ is strongly non-integrable with U, α, β, δ as in definition 3.9. Then given $\omega_1 \in U$, $N \in \mathbb{N}$, there exist $\omega_2 \in W^u_{loc}(\omega_1)$, and $\omega_3, \omega_4 \in U$ with either $\omega_3, \omega_4 \in W^u_{loc}(\alpha)$ or $\omega_3, \omega_4 \in W^u_{loc}(\beta)$, and $\omega_3 \in W^s_{loc}(\omega_1)$, $\omega_4 \in W^s_{loc}(\omega_2)$ such that $d_\theta(\iota(\omega_1), \iota(\omega_2)) = \theta^N$ and*
$$|\phi(\omega_1, \cdots, \omega_4)| \geq \frac{\delta}{2} d_\theta(\iota(\omega_1), \iota(\omega_2)).$$

Proof. Let $\tilde{\omega} = W^s_{\text{loc}}(\omega_1) \cap W^u_{\text{loc}}(\alpha)$. By the definition of strong non-integrability, we can find $\tau_3 = W^s_{\text{loc}}(\omega_1) \cap W^u_{\text{loc}}(\beta)$, $\tau_2 \in W^u_{\text{loc}}(\alpha)$ and $\tau_4 = W^u_{\text{loc}}(\beta) \cap W^u_{\text{loc}}(\tau_2)$ such that $d_\theta(\iota(\tilde{\omega}), \iota(\tau_2)) = \theta^N$
$$|\phi(\tilde{\omega}, \tau_2, \tau_3, \tau_4)| \geq \delta d_\theta(\iota(\tilde{\omega}_1), \iota(\tau_2)).$$

Set $\omega_2 = W^s_{\text{loc}}(\tau_2) \cap W^u_{\text{loc}}(\omega_1)$. Observe that
$$\phi(\tilde{\omega}, \tau_2, \tau_3, \tau_4) = \phi(\omega_1, \omega_2, \tau_3, \tau_4) - \phi(\omega_1, \omega_2, \tilde{\omega}, \tau_2)$$

Since $|\phi(\tilde{\omega}, \tau_2, \tau_3, \tau_4)| \geq \delta d_\theta(\iota(\tilde{\omega}_1), \iota(\tau_2))$, we must have one or other of $|\phi(\omega_1, \omega_2, \tau_3, \tau_4)|$, $|\phi(\omega_1, \omega_2, \tilde{\omega}, \tau_2)|$ at least $\frac{\delta}{2} d_\theta(\iota(\omega_1), \iota(\omega_2))$. Label ω_3, ω_4 accordingly. □

Using the same proof we also have

Lemma 3.11. *Let $r \in R^1(\Lambda)$. Suppose there exist $\alpha, \beta \in \Lambda$, $\delta > 0$ and an open neighbourhood U of α for which the following conditions hold.*

(a) *α, β lie in the same rectangle (I_0 or I_1) and $\beta \in W^s_{loc}(\alpha)$.*
(b) *We may choose a sequence $(\tilde{\omega}_N) \subset W^u_{loc}(\alpha) \cap U$ for which $d_\theta(\iota(\alpha), \iota(\omega_N)) = \theta^N$ and such that if we set $\tau_N = W^s_{loc}(\omega_N) \cap W^u_{loc}(\beta)$ then*

$$|\phi(\tilde{\omega}_N, \alpha, \tau_N, \beta)| \geq \delta \theta^N, \; N \in \mathbb{N}.$$

Then r is strongly non-integrable (same U, α, β and δ replaced by $\delta/2$).

Example 3.12. In this example we show that there exists at least one C^2-roof function on Λ which is strongly non-integrable. We use the coding of Λ given by $\chi : \Sigma \to \Lambda$. We define $\alpha = u_1 \cdot w_2$, $\beta = w_1 \cdot w_2$, where

$$w_2 = \cdot 00\overline{1}, \quad w_1 = \overline{1}, \quad u_1 = \overline{001^p},$$

