Bifurcation from relative periodic solutions

Claudia Wulff [*] Jeroen S. W. Lamb [†] Ian Melbourne [‡]

Institut für Mathematik I
Freie Universität Berlin
Arnimallee 2-6
14195 Berlin, Germany

Department of Mathematics
University of Houston
Houston, TX 77204-3476, USA

December 29, 1998

Abstract

Relative periodic solutions are ubiquitous in dynamical systems with continuous symmetry. Recently, Sandstede, Scheel and Wulff derived a center bundle theorem, reducing local bifurcation from relative periodic solutions to a finite dimensional problem. Independently, Lamb and Melbourne showed how to systematically study local bifurcation from isolated periodic solutions with discrete spatiotemporal symmetries.

In this paper, we show how the center bundle theorem, when combined with certain group theoretic results, reduces bifurcation from relative periodic solutions to bifurcation from isolated periodic solutions. In this way, we obtain a systematic approach to the study of local bifurcation from relative periodic solutions.

AMS subject classification. 35B32, 58F14, 34C35, 57S20
Keywords. relative periodic solutions, equivariant dynamical systems, noncompact groups

1 Introduction

Relative equilibria and relative periodic solutions occur in numerous physical experiments in which there are continuous symmetries present. For example, in excitable media there arise spirals that rigidly rotate, see [39], as well as spirals that undergo quasiperiodic meandering, see [35] and that undergo linear drift [40, 29]. The rigidly rotating spirals are examples of relative equilibria — in a rotating frame they are ordinary equilibria. The quasiperiodically meandering and linearly drifting spirals are examples of relative periodic solutions. In appropriate moving frames (a rotating frame and a translating frame respectively) they reduce to periodic solutions.

Similarly, in the Taylor-Couette experiment, wavy vortices [1] are examples of relative equilibria, whereas modulated wavy vortices [20] are examples of relative periodic solutions. Relative equilibria and relative periodic solutions arise also in flame experiments [19] (and in the associated numerics [6]), and in two dimensional convection patterns [32].

[*] Research supported by the Deutsche Forschungsgemeinschaft
[†] Supported by a Talent Stipendium of the Netherlands Organization for Scientific Research (NWO)
[‡] Supported in part by NSF Grant DMS-9704980

Transitions from relative equilibria and relative periodic solutions have been analyzed in the above settings using equivariant bifurcation theory. This is an extension to systems with symmetry of the standard local bifurcation theory for systems without symmetry. We note that in dynamical systems without symmetry, there is a complete theory of the generic local bifurcations that occur as a single bifurcation parameter is varied, see for example Guckenheimer and Holmes [22, Chapter 3]. Local bifurcations are by definition the bifurcations that occur in the neighborhood of a nonhyperbolic equilibrium or a nonhyperbolic periodic solution.

A rough sketch of the developments in equivariant bifurcation theory is given in the following table.

isolated equilibria	Golubitsky, Stewart and Schaeffer [18]
relative equilibria	Krupa [23] Fiedler, Sandstede, Scheel and Wulff [12, 33]
isolated periodic solutions	Lamb and Melbourne [25]
relative periodic solutions	this paper

Table 1: Developments in equivariant bifurcation theory from (relative) equilibria and (relative) periodic solutions.

The references in Table 1 are not intended to be complete. Extensive references to previous partial results on bifurcation from isolated periodic solutions can be found in [25]. We note also that results on bifurcation from relative periodic solutions were previously obtained by Sandstede *et al.* [34]. These results are necessarily partial since at that time the results in [25] were not available. In this paper, we show that the ideas in [25, 34], together with certain group theoretic results, yield a systematic approach to bifurcation from relative periodic solutions.

We now describe in more detail the results outlined in Table 1. The first systematic results in equivariant bifurcation theory were obtained in [18] for bifurcations from fully symmetric equilibria under the assumption that the group of symmetries Γ is a compact Lie group. In particular, such equilibria are generically isolated.

A *relative equilibrium* is a Γ-orbit in phase space that is also invariant under the flow. If we denote the flow by Φ_t, then u_0 lies on a relative equilibrium if and only if $\Phi_t(u_0) \in \Gamma u_0$ for all t. The simplest example of a relative equilibrium is a group orbit of equilibria. A relative equilibrium on which the flow is periodic is called a *rotating wave*. In general, the flow on a relative equilibrium is either quasiperiodic or unbounded. The flows on a relative equilibrium were classified algebraically by Field [15] in the case that Γ is a compact Lie group and by Ashwin and Melbourne [4] for Γ a general finite dimensional Lie group.

In the case that Γ is compact, Krupa [23] showed that, modulo drifts along continuous group orbits, the problem of bifurcation from a relative equilibrium reduces to the problem of bifurcation from an isolated equilibrium as studied in [18]. In this way, the usual center manifold theorem for equilibria [22] translates into a *center bundle* theorem for relative equilibria. This was generalized by Sandstede, Scheel and Wulff [33] to the case where Γ is possibly noncompact, under the hypothesis that the isotropy subgroup

$$\Delta = \{\gamma \in \Gamma : \gamma u_0 = u_0\},$$

of the point u_0 on the relative equilibrium is compact. At the same time, Fiedler *et al.* [12] showed how to choose convenient coordinates on the center bundle to enable the computation of the drifts arising through bifurcation.

Now we turn to bifurcation from relative periodic solutions. Recall that u_0 lies on a relative periodic solution if u_0 does not lie on a relative equilibrium and there is some time $T > 0$ such that $\Phi_T(u_0) \in \Gamma u_0$. The corresponding relative periodic solution $\mathcal{P}$ is defined to be

$$\mathcal{P} = \{\gamma\Phi_t(u_0) : \gamma \in \Gamma,\, t \in [0, T)\}.$$

The minimal choice of T is called the relative period of $\mathcal{P}$ and by rescaling time we may suppose that $T = 1$. The flows on a relative periodic solution were classified by Krupa [23] in the case that Γ is compact (see also Field [16]) and by Ashwin and Melbourne [4] in the general case.

As was the case for relative equilibria, we suppose that the isotropy subgroup Δ of the point $u_0 \in \mathcal{P}$ is compact. In Lamb and Melbourne [25], it is assumed in addition that $\dim\Gamma = \dim\Delta$. The relative periodic solution is then an ordinary periodic solution and moreover is generically isolated. A systematic approach to bifurcation from such isolated periodic solutions is presented in [25]. Once again, the problem is reduced to the problem of bifurcation from an isolated equilibrium.

A center bundle theorem for relative periodic solutions is proved in Sandstede, Scheel and Wulff [34]. The local dynamics then reduces from an infinite dimensional phase space to a finite dimensional manifold M on which the Lie group Γ acts smoothly and properly. (Recall that Γ acts properly on M if the map $(\gamma, u) \mapsto (\gamma u, u) \in M \times M$ sends closed sets to closed sets and preimages of points in $M \times M$ are compact.) We take this as our starting point and refer to [34] for a statement and discussion of the technical hypotheses behind the center bundle theorem.

Spatial and spatiotemporal symmetry

Let Γ be a finite dimensional Lie group acting smoothly and properly on the finite dimensional manifold M. Suppose that $\mathcal{P}$ is a relative periodic solution (with relative period 1) for the Γ-equivariant ODE

$$\dot{u} = \mathcal{F}(u), \tag{1.1}$$

where $\mathcal{F} : M \to TM$ is a Γ-equivariant vector field

The symmetries that leave $\mathcal{P}$ invariant come in two forms. First, there is the group of *spatial symmetries*, namely the isotropy subgroup Δ of u_0. (Since Γ acts properly on M, the isotropy subgroup Δ is automatically compact.) Second, there is the group of *spatiotemporal symmetries* Σ defined in the following way. Choose $\sigma \in \Gamma$ such that $\Phi_1(u_0) = \sigma u_0$. Then, Σ is the closed subgroup of Γ generated by Δ together with σ. Note that Δ is a normal subgroup of Σ.

We have made two choices above. The choice of $u_0 \in \mathcal{P}$ is unimportant — although the subgroups Δ and Σ depend on u_0, they are unique up to conjugacy within Γ. We shall regard u_0, and hence Δ and Σ, as being fixed.

Of more importance is the choice of σ which is defined only up to multiplication by elements of Δ (so the coset $\sigma\Delta$ is uniquely determined). Denote the Lie algebra of a Lie group G by $\mathrm{L}G$. Throughout most of this paper, we make the hypothesis

(H) There is a positive integer n such that $\sigma^n \in \exp \mathrm{L}Z(\Sigma)$,

where $Z(\Sigma) \subset \Gamma$ is the centralizer of Σ.

Remark 1.1 In Section 6, we show that this hypothesis is valid in considerable generality. For example, σ can be chosen (within $\sigma\Delta$) to satisfy Hypothesis (H) whenever Γ is an algebraic group. The class of algebraic groups includes all compact Lie groups, the Euclidean groups $\mathbf{E}(N)$, the classical groups, and so on. More generally, it is sufficient that the normalizer $N(\Delta)$ of Δ is an algebraic group.

The simplest example where Hypothesis (H) cannot be satisfied is when $\Gamma = \Sigma = \mathbb{Z}$ and $\Delta = \mathbf{1}$. See Example 6.9 for examples where Γ is connected, $\Delta = \mathbf{1}$, but Hypothesis (H) fails.

Suppose that Hypothesis (H) is satisfied. We form a semidirect product $\Delta \rtimes \mathbb{Z}_{2n}$ by adjoining to Δ an element Q of order $2n$, where $Q\delta Q^{-1} = \sigma\delta\sigma^{-1}$ for $\delta \in \Delta$. (In particular, Q^n lies in the center of $\Delta \rtimes \mathbb{Z}_{2n}$.)

