COMBINATORIAL DYNAMICS

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Abstract. Recently Stewart, Golubitsky and coworkers have formulated a general theory of networks of coupled cells. Their approach depends on groupoids, graphs, balanced equivalence relations and 'quotient networks'. We present a combinatorial approach to coupled cell systems. While largely equivalent to that of Stewart et al., our approach is motivated by ideas coming from analog computers and avoids abstract algebraic formalism.

1. Introduction

Recently Stewart et al [11, 1, 9] have developed a theory of *coupled cell systems*. This theory is designed to allow the investigation and classification of patterns of synchrony in networks of (not necessarily identical) coupled differential equations and may be regarded as an attempt to quantify the extent to which the architecture of a network of coupled differential equations determines the dynamics of the system. The formalism of the theory involves *groupoids*, *graphs*, *balanced equivalence relations* and, in particular, *quotient networks*. The theory of coupled cell networks can be thought of as a generalization of the theory of symmetric coupled cell systems (see [2, 3, 8], [4, Chapter 7]). The global symmetries of a symmetrically coupled system are now replaced by 'local' symmetries, quantified by the groupoid structure.

In this paper we present a somewhat different, though largely equivalent, formulation of the theory of coupled cell networks. We hope that our presentation may be appealing to those with more of an engineering or applications background – indeed, the formalism we adopt was motivated by previous experience working with analog computers in the context of control and simulation. An advantage of our approach is that it avoids the use of quotient networks and groupoid formalism and is highly combinatorial in character. Also, we emphasize the synthesis and construction of networks rather than the analysis of specific *given* networks. In this sense, our viewpoint is slanted more towards

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potential applications in engineering rather than to biology or neuroscience. So as to distinguish our approach from that of Stewart et al, we adopt the term 'network cell system' rather than 'coupled cell system'. While familiarity with the works of Stewart et al on coupled cell systems is not necessary for a reading of this work, we do make comparative references to their works and have included a number of 'dictionary' entries to show how our terminology corresponds to the groupoid terminology used in the papers by Stewart et al.

In general terms, our aim is to develop the conceptual basis of a network cell system rather than pursue the detailed investigation of dynamics or bifurcation in specific network cell systems. Our focus will be on minimal and slaved networks and how synchrony can be viewed as invariant under appropriate repatchings of a network cell system. Nevertheless, we do present several examples, some quite interesting from the point of view of dynamics. We refer the reader to [11, 1, 9, 6, 5] for more explicit examples involving dynamics and bifurcation.

1.1. **Analog computer model.** As motivation for our approach it may be helpful to give a brief and simplified description of the analog computer model (for a more detailed introduction, see [10]). An analog computer consists of a number of different types of "black box" that we shall henceforth refer to as *cells*. For the simplest analog computers, all the cells will be identical linear devices which either allow a sign-reversing summation or integration. Either of these operations may be realized using a high gain DC amplifier. Cells mays be included to model specific nonlinear phenomena (including multiplication and division!). These 'nonlinear' cells are typically electro-mechanical devices. Every cell in the analog computer allows a number of inputs and an 'unlimited' number of outputs. Of course, for linear devices we can always allow an unlimited number of inputs but for nonlinear devices the number of allowed inputs is usually fixed. Practically speaking, the various cells are coupled together using "patch cords". This is achieved by having a large matrix of sockets – directly connected to inputs and outputs of cells. Patchcords are then used to connect the output of one cell to the input of another – possibly the same – cell. In this way, complex networks of cells can be patched together so as to model linear and nonlinear systems.

As a simple example, the differential equation $x' = -x$ can be modeled by taking the output of an integrator cell, denoted by **I** in figure 1, into the input of the same cell. The equation $x' = x$ can be modeled using two cells – one integrator, one sign reverser, denoted by **S** in figure 1. Formally, the integrator and sign reversor cells act on an input

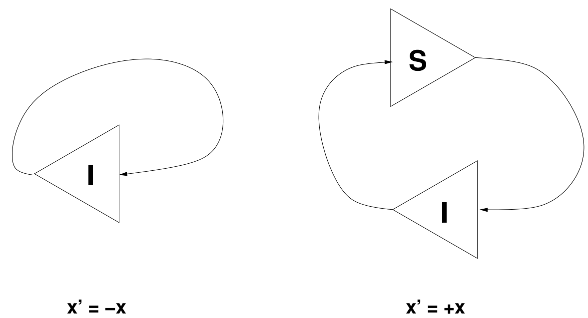

FIGURE 1. Modeling ODEs using an analog computer

$x(t)$ according to $\mathbf{I}x(t) = -\int_0^t x(s)\,ds$ and $\mathbf{S}x(t) = -x(t)$ respectively. Initial conditions can be set using a second input (not shown) to the integrator cell.

We should mention that real analog computers allow additional features such as varying the gain in a loop, and multiplication and division.

Our concept of a network cell system will differ from that of an analog computer in following way. We allow a fixed number of different *types* of cell and each type of cell will have a prescribed number of inputs of each type. When we patch a group of cells together to form a network cell system, we insist that all the inputs are filled – this is related to the fact that we are dealing with *nonlinear* devices. Just as for the analog computer, there are no restrictions on the number of outputs.

1.2. **Brief description of contents.** In sections 2, 3, we give an informal description of a number of basic examples that underlie the formalism we develop in later sections. In section 4, we give formal definitions for a network cell system and synchrony classes. Much of what we do in this and subsequent sections corresponds more closely to the multiarrow formalism of Golubitsky et al. [9] rather than the original work of Stewart et al. [11]. In section 5, we define patch equivalence, minimal subnetworks, slaved subnetworks, transitivity and component subnetworks. We develop the basic combinatorics of network cell systems and show that every network cell system is patch equivalent to a network cell system in 'normal' form (Proposition 5.16). The latter network contains a minimal subnetwork which is dynamically equivalent to the quotient network of Stewart et al [11, 9]. In our setup, it is a triviality that every solution on a minimal network determines a (synchronous) solution on the original network. In section 6, we make a more thorough study of the possible synchrony classes for a fixed network cell system. This investigation uses the concept of a balanced family of cells – related to the 'balanced equivalence relation' of Stewart et al. – and the corresponding versions of minimality, transitivity and patch equivalence. We conclude with some remarks about possible applications of our approach and mathematical differences with the work of Stewart et al [11, 9].

2. Network cell systems: Examples

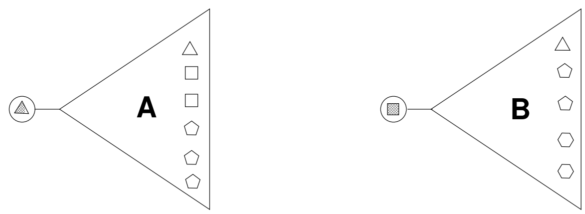

Figure 2. Cells

2.1. **Building blocks: model examples.** In figure 2, we show two symbolic representations of basic cells from which we might build a network. Each cell allows inputs – which we have denoted by triangles, squares, etc in the figure – and an output – denoted by a circle containing a single regular figure which signifies the type of output. Cell **A** has a triangle output and one input of triangle type, two of square type and three of pentagon type. On the other hand, cell **B** has a square output and allows one triangle input and two inputs each of pentagonal and hexagonal type. (We often omit the regular figures signifying input types if all the cells in a network are identical or the context makes the cell type clear.)

Typically the evolution of the state of the cell – quantified by its output – will be governed by an ordinary differential equation (discrete time evolution is also possible though we will not consider that possibility here). If we think of cell **A** as being of 'triangular type' (type is determined by the output), and denote the state of the cell by $\mathbf{u}$, then the evolution of cell **A** will be governed by a differential equation of the form

$$\dot{\mathbf{u}} = F(\mathbf{u}; \mathbf{u}_a, \mathbf{v}_b, \mathbf{v}_c, \mathbf{w}_d, \mathbf{w}_e, \mathbf{w}_f), \tag{2.1}$$

where $\mathbf{u}_a$ will be an input from a cell of triangular type, $\mathbf{v}_b, \mathbf{v}_c$ will be inputs from cell(s) of square type and $\mathbf{w}_d, \mathbf{w}_e, \mathbf{w}_f$ will be inputs from cell(s) of pentagonal type. In general, the evolution of a cell may depend on its state and so we allow F to depend on the 'internal' variable $\mathbf{u}$.

In order to interconnect cells, we use 'patchcords'. Each patchcord is terminated by two plugs of the *same* type which can be triangular, square etc. We can plug any number of patchcords with a regular n-gon plug into the output of a cell with n-gon output. The other end of the patchcord must go into an a free n-gon input socket of a cell. The order in which we plug patchcords with the *same* termination is irrelevant. In terms of the equation governing the evolution of cell **A**, this means that

$$\begin{aligned} F(\mathbf{u};\mathbf{u}_a,\mathbf{v}_b,\mathbf{v}_c,\mathbf{w}_d,\mathbf{w}_e,\mathbf{w}_f) &= F(\mathbf{u};\mathbf{u}_a,\mathbf{v}_c,\mathbf{v}_b,\mathbf{w}_d,\mathbf{w}_e,\mathbf{w}_f),\\ &= F(\mathbf{u};\mathbf{u}_a,\mathbf{v}_b,\mathbf{v}_c,\mathbf{w}_e,\mathbf{w}_f,\mathbf{w}_d), \text{ etc.} \end{aligned}$$

This symmetry in the inputs of the same type imposes constraints on the evolution of dynamics of interconnected cells. The group of input symmetries for a given cell corresponds to the *vertex group* of Stewart et al. [11, §3].

2.2. **Network cell system.** A network cell system will consist of a finite number of cells, interconnected by patchcords with no input left unfilled[1]. In figure 3, we show a simple network comprising cells $\mathbf{A}_1, \mathbf{A}_2$

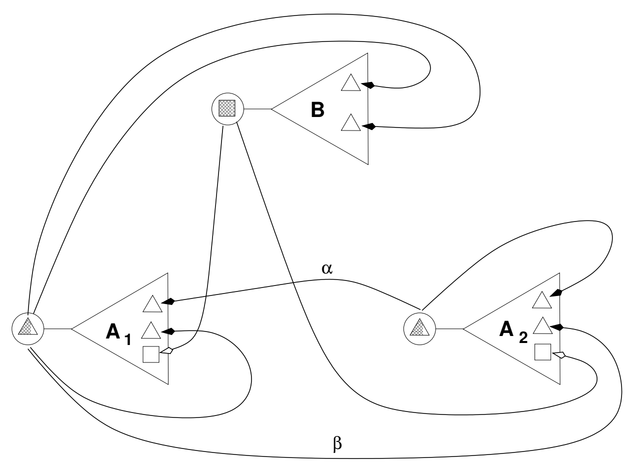

FIGURE 3. A simple network

of triangular type and a cell **B** of square type. The corresponding equations that govern the evolution of this system are given by

$$\begin{aligned} \dot{\mathbf{x}}_{A1} &= F(\mathbf{x}_{A1};\mathbf{x}_{A1},\mathbf{x}_{A2},\mathbf{x}_B),\\ \dot{\mathbf{x}}_{A2} &= F(\mathbf{x}_{A2};\mathbf{x}_{A1},\mathbf{x}_{A2},\mathbf{x}_B),\\ \dot{\mathbf{x}}_B &= G(\mathbf{x}_B;\mathbf{x}_{A1},\mathbf{x}_{A1}), \end{aligned}$$

[1]We can relax this requirement by allowing for *null* and/or *constant* cells – basically we can either 'earth' the unused inputs or feed them a constant input.

where F and G define vector fields on the phase spaces of cells of type $\mathbf{A}$ and $\mathbf{B}$ respectively.

Although the network shown in figure 3 is asymmetric, it does admit a robust family of synchronous solutions where the outputs of $\mathbf{A}_1, \mathbf{A}_2$ are equal. Observe that we can we repatch the network *without* destroying this synchrony property. See figure 4 and note that we have changed the patchcords denoted by α, β in figure 3.