and $p \geq 1$. Thus $\alpha = \overline{001^p} \cdot 00\overline{1}$, $\beta = \overline{1} \cdot 00\overline{1} \in I_{00}$. We remark that $\alpha, \beta \in W^s(\overline{1})$. Take $U = I_{00} \cap \Lambda$ and define $r \in C^\infty(\Lambda)$ by $r(X) = 1$ if $X \notin I_{00}$, and $r(X) = r(x, y) = 1 + x$, if $(x, y) \in I_{00}$, where regard $W^s(\alpha)$ as the y-axis and $W^u(\alpha)$ as the x-axis. Let $\omega_1, \omega_2 \in \mathbf{W}^u(\alpha)$ and define ω_3, ω_4 as in figure 5. We have $\sigma^{-j}(\omega_1), \sigma^{-j}(\omega_2) \in I_{00}$ iff $j = (p+2)k$, where $k \geq 0$ is a positive integer. Obviously, $\sigma^{-j}(\omega_3), \sigma^{-j}(\omega_4) \notin I_{00}$ for all $j \geq 1$. A simple computation shows that $\phi(\omega_1, \cdots, \omega_4) = \frac{1}{1+\theta^{p+2}} d_\theta(\omega_1, \omega_2)$ proving strong non-integrability.

Proposition 3.13. *Let $s \geq 2$. The set of roof functions $r \in R^s(\Lambda)$ which are strongly non-integrable is a non-empty C^2-open and C^s dense subset of $R^s(\Lambda)$.*

Proof. Openness is immediate from proposition 3.8. The previous example shows that the set of strongly non-integrable roof functions is non-empty. Finally, density is an easy argument using lemma 3.11 and the method of the previous example. □

Remark 3.14. Dolgopyat proves a similar result to proposition 3.13 in [12, §2]. However, Dologopyat's uses the symbolic structure and metric and the perturbations are different from those we use. A precursor of the type of result proved above is given in the transition from the theorem of Parry & Pollicott on stable mixing for symbolic shifts [25, §3] to a setting that can be viewed as applicable to smooth maps [13, §3.2].

Theorem 3.15. *Let $s \geq 2$. The set of C^s roof functions on Λ for which the corresponding suspension flow is exponentially mixing is C^2-open and C^s dense.*

Proof. The main result in Dolgopyat is the proof that strong non-integrability implies exponential mixing [12, lemma 2.3]. Openness and density in our context follow from proposition 3.13. □

Remark 3.16. A consequence of the results in [12] is that we have exponential error bounds in the Prime Orbit Theorem. That is, if $s \geq 2$, $r \in R^s(\Lambda)$ is strongly non-integrable and the associated suspension flow Φ has topological entropy h, then there exists $\beta > 0$ such that

$$\pi(\Phi, T) = \mathrm{Li}(e^{hT})(1 + O(e^{-\beta T})),\ T \to \infty,$$

where $\pi(\Phi, T)$ is the number of closed periodic orbits of Φ of prime period less than or equal to T and Li is the offset logarithmic integral defined by $\mathrm{Li}(t) = \int_2^t ds/\log s\ ds$. We also expect that constants depend uniformly on r, C^2 topology (see [12, §1] for more details on this point).

4. A large class of exponentially mixing flows

In this section we briefly sketch the proof of the general theorem stated in the introduction.

Theorem 4.1. *Let Σ be a mixing subshift of finite type. Then we may embed Σ in a compact differential manifold M so that*

(1) *Σ is a basic set of a C^r Axiom A diffeomorphism $f : M \to M$, $2 \leq r \leq \infty$.*
(2) *There is a C^2-open and C^r-dense subset $\mathcal{U}$ of the set of C^r-roof functions on M such that for all $r \in \mathcal{U}$ the suspension flow $\phi_t : \Sigma^r \to \Sigma^r$ is exponentially mixing.*

Proof. An old result of Williams [34] allows us to realize every subshift of finite type as a basic set of a smooth Axiom A diffeomorphism f of S^3. With our choice of Σ, the Ω limit set of f will consist of two hyperbolic fixed points, one attracting, one repelling, and two subshifts of finite type the second of which is defined by the dual of the zero-one matrix of Σ. The unstable manifolds of points in Σ will be one-dimensional. More generally, using the same construction as Williams, we may realize Σ as the basic set of a smooth diffeomorphism of any manifold of dimension at least three so that the unstable stable manifolds of points in Σ are one-dimensional. In either case, we may require that we can choose smooth coordinates on neighbourhoods of rectangles defining the subshift so that in these coordinates f is linear — exactly the situation we had in our model example – and that contraction and expansion rates are constant over the subshift. We may further assume all the contraction rates are equal. Our local linearizability assumption implies that the stable and unstable foliations of f are smooth. The arguments we gave for our model example extend without difficulty to this more general situation. □