Skew product for relative periodic solutions

We continue to suppose that $\mathcal{P}$ is a relative periodic solution for a Γ-equivariant vector field on M. As before, the spatiotemporal symmetry Σ is the closed subgroup of Γ generated by the spatial symmetry Δ together with the element σ.

Theorem 1.2 *Suppose that Hypothesis (H) is satisfied and write $\sigma^n = \exp n\xi$ where $\xi \in \mathrm{L}Z(\Sigma)$. Let $\Delta \rtimes \mathbb{Z}_{2n}$ be defined as above. In a comoving frame, moving uniformly with velocity ξ, the equations in a neighborhood of the relative periodic solution $\mathcal{P}$ lift to skew product equations on $\Gamma \times V \times S^1$ where V is a representation of the group $\Delta \rtimes \mathbb{Z}_{2n}$, and $S^1 = \mathbb{R}/2n\mathbb{Z}$. The skew product equations have the form (after reparameterizing time)*

$$\begin{aligned} \dot{\gamma} &= \gamma f_\Gamma(v,\theta), \\ \dot{v} &= f_V(v,\theta), \\ \dot{\theta} &= 1, \end{aligned} \tag{1.2}$$

where $f_\Gamma : V \times S^1 \to \mathrm{L}\Gamma$ and $f_V : V \times S^1 \to V$ are smooth vector fields satisfying $f_\Gamma(0,\theta) = \xi$, $f_V(0,\theta) = 0$, for all $\theta \in S^1$.

In addition, the skew product equations (1.2) are $\Delta \rtimes \mathbb{Z}_{2n}$-equivariant with respect to the action of $\Delta \rtimes \mathbb{Z}_{2n}$

$$\delta \cdot (\gamma, v, \theta) = (\gamma\delta^{-1}, \delta v, \theta), \;\; \delta \in \Delta, \qquad Q \cdot (\gamma, v, \theta) = (\gamma\alpha^{-1}, Qv, \theta + 1), \tag{1.3}$$

where $\alpha = \exp(-\xi)\sigma$.

Remark 1.3 (a) In the comoving frame, the relative periodic solution $\mathcal{P}$ is transformed into a group orbit (under Γ) of ordinary periodic solutions. We speak of *comoving* frames since it is necessary to pass both to corotating frames (in case $\mathcal{P}$ is a modulated rotating wave) and to cotraveling frames (in case $\mathcal{P}$ is a modulated traveling wave).

(b) The vector space V in Theorem 1.2 is defined to be a Δ-invariant cross-section to the relative periodic solution $\mathcal{P}$, see [34] and also Section 2. The representation of $\Delta \rtimes \mathbb{Z}_{2n}$ is arbitrary, in the sense that any representation can arise for an appropriate choice of manifold M and vector field $\mathcal{F}$. Furthermore, $f_\Gamma : V \times S^1 \to \mathrm{L}\Gamma$ and $f_V : V \times S^1 \to V$ are general vector fields satisfying the equivariance conditions and the restrictions at $v = 0$ in Theorem 1.2.

(c) If we replace $\Gamma \times V \times S^1$ by $\Gamma \times V$, and $\Delta \rtimes \mathbb{Z}_{2n}$ by Δ, then the skew product structure in Theorem 1.2 reduces to the skew product structure in Fiedler *et al.* [12]. Hence Theorem 1.2 is a generalization to relative periodic solutions of the results in [12] for relative equilibria.

The restrictions on f_Γ and f_V that arise from the action (1.3) of $\Delta \rtimes \mathbb{Z}_{2n}$ are easily computed to be as follows:

$$\begin{aligned} f_\Gamma(\delta v, \theta) &= \mathrm{Ad}_\delta f_\Gamma(v,\theta), \;\; \delta \in \Delta, \qquad & f_\Gamma(Qv,\theta) &= \mathrm{Ad}_\alpha f_\Gamma(v, \theta - 1), \\ f_V(\delta v, \theta) &= \delta f_V(v,\theta), \;\; \delta \in \Delta, \qquad & f_V(Qv,\theta) &= Q f_V(v, \theta - 1). \end{aligned}$$

If time is not reparameterized, the $\dot\theta$ equation takes the form $\dot\theta = f_\Theta(v,\theta)$ where

$$f_\Theta(\delta v,\theta) = f_\Theta(v,\theta), \;\; \delta\in\Delta, \qquad\qquad f_\Theta(Qv,\theta) = f_\Theta(v,\theta-1).$$

Bifurcation from relative periodic solutions

Lamb and Melbourne [25] give a systematic approach to generic bifurcation from isolated periodic solutions with spatiotemporal symmetry. It follows from Theorem 1.2 that this theory generalizes to arbitrary relative periodic solutions provided Hypothesis (H) is satisfied. Indeed the relative periodic solution $\mathcal{P}$ in Theorem 1.2 reduces to an isolated periodic solution for the (v,θ)-subsystem. Moreover, by Remark 1.3(b), the (v,θ)-subsystem is a general $\Delta\rtimes\mathbb{Z}_{2n}$-equivariant ODE possessing a periodic solution with spatial symmetry Δ and spatiotemporal symmetry $\Delta\rtimes\mathbb{Z}_{2n}$. Hence the theory of [25] applies with the group Γ replaced by the group $\Delta\rtimes\mathbb{Z}_{2n}$. In particular, modulo drifts along continuous group orbits in Γ governed by the $\dot\gamma$ equation, bifurcation from a relative periodic solution reduces to bifurcation from an isolated periodic solution.

The group $\Delta\rtimes\mathbb{Z}_{2n}$ is closely related to the group $\Delta\rtimes\mathbb{Z}_{2k}$ used in [25] to study bifurcation from isolated periodic solutions. There, the integer k is the least positive integer such that $\sigma^k\in Z(\Delta)$. Of course, k divides n. In fact, it turns out that in applying Theorem 1.2 it is sufficient to consider representations V of $\Delta\rtimes\mathbb{Z}_{2n}$ in which the generator Q of $\mathbb{Z}_{2n}$ satisfies $Q^{2k}=I$.

In other words, the (v,θ)-subsystem in Theorem 1.2 can be taken to be a general $\Delta\rtimes\mathbb{Z}_{2k}$-equivariant ODE on $V\times S^1$ where $S^1=\mathbb{R}/2k\mathbb{Z}$.

In Sections 3 and 4, we consider in detail how to use Theorem 1.2 to analyze bifurcation from relative periodic solutions. The analysis divides into three steps:

1. Study the bifurcation in the $\Delta\rtimes\mathbb{Z}_{2k}$-equivariant (v,θ)-subsystem in (1.2).

2. Substitute the solution $(v(t),\theta(t))$ from Step 1 into the $\dot\gamma$-equation in (1.2) and solve for $\gamma(t)$.

3. Interpret the solution $(\gamma(t),v(t),\theta(t))$ to the skew product equations (1.2) in the original ODE (1.1).

We note that Step 1 can be carried out (at least for codimension one bifurcations) using the theory of Lamb and Melbourne [25]. Step 2 is routine in the examples considered in this paper, but is nontrivial in general. (A similar problem arises in bifurcation from relative equilibria and is made tractable there by ideas of Fiedler and Turaev [13].)

The interpretation in Step 3 is implicit in the proof of Theorem 1.2. Note that, in particular, it is necessary to pass back from the comoving frame to the original 'laboratory' frame. For convenience, we make certain aspects of this interpretation explicit in the next proposition.

Proposition 1.4 *Suppose that $y(t)=(v(t),\theta(t))$ is a solution to the (v,θ)-subsystem and let $(\gamma(t),y(t))$ be the solution to the skew product equations (1.2) with initial condition* (id,y_0). *Let $u(t)$ be the corresponding solution for the underlying ODE (1.1). Then*

(a) The spatial symmetry of $u(t)$ (that is, the isotropy subgroup of $u(t_0)$ for each t_0) coincides with the spatial symmetry of $y(t)$.

(b) $u(t)$ is a relative periodic trajectory for the ODE (1.1) if and only if $y(t)$ is a periodic solution for the (v,θ)-subsystem.

(c) Suppose that $u(t)$ is a relative periodic solution. Then the relative period of $u(t)$ is the least integer $p\geq 1$ such that $y(p)=Q^p\delta\cdot y_0$ for some $\delta\in\Delta$. Moreover, it is then the case that $u(p)=\gamma(p)\alpha^p\delta\cdot u(0)$.

Remark 1.5 The property that $\sigma^k \in Z(\Delta)$ for some positive integer k is more generally satisfied than Hypothesis (H). (The simplest example where this occurs is $\Gamma = \Sigma = \mathbb{Z}$, $\Delta = \mathbf{1}$. In general, it is sufficient that Γ is a matrix group, see Section 6.) Hence it is reasonable to ask what aspects of Theorem 1.2 go through under the more general hypothesis

(H′) There is a positive integer k such that $\sigma^k \in Z(\Delta)$.

Suppose that Hypothesis (H′) is satisfied. In Section 7, we show that in principle bifurcation from relative periodic solutions reduces as before to bifurcation from isolated periodic solutions with Γ replaced by $\Delta \rtimes \mathbb{Z}_{2k}$. Indeed, modulo drifts along continuous group orbits, the theory is as complete as when Hypothesis (H) is satisfied. However, there is no longer a comoving frame in which $\mathcal{P}$ reduces to a periodic solution, and the computation of drifts along continuous group orbits is less transparent.

For example, parts (a) and (b) of Proposition 1.4 are unchanged under the weaker Hypothesis (H′). However, in part (c), it is necessary to change α to σ and to solve a more complicated equation for $\gamma(t)$.

Finally, in Section 7, we show how to analyze bifurcation from relative periodic solutions when Hypothesis (H′) fails, but in this generality, the integer k used to define $\Delta \rtimes \mathbb{Z}_{2k}$ depends on the specific representation of Δ on the cross-section V.