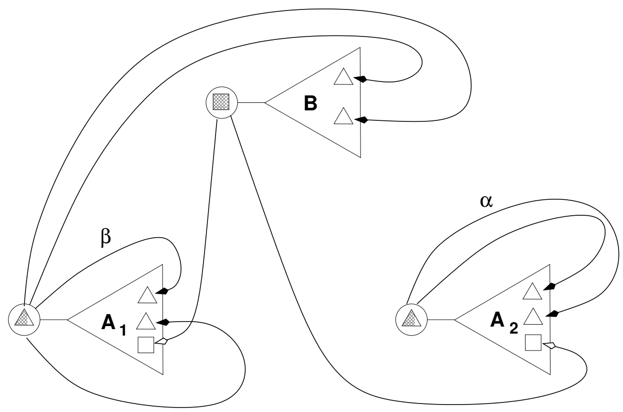

FIGURE 4. Repatching the network

It is obvious looking at figure 4 that the cell $\mathbf{A}_2$ has no influence on the rest of the network. We refer to a subnetwork of this type as a *slaved network* (we give a precise definition later). If we remove the slaved network, the remaining collection of cells continues to be an admissible network – all the inputs are filled, see figure 5. In this instance it is easy to see that there are no repatchings that allow us to further reduce the number of cells in the network. The network shown in figure 5 corresponds[2] to the quotient network of Golubitsky et al. [9]. We call a network that cannot be reduced in size by repatching and excision of slaved subnetworks, a *minimal network*. The equations governing the evolution of the minimal network shown in figure 5 are

$$\begin{aligned} \dot{\mathbf{x}}_A &= F(\mathbf{x}_A; \mathbf{x}_A, \mathbf{x}_A, \mathbf{x}_B), \\ \dot{\mathbf{x}}_B &= G(\mathbf{x}_B; \mathbf{x}_A, \mathbf{x}_A), \end{aligned}$$

Any solution $(\mathbf{x}_A(t), \mathbf{x}_B(t))$ for the minimal network determines a synchronous solution $(\mathbf{x}_{A1}(t), \mathbf{x}_{A2}(t), \mathbf{x}_B(t)) = (\mathbf{x}_A(t), \mathbf{x}_A(t), \mathbf{x}_B(t))$ of the original network and conversely.

[2]Strictly speaking it is not a 'quotient'. An analogy would be that the vector space complement of a subspace $\mathbb{H} \subset \mathbb{E}$ corresponds to the quotient space $\mathbb{E}/\mathbb{H}$

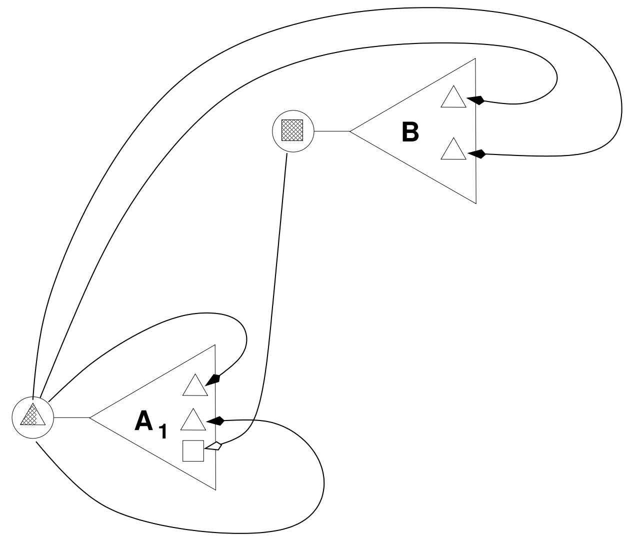

FIGURE 5. Minimal network

In figure 6, we show two repatchings of an eight cell network comprised of two different types of cells. Later we shall show that the first network admits multiple synchronous states. Observe that the first repatching results in a 'circular' network, while the second repatching results in four disjoint *identical* two-cell networks, each consisting of a B cell and a slaved A cell.

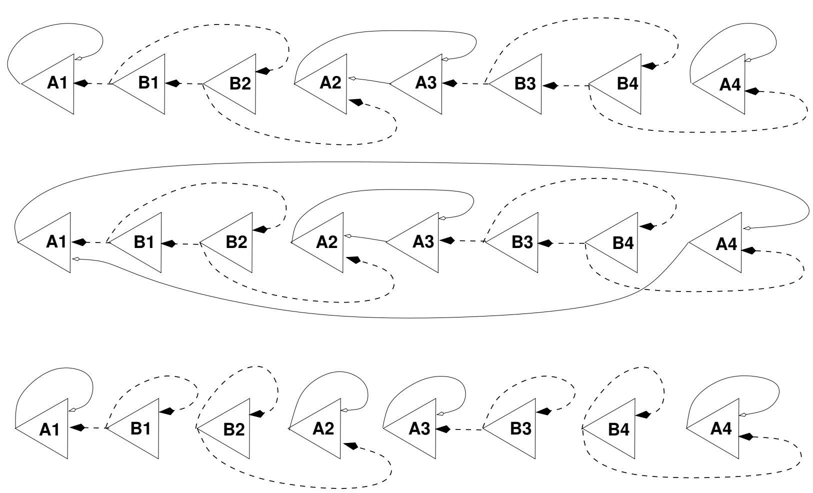

FIGURE 6. Two repatchings of a linear network

However we repatch the original network, it will always be the case that any solution $(\mathbf{x}_A(t), \mathbf{x}_B(t))$ of one of the two-cell networks determines a synchronous solution $\mathbf{x}_{A1}, \dots, \mathbf{x}_{B4}$ of the full eight cell network (possibly repatched) by setting $\mathbf{x}_{Ai} = \mathbf{x}_A$, $\mathbf{x}_{Bi} = \mathbf{x}_B$, $i = 1, \dots, 4$.

3. Multiple synchronous states

As was suggested in the discussion of the first network shown in figure 6, it is possible for a network cell system to exhibit more than the 'maximal' synchronous state where all cells of the same type are synchronous. In order to discuss more general synchronous solutions, it is helpful to adopt the following

Notational conventions.

Suppose that a network cell system is comprised of cells of types A, B, C, $\dots$... We label individual cells of the same type according to $A1, A2, \dots$. By a *synchrony class* we shall mean a group of cells that can admit a set of nontrivial synchronous solutions determined by the architecture of the network[3]. Typically, we write a synchrony class $\mathcal{S}$ in the form $\{A_1, \dots, A_s \,\|\, B_1, \dots, B_t \,\|\, \dots\}$. This notation is to be interpreted as meaning that the cells $A_1, \dots, A_s$ are synchronous (and therefore are identical cells), $B_1, \dots, B_t$ are synchronous (and therefore are identical cells) and so on. It is understood that the A and B-cells may or not be identical. If they are identical, it is *not* required that the A- and B-cells are synchronous. Indeed, we use the notation $\{A_1, \dots, A_t, B_1, \dots, B_s\}$ to signify that the cells $A_1, \dots, B_t$ are synchronous (and so A and B cells must be identical).

It is easy to verify that the first network shown in Figure 6 has nine synchrony classes:
$\{B1, B2\}$, $\{B3, B4\}$, $\{B1, B2 \,\|\, B3, B4\}$, $\{B1, B2, B3, B4\}$,
$\{A1, A4 \,\|\, B1, B2, B3, B4\}$, $\{A1, A2, A3, A4 \,\|\, B1, B2, B3, B4\}$,
$\{A1, A2, A3 \,\|\, B1, B2, B3, B4\}$, $\{A2, A3, A4 \,\|\, B1, B2, B3, B4\}$,
and $\{A2, A3 \,\|\, B1, B2, B3, B4\}$.

We briefly examine one of the synchrony classes: $\{B1, B2 \,\|\, B3, B4\}$. In figure 7, we have grouped the B-cells into two blocks $G1$ and $G2$. Note that $G1$ receives no inputs from cells outside $G1$. Similarly for $G2$. It follows that in this case, we only need concern ourselves with the subnetwork cell systems determined by $G1$ and $G2$. We can repatch $G1$ to obtain the minimal networks shown in figure 8. The differential equations for these minimal networks will be of the form

$$\dot{\mathbf{x}} = F(\mathbf{x}; \mathbf{x}). \tag{3.2}$$

[3]The term synchrony class corresponds to the *polydiagonal subspace* of [11, §6]. Synchrony classes will be *robustly polysynchronous* [11, §6],[9, §4].

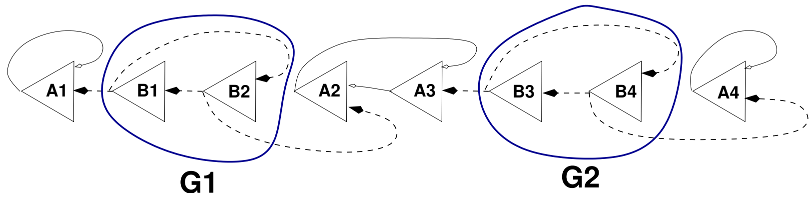

FIGURE 7. The synchrony class $\{B1, B2 \,\|\, B3, B4\}$

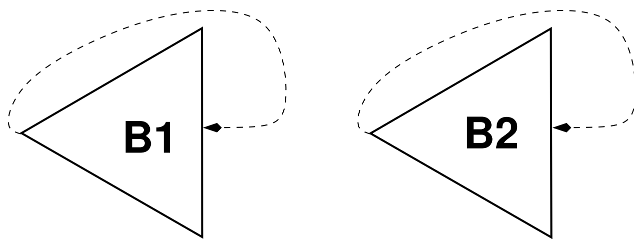

FIGURE 8. Minimal networks associated to $G1$

Any solution $\mathbf{x}(t)$ of (3.2) will determine a synchronous solution of the $G1$ subnetwork by setting $\mathbf{x}_{B1} = \mathbf{x}_{B2} = \mathbf{x}$. Similarly for the $G2$ subnetwork (using a different solution of (3.2)). Allowing the A-cells to evolve according to B-inputs we arrive at equations governing the synchronous solutions in the class $\{B1, B2 \,\|\, B3, B4\}$.

For this rather simple example, we imposed a repatching rule that restricted repatching to be within specified subsets of cells. Later, we will allow repatching where the input and output of patchcords are constrained to lie in (possibly) different subsets of a *balanced partition* of the network. These repatching rules will allow us to identify – in theory at least – all possible synchrony classes in a given network and determine the associated dynamics using minimal models. Our approach may be compared with the use by Stewart et al [11, 9] of 'balanced equivalence relations' and 'multicolour formalism'.

4. Network cell systems

In this section we give a formal description of a network cell system built from k different types of cell. In order to do this, we need to describe rules for the time evolution of each of the k individual cell types as well as the 'connection matrix' for the entire network. We shall assume that the evolution of each cell is governed by an ordinary differential equation. This differential equation will depend on parameters (corresponding to cell outputs). The connection matrix of the system specifies the connections between cells in the network. When we repatch the network, we change the connection matrix but not the individual cells.

We adopt the convention that if $a \leq b \in \mathbb{N}$, then $[a,b]_{\mathbb{N}} = [a,b] \cap \mathbb{N}$.

4.1. **Cell dynamics.** We assume k different cell types. We denote the phase space for a cell of type j by V_j, where V_j will be a finite dimensional vector space, $j \in [1,k]_{\mathbb{N}}$.

For $j \in [1,k]_{\mathbb{N}}$, let $\mathbf{p}^j = (p^i_j) \in \mathbb{N}^k$. The integers p^i_j, $i \in [1,k]_{\mathbb{N}}$, give the total number of inputs that a cell of type j receives from cells of type i. In particular, $\Sigma_i p^i_j = p_j$ is the total number of inputs for a cell of type j.

Suppose that for each $j \in [1,k]_{\mathbb{N}}$, we are given a parametrized family

$$F^j : V_j \times \oplus_{i=1}^k V_i^{p^i_j} \to V_j \tag{4.3}$$

of vector fields on V_j. We assume that F^j is symmetric in each of the variables $\mathbf{x}^i = (x^i_1, \ldots, x^i_{p^i_j}) \in V_i^{p^i_j}$. More precisely, each F^j will be invariant with respect to the natural action on $\oplus_{i=1}^k V_i^{p^i_j}$ of the product $S(j) = S_{p^1_j} \times \ldots \times S_{p^k_j}$ of symmetric groups.

Remark 4.1. The symmetry we require of the F^j corresponds to symmetry of the patchcord examples described in the previous section: if we insert a triangular plug into an input of a particular cell, it does not matter which triangular socket we use. These local 'socket symmetries' possessed by a cell have a major impact on the dynamics that can occur robustly in a network of cells and correspond to the vertex groups of Stewart et al. [11].

