Preface

This book is about the geometric theory of smooth dynamical systems that are symmetric (equivariant) with respect to a Lie group of transformations. The project started with a series of lectures I gave on equivariant bifurcation theory and equivariant transversality while I was a visiting Leverhulme professor at Imperial College, London, during the academic year 2004–2005. At the invitation of Laurent Chaminade and Imperial College Press, the lecture notes evolved into a more comprehensive work on symmetric dynamics. One special reason I had for undertaking this project was to give a careful and systematic introduction to equivariant transversality which was geared towards applications in dynamics. The theory of equivariant transversality was developed independently in the mid 1970's by Ed Bierstone and myself. My interest was in applications to equivariant dynamics, Bierstone's was in extending Mather's stability theory to equivariant maps. Much later, it turned out that equivariant transversality had powerful applications to equivariant bifurcation theory, many of which are described in this text. More recently, my host at Imperial, Jeroen Lamb, together with Mark Roberts and Luciano Buono, had started work on applying methods based on equivariant transversality to reversible equivariant dynamical systems. Their work resonated with an old (and much deferred) project of mine to develop equivariant transversality in the context of equivariant Hamiltonian systems. All in all, it seemed timely to write a systematic introduction to equivariant transversality that was geared to applications in dynamics and that would form a good basis for potential extensions to reversible and Hamiltonian dynamics.

The emphasis in this book is theoretical. While I give no applications to the physical sciences, I believe many of the methods developed have the potential for significant applications. For the reader interested in applications, I recommend the fine texts by Golubitsky, Schaeffer & Stewart [84,85] and Golubitsky & Stewart [87] which also contain comprehensive bibliographies of applications. Of course, the texts by Golubitsky *et al.* include a theory of local equivariant bifurcation. However, their singularity theory approach is quite different from the techniques used in this book.

The ten chapters of the book naturally group together: Chapters 1, 2 and 3 cover prerequisites; Chapters 4 and 5 form an elementary introduction to equivariant bifurcation theory; Chapters 6 and 7 develop the theory of equivariant transversality and its applications to equivariant bifurcation theory (mainly for finite groups); Chapter 8 describes the general theory of equivariant dynamical systems which are equivariant with respect to a compact Lie group and includes both the local theory of relative equilibria and periodic orbits as well as global theorems of the Kupka-Smale type. This work is developed further in Chapter 10 where we give an introduction to the bifurcation theory of equivariant maps and relative periodic orbits. Finally, Chapter 9 gives an overview of that part of the global theory of equivariant dynamical systems that does not depend on equivariant transversality. It remains to write the chapter on global dynamics depending on equivariant transversality.

In a little more detail, the first three chapters are introductory and cover prerequisites on groups, representations and smooth G-manifolds respectively. The level is sharply graded in these chapters — for example, the first chapter on groups progresses rapidly from the definition of a group up to topological and Lie groups, including a proof of the theorem that a closed subgroup of a Lie group is Lie. The more advanced material in these chapters is, however, not used until later in the book and readers are strongly advised to skim quickly through the first three chapters so as to familiarize themselves with notational conventions and refer back as needed (there is an index of notational conventions at the end of the book).

Chapter 4 is an introduction to steady state equivariant bifurcation theory on a G-representation, where G is a finite group. After carefully setting up the formalism of branching patterns, the chapter includes a large number of examples which are related to the standard irreducible representations of the hyperoctahedral and symmetric groups. Among the few non-elementary results in the chapter is a general theorem that gives a large class of examples (related to the hyperoctahedral and symmetric groups) where the *Maximal Isotropy Subgroup Conjecture* (MISC) fails. We also show that the examples studied in this chapter are all 2- or 3-determined. The proof, which uses results on constructible sets, is given in an appendix at the end of the chapter.

Chapter 5 is devoted to a study of some of the rich dynamics that can be generated in equivariant bifurcations. We start with a proof of the invariant sphere theorem. This result is used many times throughout the chapter and allows us to reduce to studying dynamics on a compact space — in this case a sphere. Apart from some more theoretical sections on the equivariant Hopf bifurcation that use blowing-up techniques from complex algebraic geometry, the remainder of the chapter is devoted to specific examples of dynamics that can be generated in equivariant bifurcations. These examples include steady state bifurcations generating branches of homoclinic cycles and periodic orbits and the generation of complex dynamics in low dimensional steady state bifurcation ('instant chaos' and Shilnikov networks). Together, Chapters 4 and 5 constitute an elementary introduction to equivariant bifurcation theory which includes a substantial set of interesting examples as well as basic results and terminology.