Structure of the center bundle

Suppose that V is a representation of Δ. Consider the free action of $\Gamma \times \Delta$ on $\Gamma \times V$ where Γ acts by left multiplication on the Γ-component, and $\delta \in \Delta$ acts as

$$\delta \cdot (\gamma, v) = (\gamma\delta^{-1}, \delta v).$$

Since Δ is compact and acts freely, we can form the quotient manifold

$$\Gamma \times_\Delta V = (\Gamma \times V)/\Delta.$$

Fiedler *et al.* [12] show that if the vector field $\mathcal{F} : M \to TM$ has a relative equilibrium with spatial symmetry Δ, then there is a neighborhood U of the relative equilibrium that is diffeomorphic to $\Gamma \times_\Delta V$. The vector field restricted to U lifts to a $\Gamma \times \Delta$-equivariant vector field on $\Gamma \times V$. The equivariance conditions translate into the Δ-equivariant skew product structure discussed in Remark 1.3(c).

We now state a generalization of this structure for relative periodic solutions.

Theorem 1.6 *Suppose that Hypothesis (H) is satisfied and write $\sigma^n = \exp n\xi$ where $\xi \in \mathrm{L}Z(\Sigma)$. Form the group $\Delta \rtimes \mathbb{Z}_{2n}$ as before. Then, there is a neighborhood U of the relative periodic solution $\mathcal{P}$ such that, in a comoving frame, moving uniformly with velocity ξ,*

$$U \cong \frac{(\Gamma \times_\Delta V) \times S^1}{\mathbb{Z}_{2n}} \cong \frac{\Gamma \times V \times S^1}{\Delta \rtimes \mathbb{Z}_{2n}},$$

where V is a representation of the group $\Delta \rtimes \mathbb{Z}_{2n}$, and $S^1 = \mathbb{R}/2n\mathbb{Z}$.

The action of $\Delta \rtimes \mathbb{Z}_{2n}$ on $\Gamma \times V \times S^1$ is given by

$$\delta \cdot (\gamma, v, \theta) = (\gamma\delta^{-1}, \delta v, \theta),\ \delta \in \Delta, \quad Q \cdot (\gamma, v, \theta) = (\gamma\alpha^{-1}, Qv, \theta + 1),$$

where $\alpha = \exp(-\xi)\sigma$.

Remark 1.7 The bundle in Theorem 1.6 is $2n$-periodic. In certain instances, it is possible to obtain an n-periodic bundle, see Remark 2.5. In that case, Theorems 1.2 and 1.6 go through with $2n$ replaced by n.

The remainder of the paper can be outlined as follows. In Section 2, we sketch the results of [34] and then prove Theorems 1.2 and 1.6, and Proposition 1.4. In Section 3 and 4, we consider several applications of the results in Sections 2.

The remaining sections of the paper are not required to understand the applications but serve to illuminate further the structure and the hypotheses in our main results.

In Section 5, we investigate in more depth the structure of relative periodic solutions that arise in equivariant dynamical systems, building upon work of [23, 16, 4]. In particular, when Hypothesis (H) is satisfied, we define the index m of a relative periodic solution and we prove the existence of a comoving frame in which the skew product equations are $2m$-periodic (the optimal periodicity that can be obtained in general).

In Section 6, we prove that σ can be chosen to satisfy Hypothesis (H) if Γ is an algebraic group. However, in Section 7, we prove also that bifurcation from relative periodic solutions for *any* symmetry group Γ reduces, modulo drifts along continuous group orbits, to bifurcation from isolated periodic solutions as studied in [25].

2 Relative periodic solutions and skew products

In this section, we prove the results stated in the introduction. In Subsection (a), we prove a result about twisted equivariant maps. In Subsection (b), we review the skew product construction of Sandstede *et al.* [34]. Theorems 1.2 and 1.6 are proved in Subsection (c).

(a) Twisted equivariant linear maps

Suppose that Γ is a finite dimensional Lie group with compact subgroup Δ. Let $\sigma \in N(\Delta)$. Then σ induces an automorphism $\phi \in \mathrm{Aut}(\Delta)$ defined by

$$\phi(\delta) = \sigma^{-1}\delta\sigma.$$

Suppose that Δ acts orthogonally on a finite dimensional vector space X. Following [25], we say that a linear map $L : X \to X$ is *twisted equivariant* if

$$L\delta = \phi(\delta)L,$$

for all $\delta \in \Delta$. (In [27], such a map is said to be *k-symmetric* where k is least such that ϕ^k is the identity automorphism of Δ.)

Proposition 2.1 *Suppose that $L : X \to X$ is a twisted equivariant nonsingular linear map. Then there is a twisted equivariant orthogonal map $A : X \to X$ such that $A^{-1}L$ is Δ-equivariantly isotopic to the identity.*

Proof. By polar decomposition, we can write L uniquely as $L = AB$ where A is orthogonal and B is positive definite. Indeed, B is defined as the unique positive definite square root of $B^2 = L^T L$.

We claim that B is equivariant and A is twisted equivariant. It follows from the orthogonality of the action of Δ on X that $L^T L$ commutes with Δ. But then for each $\delta \in \Delta$, the positive definite matrix $B' = \delta B\delta^{-1}$ satisfies $(B')^2 = L^T L$. Since B is the unique positive definite square root of

$L^T L$, it follows that $B = B' = \delta B \delta^{-1}$ so that B commutes with Δ. Since L is twisted equivariant and B is equivariant, we have that $A = LB^{-1}$ is twisted equivariant, proving the claim.

Since B is positive definite and equivariant, it follows that $J_t = (1-t)I + tB$ is positive definite and equivariant for all $t \in [0,1]$. In particular, B is equivariantly isotopic to the identity. ∎

Lemma 2.2 *Let Δ be a compact Lie group acting orthogonally on X and suppose that $\phi \in \mathrm{Aut}(\Delta)$ is an automorphism of finite order k. Let $L : X \to X$ be a twisted equivariant nonsingular linear map. Then there is a twisted equivariant orthogonal map $A : X \to X$ such that $A^{2k} = I$ and $A^{-1}L$ is Δ-equivariantly isotopic to the identity.*

Proof. By Proposition 2.1, we may first reduce to the case where L is orthogonal.

The map L^k is equivariant in the usual sense. Let μ denote an eigenvalue of L^k and denote the corresponding eigenspace by E_μ. Note that E_μ is Δ-invariant and is also invariant under L. To prove the lemma, it is sufficient to restrict to E_μ.

If $\mu = \pm 1$, then we have $L^{2k} = I$ and it suffices to take $A = L$. Otherwise μ lies on the unit circle, but μ is not real. Let $\lambda \in \mathbb{C}$ denote a k'th root of μ and define $A = \lambda^{-1}L$. Then A is twisted equivariant, and $A^{-1}L$ is equivariantly isotopic to the identity. Moreover, $A^k = I$. ∎

(b) The skew product construction of [34]

We consider a finite dimensional Lie group Γ acting smoothly and properly on a finite dimensional manifold M. Let $\mathcal{F}$ be a Γ-equivariant vector field on M with flow Φ_t and suppose that $\mathcal{P}$ is a relative periodic solution. Choose $u_0 \in \mathcal{P}$. Without loss, we have that $\Phi_1(u_0) = \sigma u_0$ for some $\sigma \in \Gamma$ and $\Phi_t(u_0) \notin \Gamma u_0$ for $0 < t < 1$. Let Δ and Σ denote the spatial and spatiotemporal symmetry groups of $\mathcal{P}$, so Δ is compact and Σ is the closed subgroup of Γ generated by Δ and σ.

In [34], it was shown how the flow in a neighborhood U of the relative periodic solution $\mathcal{P}$ can be written as a skew product flow on a space of the form $\Gamma \times V \times \mathbb{R}$. Here, V is a Δ-invariant cross-section to $\mathcal{P}$ and $\theta \in \mathbb{R}$ plays the role of the phase along the relative periodic solution. Note that V is a Poincaré section which is transverse not only to the time orbit of u_0, but also to the group orbit of u_0.

Roughly speaking, the idea in [34] is to construct a family of cross-sections V_θ around the relative periodic solution and to use the linearized flow $(\mathrm{D}\Phi_\theta)_{u_0}$ to define coordinates on V_θ. First, write

$$T_{\Phi_\theta(u_0)}M = T_{\Phi_\theta(u_0)}\mathcal{P} \oplus V_\theta, \tag{2.1}$$

where $V_0 = V$ and the cross-sections V_θ are Δ-invariant and depend smoothly on θ. Let $P_\theta : T_{\Phi_\theta(u_0)}M \to V_\theta$ denote the associated family of projections. By construction, the projections are Δ-equivariant and depend smoothly on θ. Moreover, $(\mathrm{D}\Phi_\theta)_{u_0}$ restricts to a Δ-equivariant map $P_\theta(\mathrm{D}\Phi_\theta)_{u_0} : V \to V_\theta$.

The idea of the proof can be seen by glancing at the submersion τ defined below in equation (2.3), but with the term J_θ omitted. For various technical reasons, we require the following lemma.