We regard the vector field F^j as defining a *cell* of *type* j. In the sequel we frequently regard cell and associated differential equation as synonymous. The evolution of the cell will be governed by the differential equation

$$\dot{x} = F^j(x; \mathbf{x}^1, \ldots, \mathbf{x}^k), \tag{4.4}$$

where $\mathbf{x}^i = (x^i_1, \ldots, x^i_{p^i_j}) \in V_i^{p^i_j}$, $i \in [1,k]_{\mathbb{N}}$. The variable $x \in V_j$ defines the *state* of the cell and may be regarded as an *internal* variable for the cell defined by (4.4). We call the variables $x^i_\ell \in V_i$ *input* variables.

4.2. **Connection data.** Suppose that we are given a fixed set $\mathcal{C} = \{F^j \mid 1 \leq j \leq k\}$ of k distinct cells.

We describe the concept of a network cell system modeled on $\mathcal{C}$. The number of cells of each type is specified by a vector $\mathbf{m} \in \mathbb{N}^k$, $\mathbf{m} \neq \mathbf{0}$. The pair $(\mathcal{C}, \mathbf{m})$ represents a set of $|\mathbf{m}| = m^1 + \ldots + m^k$ cells containing

m^j-cells of type j, $j \in [1,k]_{\mathbb{N}}$. We order cells of each type in $(\mathcal{C},\mathbf{m})$ and let c_i^j denote the ith cell of type j, $i \in [1,m^j]_{\mathbb{N}}$, $j \in [1,k]_{\mathbb{N}}$. We refer to the pair $(\mathcal{C},\mathbf{m})$ as a 'set of $\mathcal{C}$-cells'.

Definition 4.2. A network cell system modeled on $\mathcal{C}$ is a triple $\mathcal{N} = (\mathcal{C},\mathbf{m},\mathbf{P})$ where

(1) $(\mathcal{C},\mathbf{m})$ is a set of $\mathcal{C}$-cells.
(2) $\mathbf{P} = \{p_{rs}^{ij} \in [0,p_j^i]_{\mathbb{N}} \mid i,j \in [1,k]_{\mathbb{N}}, s \in [0,m^j]_{\mathbb{N}}, r \in [0,m^i]_{\mathbb{N}}\}$.
(3) $\Sigma_{r=1}^{m^i} p_{rs}^{ij} = p_j^i$ is independent of s, all i,j.

The array $\mathbf{P}$ gives the *connection matrix* for $\mathcal{N}$.

Remarks 4.3. (1) Suppose that $\mathcal{N} = (\mathcal{C},\mathbf{m},\mathbf{P})$ is a network cell system. The system will comprise a total of $|\mathbf{m}|$ cells. There will be m^j cells in $\mathcal{N}$ of type j, $1 \le j \le k$. The array $\mathbf{P}$ describes the number and type of connections between different cells. More precisely, The integer p_{rs}^{ij} gives the number of inputs the sth cell of type j receives from the rth cell of type i. Thus $p_j^i = \Sigma_{r=1}^{m^i} p_{rs}^{ij}$ gives the total number of inputs of type i into any cell of type j, and $\Sigma_{i,s} p_{rs}^{ij} = \Sigma_i p_j^i = p_j$ is the total number of inputs for any cell of type j. Implicit in condition (3) of the definition is the requirement that no inputs are left unfilled. In figure 9, we show the set of inputs to the sth. cell of type j.
(2) Abusing notation, we often identify $\mathcal{N}$ with the cells $\{c_i^j \mid i \in [1,m^j]_{\mathbb{N}}, j \in [1,k]_{\mathbb{N}}\}$ that comprise $\mathcal{N}$. In particular, whenever we write $\mathcal{S} \subset \mathcal{N}$, this is to be interpreted as meaning that $\mathcal{S}$ is a subset of the cells comprising $\mathcal{N}$.

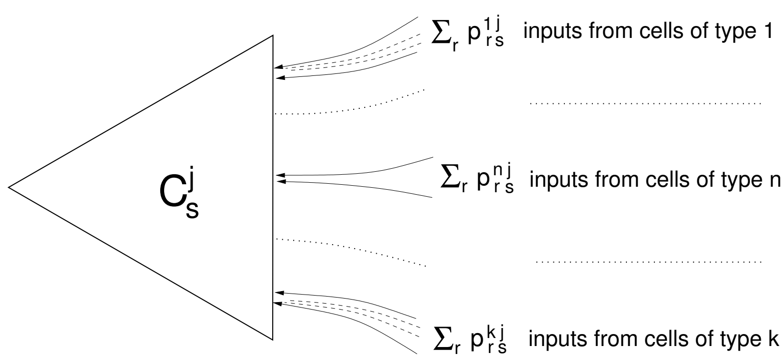

FIGURE 9. Inputs to the cell c_s^j.

Subnetwork of cells of given type. If $\mathcal{N} = (\mathcal{C}, \mathbf{m}, \mathbf{P})$ is a network cell system, and $j \in [1,k]_{\mathbb{N}}$, we define

$$\begin{aligned} \mathcal{N}^j &= (\{F^j\}, m^j, \mathbf{P}^j), \\ \mathbf{P}^j &= \{p^{jj}_{rs} \in [0, p^j_j]_{\mathbb{N}} \mid r, s \in [0, m^j]_{\mathbb{N}}\}. \end{aligned}$$

The set $\mathcal{N}^j$ represents the cells of type j together with all connections between cells of type j. In general, $\mathcal{N}^j$ will not be a network cell system. As noted above (remarks 4.3(2)), we will sometimes identify $\mathcal{N}^j$ with the set of cells $\{c^j_s \mid s \in [1, m^j]_{\mathbb{N}}\}$ that comprise $\mathcal{N}^j$. Similarly, if $\mathcal{S} \subset \mathcal{N}$, we let $\mathcal{S}^j$ denote the subset of $\mathcal{S}$ consisting of cells in $\mathcal{S}$ which are of type j. Note that, as far as is practical, we always use superscripts to specify cell type.

4.3. **Dynamics on a network cell system.** We now describe dynamics for the network cell system $\mathcal{N} = (\mathcal{C}, \mathbf{m}, \mathbf{P})$.

The phase space for $\mathcal{N}$ will be the vector space $\mathbf{V}$ defined by

$$\mathbf{V} = \oplus_{i=1}^k V_i^{m^i}.$$

We define a vector field $\mathbf{F}$ on $\mathbf{V}$ so that the evolution of the network cell system $\mathcal{N}$ is determined by the solutions of the differential equation $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ on $\mathbf{V}$. In order to do this, it suffices to specify the component of $\mathbf{F}$ that governs the evolution of each cell in the network. The differential equations governing the evolution of individual cells in the network are given by

$$\dot{x}^j_s = F^j(x^j_s; (\mathbf{x}^1)^{\mathbf{P}^{1j}_s}, \ldots, (\mathbf{x}^k)^{\mathbf{P}^{kj}_s}), \;\; 1 \le s \le m^j, \; 1 \le j \le k, \tag{4.5}$$

where

$$(\mathbf{x}^i)^{\mathbf{P}^{ij}_s} = ((x^i_1)^{p^{ij}_{1s}}, \ldots, (x^i_{m^i})^{p^{ij}_{m^i s}}), \; 1 \le i \le k.$$

With this definition of $\mathbf{F}$, we say that (4.5) are the equations for the network cell system $\mathcal{N}$.

Example 4.4. We recall the equations for the network cell system shown in figure 3. For this system we have $k = 2$. We regard the cells $\mathbf{A}_1$, $\mathbf{A}_2$ as being of type 1, and the cell $\mathbf{B}$ as being of type 2. In this case, we have $p^1_1 = 2$, $p^2_1 = 1$, $p^1_2 = 2$, $p^2_2 = 0$. We assume that the phase space for type 1 cells is V_1, and for type 2 cells is V_2. The family of differential equations for the network is given by

$$\begin{aligned} \dot{x}^1_1 &= F_1(x^1_1; x^1_1, x^1_2, x^2_1), \\ \dot{x}^1_2 &= F_1(x^1_2; x^1_2, x^1_1, x^2_1), \\ \dot{x}^2_1 &= F_2(x^2_1; x^1_1, x^1_1), \end{aligned}$$

where $F_1 : V_1 \times (V_1^2 \oplus V_2) \to V_1$, $F_2 : V_2 \times V_1^2 \to V_2$, and F_1, F_2 satisfy $F_1(v; x_1, x_2, z) = F_1(v; x_2, x_1, z)$, $F_2(w; x_1, x_2) = F_2(w; x_2, x_1)$, all $x_1, x_2, v \in V_1$, $w, z \in V_2$.

4.4. **Synchrony classes.** In this section we give a definition of synchrony class for network cell systems. Our definition is chosen so that synchrony classes depend on the *architecture* of the system – that is, on the connection matrix $\mathbf{P}$ – and do not depend on explicit knowledge of the vector fields F_j used to model the evolution of each cell.

Let $\mathbf{S} = \{\mathcal{S}_\ell \mid \ell \in [1, L]_\mathbb{N}\}$ be a family of mutually disjoint non-empty subsets of the network cell system $\mathcal{N} = (\mathcal{C}, \mathbf{m}, \mathbf{P})$. We define the subspace $\mathbf{V_S}$ of $\mathbf{V}$ by

$$\mathbf{V_S} = \{(\mathbf{x}^1, \ldots, \mathbf{x}^k) \mid x_a^j = x_b^j, \;\; a, b \in \mathcal{S}_\ell^j, \ell \in [1, L]_\mathbb{N}, j \in [1, k]_\mathbb{N}\}.$$

Definition 4.5. The family $\mathbf{S}$ defines a *synchrony class* for $\mathcal{N}$ if $\mathbf{V_S}$ is invariant by the flow of (4.5). We refer to $\mathbf{V_S}$ as a *synchrony subspace* for $\mathcal{N}$ and any solution with initial condition lying in $\mathbf{V_S}$ as $\mathbf{S}$-*synchronized*.

As it stands, this definition a little imprecise since the invariance of $\mathbf{V_S}$ may occur because of some "accidental" properties of the vector fields F_j.

Example 4.6. Consider the four cell system defined for $(x_1, \ldots, x_4) \in \mathbb{R}^4$ by

$$\begin{aligned}
\dot{x}_1 &= F_1(x_1; x_2, x_3),\\
\dot{x}_2 &= F_1(x_2; x_1, x_4),\\
\dot{x}_3 &= F_2(x_3; x_3),\\
\dot{x}_4 &= F_3(x_4; x_3, x_4).
\end{aligned}$$

This system does not have any synchrony classes. However, if we define $F_1(x; u, v) = f(x) + u + (x - u)^2 v$, then the subspace of $\mathbb{R}^4$ defined by $x_1 = x_2$ is an invariant subspace. Of course, for "generic" F, $x_1 = x_2$ will not be an invariant subspace.

There are several ways to proceed. First, we could simply require that $\mathbf{V_S}$ was an invariant subspace for generic choices of $F_1, \ldots, F_k$ (satisfying the symmetry conditions on inputs of like type) or that the invariance of $\mathbf{V_S}$ was stable under perturbations of $F_1, \ldots, F_k$. More formally, we could work in terms of variable substitutions – replacing the variables in each $\mathcal{S}_\ell^j$ by a single new variable and requiring that the original equations define a unique equation in the new variables.

Providing we work symbolically, none of this will cause us any problems. Hence, rather than go through the somewhat tedious and unenlightening algebra involved in formalizing these ideas, we defer the matter for the moment. Later, in section 6, we will introduce the idea of a balanced partition of $\mathcal{N}$ and show that every balanced partition determines a unique synchrony class and conversely.

Remarks 4.7. (1) In definition 4.5, if $\mathcal{S}_\ell^j = \emptyset$ or $\mathcal{S}_\ell^j$ consists of one cell, then no restrictions are placed on $\mathbf{x}^j$. If every $\mathcal{S}_\ell \in \mathbf{S}$ consists of only one cell, or if all the cells in $\mathbf{S}$ have different type, then $\mathbf{S}$ defines the *null* or *minimal* synchrony class for $\mathcal{N}$.
(2) A synchrony class $\mathbf{S}$ can always be extended to a partition $\mathbf{S}'$ of $\mathcal{N}$ by defining $\mathbf{S}' = \mathbf{S} \cup \{\{c\} \mid c \notin \cup_\ell \mathcal{S}_\ell\}$. We have $\mathbf{V}_{\mathbf{S}'} = \mathbf{V}_{\mathbf{S}}$ and $\mathbf{S}'$ is a synchrony class. Often, in examples, it is preferable to suppose that a synchrony class contains no singleton cells. In more theoretical discussions, it is usually easiest to allow for singleton cells and assume that the synchrony class defines a partition of $\mathcal{N}$. Indeed, this is the approach we adopt in section 6.
(3) The definition allows for there to be multiple sets $\mathcal{S}_\ell$ consisting of cells of the same type. However, for the examples and results in this section it will always be the case that $\mathbf{S}$ consists of a single subset $\mathcal{S}$. Thus, we may and shall identify $\mathbf{S}$ with $\mathcal{S}$ and write $\mathbf{S} = \dot{\bigcup}_{j=1}^{k} \mathcal{S}^j$ without ambiguity.