Remarks 4.2. (1) If $f : M \to M$ is a smooth diffeomorphism of a compact manifold then there is a C^0-small perturbation of f to an Axiom A diffeomorphism $\hat{f} : M \to M$ with Ω limit set consisting of hyperbolic periodic points and subshifts of finite type [31]. Just as in the proof of theorem 4.1 we may assume local linearizability. Theorem 4.1 extends to the suspension flows of the mixing subshifts in the Ω limit set of $\hat{f}$.
(2) As in the case of our model example, the key assumption is that the foliations are C^2 or better (later we indicate how this assumption may be weakened).
(3) In our sketch proof of theorem 4.1, we make the simplifying assumption that contraction rates are not only constant but are equal. However, it should be possible, and interesting (see section 6) to remove the restriction that the rates are all equal using the approach described in [13].

5. An example

Let $f : \mathbb{T} \to \mathbb{T}$ denote the cat map (or, for that matter, any algebraic orientation preserving Anosov diffeomorphism of an n-torus, $n \geq 2$). For $s \geq 2$, let $R^s(\mathbb{T})$ denote the set of C^s roof functions on $\mathbb{T}$. If q is a homoclinic point of the unique fixed point 0 of f, then it follows by Smale's theorem (see [20, Exercise 6.5.1, pg. 278]) that there exists an embedded hyperbolic locally maximal subshift of finite type $\Lambda \subset \mathbb{T}$ containing q. The stable and unstable foliations for Λ are 'linear' and satisfy the hypotheses needed for theorem 4.1. Hence for $\infty > s \geq 2$, there exists an open and dense subset U_q of $R^s(\mathbb{T})$ such that if $r \in R^s(\mathbb{T})$ then the corresponding suspension flow $\Phi_t^r : \mathbb{T} \to \mathbb{T}$ restricts to a exponentially mixing flow on Λ. There are a countable set of distinct orbits of homoclinic points of 0. Let $q_1, q_2, \cdots$ be a set of representative homoclinic points so that every homoclinic point of 0 lies on the f-orbit of some q_i. For $n \geq 1$, let Λ_n be an embedded subshift of finite type containing q_n (note that Λ_n contains the f-orbit of q_n by the f-invariance of Λ_n). Set $U = \cap_{n \geq 1} U_{q_n} \subset R^s(\mathbb{T})$. Observe that U is a residual subset of $R^s(\mathbb{T})$ and that for all $r \in U$, the suspension flow $\Phi_t^r : \mathbb{T} \to \mathbb{T}$ restricts to a exponentially mixing flow on Λ_n, $n \geq 1$.

It is not hard to extend this argument to provide a residual subset of $R^s(\mathbb{T})$ that gives exponential mixing for a subshift of finite type associated to every homoclinic point of every periodic point of f. In any case, even the restriction to homoclinic points of 0 yields very strong mixing conditions for Φ^r on $\cup_{n\geq 1}\Lambda_n$ — a dense subset of $\mathbb{T}$. While these observations certainly do *not* prove there exist $r \in R^s(\mathbb{T})$ for which Φ^r is exponentially mixing, it seems possible that good uniform estimates on exponential mixing for $\Phi^r|\Lambda_n$ may yield strong results on the distribution of poles of the the zeta-function of the flow — at least for the case of the measure of maximal entropy. This suggests that it is reasonable to conjecture that, for generic $r \in R^s(\mathbb{T})$, Φ^r will be exponentially mixing.