Lemma 2.3 *Let $L = P_0\sigma^{-1}(\mathrm{D}\Phi_1)_{u_0} : V \to V$.*
(a) There is an orthogonal map $A : V \to V$ and a smooth family of Δ-equivariant nonsingular linear maps $J_\theta : V \to V$, $\theta \in \mathbb{R}$, such that

$$J_0 = I, \qquad LJ_{\theta+1} = J_\theta A,\ \theta \in \mathbb{R}. \tag{2.2}$$

(b) The projections P_θ can be chosen so that

$$P_{\theta+1}\sigma = \sigma P_\theta,\ \theta \in \mathbb{R}.$$

Proof. Note that $L : V \to V$ is twisted equivariant as defined in Subsection (a). That is, $L\delta = \phi(\delta)L$ for all $\delta \in \Delta$, where $\phi \in \mathrm{Aut}(\Delta)$ is given by $\phi(\delta) = \sigma^{-1}\delta\sigma$. It follows from Proposition 2.1 that there is a twisted equivariant orthogonal map $A : V \to V$ such that $A^{-1}L$ is Δ-equivariantly isotopic to the identity. In particular, there is a smooth family of Δ-equivariant nonsingular linear maps $J_\theta : V \to V$, $\theta \in [0,1]$, such that $J_0 = I$ and $LJ_1 = A$. This family extends uniquely to a continuous family J_θ, $\theta \in \mathbb{R}$, satisfying equation (2.2). Modifying J_θ near $\theta = 0$ and $\theta = 1$, we obtain a smooth family J_θ, $\theta \in \mathbb{R}$, proving part (a).

Next, we prove part (b). Let $\langle\, , \rangle$ be a Δ-invariant inner product on $T_{u_0}M$ and define P_0 to be the orthogonal projection onto V. We repeat the argument above, but applied to the twisted equivariant map $\widetilde{L} = \sigma^{-1}(\mathrm{D}\Phi_1)_{u_0} : T_{u_0}M \to T_{u_0}M$ to obtain an orthogonal map $\widetilde{A} : T_{u_0}M \to T_{u_0}M$ and a smooth family of Δ-equivariant nonsingular linear maps $\widetilde{J}_\theta : T_{u_0}M \to T_{u_0}M$ such that

$$\widetilde{J}_0 = I, \qquad \widetilde{L}\widetilde{J}_{\theta+1} = \widetilde{J}_\theta\widetilde{A},\ \theta \in \mathbb{R}.$$

Let $\langle\, , \rangle_\theta$ be the Δ-invariant inner product on $T_{\Phi_\theta(u_0)}M$ defined by

$$\langle v, w\rangle_\theta = \langle \widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}v, \widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}w\rangle.$$

We compute that

$$\begin{aligned}(\mathrm{D}\Phi_{\theta+1})_{u_0} &= (\mathrm{D}\Phi_\theta)_{\Phi_1(u_0)}(\mathrm{D}\Phi_1)_{u_0} = (\mathrm{D}\Phi_\theta)_{\sigma u_0}(\mathrm{D}\Phi_1)_{u_0} \\ &= \sigma(\mathrm{D}\Phi_\theta)_{u_0}\sigma^{-1}(\mathrm{D}\Phi_1)_{u_0} = \sigma(\mathrm{D}\Phi_\theta)_{u_0}\widetilde{L}.\end{aligned}$$

Hence

$$\widetilde{J}_{\theta+1}^{-1}(\mathrm{D}\Phi_{\theta+1})_{u_0}^{-1} = \widetilde{J}_{\theta+1}^{-1}\widetilde{L}^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\sigma^{-1} = \widetilde{A}^{-1}\widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\sigma^{-1}.$$

Using the orthogonality of $\widetilde{A}$, it follows that

$$\begin{aligned}\langle v, w\rangle_{\theta+1} &= \langle \widetilde{J}_{\theta+1}^{-1}(\mathrm{D}\Phi_{\theta+1})_{u_0}^{-1}v, \widetilde{J}_{\theta+1}^{-1}(\mathrm{D}\Phi_{\theta+1})_{u_0}^{-1}w\rangle \\ &= \langle \widetilde{A}^{-1}\widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\sigma^{-1}v, \widetilde{A}^{-1}\widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\sigma^{-1}w\rangle \\ &= \langle \widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\sigma^{-1}v, \widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\sigma^{-1}w\rangle \\ &= \langle \sigma^{-1}v, \sigma^{-1}w\rangle_\theta.\end{aligned}$$

Finally, define V_θ to be the orthogonal complement to $T_{\Phi_\theta(u_0)}\mathcal{P}$ in $T_{\Phi_\theta(u_0)}M$ with respect to the inner product $\langle\, , \rangle_\theta$, and let $P_\theta : T_{\Phi_\theta(u_0)}M \to V_\theta$ be the orthogonal projection. Then we have $P_{\theta+1}\sigma = \sigma P_\theta$ as required. ∎

Now, consider the submersion $\tau : \Gamma \times V \times \mathbb{R} \to M$ defined by

$$\tau(\gamma, v, \theta) = \gamma\big(\Phi_\theta(u_0)\,,\, P_\theta(\mathrm{D}\Phi_\theta)_{u_0}J_\theta v\big). \tag{2.3}$$

It follows from Δ-equivariance of P_θ, $(\mathrm{D}\Phi_\theta)_{u_0}$ and J_θ, and Δ-invariance of the points $\Phi_\theta(u_0) \in \mathcal{P}$, that $\tau(\gamma\delta, v, \theta) = \tau(\gamma, \delta v, \theta)$ for all $\delta \in \Delta$.

As in the proof of Lemma 2.3(b), we have $(\mathrm{D}\Phi_{\theta+1})_{u_0} = \sigma(\mathrm{D}\Phi_\theta)_{u_0}L$, so that

$$\begin{aligned}\tau(\gamma, v, \theta+1) &= \gamma\big(\phi_{\theta+1}(u_0)\,,\, P_{\theta+1}(\mathrm{D}\Phi_{\theta+1})_{u_0}J_{\theta+1}v\big) \\ &= \gamma\big(\phi_\theta(\sigma u_0)\,,\, P_{\theta+1}\sigma(\mathrm{D}\Phi_\theta)_{u_0}LJ_{\theta+1}v\big) \\ &= \gamma\big(\sigma\phi_\theta(u_0)\,,\, \sigma P_\theta(\mathrm{D}\Phi_\theta)_{u_0}J_\theta Av\big) \\ &= \gamma\sigma\big(\phi_\theta(u_0)\,,\, P_\theta(\mathrm{D}\Phi_\theta)_{u_0}J_\theta Av\big) \\ &= \tau(\gamma\sigma, Av, \theta).\end{aligned}$$

Define the orthogonal map $Q = A^{-1}$. To summarize, we have the identifications

$$\tau(\gamma\delta, v, \theta) = \tau(\gamma, \delta v, \theta), \quad \delta \in \Delta, \qquad \tau(\gamma, v, \theta + 1) = \tau(\gamma\sigma, Q^{-1}v, \theta). \tag{2.4}$$

Remark 2.4 If the term J_θ is absent in the definition (2.3) of τ, then the presence of contracting and expanding eigenvalues for L may imply that the image of τ is a nonuniform neighborhood of $\mathcal{P}$ as $\theta \to \pm\infty$.

Next, we introduce an action of a group $\Gamma \times (\Delta \rtimes \mathbb{Z})$ on $\Gamma \times V \times \mathbb{R}$ where Γ acts as left multiplication on the Γ component and the action of $\Delta \rtimes \mathbb{Z}$ is given by

$$(\gamma, v, \theta) \mapsto (\gamma\delta^{-1}, \delta v, \theta), \quad \delta \in \Delta, \qquad (\gamma, v, \theta) \mapsto (\gamma\sigma^{-1}, Qv, \theta + 1).$$

It is immediate that τ is Γ-equivariant. It follows from the identifications (2.4) that τ induces a Γ-equivariant map

$$\tau \;:\; \frac{\Gamma \times V \times \mathbb{R}}{\Gamma \times (\Delta \rtimes \mathbb{Z})} \to M.$$

As shown in [34], this is an equivariant diffeomorphism onto a uniform neighborhood U of the relative periodic solution.

The Γ-equivariant vector field on the neighborhood U lifts to a $\Gamma \times (\Delta \rtimes \mathbb{Z})$-equivariant vector field on $\Gamma \times V \times \mathbb{R}$. The Γ-equivariance is equivalent to saying that the lifted vector field has the skew product form

$$\dot{\gamma} = \gamma f_\Gamma(v, \theta), \qquad \dot{v} = f_V(v, \theta), \qquad \dot{\theta} = f_\Theta(v, \theta), \tag{2.5}$$

where $f_\Gamma(0, \theta) = 0$, $f_V(0, \theta) = 0$, for all $\theta \in \mathbb{R}$.

We end this subsection by demonstrating that the skew product equations (2.5) are general equations satisfying the equivariance conditions and the restrictions at $v = 0$ (cf. Remark 1.3(b)). We continue to suppose that $\mathcal{F} : M \to TM$ is a fixed Γ-equivariant vector field with flow Φ_t and relative periodic solution $\mathcal{P}$ of relative period 1. As usual, we fix a point $u_0 \in \mathcal{P}$ and write $\Phi_1(u_0) = \sigma u_0$. As described in this subsection, we construct a submersion $\tau : \Gamma \times V \times \mathbb{R} \to U$ where U is a uniform neighborhood of $\mathcal{P}$. Note that τ depends on the flow Φ_t and hence on the underlying vector field $\mathcal{F}$.

Now, let $\widetilde{\mathcal{F}} : U \to TU$ be a general Γ-equivariant vector field defined on the neighborhood U of the relative periodic solution $\mathcal{P}$ and satisfying $\widetilde{\mathcal{F}}|_\mathcal{P} = \mathcal{F}|_\mathcal{P}$. The fixed submersion τ (defined in terms of $\mathcal{F}$) gives a one-to-one correspondence between the Γ-equivariant vector fields $\widetilde{\mathcal{F}} : U \to TU$ with relative periodic solution $\mathcal{P}$ and $\Gamma \times (\Delta \rtimes \mathbb{Z})$-equivariant skew product equations

$$\dot{\gamma} = \gamma \widetilde{f}_\Gamma(v, \theta), \qquad \dot{v} = \widetilde{f}_V(v, \theta), \qquad \dot{\theta} = \widetilde{f}_\Theta(v, \theta),$$

where $\widetilde{f}_\Gamma(0, \theta) = 0$, $\widetilde{f}_V(0, \theta) = 0$, for all $\theta \in \mathbb{R}$.