Examples 4.8. (1) If $\mathcal{N}$ is a network cell system $\mathbf{S} = \{\mathcal{N}^1 \parallel \ldots \parallel \mathcal{N}^k\}$ is always a synchrony class: the *maximal* synchrony class of $\mathcal{N}$. The existence of this synchrony class follows rather simply because of our emphasis on cell type. Of course, if there are no two cells of the same type then there are no nontrivial synchronous solutions. (The maximal synchrony class corresponds to the *coarsest balanced equivalence relation* $\bowtie^\star$ described in Golubitsky et al. [9, Appendix].)
(2) If we define $\mathcal{S} = \{A1, A2\}$, then $\mathcal{S}$ is a synchrony class for the networks shown in Figure 3, 4 (we could also have taken $\mathcal{S} = \{A1, A2 \parallel B\}$, the maximal synchrony class of the network).
(3) In the example discussed in section 3, every synchrony class contained at least one synchronous B-pair (notation of Figure 6). In Figure 10 we show an example of an asymmetric network containing two types of cell which can synchronize independently.

The network has three synchrony classes: $\{B1, B2\}$, $\{A1, A2, A3\}$, and $\{A1, A2, A3 \parallel B1, B2\}$. This type of network can admit robust heteroclinic cycles, see [5] and section 6.

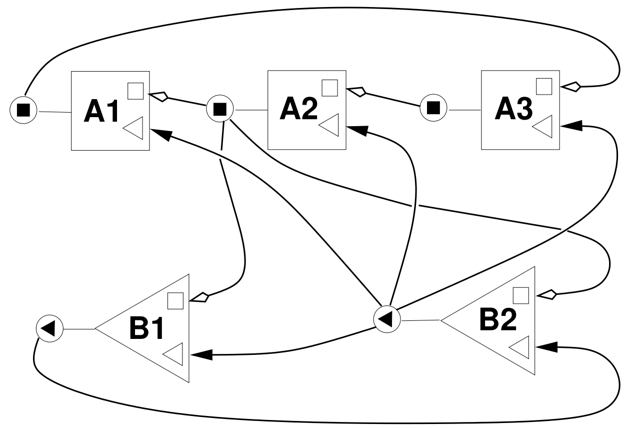

FIGURE 10. A network with "independent" synchronous states

Lemma 4.9. *Suppose that the network cell system $\mathcal{N} = (\mathcal{C}, \mathbf{m}, \mathbf{P})$ has synchrony class $\mathcal{S}$. Then for all cells $c_s^j \in \mathcal{S}$, $c_r^i \notin \mathcal{S}$, we have*

$$p_{r\bar{s}}^{ij} = p_{rs}^{ij}, \quad \text{all } c_{\bar{s}}^j \in \mathcal{S}, \quad i, j \in [1, k]_{\mathbb{N}}.$$

Moreover, if we have a subset $\mathcal{K}$ of cells for which this condition holds, then $\mathcal{K}$ is a synchrony class for $\mathcal{N}$.

Proof. If the cell $c_s^j \in \mathcal{S}$ receives k inputs from a cell $c_r^i \notin \mathcal{S}$, then all type j cells in $\mathcal{S}$ must receive k inputs from c_r^i – else $\mathcal{S}$ could not be a synchrony class for $\mathcal{N}$. The converse follows equally simply. □

Remarks 4.10. (1) It is required in Lemma 4.9 that the synchrony class consist of a single subset of $\mathcal{N}$. Note that if $\mathcal{S} = \mathcal{N}$, the lemma imposes no restrictions and we recover the maximal synchrony class of $\mathcal{N}$.
(2) The lemma may be regarded as giving a satisfactory formal definition of synchrony class for the case when the synchrony class is given by a single subset of $\mathcal{N}$. In section 6, we address the case when the synchrony class is given in terms of multiple disjoint subsets of $\mathcal{N}$.

5. COMBINATORICS

In this section we start to explore the invariants of a network cell system under repatching.

Definition 5.1. We say two network cell systems $\mathcal{N} = (\mathcal{C}, \mathbf{m}, \mathbf{P})$, $\mathcal{M} = (\mathcal{D}, \mathbf{n}, \mathbf{Q})$ are *patch equivalent* if by repatching the connections we can change $\mathcal{N}$ to $\mathcal{M}$.

If $\mathcal{N}$ and $\mathcal{M}$ are patch equivalent then it is obvious (count inputs – in 1:1 correspondence with connections) that $\mathcal{C} = \mathcal{D}$ and $\mathbf{m} = \mathbf{n}$. Further, we must have $p^i_j = q^i_j$, all $i, j \in [1, k]_{\mathbb{N}}$. Indeed, patch equivalence of two network cell systems is equivalent to requiring that the systems have the same number of cells of each type and the same underlying dynamics. In particular, corresponding cells have the same number of inputs of each type. The next lemma summarizes this discussion.

Lemma 5.2. *The network cell systems* $(\mathcal{C}, \mathbf{m}, \mathbf{P})$*,* $(\mathcal{D}, \mathbf{n}, \mathbf{Q})$ *are patch equivalent if only only if* $\mathcal{C} = \mathcal{D}$ *and* $\mathbf{m} = \mathbf{n}$*.*

Example 5.3. Following [9], we say that a network cell system $\mathcal{N} = (\mathcal{C}, \mathbf{m}, \mathbf{P})$ is *homogeneous* if $\mathcal{C}$ contains exactly one cell. In figure 11, we show a homogeneous network containing three cells. This is patch equivalent to a disconnected network consisting of three cells – see figure 11. We remark that similar results holds for any homogeneous network and that the single cell system is dynamically equivalent to the quotient network in the sense of [9, 11]. Any solution $\mathbf{x}(t)$ determines a synchronous solution $(\mathbf{x}(t), \mathbf{x}(t), \mathbf{x}(t))$ for $\mathcal{N}$.

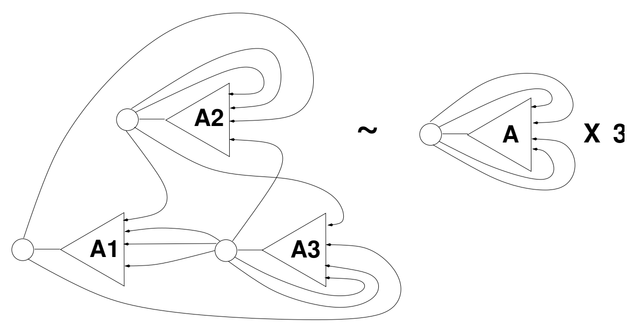

FIGURE 11. Patch equivalence for a homogeneous network

Definition 5.4. Suppose that $\mathcal{N}$ is a network cell system and $\mathcal{S} \subset \mathcal{N}$.
(1) We say that $\mathcal{S}$ is a *slaved subnetwork* $\mathcal{N} \setminus \mathcal{S} \to \mathcal{S}$ of $\mathcal{N}$ if

- (a) $p^{ij}_{rs} = 0$ for all cells $c^i_r \in \mathcal{S}$, $c^j_s \in \mathcal{N} \setminus \mathcal{S}$ (there are no connections from cells in $\mathcal{S}$ to cells in $\mathcal{N} \setminus \mathcal{S}$),
- (b) There exists at least one connection from $\mathcal{N} \setminus \mathcal{S}$ to $\mathcal{S}$.

(2) If $\mathcal{K}$ is a subnetwork cell system of $\mathcal{N}$, we say that a slaved subnetwork $\mathcal{S}$ is $\mathcal{K}$-slaved, written $\mathcal{K} \to \mathcal{S}$, if $\mathcal{S} \cup \mathcal{K}$ is a subnetwork cell system of $\mathcal{N}$ and $\mathcal{S}$ is a slaved subnetwork of $\mathcal{S} \cup \mathcal{K}$.

Remark 5.5. If $\mathcal{N} \setminus \mathcal{S} \to \mathcal{S}$ then there are no outputs from cells of $\mathcal{S}$ to cells in $\mathcal{N} \setminus \mathcal{S}$. A slaved subnetwork $\mathcal{N} \setminus \mathcal{S} \to \mathcal{S}$ is never a (sub) network cell system. On the other hand, if $\mathcal{N} \setminus \mathcal{S} \to \mathcal{S}$, then $\mathcal{N} \setminus \mathcal{S}$ has the structure of a network cell system (induced from that on $\mathcal{N}$).

Definition 5.6. We say that the network cell system $\mathcal{N}$ is *minimal* if every network cell system $\mathcal{N}^\star$ patch equivalent to $\mathcal{N}$ contains no slaved subnetwork.

Definition 5.7. Suppose that $\mathcal{N}$ is a network cell system and $\mathcal{S} \subset \mathcal{N}$. We say that $\mathcal{S}$ is a *disconnected* subnetwork if we can write $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1, \mathcal{S}_2$ are proper subsets of $\mathcal{S}$ and there are no connections between cells in $\mathcal{S}_1$ and $\mathcal{S}_2$. If $\mathcal{S}$ is not disconnected, we say $\mathcal{S}$ is *connected*.

Definition 5.8. Suppose that $\mathcal{N}$ is a network cell system and $\mathcal{S} \subset \mathcal{N}$. We say that $\mathcal{S}$ is *strongly connected* if $\mathcal{S}$ is connected and it is not possible to disconnect $\mathcal{S}$ by repatching the subnetwork $\mathcal{S}$.

Example 5.9. In figure 12, we show two examples of subnetworks. The first example figure 12(a) consists of three cells and is strongly connected. The second example, figure 12(b), consists of two cells and is connected but not strongly connected (we can repatch $\{A1, A2\}$ so that the output of Ai goes to Ai, $i = 1, 2$).

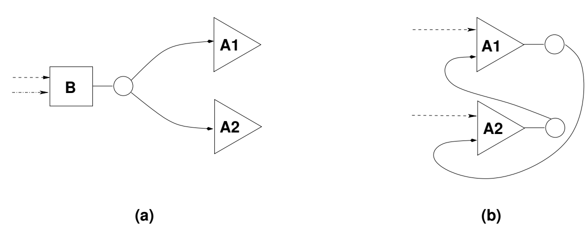

FIGURE 12. Connectedness of a subnetwork

Definition 5.10. Suppose that $\mathcal{N}$ is a network cell system and $\mathcal{Q} \subset \mathcal{N}$. We say that $\mathcal{Q}$ is a *component* of $\mathcal{N}$ if

(1) There are no connections between cells in $\mathcal{Q}$ and cells in $\mathcal{N} \setminus \mathcal{Q}$ (either from or to $\mathcal{Q}$).
(2) $\mathcal{Q}$ contains at least one cell.
(3) $\mathcal{Q}$ is strongly connected.

We say that $\mathcal{N}$ has k components if we can write $\mathcal{N}$ as a disjoint union $\dot{\bigcup}_i \mathcal{N}_i$, where each $\mathcal{N}_i$ is a component of $\mathcal{N}$. We say $\mathcal{N}$ is *connected* if it has one component.

Remark 5.11. If $\mathcal{Q}$ is a component of the network cell system $\mathcal{N}$, then $\mathcal{Q}$ is a network cell system with connection matrix defined by restricting the connection matrix of $\mathcal{N}$ to $\mathcal{Q}$.

Definition 5.12. A network cell system $\mathcal{N}$ is *transitive* if for all $i, j \in [1, k]$, there exist cells c_r^i, c_s^j such that there is a (directed) path of connections from c_r^i to c_s^j.