Chapter 6 is an introduction to equivariant transversality. I tried to write this chapter so that it is readable by those who do not have a strong background in stratification theory and differential analysis. I have also written the development in a way that gives the essential structure needed for applications to bifurcation theory relatively early on. More general results, such as equivariant transversality openness and density theorems, are covered towards the end of the chapter.

Chapter 7 applies equivariant transversality to the study of bifurcations equivariant with respect to a finite group. Perhaps surprisingly, it follows easily from our methods that the topological stability of the branching pattern of an equivariant bifurcation does not depend on the invariant structure — only on the equivariants (similar results hold for general compact Lie groups). For the discussion of stabilities along branches, we develop the necessary part of Bierstone's equivariant jet transversality theory and then prove more refined stability and determinacy theorems. We conclude the chapter with a discussion of higher order versions of equivariant transversality and show how our results generalize more or less immediately to general compact groups and bifurcation of equilibrium group orbits.

Chapter 8 is an introduction to the qualitative theory of smooth vector fields and diffeomorphisms equivariant with respect to a compact Lie group. At the local level we give a classification of dynamics on relative fixed and periodic sets (for diffeomorphisms) and relative equilibria and periodic orbits for flows. Next follows the local stability theory and then the local and global stable manifold theory. We conclude the chapter with the statement and (abbreviated) proof of equivariant versions of the Kupka-Smale theorems.

Chapter 9 describes some basic classes of equivariant dynamical system on a compact G-manifold. After a quick review of skew products, we consider G-invariant Morse functions and obtain special handlebundle decompositions of a G-manifold using a restricted class of G-Morse functions. We investigate equivariant versions of topological Markov chains (subshifts of finite type) and, using our results on handlebundle decompositions, obtain equivariant versions of the Shub-Smale C^0-density and isotopy theorems. The chapter concludes with results on solenoidal attractors (after Williams) and equivariant versions of Anosov diffeomorphisms and flows. Equivariant transversality plays virtually no role in the chapter. Indeed, as indicated above, it remains to obtain large classes of dynamically interesting examples which possess robustly non-transverse, equivariantly transverse, intersections of invariant manifolds that go beyond the well-known examples on homoclinic and heteroclinic cycles.

In Chapter 10 we extend our results based on equivariant transversality to include equivariant bifurcation to relative equilibria and the equivariant Hopf bifurcation. We also present the general theory for bifurcation of smooth families of equivariant maps and, following the recent work of Lamb, Melbourne and Wulff, conclude by indicating how our results may be applied to equivariant bifurcation from relative periodic orbits.

Chapters 4 to 10 all include brief notes on the history of the topics covered in the chapter.

Many of the results described in this work resulted from rewarding and enjoyable collaborations with mathematicians in Australia, the UK, Europe and North America. Although I investigated equivariant dynamics in my thesis (1970) and developed equivariant transversality in the mid 1970's, the work on equivariant bifurcation theory did not start until the Armidale meeting of the Australian Mathematical Society in 1987. It was at this meeting that Marty Golubitsky suggested to Roger Richardson and I that a good project to investigate would be the Maximal Isotropy Subgroup Conjecture. Although not a bifurcation theorist, Roger was an expert in representation theory and algebraic group actions and we both got interested in investigating the conjecture in cases where the invariant structure was well-known and simple: finite reflection groups. We ended up independently finding infinite families of counterexamples to the conjecture, promptly joined forces and wrote a number of papers on symmetry breaking which form the basis of Chapter 4. During the period 1987–1996, I developed new methods in equivariant bifurcation theory based on equivariant transversality. This work benefited enormously from many contacts and conversations with Ed Bierstone and Gerry Schwarz. Several of the main results in Chapter 5 on steady state and the Hopf bifurcation were joint work with Jim Swift done over the period 1989–94. In Chapter 5 I also report on Manuela Aguiar's PhD thesis work on chaos in 'Shilnikov networks'. The work on solenoidal attractors described in Chapter 9 is a small part of a long, continuing and wide-ranging collaboration with Ian Melbourne and Matt Nicol. Finally, in Chapter 10, I introduce some recent and, in my opinion, very rich and interesting work by Jeroen Lamb, Ian Melbourne and Claudia Wulff on bifurcation from relative periodic orbits.

Over the years I have learnt and benefitted much from conversations with many dynamicists, geometers and algebracists. Aside from those mentioned above, I would particularly like to record my thanks and gratitude to Peter Ashwin, Luciano Buono, Sofia Castro, Pascal Chossat, Jim Damon, Michael Dellnitz, Ana Dias, Stefan van Gils, John Guckenheimer, Bob Howlett, Martin Krupa, Tzee-Char Kuo, Isabel Labouriau, Reiner Lauterbach, Pierre Milman, Mark Roberts, Ian Stewart, Andrew Török, and Bob Williams.

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