Hence, by perturbing the underlying vector field $\mathcal{F}$, but keeping the submersion τ fixed, we obtain arbitrary equivariant perturbations of the skew product equations. It is in this sense that the equations (2.5) are general.

(c) Proof of Theorems 1.2 and 1.6

Now, suppose that $\sigma^n \in \exp \mathrm{L}Z(\Sigma)$ for some $n \geq 1$. We make two modifications to the construction in Subsection (b).

The first modification involves the choice of Q. Since $\sigma^n \in Z(\Delta)$, the induced automorphism $\phi \in \mathrm{Aut}(\Delta)$ satisfies $\phi^n = \mathrm{Id}$. It follows from Lemma 2.2 that the isotopy J_θ can be chosen so that the orthogonal map $A = Q^{-1}$ in equation (2.2) has finite order, indeed $Q^{2n} = I$. In the remainder of this subsection, $\tau : \Gamma \times V \times \mathbb{R} \to M$ denotes the submersion in Subsection (b) but with the new isotopy J_θ and orthogonal map Q.

The second modification is to pass to a convenient comoving frame. Write $\sigma^n = \exp n\xi$ where $\xi \in \mathrm{L}Z(\Sigma)$. Define $\alpha = \exp(-\xi)\sigma$, so that α has order n. We define the new submersion

$$\tau^{\mathrm{new}}(\gamma, v, \theta) = \tau(\gamma \exp(-\theta\xi), v, \theta). \tag{2.6}$$

Note that τ^{new} remains Γ-equivariant, since Γ acts on the left.

Since $\tau(\gamma\delta, v, \theta) = \tau(\gamma, \delta v, \theta)$ for all $\delta \in \Delta$, and $\xi \in \mathrm{L}Z(\Delta)$, it is immediate that $\tau^{\mathrm{new}}(\gamma\delta, v, \theta) = \tau^{\mathrm{new}}(\gamma, \delta v, \theta)$ for all $\delta \in \Delta$. Similarly, we compute that

$$\begin{aligned}
\tau^{\mathrm{new}}(\gamma, v, \theta + 1) &= \tau(\gamma \exp(-\xi)\exp(-\theta\xi), v, \theta + 1) \\
&= \tau(\gamma \exp(-\xi)\exp(-\theta\xi)\sigma, Q^{-1}v, \theta) \\
&= \tau(\gamma\alpha \exp(-\theta\xi), Q^{-1}v, \theta) \\
&= \tau^{\mathrm{new}}(\gamma\alpha, Q^{-1}v, \theta).
\end{aligned}$$

Since α has order n and $Q^{2n} = I$, it follows that $\tau^{\mathrm{new}}(\gamma, v, \theta + 2n) = \tau^{\mathrm{new}}(\gamma, v, \theta)$. Hence τ^{new} induces a Γ-equivariant submersion

$$\tau^{\mathrm{new}} : \Gamma \times V \times S^1 \to M,$$

where $S^1 = \mathbb{R}/2n\mathbb{Z}$.

Since $Q = A^{-1}$ where A is twisted equivariant, it follows that $Q\delta Q^{-1} = \sigma\delta\sigma^{-1}$ for $\delta \in \Delta$. Hence Δ and Q generate the compact group $\Delta \rtimes \mathbb{Z}_{2n}$ defined in the introduction. Moreover, we have a fixed-point free action of $\Delta \rtimes \mathbb{Z}_{2n}$ on $\Gamma \times V \times S^1$ given by

$$(\gamma, v, \theta) \mapsto (\gamma\delta^{-1}, \delta v, \theta), \;\; \delta \in \Delta, \qquad (\gamma, v, \theta) \mapsto (\gamma\alpha^{-1}, Qv, \theta + 1).$$

By the arguments described in Subsection (b), it follows that τ induces a Γ-equivariant diffeomorphism

$$\tau : \frac{(\Gamma \times_\Delta V) \times S^1}{\mathbb{Z}_{2n}} \cong U,$$

where U is a neighborhood of the relative periodic solution and $\Gamma \times_\Delta V = (\Gamma \times V)/\Delta$. This completes the proof of Theorem 1.6.

As before, the Γ-equivariant vector field on U lifts to a $\Gamma \times (\Delta \rtimes \mathbb{Z}_{2n})$-equivariant vector field on $\Gamma \times V \times S^1$. The skew product structure (1.2) follows again from Γ-equivariance. Hence we have proved Theorem 1.2.

Finally, we note that the coordinates on the new bundle are related to the coordinates on the old bundle by the Γ-equivariant transformation

$$\gamma^{\mathrm{new}} = \gamma \exp(\theta\xi).$$

The relationship between the vector fields on the two bundles is given by

$$f_\Gamma^{\mathrm{new}}(v, \theta) = \mathrm{Ad}_{\exp(-\theta\xi)} f_\Gamma(v, \theta) + \xi. \tag{2.7}$$

In particular, $f_\Gamma^{\rm new}(0,\theta) = \xi$ for all $\theta \in \mathbb{R}$. To verify equation (2.7), we compute that

$$\begin{aligned}
\dot\gamma^{\rm new} &= \dot\gamma \exp(\theta\xi) + \gamma \exp(\theta\xi)\dot\theta\xi \\
&= \gamma\{f_\Gamma(v,\theta)\exp(\theta\xi) + \exp(\theta\xi)\xi\} \\
&= \gamma^{\rm new}\{\exp(-\theta\xi) f_\Gamma(v,\theta)\exp(\theta\xi) + \xi\} \\
&= \gamma^{\rm new} f_\Gamma^{\rm new}.
\end{aligned}$$

Remark 2.5 The $2n$-periodicity in Theorems 1.2 and 1.6 holds in complete generality and depends only on the group-theoretic integer n in Hypothesis (H). However, the methods of the present subsection often lead to an n-periodic bundle. This is the case when Q can be chosen in Lemma 2.2 so that the order of Q divides n.

In fact, by Lemma 2.2, we have $Q^{2k} = I$, where k is the least integer such that $\sigma^k \in Z(\Delta)$. Hence k divides n.

In particular, suppose that n/k is even. Then $2k$ divides n and we obtain n-periodicity in complete generality.

Next, suppose that n/k is odd and that Δ acts faithfully on the cross-section V. Then Q cannot have order $j < k$ since Q^j is not equivariant. It follows that Q can be chosen to have order either k or $2k$, and we obtain an n-periodic bundle if and only if Q can be chosen to have order k. The proof of Lemma 2.2 shows that the only obstruction is negative eigenvalues of Q^k. However, the precise conditions under which we can choose Q of order k are somewhat subtle and will not be given here.

A third possibility arises when the action of Δ is not faithful, so the effective order of the automorphism ϕ (modulo the kernel of the representation of Δ) is reduced. For example, suppose that $k = n$ even, and that Δ acts trivially on V. Then $Q|_V$ is equivariant and can be chosen to have order 2. Hence the order of $Q|_V$ divides n and we obtain an n-periodic bundle (even though n/k is odd, and regardless of the eigenvalues of $Q|_V$).

Proof of Proposition 1.4 Write $u(t) = \tau(\gamma(t), y(t)) = \tau(\gamma(t), v(t), t)$. In particular, $u(0) = \tau(\mathrm{id}, v_0, 0)$.

The spatial symmetry of $u(t)$ is a subgroup of Δ and is independent of t, so it suffices to compare the isotropy subgroups of $u(0)$ and v_0. Let $\delta \in \Delta$. Then

$$\delta u(0) = \delta\tau(\mathrm{id}, v_0, 0) = \tau(\delta, v_0, 0) = \tau(\mathrm{id}, \delta v_0, 0).$$

It follows that $\delta u(0) = u(0)$ if and only if $\delta v_0 = v_0$ proving part (a).

Next, we prove parts (b) and (c). Suppose that $p \in \mathbb{Z}$. Then

$$u(p) = \tau(\gamma(p), v(p), p) = \tau(\gamma(p)\alpha^p, Q^{-p}v(p), 0).$$

If $u(t)$ has relative period p, then $u(p) = gu(0)$ for some $g \in \Gamma$. That is,

$$\tau(\gamma(p)\alpha^p, Q^{-p}v(p), 0) = g\tau(\mathrm{id}, v_0, 0) = \tau(g, v_0, 0) = \tau(g\delta^{-1}, \delta v_0, 0),$$

for all $\delta \in \Delta$. It follows that $Q^{-p}v(p) = \delta v_0$ for some $\delta \in \Delta$ and hence that $g = \gamma(p)\alpha^p\delta$. This calculation shows that if $u(t)$ has relative period p, then $v(t)$ has relative period p and the converse is easier to see. Moreover, the calculation shows that $v(t)$ has spatiotemporal symmetry $Q^p\delta$ if and only if $u(t)$ has spatiotemporal symmetry $\gamma(p)\alpha^p\delta$. ∎

3 NonHopf bifurcation from relative periodic solutions

In this section, and in the following section, we give examples illustrating how to study bifurcation from relative periodic solutions.

As described in the introduction, the first step is to study bifurcation from isolated periodic solutions in the (v, θ)-subsystem on $V \times S^1$, where $S^1 = \mathbb{R}/2n\mathbb{Z}$. The (v, θ)-subsystem is $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant, where the group $\Delta \rtimes \mathbb{Z}_{2k}$ is generated by the spatial symmetries Δ of the underlying relative periodic solution $\mathcal{P}$ together with an element Q of order $2k$. Note that $\mathcal{P}$ reduces to a periodic solution $\{0\} \times S^1 \subset V \times S^1$ with spatial symmetry Δ and spatiotemporal symmetry $\Delta \rtimes \mathbb{Z}_{2k}$.