Lemma 5.13. (1) *Transitivity is an invariant of patch equivalence.*
(2) *Every minimal network is transitive.*

Proof. Trivial. □

Let $\bar{\mathcal{C}} = (\mathcal{C}, \mathbf{1}, \mathbf{C})$ be the network cell system which consists of one cell of each type. In this case, the array of connections $\mathbf{C}$ is uniquely determined. Henceforth, we shall always assume that our cells $\mathcal{C}$ are such that $\bar{\mathcal{C}}$ is connected.

Lemma 5.14. (1) *$\bar{\mathcal{C}}$ contains a finite set of disjoint minimal subnetworks $\mathbf{M}(\mathcal{C}) = \{\mathcal{C}_i \mid 1 \leq i \leq c\}$. The $\mathcal{C}_i$ are uniquely determined up to order.*
(2) *If $\bar{\mathcal{C}}$ transitive, $\bar{\mathcal{C}}$ is minimal and $\mathbf{M}(\mathcal{C}) = \{\mathcal{C}\}$.*
(3) *There exists a partition of $\mathcal{C} \setminus \cup_i \mathcal{C}_i$, into sets $\mathcal{S}_j$, $1 \leq j \leq d$, such that*
  (a) *Each $\mathcal{S}_j$ is strongly connected.*
  (b) *For each $j \in [1, d]_{\mathbb{N}}$, there exists a subset $p(j) \subset [1, c]_{\mathbb{N}}$ for which $\mathcal{S}_j$ is $\cup_{i \in p(j)} \mathcal{C}_i$-slaved.*
  (c) *There are no connections between cells in $\mathcal{S}_j$ and $\mathcal{S}_{j'}$, $j \neq j'$.*
  (d) *Properties (a,b,c) uniquely characterize the $\mathcal{S}_j$.*

Proof. We prove 1 by induction on the number of cells k in $\mathcal{C}$. The result is trivial if $k = 1$. So suppose the result is known for all network cell systems with fewer than k cell types. Let $\mathcal{S}$ be any nonempty proper subset of $\mathcal{C}$. Either there exists a connection from a cell of $\mathcal{C} \setminus \mathcal{S}$ to $\mathcal{S}$ or not. If there exists a connection, then $\mathcal{S}$ is not minimal. If this is so for all proper subsets $\mathcal{S}$, then $\bar{\mathcal{C}}$ is minimal and we are done. If there is no such connection, then $\mathcal{S}$ defines a network cell system with fewer than k cells and so, by the inductive hypothesis, $\mathcal{S}$ contains a minimal subnetwork. It follows that $\bar{\mathcal{C}}$ contains a maximal set $\{\mathcal{C}_i \mid 1 \leq i \leq c\}$ of minimal subnetworks. Since a minimal subnetwork is transitive, Lemma 5.13, it follows that the minimal subnetworks we have constructed are unique and mutually disjoint, proving 1.

We define an equivalence relation $\sim$ on cells in $\mathcal{C} \setminus \cup_i \mathcal{C}_i$ by requiring that $c \sim c'$ if there exists a connected subnetwork $\mathcal{S} \subset \mathcal{C} \setminus \cup_i \mathcal{C}_i$ which contains c, c'. Let $\{\mathcal{S}_j \mid 1 \leq j \leq d\}$ denote the resulting partition of $\mathcal{C} \setminus \cup_i \mathcal{C}_i$. Associated to each $j \in [1, d]_{\mathbb{N}}$, we let $p(j) \subset [1, c]_{\mathbb{N}}$ be the set of indices i for which there exists a connection from a cell in $\mathcal{C}_i$ to a cell in $\mathcal{S}_j$. It is obvious that the set $\{\mathcal{S}_j\}$ satisfies all the required properties. $\square$

Example 5.15. In figure 13, we show a 5-cell connected network. This network is connected but not transitive. It contains two minimal subnetworks $\mathcal{C}_1 = \{\mathbf{A}, \mathbf{B}\}$, and $\mathcal{C}_2 = \{\mathbf{E}\}$. The subnetwork $\{\mathbf{C}\}$ is $\mathcal{C}_1 \cup \mathcal{C}_2$-slaved, and the subnetwork $\{\mathbf{D}\}$ is $\mathcal{C}_2$-slaved.

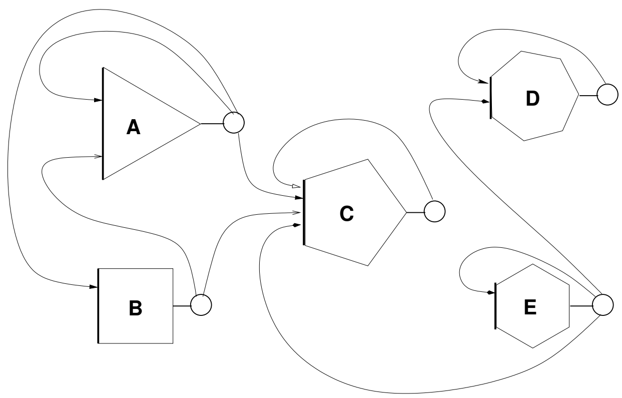

FIGURE 13. Decomposition of a 5-cell network

We now prove a version of Lemma 5.14 that holds for general network cell systems and shows that, up to permutation of certain connections, every network cell system has a 'normal' form. The important part of this result is the description of the minimal subnetworks. We make little use of the remaining statements in the sequel and we only include them for completeness and comparison with Lemma 5.14.

Proposition 5.16. *Suppose that* $\mathcal{N} = (\mathcal{C}, \mathbf{M}, \mathbf{P})$ *is a network cell system and that* $m^j > 0$, $j \in [1, k]_{\mathbb{N}}$. *Then* $\mathcal{N}$ *is patch equivalent to a network cell system* $\mathcal{N}^\star$ *where*

(1) $\mathcal{N}^\star$ *contains a finite set of mutually disjoint minimal subnetworks* $\mathcal{M}_i$, $1 \le i \le t$. *The* $\mathcal{M}_i$ *are unique up to order. As sets,* $\{\mathcal{M}_i \mid 1 \le i \le t\} = \mathbf{M}(\mathcal{C})$.
(2) *There exists a partition of* $\mathcal{N}^\star \setminus \cup_i \mathcal{M}_i$, *into sets* $\mathcal{S}_j$, $1 \le j \le d$, *such that*
  (a) *Each* $\mathcal{S}_j$ *is strongly connected.*
  (b) *For each* $j \in [1, d]_{\mathbb{N}}$, *there exists a subset* $p(j) \subset [1, t]_{\mathbb{N}}$ *for which* $\mathcal{S}_j$ *is* $\cup_{i \in p(j)} \mathcal{M}_i$*-slaved.*
  (c) *There are no connections between cells in* $\mathcal{S}_j$ *and* $\mathcal{S}_{j'}$, $j \neq j'$.
  (d) *If* $\mathcal{S}_j$ *contains cells of type* ℓ, *then there are no connections from cells of type* ℓ *in* $\mathcal{N}^\star \setminus \mathcal{S}_j$ *to cells in* $\mathcal{S}_j$.

(e) *Properties (a,b,c,d) uniquely characterize the* $\mathcal{S}_j$*.*

(3) *If* $\mathcal{N}$ *is transitive, then* $t = \min_j m^j$*, and the minimal subnetworks* $\mathcal{M}_i$ *are all patch equivalent and contain exactly* k *cells. In particular, if the network is homogeneous, there are exactly* m_1 *minimal subnetworks, each containing one cell.*

Proof. We proceed by a double induction on the number of cells $M = |\mathbf{m}|$ and the number k of different types of cell.

Suppose $k = 1$. Then $\mathcal{N}$ is patch equivalent to the network cell system $\mathcal{N}^\star$ with M components, each of which consists of a single cell with every output of the cell going to an input of the cell. Hence the proposition is true if $k = 1$. Suppose now that we have verified the proposition for network cell systems containing fewer than M cells and at most k distinct types of cell. Since every cell type occurs at least once, it is easy to see that $\mathcal{N}$ is patch equivalent to a network cell system $\mathcal{N}_1$ which has a connected subnetwork $\mathcal{Q}$ containing k cells, one of each type (cf lemma 5.14), and no connections from cells in $\mathcal{N}_1 \setminus \mathcal{Q}$ to cells in $\mathcal{Q}$. Suppose first that $\mathcal{N}_1 \setminus \mathcal{Q}$ contains at least one cell of each type or that there are no outputs from cells in $\mathcal{Q}$ to cells in $\mathcal{N}_1 \setminus \mathcal{Q}$. In the former case, we may repatch so that there are no cells in $\mathcal{N}_1 \setminus \mathcal{Q}$ which receive an output from a cell in $\mathcal{Q}$. Consequently, $\mathcal{N}_1 \setminus \mathcal{Q}$ is a network cell system and we may apply the inductive hypothesis to $\mathcal{N}_1 \setminus \mathcal{Q}$ to find a network cell system $\mathcal{N}_1^\star$ which is patch equivalent to $\mathcal{N}_1 \setminus \mathcal{Q}$ and satisfies all the conditions of the proposition. Since $\mathcal{N}$ is patch equivalent to $\mathcal{N}_1^\star \cup \mathcal{Q}$, the result now follows from the inductive hypothesis and an application of lemma 5.14 to $\mathcal{Q}$. Suppose instead that there are outputs from $\mathcal{Q}$ to cells in $\mathcal{N} \setminus \mathcal{Q}$. We can always repatch so that the outputs come from cells in $\mathcal{Q}$ which do not appear in $\mathcal{N}_1 \setminus \mathcal{Q}$. Suppose there are q of these cells. Necessarily $q < k$. We now define a new network cell system $\overline{\mathcal{N}}$ that consists of $\mathcal{N} \setminus \mathcal{Q}$. together with q new dummy cells, without inputs, which fill in the inputs that originally came from $\mathcal{Q}$. The network cell system $\overline{\mathcal{N}}$ has fewer than M cells, at most k types, and so we may apply the inductive hypothesis. We then reinsert $\mathcal{Q}$ in the obvious way and apply lemma 5.14 so as to obtain the required patch equivalence between $\mathcal{N}$ and a network $\mathcal{N}^\star$ satisfying the hypotheses of the proposition. This completes the first inductive step.

Next suppose that we have verified the proposition for $k > 1$, no restriction on M. Let $\mathcal{N}$ be a network cell system with $k+1$ different types of cell. Certainly, we must have $M \geq k+1$. If $M = k+1$, the result follows by Lemma 5.14. If $M > k+1$, we proceed as before. We leave the verification of the uniqueness statements as an easy exercise for the reader. □

Remark 5.17. If the network cell system $\mathcal{N}^\star$ satisfies the hypotheses of proposition 5.16, we say that $\mathcal{N}^\star$ is the *normal form* of $\mathcal{N}$. The normal form is unique up to permutation of connections from the minimal networks to slaved subnetworks.

Examples 5.18. (1) The network of figure 3 is transitive and is patch equivalent to the network $\mathcal{N}^\star$ shown in figure 4. Note that the cells $\mathbf{A}_1, \mathbf{B}$ define a minimal subnetwork $\mathcal{M}_1$ of $\mathcal{N}^\star$ and $\{\mathbf{A}_2\}$ is an $\mathcal{M}_1$-slaved subnetwork of $\mathcal{N}^\star$.
(2) The networks shown in figure 6 are not transitive. They are all patch equivalent to the third network shown in figure 6. This network has four components, all minimal and patch equivalent.
(3) The network shown in figure 10 is transitive and patch equivalent to the network $\mathcal{N}^\star$ of figure 14. The network $\mathcal{N}^\star$ contains two minimal subnetworks $\mathcal{M}_1 = \{\mathbf{A}_1, \mathbf{B}_1\}$, and $\mathcal{M}_2 = \{\mathbf{A}_2, \mathbf{B}_3\}$. The subnetwork $\mathbf{A}_3$ is $\mathcal{M}_2$-slaved.