Before leaving this example, observe that the issue of *stability* of exponential mixing for the suspension flow is tricky; even at the level of suspended subshift dynamics and ignoring issues about regularity of the temporal distance function. The reason is that there is a resonance in the eigenvalues of f: $\lambda\mu = 1$. Consequently, the methods we used in section 3 to prove exponential mixing, all immediately break down as we cannot assume the contraction and expansion rates are bunched and separated from each other. A reflection of this problem is found in the arguments used in [14] to prove stability of mixing and rapid mixing for hyperbolic flows. In the resonant case, mixing is 'faster' (see [13] for asymptotics) but the arguments for stability break down. Hence, non-resonance conditions are assumed in the analysis in [14]. To emphasize this point, the methods used in [14] do not directly show that any suspension flow of the cat map or the Smale horseshoe with $\lambda\mu = 1$ is *stably* mixing or rapid mixing! It is conceivable that one of the reasons it has been difficult to prove stability of exponential mixing is that the geometric examples of contact Anosov flows exhibit resonances (in Floquet multipliers of periodic orbits) which break down under perturbation and thus break uniformity of the constants associated to the mixing rates. Of course, these resonances are not directly seen in existing proofs of exponential mixing and so these observations should be taken as speculative.

6. Extensions and generalizations

We conclude with some remarks about what happens when we perturb the base dynamics. Let $s \geq 2$ and suppose that Λ is a locally maximal hyperbolic subshift of finite type for a C^s-diffeomorphism of the differential manifold M. We denote the corresponding splitting of $T_\Lambda M$ by $E^s \oplus E^u$. For simplicity of exposition, we assume that $\text{codim}(E^s) = 1$. Fix a Riemannian metric on M. We review the definition of the *bunching parameter* (for more details, see [17, chapter 3, pages 262–3]). Given $p \in \Lambda$, there exists $C > 0$, and $\mu_f(p) \leq \mu_s(p) < 1 < \lambda_s(p) \leq \lambda_f(p)$ such that for $n \in \mathbb{N}$ we have

$$\begin{aligned} C^{-1}\mu_f^n(p)\|v\| &\leq \|Df^n(v)\| \leq C\mu_s^n(p), \\ C^{-1}\lambda_f^{-n}(p)\|u\| &\leq \|Df^{-n}(u)\| \leq C\lambda_s^{-n}(p)\|u\|, \end{aligned}$$

where $v \in E^s_p$, $u \in E^u_p$. We refer to $\mu_s(p), \mu_f(p)$ as slow and fast contraction rates at p. Similarly for the expansion rates at p. We define the bunching parameter

$$B^s(f) = \sup\{\inf_{p\in\Lambda}(\log\lambda_s(p) - \log\mu_s(p))/\log\lambda_f(p)\},$$

where the supremum is over all possible choices of the fast and slow contraction and expansion rates. In our case we are assuming that $\text{codim}(E^s) = 1$ and so we can work with just one expansion rate, say $\lambda(p)$, $p \in \Lambda$. Hence

$$B^s(f) = \sup\{\inf_{p\in\Lambda}\{1 - (\log\mu_s(p)/\lambda(p))\}.$$

Provided $B^s(f) \notin \mathbb{N}$, the stable foliation is of class $C^{B^s(f)}$ [17, chapter 3]. In particular, if $\mu_s(p)\lambda(p) < 1$ for all $p \in \Lambda$, then the stable foliation is of class C^{2+}.

Remark 6.1. If the dimension of M is two and $\Lambda \subset M$ is hyperbolic and locally maximal, then the stable and unstable foliations of Λ are of class C^{1+} [17, chapter 3]. However, both foliations will generally not be of class C^2. Moreover, when we look at suspension flows, regularity of the *strong* stable and unstable foliations will typically not be C^1 (see [15] for more details and references on this point). If $\Lambda \neq M$, for example if Λ is a subshift of finite type, then the regularity of the foliations may be higher than that given by standard results [22]. However, as far as I am aware, it is not completely clear at this time what the optimal regularity results are for subshifts of finite type. For Anosov diffeomorphisms, regularity may be stably very poor [18].

Henceforth assume there exists $\alpha < 1$ such that $\mu_s(p)\lambda(p) \leq \alpha$ for all $p \in \Lambda$. Now choose a cover $\mathcal{R} = \{R_1, \cdots, R_q\}$ of Λ by disjoint closed rectangles defining a Markov partition for Λ. This fixes a symbolic coding for Λ and a representation of Λ as a subshift S of finite type on q symbols. Fix a conjugacy $h : \Lambda \to S$. By our assumption that $\mu_s(p)\lambda(p) \leq \alpha$ for all $p \in \Lambda$, we may assume that each R_j is chosen so that it has a C^2-trivial stable foliation extending the local stable sets of Λ. Note that there is no such such result for the unstable lamination which conceivably could be quite irregular and not C^1 or even Lipschitz.