In principle, it is possible to apply the results of [25] to the (v, θ)-subsystem. In practice, this is slightly confusing due to the presence of additional structure in the (v, θ)-subsystem. For example, the phase space $V \times S^1$ is a trivial bundle over the periodic solution which was not assumed in [25]. Moreover, an important construction in [25] is a group $\Delta \rtimes \mathbb{Z}_{2k}$ which coincides here (but not in [25]) with the spatiotemporal symmetry in the (v, θ)-subsystem. For these reasons, we repeat certain calculations from [25] in the following exposition.

Some notational confusion arises from the fact that $\Delta \rtimes \mathbb{Z}_{2k}$ acts both on $V \times S^1$ and on the cross-section $V \cong V \times \{0\} \subset V \times S^1$. To avoid confusion, we now use Q to denote the action of Q on $V \times S^1$ and Q_V to denote the action of Q on V. Thus $Q \cdot (v, \theta) = (Q_V v, \theta + 1)$. (Note that Q here corresponds to σ in [25] and that there is no analogue of Q_V in [25].)

Following the notation of [25] (with σ replaced by Q), we let $g^{(1)} : V \to QV$ be the first hit map for the flow in the (v, θ)-subsystem. (Alternatively, we could write $g^{(1)} : V \times \{0\} \to V \times \{1\}$, where $V \times \{0\}$, $V \times \{1\} \subset V \times S^1$ are successive cross-sections along the periodic solution.) Bifurcations in the (v, θ)-subsystem are governed by eigenvalues of the twisted equivariant linear map $Q^{-1}(\mathrm{D}g^{(1)})_0 : V \to V$ (cf. $L = \sigma^{-1}(\mathrm{D}g^{(1)})_0$ in [25]). In fact, this linear map coincides with the twisted equivariant linear map $L = P_0 \sigma^{-1}(\mathrm{D}\Phi_1)_{u_0}|_V$ from Section 2(b):

Proposition 3.1 $L = Q^{-1}(\mathrm{D}g^{(1)})_0$.

Proof. Write the time-one map $\Phi_1 : \{\mathrm{id}\} \times V \times \{0\} \to \{\mathrm{id}\} \times V \times \{1\}$ in the (γ, v, θ) coordinates

$$\Phi_1(\mathrm{id}, v, 0) = (\Phi_1^\gamma(\mathrm{id}, v, 0), \Phi_1^v(\mathrm{id}, v, 0), 1).$$

Using the identification $(\gamma\sigma^{-1}, v, \theta + 1) \sim (\gamma, Q_V^{-1} v, \theta)$, we compute that

$$\begin{aligned} \sigma^{-1}\Phi_1(\mathrm{id}, v, 0) &= (\sigma^{-1}\Phi_1^\gamma(\mathrm{id}, v, 0), \Phi_1^v(\mathrm{id}, v, 0), 1), \\ &= (\mathrm{Ad}_{\sigma^{-1}}\Phi_1^\gamma(\mathrm{id}, v, 0)\sigma^{-1}, \Phi_1^v(\mathrm{id}, v, 0), 1), \\ &= (\mathrm{Ad}_{\sigma^{-1}}\Phi_1^\gamma(\mathrm{id}, v, 0), Q_V^{-1}\Phi_1^v(\mathrm{id}, v, 0), 0). \end{aligned}$$

It follows that

$$L = P_0\sigma^{-1}(\mathrm{D}\Phi_1)_{(\mathrm{id},0,0)}|_V = Q_V^{-1}(\mathrm{D}_v\Phi_1^v)_{(\mathrm{id},0,0)}.$$

On the other hand,

$$Q^{-1}g^{(1)}(v, 0) = Q^{-1}(\Phi_1^v(\mathrm{id}, v, 0), 1) = (Q_V^{-1}\Phi_1^v(\mathrm{id}, v, 0), 0)$$

so that $Q^{-1}(\mathrm{D}g^{(1)})_{0,0} = Q_V^{-1}(\mathrm{D}_v\Phi_1^v)_{(\mathrm{id},0,0)} = L$. ∎

By center manifold reduction, we may suppose without loss that V is the center subspace of L. Define Δ_L to be the closed group generated by the actions of Δ and L on V.

Theorem 3.2 ([25, Theorem 3.4]) *Suppose that the periodic solution in the (v,θ)-subsystem undergoes bifurcation. Generically, the center subspace V of L is an irreducible representation of Δ_L. Moreover, either*

NonHopf: *V is absolutely irreducible, or*

Hopf: *V is irreducible of complex type.*

Remark 3.3 We recall that if V is a real irreducible representation of a group G, then the space of G-equivariant linear maps is a real division ring and hence is isomorphic to the reals $\mathbb{R}$, complexes $\mathbb{C}$ or quaternions $\mathbb{H}$. Such representations are said to be irreducible of real, complex, or quaternionic type respectively. In alternative terminology, irreducible representations of real type are said to be absolutely irreducible.

We now concentrate on the nonHopf case, leaving the Hopf case until Section 4.

Proposition 3.4 *Suppose that Δ_L acts absolutely irreducibly on V. Then $L = Q_V^{-1}$ and $\Delta_L \cong \Delta \rtimes \mathbb{Z}_{2k}$.*

Proof. Observe that L and Q_V^{-1} are twisted equivariant so that $Q_V L : V \to V$ is an equivariant linear map. Therefore, $Q_V L = \alpha I$ where $\alpha \in \mathbb{R}$. It follows that Q_V and L commute and hence $\alpha^{2k} I = (Q_V L)^{2k} = Q_V^{2k} L^{2k} = L^{2k}$. Since V is the center subspace of L, it follows that $\alpha = \pm 1$. Finally, $QL = (\mathrm{D}g^{(1)})_0$ which is equivariantly isotopic (by the flow) to the identity on V, ruling out the possibility that $\alpha = -1$. ∎

Next, we define a Δ-equivariant diffeomorphism $h : V \to V$ by writing $g^{(1)}(v, 0) = (h(v), 1)$.

Proposition 3.5 (cf. [25, Lemma 4.4]) *Up to arbitrarily high order, coordinates can be chosen so that h is Δ_L-equivariant. Moreover, h can be regarded, to arbitrarily high order, as a general Δ_L-equivariant diffeomorphism satisfying $h(0) = 0$ and $(\mathrm{D}h)_0 = I$.*

Proof. We have already observed that h is Δ-equivariant (to all orders). Hence $f = Lh : V \to V$ is twisted equivariant. Moreover, L is semisimple (indeed $L^{2k} = I$). Hence it follows from Birkhoff normal form theory for twisted equivariant maps [24] that f can be transformed (by changes of coordinates that preserve twisted equivariance) so that f is L-equivariant up to any specified order in its Taylor expansion. In particular, $h = L^{-1} f$ is Δ_L-equivariant up to any specified order.

It is clear that $h(0) = 0$, and the definition $L = Q^{-1}(\mathrm{D}g^{(1)})_0$ reduces to $L = Q_V^{-1}(\mathrm{D}h)_0$ so that $(\mathrm{D}h)_0 = I$ by Proposition 3.4. ∎

The bifurcation theory for such a diffeomorphism h is given in [9]. Tracing back through the definitions as in [25] yields the bifurcation theory for the isolated periodic solution in the (v,θ)-subsystem.

The three step algorithm in the introduction now becomes a five step algorithm:

1(i). Enumerate the absolutely irreducible representations of $\Delta_L \cong \Delta \rtimes \mathbb{Z}_{2k}$.

1(ii). Study the bifurcations of a Δ_L-equivariant diffeomorphism $h : V \to V$ satisfying $h(0) = 0$ and $(dh)_0 = I$.

1(iii). Read off the results for bifurcation from the isolated periodic solution using the identifications in [25].

2. Solve for $\gamma(t)$ in the $\dot{\gamma}$-equation in (1.2).

3. Interpret the results for bifurcation from the relative periodic solution $\mathcal{P}$ in the original ODE (1.1).

By concentrating on specific aspects of the bifurcation theory for h in Step 1(ii), we can state a general result about solutions bifurcating from $\mathcal{P}$ in Step 3. In particular, if $J \subset \Delta \rtimes \mathbb{Z}_{2k}$ is an axial isotropy subgroup (that is, the fixed-point subspace of J is one dimensional), then the equivariant branching lemma [18, 9] guarantees the existence of branches of fixed points with isotropy J for the diffeomorphism h. Concentrating attention on the axial isotropy subgroups, we have the following result for bifurcation from relative periodic solutions.

Proposition 3.6 *Let $\mathcal{P}$ be a relative periodic solution for the Γ-equivariant ODE (1.1) on M, with spatial symmetry Δ and spatiotemporal symmetry Σ generated by Δ and σ.*

Suppose that there is a nondegenerate nonHopf bifurcation in the $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant subsystem. In particular $\Delta_L \cong \Delta \rtimes \mathbb{Z}_{2k}$ acts absolutely irreducibly on E^c. Suppose that J is an axial isotropy subgroup of Δ_L and let $p \geq 1$ be least such that $L^{-p}\delta \in J$ for some $\delta \in \Delta$.

Then there is a branch of relative periodic solutions $\mathcal{P}^{\mathrm{bif}}$ for the ODE (1.1) on M with relative period p, spatial symmetry $\Delta^{\mathrm{bif}} = J \cap \Delta$, and spatiotemporal symmetry Σ^{bif} generated by Δ^{bif} and $\gamma\alpha^p\delta$, where γ is an element of $Z(\Delta^{\mathrm{bif}})$ close to $\exp p\xi$.