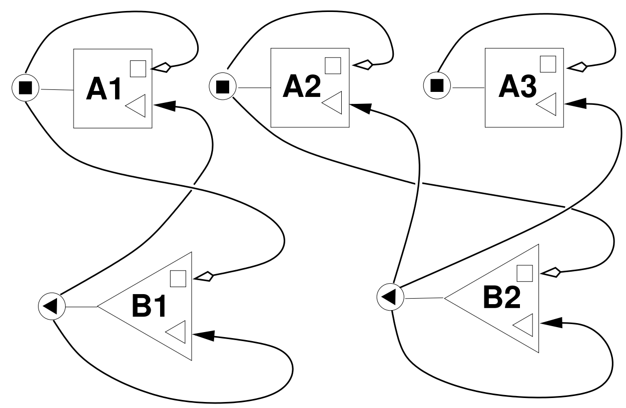

FIGURE 14. Minimal and slaved subnetworks

Proposition 5.19. *Suppose that the network cell system* $\mathcal{N}$ *has normal form* $\mathcal{N}^\star$. *Each of the minimal subnetworks of* $\mathcal{N}^\star$ *given by proposition 5.16, determines a (unique) synchrony class of the network* $\mathcal{N}$. *Moreover, each set of slaved networks also determines a synchrony class.*

Proof. It follows from proposition 5.16 that each minimal subnetwork of $\mathcal{N}^\star$ is equal to a unique $\mathcal{C}_i \in \mathbf{M}(\mathcal{C})$. If $\mathcal{C}_i$ contains cells of types $j_1 < \ldots < j_s$, then we may define the synchrony class $\mathbf{S}_i$ to consist of all the cells in $\mathcal{N}$ which are of types $j_1, \ldots, j_s$. If $\mathcal{S}_j$ is $\cup_{i \in p(j)} \mathcal{M}_i$-slaved, then we define the associated synchrony class to consist of all the cells in $\mathcal{N}$ which are of the types occurring in $\mathcal{S}_j \cup \cup_{i \in p(j)} \mathcal{M}_i$. Similarly, any set of slaved networks determines a synchrony class. □

Example 5.20. Suppose that $\mathcal{N}$ is a network cell system with $\bar{\mathcal{C}}$ equal to the network shown in figure 13. It follows from proposition 5.19, that $\mathcal{N}$ has at the least the following synchrony classes: (a) All A-cells synchronized and all B-cells synchronized, (b) all E cells synchronized; (c) all cell types synchronized except cells of type D; (d) all E cells synchronized and all D-cells synchronized; (e) all cell types synchronized.

Remark 5.21. Although Proposition 5.19 gives a relatively weak result, it does, as the previous example shows, give nontrivial minimal information about synchrony classes for a network. Nonetheless, it is the best we can do without additional structural assumptions on the network $\mathcal{N}$. We address this issue in the next section.

Example 5.22. In figure 15 we show an eight cell network $\mathcal{N}$ with normal form $\mathcal{N}^\star$. Each minimal subnetwork of $\mathcal{N}^\star$ consists of a single type **A**-cell. Consequently, $\{A1, A2, A3\}$ is a synchrony class for $\mathcal{N}$.

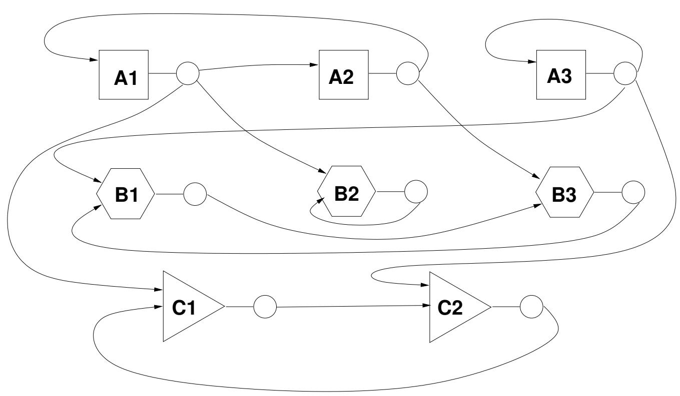

FIGURE 15. An eight cell network

In $\mathcal{N}^\star$ we find slaved networks of the form $A \to C$ and $A \to B$. Consequently, we have the synchrony classes $\{A1, A2, A3 \,\|\, B1, B2, B3\}$, $\{A1, A2, A3 \,\|\, C1, C2\}$ as well as the whole network $\mathcal{N}$.

6. Synchrony subspaces

Throughout this section $\mathcal{N} = (\mathcal{C}, \mathbf{m}, \mathbf{P})$ will denote a fixed network cell system. We assume that $\mathcal{N}$ is connected and that $m^j > 0$, all $j \in [1, k]_{\mathbb{N}}$.

Let c, c' be cells in $\mathcal{N}$. We let $p(c, c')$ denote the number of connections *from* c' *to* c. In terms of $\mathbf{P}$, if c is the sth cell of type j and c' is the rth cell of type i, then $p(c, c') = p^{ij}_{rs}$. In general, $p(c, c')$ will depend on r, s as well as the types i and j.

Definition 6.1. Let $\mathcal{S} \subset \mathcal{N}^j$ and $\mathcal{T} \subset \mathcal{N}^\ell$ be nonempty sets of cells of the same type. We say $\mathcal{S}$ is *$\mathcal{T}$-balanced* if for all $c \in \mathcal{S}$, $\sum_{e \in \mathcal{T}} p(c, e) = k_{j\ell}$ depends only on j, ℓ (not on the choice of $c \in \mathcal{S}$).

Remarks 6.2. (1) If b is a cell in $\mathcal{N} \setminus \mathcal{S}$, then $\mathcal{S}$ is $\{b\}$-balanced if and only if for every $c \in \mathcal{S}$, $p(c, b)$ depends only on j and b (and not on the choice of $c \in \mathcal{S}$). Note that $\{b\}$ is trivially $\mathcal{S}$-balanced.
(2) It is easy to extend Definition 6.1 to allow for arbitrary subsets $\mathcal{S}, \mathcal{T}$ of $\mathcal{N}$. We then have that $\mathcal{S}$ is $\mathcal{T}$-balanced if and only if $\mathcal{S}^j$ is $\mathcal{T}^\ell$ balanced for all $j, \ell \in [1, k]_{\mathbb{N}}$. However, in this section we shall always assume that when $\mathcal{S}$ is $\mathcal{T}$-balanced, $\mathcal{S}$ and $\mathcal{T}$ both consist of cells of the same type.

Definition 6.3. [cf [9]] Let $\mathbf{S} = \{\mathcal{S}_i \mid 1 \leq i \leq L\}$ be a partition of the network cell system $\mathcal{N}$ into disjoint sets of cells of the same type. We say $\mathbf{S}$ is *balanced* or a *balanced partition* if each $\mathcal{S}_i$ is $\mathcal{S}_j$-balanced, $1 \leq i, j \leq L$.

Remark 6.4. Instead of requiring that $\mathbf{S}$ defines a partition of $\mathcal{N}$, we could equally have required that the sets $\mathcal{S}_i$ all contain at least two cells. Under this assumption, we would add the condition that for all $i \in [1, L]_{\mathbb{N}}$, $c \in \mathcal{S}_i$ and every cell $b \notin \cup_i \mathcal{S}_i$, $\mathcal{S}_i$ is $\{b\}$-balanced (see remarks 6.2(1)) If this extra condition holds, we refer to $\mathbf{S}$ as a *balanced family*. Conversely, given a balanced family, we can always adjoin the single cell sets $\mathcal{S}_c = \{c\}$, $c \notin \mathbf{S}$, to obtain a balanced partition of $\mathcal{N}$. In the sequel, we often work with balanced families and assume that the sets $\mathcal{S}_i$ all contain at least two cells. Implicit in this description will be the requirement that $\mathcal{S}_i$ is $\{b\}$-balanced, $b \notin \cup_i \mathcal{S}_i$.

Examples 6.5. (1) Referring to the network of figure 10, let $\mathcal{S}_1 = \{A1, A2, A3\}$, $\mathcal{S}_2 = \{B_1, B2\}$, $\mathcal{S}_3 = \{A1, A2, A3, B1, B2\}$. In this case the families $\{\mathcal{S}_1\}$, $\{\mathcal{S}_2\}$, $\{\mathcal{S}_1, \mathcal{S}_2\}$ and $\{\mathcal{S}_3\}$ are all balanced.
(2) Referring to the network of figure 11, the family $\mathcal{S} = \{A1, A2\}$ is balanced. However, neither $\{A2, A3\}$ nor $\{A1, A3\}$ is balanced.
(3) Our final example comes from [9]. In figure 16, we show a homogeneous network of four cells. Obviously $\mathbf{S}_1 = \{A0, A1, A2, A3, A4\}$ is balanced. If we define $\mathcal{S}_1 = \{A1, A3\}$, $\mathcal{S}_2 = \{A2, A4\}$, then $\mathbf{S}_2 = \{\mathcal{S}_1, \mathcal{S}_2\}$ is also a balanced family. There are no other balanced families for this network (granted the conventions of remark 6.4).

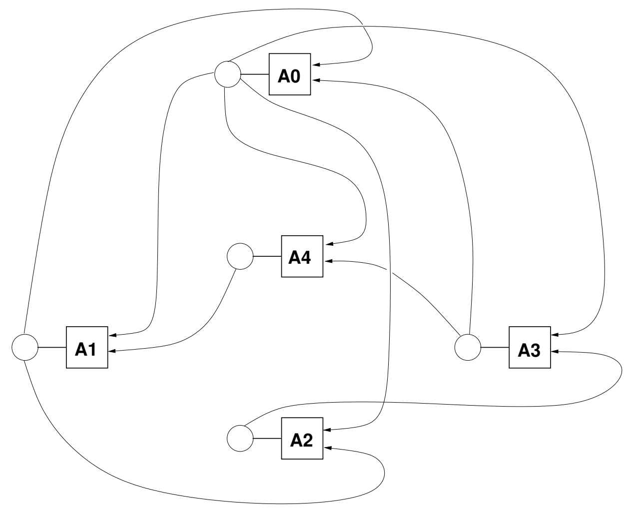

FIGURE 16. A homogeneous network of four cells

Proposition 6.6. *Every balanced family of subsets of $\mathcal{N}$ naturally determines a synchrony class for $\mathcal{N}$. Conversely, every synchrony class* $\mathbf{S}$ *determines a unique balanced family with associated synchrony class* $\mathbf{S}$.

Proof. Suppose that $\mathbf{S} = \{\mathcal{S}_\ell \mid 1 \leq \ell \leq L\}$ is a balanced family. Let $\ell \in [1, L]_{\mathbb{N}}$ and suppose that $\mathcal{S}_\ell \subset \mathcal{N}^j$. Define

$$\mathbf{V}_\ell = \{(\mathbf{x}^1, \ldots, \mathbf{x}^k \mid x^j_a = x^j_b,\ a, b \in \mathcal{S}_\ell\}.$$

Set $\mathbf{V}_\mathbf{S} = \cap_\ell \mathbf{V}_\ell$. It follows immediately from the definition of balanced that $\mathbf{V}_\mathbf{S}$ is invariant by the flow of the dynamical system defined by $\mathcal{N}$. The converse is equally straightforward. □

Remarks 6.7. (1) It follows from Proposition 6.6 that if $\mathbf{S} = \{\mathcal{S}_\ell \mid 1 \leq \ell \leq L\}$ is a balanced family, then the corresponding synchrony class of the network cell system is obtained by requiring cells within each $\mathcal{S}_\ell$ to be synchronous. Of course, if we had worked in terms of a balanced *partition*, this would not impose conditions on the singleton cells.
(2) The conditions for a family $\mathbf{S}$ to be balanced can obviously be written in terms of the connection matrix $\mathbf{P}$. This gives us a way of defining a synchrony class purely in terms of the connection matrix $\mathbf{P}$. It also suggests the possibility of developing computer based search algorithms to determine all synchrony classes in medium sized networks. In this direction, a useful observation is that if $\{\mathcal{S}_\ell \mid 1 \leq \ell \leq L\}$ is a balanced family, then $\mathcal{S}_\ell$ must be $\mathcal{S}_\ell$-balanced, $1 \leq \ell \leq L$.

6.1. **Order structure on synchrony classes.** Most of what we say is closely related to [9, Appendix]. We start by defining an order on the set of all partitions of a set.

Definition 6.8. Given partitions $\mathcal{P}_1, \mathcal{P}_2$ of a set, we write $\mathcal{P}_1 > \mathcal{P}_2$ if each component of $\mathcal{P}_2$ is contained within some component of $\mathcal{P}_1$.

Lemma 6.9. *Given two synchrony classes* $\mathbf{S}$, $\mathbf{T}$, $\mathbf{S} > \mathbf{T}$ *if and only if* $\mathbf{V_S} \subset \mathbf{V_T}$.