If $x, y \in R_j \cap \Lambda$, set $z = [x, y] = W^s_{\text{loc}}(x) \cap W^u_{\text{loc}}(y) \in R_j \cap \Lambda$ and $z' = [y, x]$. Define a metric ρ on $R_j \cap \Lambda$ by

$$\rho(x, y) = \max\{d_s(x, z), d_u(z, y), d_u(x, z'), d_s(z', y)\},$$

where d_s measures the Riemannian distance between x and y along $W^s_{\text{loc}}(x)$ (induced Riemannian metric on $W^s(x)$). Similarly for d_u. If $x, y \in \Lambda$ belong to different rectangles, set $\rho(x, y) = 1$.

Remark 6.2. Using the conjugacy h, we obtain an induced metric $\tilde{\rho}$ on S. The metric $\tilde{\rho}$ satisfies an estimate $cd_\theta \leq \tilde{\rho} \leq Cd_{\theta'}$, where $c, C > 0$ and $0 < \theta < \theta' < 1$ can be estimated in terms of contraction and expansion rates.

Let $C_\rho(\Lambda)$ denote the space of ρ-Lipschitz maps $f : \Lambda \to \mathbb{R}$ and $C_\rho(\Lambda^+)$ the subspace consisting of maps depending only on future coordinates (defined via the symbolic coding). Just as in section 2, we define a norm $\| \|_\rho$ which gives $C_\rho(\Lambda)$ the structure of a Banach space. Obviously, $C^1(\Lambda) \subset C_\rho(\Lambda)$. More interestingly, we may prove the following extension of proposition 3.3.

Proposition 6.3. *Given $r \in C^2(\Lambda)$, define $h : \Lambda \to \mathbb{R}$ by*

$$h(X) = \sum_{n=0}^{\infty} r(f^n(X)) - r(f^n(\bar{X})),\ X \in \Lambda.$$

(a) *The series for $h(X)$ converges uniformly on Λ and we have $\|h\|_0 \leq C\|r\|_1$.*
(b) *$h \in C_\rho(\Lambda)$.*

If we let $\Theta : C^2(\Lambda) \to C_\rho(\Lambda)$ denote the linear map defined by $\Theta(r) = h \circ f - h$, then Θ is continuous with respect to the C^2-topology on $C^2(\Lambda)$ and $\| \|_\rho$-topology on $C_\theta(\Lambda)$.

Proof. The proof is similar to that of proposition 3.3 and makes use of the C^2 triviality of the stable foliation together with our assumption that $\lambda(p)\mu_s(p) \leq \alpha < 1$ for all $p \in \Lambda$, and our definition of ρ. □

Corollary 6.4. *There is a continuous linear map $C^2(\Lambda) \to C_\rho(\Lambda^+)$, $r \mapsto \tilde{r}$, such that $\tilde{r}$ is cohomologous to $\tilde{r}$.*

Proof. Define $\tilde{r} = r + \Theta(r)$. □

Along exactly the same lines as in section 3, we can define the temporal distance function and strong non-integrability. We have analogs of proposition 3.8 and lemmas 3.10, 3.11. Because we are assuming that the unstable lamination is 1-dimensional, it is also relatively straightforward to extend proposition 3.13 (openness and density of strongly non-integrable roof functions in $R^2(\Lambda) \subset C^2(\Lambda)$). Notice, however, that for this extension we need to work on an unstable leaf $W^u(\alpha)$ with maximal expansion rate. This can be achieved by a C^2-small perturbation to obtain a periodic point p with maximal expansion rate and corresponding $\alpha \in W^s(p)$. The question then arises whether or not Dolgopyat's method [12] applies when we use the induced metric $\tilde{\rho}$ on the symbolic space. Our feeling is that these methods should still apply. However, we have not carried out a careful analysis because we feel a better approach is likely to be provided by some of the newer methods now available (see our comments in the introduction and note that we are particularly interested in removing the regularity assumptions on the stable foliation).

In conclusion, we hope that the analysis and examples we have presented give evidence for the existence at least of locally dense sets of exponentially mixing hyperbolic suspension flows and new reasons as to why it may be difficult to obtain results on exponential mixing by perturbing away from the class of contact Anosov flows.

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