Proof. Since J is axial, there is a branch of fixed points with isotropy J for the diffeomorphism h. By [25, Lemma 4.7], there is a periodic solution $y(t)$ for the (v, θ)-subsystem with spatial symmetry $J \cap \Delta$ and satisfying $y(p) = Q^p\delta y(0)$, where $p \geq 1$, $\delta \in \Delta$, and p is least. It follows from Proposition 1.4 that there is a relative periodic solution $u(t)$ for the underlying ODE (1.1) with spatial symmetry $J \cap \Delta$ and relative period p. Moreover, by Proposition 1.4, $u(p) = \gamma(p)\alpha^p\delta u(0)$, where $\gamma(t)$ is the solution to the $\dot{\gamma}$ equation in the skew product equations (1.2) with initial condition $\gamma(0) = \mathrm{id}$.

The solution $y(t)$ depends smoothly at least on $\sqrt{\lambda}$ (where λ is the bifurcation parameter), so that solutions $\gamma(t, \lambda)$ to the equation

$$\dot{\gamma} = \gamma f_\Gamma(y(t, \lambda), t, \lambda)$$

depend smoothly on $\sqrt{\lambda}$. Since $f_\Gamma(0, t, 0) \equiv \xi$, it follows that $\gamma(t, 0) = \exp t\xi$ and hence $\gamma(p, \lambda)$ is close to $\exp p\xi$.

The condition $f_\Gamma(\delta v, \theta) = \mathrm{Ad}_\delta f_\Gamma(v, \theta)$ implies that $f_\Gamma(y(t, \lambda), t, \lambda) \in \mathrm{L}Z(\Delta^{\mathrm{bif}})$ for all $t, \lambda \in \mathbb{R}$. Hence $\gamma(t, \lambda) \in Z(\Delta^{\mathrm{bif}})$. ∎

Remark 3.7 (a) For each fixed value of the bifurcation parameter λ, the element γ in Proposition 3.6 can be viewed as a *general* element of $Z(\Delta^{\mathrm{bif}})$ close to $\exp p\xi$. This is true for rather general reasons, see Field [14]. In this particular context, generality of the element γ is easily seen from the form of f_Γ.

In particular, if $Z(\Delta^{\mathrm{bif}})^0 = \mathbf{1}$, then $\dot{\gamma} \equiv 0$, whereas if $Z(\Delta^{\mathrm{bif}})^0 \neq \mathbf{1}$, then generically $\dot{\gamma} \not\equiv 0$.

(b) Although the element γ in Proposition 3.6 can be viewed as a general element of $Z(\Delta^{\mathrm{bif}})$ close to $\exp p\xi$, the dependence of γ on various parameters in the problem may be of significance for applications. This is particularly true of Hopf bifurcation, see Section 4. Even for nonHopf bifurcation, the dependence of γ on the bifurcation parameter λ has implications for the motion of the corresponding patterns in physical space (cf. [17]). See the examples below.

(c) In our examples, we concentrate attention on the existence of relative periodic solutions arising from axial isotropy subgroups $J \subset \Delta \rtimes \mathbb{Z}_{2k}$. More detailed bifurcation results (such as existence of relative periodic solutions corresponding to nonaxial isotropy subgroup J, stabilities, and so on) follow similarly from a more detailed analysis of the associated bifurcation for the diffeomorphism h.

Period preserving and period doubling bifurcations In classical bifurcation theory from periodic solutions, nonHopf bifurcations may be *period preserving* (where the period of the bifurcating periodic solutions are approximately equal to the period of the underlying periodic solution), and *period doubling* (where the period of the bifurcating periodic solutions are approximately twice the period of the underlying periodic solution).

This dichotomy extends to nonHopf bifurcation from isolated periodic solutions in systems with symmetry, as shown in [25]. More precisely, if we restrict attention to periodic solutions corresponding to fixed points of the diffeomorphism h, then generically, either all of the bifurcating periodic solutions have approximately the same absolute period as the underlying periodic solution, or all of the bifurcating periodic solutions have approximately twice the absolute period of the underlying periodic solution. (The relative period of the bifurcating solutions may in general be approximately any multiple of the period of the underlying periodic solution.)

The dichotomy of period preserving/doubling extends also to bifurcation from relative periodic solutions under certain circumstances. First suppose that the underlying relative periodic solution is composed of periodic solutions which then have a well-defined absolute period m. Indeed, m is the least positive integer such that $\Phi_m(u_0) = u_0$. Equivalently, $\sigma^m u_0 = u_0$, so m is the least positive integer such that $\sigma^m \in \Delta$.

Lemma 3.8 *Let $\mathcal{P}$ be composed of periodic solutions of absolute period m, and suppose that $\mathcal{P}$ undergoes generic nonHopf bifurcation. Define the diffeomorphism $h : V \to V$ and suppose that $h(v) = v$ for some $v \in V$. Let ψ_t be the flow on $(\Gamma \times V \times \mathbb{R})/(\Delta \rtimes \mathbb{Z}_{2n})$. Then $\psi_m(e, v, 0) = (\gamma, \pm v, 0)$ where γ is near identity.*

Proof. We have

$$\psi_m(e, v, 0) = (\gamma, h^m(v), m) = (\gamma, v, m) \sim (\gamma\sigma^m, Q_V^{-m} v, 0) \sim (\gamma, \sigma^m Q_V^{-m} v, 0),$$

the last equality following from the fact that $\sigma^m \in \Delta$. A computation shows that $\sigma^m Q_V^{-m} : V \to V$ is Δ_L-equivariant, so it follows from absolute irreducibility that $\sigma^m Q_V^{-m} = \pm I$. ∎

The near-identity element γ corresponds to slow drifts along continuous group orbits. Hence, it follows from Lemma 3.8 that we have period preserving or period doubling, modulo slow drifts.

Next, suppose that the underlying relative periodic solution $\mathcal{P}$ is not a periodic solution. Passing to a comoving frame as in Theorem 1.2, the relative periodic solution becomes periodic, and Lemma 3.8 is valid in the comoving frame. As defined, the notion of period preserving/doubling is dependent on the choice of comoving frame. However, in the examples considered in this section (in Subsections (b) and (c)), there are natural choices and period preserving/doubling is well-defined with respect to these natural choices.

In the remainder of this section, we consider examples of nonHopf bifurcation from discrete rotating waves (Subsection (a)), modulated rotating waves (Subsection (b)), and modulated traveling waves (Subsection (c)).

(a) NonHopf Bifurcation from discrete rotating waves

In this subsection, we consider examples of nonHopf bifurcation from discrete rotating waves. For the purposes of this paper, a discrete rotating wave [11] is a relative periodic solution that is foliated by periodic solutions and which remains so under perturbation. That is, there are no drifts along continuous group orbits in Γ. In contrast to the setting in [25] (*isolated* discrete rotating waves), we do not require that $\dim \Gamma = \dim \Delta$.

In Examples 3.9 and 3.10, we show how to apply directly the theoretical results of [25] described above. In Example 3.11, we show how to use the examples detailed in [25].

Since we restrict to bifurcations from discrete rotating waves in this subsection, we have $\alpha = \sigma$ and $\xi = 0$ in Proposition 3.6.

Example 3.9 Let $\Gamma = \mathbf{O}(2)$, $\Sigma = \mathbb{D}_2$ and $\Delta = \mathbb{D}_1$. Discrete rotating waves with this symmetry arise, for example, as standing wave solutions in Hopf bifurcation from an $\mathbf{O}(2)$-invariant equilibrium, see [18].

Write $\mathbb{D}_2 = \mathbb{D}_2(R_\pi, \kappa)$ and choose $\mathbb{D}_1 = \mathbb{D}_1(\kappa)$, $\sigma = R_\pi$. We have $k = 1$, $n = 2$, and $\mathcal{P}$ is a discrete rotating wave of relative period 1 and absolute period 2.

Since $\Delta \rtimes \mathbb{Z}_{2k} = \mathbb{D}_1 \times \mathbb{Z}_2$ is abelian, the absolutely irreducible representations V are one dimensional. Let V_+ denote the trivial irreducible representation of $\mathbb{D}_1$ and let V_- denote the nontrivial irreducible representation of $\mathbb{D}_1$. There are four cases to consider: $V = V_\pm$, $L = \pm 1$. Our results are tabulated in Table 2.

Space	L	Δ^{bif}	p	$\sigma^p\delta$	$Z(\Delta^{\mathrm{bif}})^0$	Σ^{bif}	Remarks
V_+	1	$\mathbb{D}_1(\kappa)$	1	R_π	$\mathbf{1}$	$\mathbb{D}_2(R_\pi,\kappa)$	saddle-node
V_+	-1	$\mathbb{D}_1(\kappa)$	2	id	$\mathbf{1}$	$\mathbb{D}_1(\kappa)$	
V_-	1	$\mathbf{1}$	1	R_π	$\mathbf{SO}(2)$	$\mathbf{SO}(2)$	MRW
V_-	-1	$\mathbf{1}$	1	$R_\pi\kappa$	$\mathbf{SO}(2)$	$\mathbb{D}_1(R_{\pi+\varphi}\kappa)$	

Table 2: Spatiotemporal symmetry of bifurcating solutions in nonHopf bifurcation from a discrete rotating wave with $\Gamma = \mathbf{O}(2)$, $\Sigma = \mathbb{D}_2$, $\Delta = \mathbb{D}_1$. All bifurcations are period preserving. All bifurcations are pitchforks of discrete rotating waves unless stated otherwise. The angle φ in the case V_-, $L = -1$ is a general angle close to 0. Notation: MRW = modulated rotating wave

Verification of the entries in Table 2 Since $\dim V = 1$, there is a unique axial subgroup $J \subset \Delta \rtimes \mathbb{Z}_{2k}$, namely the subgroup of $\mathbb{D}_1 \times \mathbb{Z}_2$ that acts trivially on V. The subgroups J together with the data p and $L^{-p}\delta$ in the (v,θ)-subsystem are listed in Table 3. Applying Proposition 3.6, it is now straightforward to verify the entries in Table 2. Note that in the V_+ cases, we have $Z(\Delta^{\mathrm{bif}})^0 = \mathbf{1}$ so that $\dot\gamma \equiv 0$. Hence the bifurcating solutions are discrete rotating waves (periodic solutions with no drift).