Proof. Immediate from Proposition 6.6 and the definition of $>$. □

Remark 6.10. The relation $>$ defines a partial order on the set of synchrony classes. Note the maximal synchrony class is the unique maximal element for the partial order $>$. The order $>$ corresponds to the order $\prec$ defined in [9, Appendix]. There is a unique minimal class defined by desynchronization of all cells and with corresponding invariant subspace $\mathbf{V}$. This class corresponds to the *finest balanced equivalence relation* defined in [9, Appendix].

Definition 6.11 (cf [9, Appendix])**.** Suppose that $\mathbf{S}$, $\mathbf{T}$ are balanced partitions of $\mathcal{N}$. Let $\mathbf{S} \vee \mathbf{T}$, $\mathbf{S} \wedge \mathbf{T}$ denote the balanced partitions characterized by the following properties.

(a) $\mathbf{S} \vee \mathbf{T}$ is the smallest balanced partition $\mathbf{B}$ satisfying $\mathbf{B} > \mathbf{S}, \mathbf{T}$.
(b) $\mathbf{S} \wedge \mathbf{T}$ is the largest balanced partition $\mathbf{B}$ satisfying $\mathbf{S}, \mathbf{T} > \mathbf{B}$.

The next result is immediate from the definitions.

Lemma 6.12. (a1) $\mathbf{V_{S\vee T}} = \mathbf{V_S} \cap \mathbf{V_T}$.
(b1) $\mathbf{V_{S\wedge T}} = \cap_\mathbf{U} \mathbf{V_U}$, *where the intersection is over all synchrony classes* $\mathbf{U}$ *such that* $\mathbf{V_U} \supset \mathbf{V_S} \cup \mathbf{V_T}$.

Remarks 6.13. (1) It is possible to give a natural description of $\mathbf{S} \vee \mathbf{T}$ independent of considerations of balanced structure. We define a symmetric and reflexive relation $\sim'$ on $\mathcal{N}$ as follows. If $c, d \in \mathcal{N}$ are cells, then $c \sim' \bar{c}$ if and only if either $c, \bar{c}$ lie in the same component of $\mathbf{S}$ or they lie in the same component of $\mathbf{T}$. Observe that if $c \sim' \bar{c}$, then the cells c and $\bar{c}$ are of the same type. We define the equivalence relation $\sim$ on $\mathcal{N}$ by requiring that $c \sim \bar{c}$ if and only if there exist cells $c = c_1, \ldots, c_s = \bar{c}$ such that $c_i \sim' c_{i+1}$, $1 \leq i < s$. Then $\mathbf{S} \vee \mathbf{T}$ is the partition associated to $\sim$. In particular, $\mathbf{S} \vee \mathbf{T}$ is the smallest partition $\mathbf{B}$ satisfying $\mathbf{B} > \mathbf{S}, \mathbf{T}$.
(2) In general, $\mathbf{S} \wedge \mathbf{T}$ will not be the largest partition $\mathbf{B}$ satisfying $\mathbf{S}, \mathbf{T} > \mathbf{B}$. In fact this partition will generally not be balanced (for an example, see Examples 6.21(1)).
(3) The set of all synchrony classes has the structure of a lattice under

the operations $\vee, \wedge$. We refer to [9, Appendix] for more details on this point.

Example 6.14. Suppose that the network cell system $\mathcal{N}$ has a pair of synchrony classes $\mathcal{S} = \{A_1, \ldots, A_u\}$ and $\mathcal{T} = \{B_1, \ldots, B_v\}$. If the two sets $\{A_1, \ldots, A_u\}$, $\{B_1, \ldots, B_v\}$ are *disjoint*, then

$$\mathcal{S} \vee \mathcal{T} = \{A_1, \ldots, A_v \| B_1, \ldots, B_v\},$$

and $\mathcal{S} \wedge \mathcal{T}$ is the minimal synchrony class. On the other hand, if $\{A_1, \ldots, A_u\}$, $\{B_1, \ldots, B_v\}$ are not disjoint (and so necessarily both $\mathcal{S}$ and $\mathcal{T}$ consist of cells of the same type), then

$$\mathcal{S} \vee \mathcal{T} = \{A_1, \ldots, A_u\} \cup \{B_1, \ldots, B_v\},$$

and $\mathcal{S} \wedge \mathcal{T} = \{A_1, \ldots, A_u\} \cap \{B_1, \ldots, B_v\}$. In this case $\mathcal{S} \wedge \mathcal{T}$ is the largest partition satisfying $\mathbf{S}, \mathbf{T} > \mathcal{S} \wedge \mathcal{T}$.

6.2. **Patch equivalence for balanced partitions.**

Definition 6.15. Suppose that $\mathbf{S} = \{\mathcal{S}_\ell \mid 1 \leq \ell \leq L\}$ is a balanced partition of cells for the network cell system $\mathcal{N}$. We say that $\mathcal{N}$ is patch $\mathbf{S}$-equivalent to the network cell system $\mathcal{N}^\star$ if

(1) $\mathcal{N}$ is patch equivalent to $\mathcal{N}^\star$.
(2) For all $\ell, p \in [1, L]_\mathbb{N}$, the total number of connections from cells in $\mathcal{S}_\ell$ to cells in $\mathcal{S}_p$ is the same for $\mathcal{N}$ and $\mathcal{N}^\star$.

Remarks 6.16. (1) Definition 6.15 implies that when we repatch a connection from $\mathcal{S}_\ell$ to $\mathcal{S}_p$, we do not move the output plug outside of $\mathcal{S}_\ell$ nor the input plug outside of $\mathcal{S}_p$.
(2) If $\mathcal{N}$ is patch $\mathbf{S}$-equivalent to the network cell system $\mathcal{N}^\star$, then $\mathcal{N}^\star$ is necessarily $\mathbf{S}$-balanced.

Example 6.17. We continue with the notation and assumptions of examples 6.5(3). The network of figure 16 is $\mathbf{S}_2$-equivalent to the network of shown in figure 17, Note that $\mathbf{M}_1 = \{A0, A1, A2\}$ is minimal with respect to $\mathbf{S}_2$-equivalence and that $\{A3, A4\}$ is $\mathbf{M}_1$-slaved. The minimal subnetwork $\mathbf{M}_1$ corresponds to the quotient networks defined in [9]. In our situation, it is trivial that every solution defined on $\mathbf{M}_1$ corresponds to an $\mathbf{S}_2$-synchronous solution for the original network.

Definition 6.18. Let $\mathbf{B} = \{\mathcal{B}_\ell \mid \ell \in [1, L]_\mathbb{N}\}$ be a balanced family of subsets of the network cell system $\mathcal{N}$.

(1) The network $\mathcal{N}$ is $\mathbf{B}$-*minimal* if every network cell system $\mathcal{N}^\star$ which is $\mathbf{B}$-patch equivalent to $\mathcal{N}$ contains no slaved subnetwork.

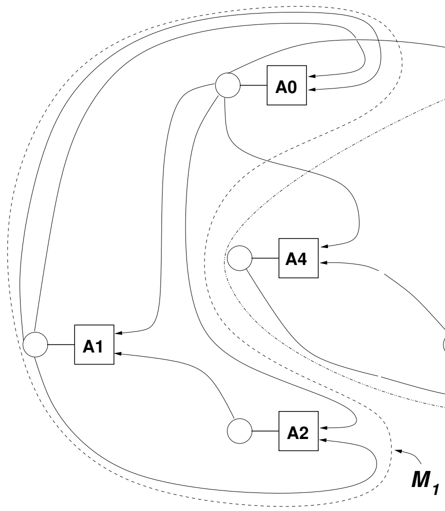

FIGURE 17. Repatching the network of figure 16

(2) Assuming that $\mathbf{B}$ defines a partition of $\mathcal{N}$, the network $\mathcal{N}$ is $\mathbf{B}$-*transitive* if for all $\ell, p \in [1, L]_{\mathbb{N}}$, there is a (directed) path of connections from a cell in $\mathcal{B}_\ell$ to a cell in $\mathcal{B}_p$.
(3) We say that $\mathcal{S} \subset \mathcal{N}$ is *strongly* $\mathbf{B}$-*connected* if $\mathcal{S}$ is connected and it is not possible to disconnect $\mathcal{S}$ by $\mathbf{B}$-repatching the subnetwork $\mathcal{S}$.

Example 6.19. The subnetwork $\mathbf{M}_1$ of figure 17 is $\mathbf{S}_2$-minimal and $\mathbf{S}_2$-transitive. The network of figure 16 is $\mathbf{S}_2$-transitive.

Theorem 6.20. *Let $\mathcal{N} = (\mathcal{C}, \mathbf{m}, \mathbf{P})$ be a network cell system with $m^j > 0$, $j \in [1, k]_{\mathbb{N}}$. Suppose that $\mathbf{B} = \{\mathcal{B}_\ell \mid 1 \leq \ell \leq L\}$ is a balanced family of subsets of $\mathcal{N}$. Then $\mathcal{N}$ is patch $\mathbf{B}$-equivalent to $\mathcal{N}^\star$ where*

(1) *$\mathcal{N}^\star$ contains a finite set of mutually disjoint $\mathbf{B}$-minimal subnetworks $\mathcal{M}_i$, $1 \leq i \leq t$. The $\mathcal{M}_i$ are unique up to order.*
(2) *There exists a partition of $\mathcal{N}^\star \setminus \cup_i \mathcal{M}_i$, into sets $\mathcal{S}_j$, $1 \leq j \leq d$, such that*
    (a) *Each $\mathcal{S}_j$ is strongly $\mathbf{B}$-connected.*
    (b) *For each $j \in [1, d]_{\mathbb{N}}$, there exists a subset $p(j) \subset [1, t]_{\mathbb{N}}$ for which $\mathcal{S}_j$ is $\cup_{i \in p(j)} \mathcal{M}_i$-slaved.*
    (c) *There are no connections between cells in $\mathcal{S}_j$ and $\mathcal{S}_{j'}$, $j \neq j'$.*
    (d) *If $\mathcal{S}_j$ contains cells in $\mathcal{B}_\ell$, then there are no connections from cells in $\mathcal{B}_\ell \cap (\mathcal{N}^\star \setminus \mathcal{S}_j)$ to cells in $\mathcal{S}_j$.*
    (e) *Properties (a,b,c,d) uniquely characterize the $\mathcal{S}_j$.*

(3) *If $\mathcal{N}$ is* **B**-*transitive, the minimal subnetworks $\mathcal{M}_i$ are all* **B**-*patch equivalent and contain exactly $L+R$ cells, where R is the number of cells in the set $\mathcal{N} \setminus \cup_\ell \mathcal{B}_\ell$.*

Proof. We use the partition of $\mathcal{N}$ determined by **B** and declare that cells lying in different sets of the partition have different types. The proof of the theorem then follows using Proposition 5.16. □

Examples 6.21. (1) In figure 18 we show a network $\mathcal{N}$ of six identical cells, each cell with two inputs. If we set $\mathbf{B} = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$, where $\mathcal{B}_1 = \{A1, A2\}$, $\mathcal{B}_2 = \{B1, B2\}$, $\mathcal{B}_3 = \{C1, C2\}$, then **B** is a balanced family of subsets of $\mathcal{N}$ and $\{A1, A2 \,\|\, B1, B2 \,\|\, C1, C2\}$ is a synchrony class. Obviously, $\mathcal{N}$ is **B**-transitive.

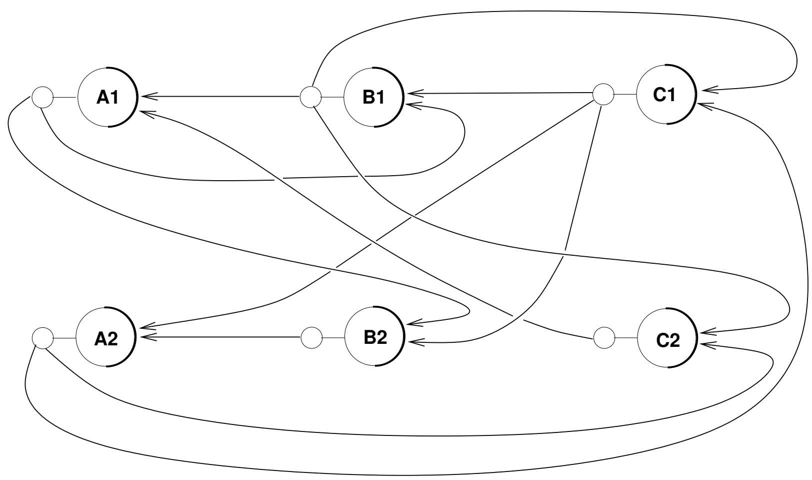

FIGURE 18. An asymmetric network of identical cells

Similarly, the sets $\{B1, B2\}$, $\{C1, C2\}$, and $\{B1, B2 \,\|\, C1, C2\}$ define synchrony classes for $\mathcal{N}$ associated to the balanced families $\{\mathcal{B}_2\}$, $\{\mathcal{B}_3\}$ and $\{\mathcal{B}_2, \mathcal{B}_3\}$ respectively. We emphasize that even though the cells are all assumed to be identical, we do not assume that the B- and C-cells in $\{B1, B2 \,\|\, C1, C2\}$ are synchronous. Indeed, $\{B1, B2, C1, C2\}$ is *not* a synchrony class. There are two other synchrony classes (balanced families). There is the maximal synchrony class $\{A1, A2, B1, B2, C1, C2\}$ where the cells A, B and C cells are all synchronous, and there is the synchrony class: $\{A1, A2 \,\|\, B1, C1 \,\|\, B2, C2\}$ associated to the unique symmetry of the system: $A1 \longleftrightarrow A2$, $B1 \longleftrightarrow C1$, $B2 \longleftrightarrow C2$.

It follows from the theory in [5] that this type of network cell system can admit robust heteroclinic cycles. More specifically, let V_{BC}, V_B and V_C denote the invariant subspaces respectively defined by the synchrony classes $\{B1, B2 \,\|\, C1, C2\}$, $\{B1, B2\}$, $\{C1, C2\}$. It is shown

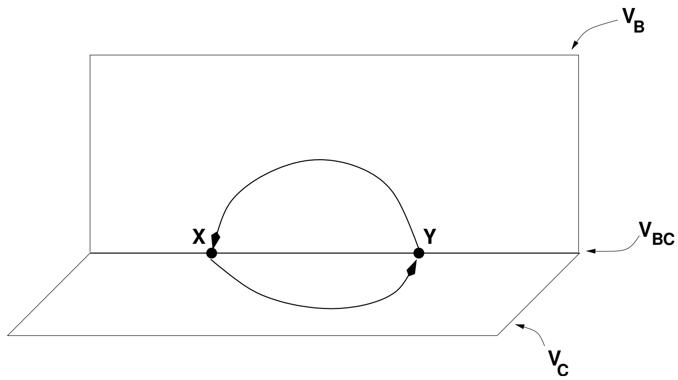

FIGURE 19. Heteroclinic cycles

in [5] that if we assume that the cells are governed by one-dimensional dynamics (dimension(V) = 1) then there are choices for the underlying dynamics F which give a robust heteroclinic cycle between equilibria $X, Y \in V_{BC}$. We refer to figure 19 and note that the dimension of the unstable manifolds of X and Y will be one, with $W^u(X) \subset V_C$, $W^u(Y) \subset V_B$. Noting that, through loss of stability, asymptotically stable cycles often give rise to nearby attracting limit cycles, this network has the possibility of exhibiting periodic solutions corresponding to an oscillation between (approximately) synchronous B and C states.

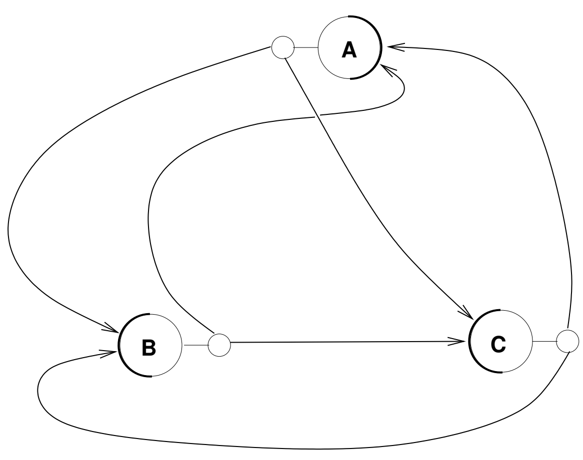

FIGURE 20. **B**-minimal network

The network $\mathcal{N}$ is patch **B**-equivalent to $\mathcal{N}^\star$, where $\mathcal{N}^\star$ consists of two identical **B**-minimal networks, see figure 20. Observe that the

B-minimal network has $\mathbf{D}_3$-symmetry. Any fully $\mathbf{D}_3$-symmetric solution to the minimal **B**-minimal network equations determines a synchronous solution lying in the $\{A1, A2, B1, B2, C1, C2\}$ synchrony subspace. In general, an *asymmetric* solution of the **B**-minimal network equations determines a solution lying in the $\{A1, A2 \,\|\, B1, B2 \,\|\, C1, C2\}$ synchrony subspace. One way of obtaining such a solution is via a $\mathbf{D}_3$-equivariant Hopf bifurcation from a fully symmetric equilibrium of the **B**-minimal network. For suitable model equations, this solution will determine three pairs of periodic solutions for the original network, each differing by a phase which is a multiple of $2\pi/3$ [7]. That is, each of cell pairs $\{A1, A2\}$, $\{B1, B2\}$, $\{C1, C2\}$ will oscillate synchronously with the same frequency but each pair will be $\pm 2\pi/3$ out of phase with the other pairs. That solutions of this type exist is perhaps not obvious from a casual glance at the network of figure 18. We refer to [9, §5] for another example of this type that depends on an interesting variation of the Hopf bifurcation argument. The differential equations for dynamics on the **B**-minimal network are given by the $\mathbf{D}_3$-equivariant system

$$\begin{aligned} \dot{x}_A &= F(x_A; x_B, x_C), \\ \dot{x}_B &= F(x_B; x_A, x_C), \\ \dot{x}_C &= F(x_C; x_A, x_B). \end{aligned}$$

(Note that the map $F(x; u, v)$ is symmetric in u, v – see section 4).

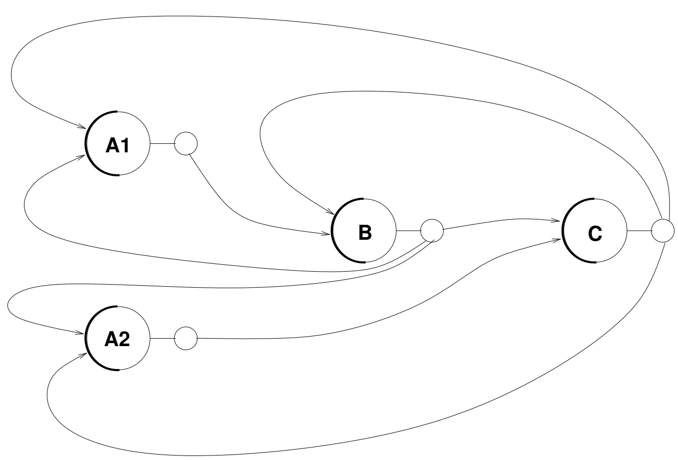

FIGURE 21. $\mathbf{B}'$-minimal network

If we define $\mathbf{B}' = \{\mathcal{B}_2, \mathcal{B}_3\}$, then $\mathbf{B}'$ defines the synchrony class $\{B1, B2 \,\|\, C1, C2\}$. The network $\mathcal{N}$ is $\mathbf{B}'$ transitive. We show the corresponding unique minimal network in figure 21.

The differential equations for dynamics on the $\mathbf{B}'$-minimal network are given by

$$\begin{aligned}\dot{x}_{A1} &= F(x_{A1}; x_B, x_C),\\ \dot{x}_{A2} &= F(x_{A2}; x_B, x_C),\\ \dot{x}_B &= F(x_B; x_{A1}, x_C),\\ \dot{x}_C &= F(x_C; x_{A2}, x_B).\end{aligned}$$

Here F is the vector field defining the dynamics on the cells. This system of equations is equivariant with respect to the order two group generated by $(x_{A1}, x_{A2}, x_B, x_C) \mapsto (x_{A2}, x_{A1}, x_C, x_B)$.
(2) Consider the network $\mathcal{N}$ of figure 10. Let $\mathcal{B}_i = \{Ai\}$, $i = 1, 2, 3$ and $\mathcal{B}_4 = \{B1, B2\}$. Then (see examples 6.5(1)), $\mathbf{B} = \{\mathcal{B}_i \mid i = 1, \dots, 4\}$ is a balanced partition of of $\mathcal{N}$. Clearly, $\mathcal{N}$ is $\mathbf{B}$-transitive. It follows that (up to $\mathbf{B}$-patch equivalence) there is just one $\mathbf{B}$-minimal network. See figure 22.

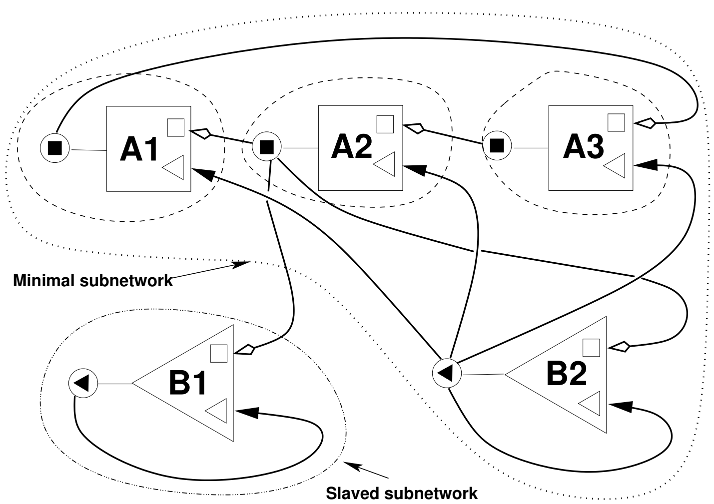

FIGURE 22. Repatching the network of figure 10

The equations for the minimal network are

$$\begin{aligned}\dot{x}_{A1} &= F_1(x_{A1}; x_{A2}, x_{B2}),\\ \dot{x}_{A2} &= F_1(x_{A2}; x_{A3}, x_{B2}),\\ \dot{x}_{A3} &= F_1(x_{A3}; x_{A1}, x_{B2}),\\ \dot{x}_{B2} &= F_2(x_{B2}; x_{A2}, x_{B2}).\end{aligned}$$

Any solution of these equations gives a solution of the original network where $B1$ and $B2$ are synchronized. Of course, since the equations for the minimal network are $\mathbb{Z}_3$-equivariant (under cyclic permutations of the A-cells), we can have solutions of the original network where $B1$ and $B2$ are synchronized and the cells $A1$, $A2$ and $A3$ are $2\pi/3$ out of phase. This can be achieved via bifurcation from solutions where $A1$, $A2$ and $A3$ are synchronous.

7. Concluding comments

The emphasis in this paper has been on obtaining a reformulation of the concept of coupled cell network that avoids some of the algebraic superstructure developed in [11, 9].

One advantage of the approach developed here is that it suggests more of an 'engineering' approach to coupled cell systems. That is, given a number of basic units (cells or operational amplifiers), design in a simple way circuits that possess certain robust patterned dynamics. Even the relatively simple six cell system described in the previous example, displays a remarkable variety of structured dynamics. Practically, there are issues of determining the stability of synchronous solutions and heteroclinic cycles. The resolution of these issues depends to a greater or lesser extent on understanding the functional structure of vector fields defined on coupled cell systems. For synchronous solutions, this requires a study of a minimal network, for asynchronous solutions (for example, attracting heteroclinic cycles), information is required about the functional structure on the entire network (see [5]).

It should be noted that there are no restrictions on the minimal networks. That is, every network cell system $\mathcal{M}$ may be regarded as the minimal network of a (larger) network cell system $\mathcal{N}$. Each cell $b_j \in \mathcal{M}$ will determine a subset $\mathcal{B}_j \subset \mathcal{N}$ such that $\mathbf{B} = \{\mathcal{B}_j\}$ is a balanced family of subsets of $\mathcal{N}$.

Finally a few comments about the mathematical differences between the two approaches to coupled cell systems. Stewart and coworkers *fix* a coupled cell system and then quantify the information using groupoids and graphs. Using various types of equivalence relation on the groupoid, it is then possible to define quotients which encapsulate dynamics on synchronous subspaces. On the other hand, we start with a coupled cell network and then find the invariants under various types of repatching of the network. This leads naturally to the concepts of minimal and slaved networks that determine dynamics on synchronous subspaces. Whatever approach one uses, the eventual equations that determine dynamics are the same.

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