In the V_- cases, we have $Z(\Delta^{\mathrm{bif}})^0 = \mathbf{SO}(2)$. When $L = 1$, Σ^{bif} is generated by $R_{\pi+\phi}$ where ϕ is a general, typically irrational, angle close to 0. Generically, we have two-frequency quasiperiodic dynamics. Moreover, the solution reduces to a periodic solution in a corotating frame so this is a modulated rotating wave.

Finally, in the case V_-, $L = -1$, Σ^{bif} is generated by $R_{\pi+\phi}\kappa$ which is a reflection. In particular, there is no net drift, even though $\dot\gamma \not\equiv 0$, and we obtain a discrete rotating wave. Observe that the angle ϕ, and hence the axis of reflection, depends on the bifurcation parameter λ (with $\sqrt{\lambda}$ dependence).

Next, we verify that the bifurcations are all period preserving. We have seen that the underlying relative periodic solutions have absolute period $m = 2$. Applying Lemma 3.8, the bifurcating solutions have absolute period 2 or 4.

Observe that p is the relative period of the bifurcating solution. Let $q \geq 1$ be least such that $(\sigma^p\delta)^q \in \Delta^{\rm bif}$. Then the absolute period modulo slow drifts is given by pq. It is easy to compute that $pq = 2 = m$ in all cases, so that the bifurcations are period preserving.

Space	L	J	p	$L^{-p}\delta$
V_+	1	$J = \mathbb{D}_1(\kappa) \times \mathbb{Z}_2(L)$	1	L^{-1}
V_+	-1	$J = \mathbb{D}_1(\kappa)$	2	id
V_-	1	$J = \mathbb{Z}_2(L)$	1	L^{-1}
V_-	-1	$J = \mathbb{Z}_2(L\kappa)$	1	$L^{-1}\kappa$

Table 3: Data for $\Gamma = \mathbf{O}(2)$, $\Sigma = \mathbb{D}_2$, $\Delta = \mathbb{D}_1$ in the (v,θ)-subsystem

Patterns in physical space Next, we consider motions along continuous $\mathbf{SO}(2)$ group orbits and the consequences for patterns in physical space.

Write the group variable $\gamma \in \Gamma$ as $\varphi \in \mathbf{SO}(2)$. Since $\mathbf{SO}(2)$ is abelian, the skew product equations reduce to

$$\dot{\varphi} = f_\varphi(v,\theta,\lambda), \quad \dot{v} = f_v(v,\theta,\lambda), \quad \dot{\theta} = 1,$$

where $f_\varphi(0,\theta,0) = 0$.

As mentioned above, $\dot{\varphi} \equiv 0$ in the V_+ cases. Hence we focus on the V_- cases. First, suppose that $L = -1$. We have

$$f_\varphi(-v,\theta) = -f_\varphi(v,\theta), \qquad f_\varphi(v,\theta+1) = -f_\varphi(v,\theta).$$

In the (v,θ)-subsystem, we have a pitchfork bifurcation of periodic solutions $v(t,\lambda)$ with $\sqrt{\lambda}$ dependence on the bifurcation parameter. The spatiotemporal symmetry in the (v,θ)-subsystem is given by $v(t+1,\lambda) = L^{-1}\kappa v(t,\lambda) = v(t,\lambda)$. The $\dot{\varphi}$-equation becomes

$$\dot{\varphi} = f_\varphi(v(t,\lambda),t,\lambda) = g(t,\lambda),$$

where $g(t+1,\lambda) = -g(t,\lambda)$. In particular, g and hence φ can be written as 2-periodic Fourier series with only odd coefficients. (In particular, there is no constant term in the Fourier series of g and so no mean drift.) On the other hand, generically $\varphi(t,\lambda) = \sqrt{\lambda}\psi(t,\lambda) + O(\lambda)$, where $\psi(t,\lambda)$ is 2-periodic.

The interpretation for patterns in physical space (in the spirit of [17]) is as follows. Prior to bifurcation, there is a large-scale 2-periodic solution with spatial reflection symmetry κ (and spatiotemporal symmetry R_π). In particular, the periodic solution possesses a time-independent axis of reflection symmetry.

After bifurcation, superimposed on the large scale 2-periodic behavior is the 2-periodic rigid rotation $\varphi(t)$ and the 1-periodic 'shape change' $v(t)$. Suppose that we ignore the effects of the shape change. Then the reflection axis of the large-scale periodic solution slowly rotates rigidly first in one direction and then in the reverse direction. This is analogous to the "reversing wave" bifurcation studied by [28] (see also [32] where such states are called "pulsating waves", and [17]). As argued in [17], the effect of the shape change is relatively insignificant near bifurcation. The behavior observed in physical space is approximately as described — approximately rigid slow rotation of an approximate axis of symmetry, with the speed of rotation varying periodically with mean speed zero.

There are two significant differences between the situation that we have just described and the situations in [17, 28]. The first difference is that it is not the entire pattern in physical space that is seen to approximately rigidly rotate, only the approximate axis of reflection (since the underlying large scale periodic behavior swamps the fine details of the bifurcation). The second difference is that the rigid rotation $\varphi(t)$ possesses all odd harmonics at order $\sqrt{\lambda}$, in contrast to the simple harmonic oscillation at order $\sqrt{\lambda}$ for the reversing waves [28].

In the case V_-, $L = 1$, the function f_φ is odd in v as before, but is now 1-periodic in θ. Again, we compute from Proposition 3.6 that $v(t, \lambda)$ is 1-periodic: $v(t+1, \lambda) = L^{-1}v(t, \lambda) = v(t, \lambda)$. There are no further restrictions, so the $\dot\varphi$ equation has the form $\dot\varphi = g(t, \lambda)$ where $g(t, \lambda)$ is 1-periodic. Hence the approximate reflection symmetry of the bifurcating modulated rotating wave appears to undergo slow rigid rotation with nonuniform speed and with nonzero mean speed.

Example 3.10 Let $\Gamma = \mathbf{E}(2)$, $\Sigma = \mathbb{D}_2$ and $\Delta = \mathbb{D}_1$. Again, we have $k = 1$, $n = 2$, and $\mathcal{P}$ is a discrete rotating wave of relative period 1 and absolute period 2. The results for nonHopf bifurcation from $\mathcal{P}$ are given in Table 4.

Space	L	Δ^{bif}	p	$\sigma^p\delta$	$Z(\Delta^{\mathrm{bif}})^0$	Σ^{bif}	Remarks
V_+	$+1$	$\mathbb{D}_1(\kappa)$	1	R_π	$\mathbb{R}$	$\mathbb{D}_2$	saddle-node
V_+	-1	$\mathbb{D}_1(\kappa)$	2	id	$\mathbb{R}$	$\mathbb{D}_1 \times \mathbb{Z}$	MTW
V_-	1	$\mathbf{1}$	1	R_π	$\mathbf{SE}(2)$	$\mathbf{SO}(2)$	MRW
V_-	-1	$\mathbf{1}$	1	$R_\pi\kappa$	$\mathbf{SE}(2)$	$\mathbb{Z}$	MTW

Table 4: Spatiotemporal symmetry of bifurcating solutions in nonHopf bifurcation from a discrete rotating wave with $\Gamma = \mathbf{E}(2)$, $\Sigma = \mathbb{D}_2$, $\Delta = \mathbb{D}_1$. The entries for Σ^{bif} are given only up to conjugacy. All bifurcations are period preserving. All bifurcations are pitchforks of discrete rotating waves unless stated otherwise. Notation: MRW = modulated rotating wave, MTW = modulated traveling wave

The verification of the entries in Table 4 is similar to that for Table 2. In particular, the entries in Table 3 are unchanged. The main difference is that we have $Z(\Delta^{\mathrm{bif}})^0 = \mathbb{R}$ in the V_+ cases and $Z(\Delta^{\mathrm{bif}})^0 = \mathbf{SE}(2)$ in the V_- cases.

It is no longer convenient to tabulate the generators of Σ^{bif} in Table 4, but these are easily deduced from the other entries in the table. In particular, by Proposition 3.6, $\Sigma^{\mathrm{bif}}/\Delta^{\mathrm{bif}}$ is generated by an element of the form $\gamma\sigma^p\delta$ where γ is an element of $Z(\Delta^{\mathrm{bif}})^0$ close to $\exp p\xi$. In the case V_+, $L = 1$, $\Sigma^{\mathrm{bif}}/\Delta^{\mathrm{bif}}$ is generated by a 2-fold rotation about a point slightly displaced from the center of rotation for R_π, but still lying on the κ reflection axis. In the case V_+, $L = -1$, the spatiotemporal symmetry is generated by a translation parallel to the κ reflection axis. In the case V_-, $L = 1$, we have rotation by an irrational angle close to π about a slightly displaced origin. In the case V_-, $L = -1$, we have glide reflection with axis approximately perpendicular to the κ reflection axis.

Note that in the cases $L = -1$, the spatiotemporal symmetry is generated by a translation and a glide reflection respectively, so that we obtain modulated traveling waves. These reduce to periodic solutions in a cotraveling frame.

Finally, we consider interpretations for physical space. In the case V_-, $L = 1$, the approximate axis of reflection of the bifurcating modulated rotating wave slowly rigidly rotates with nonuniform speed as before, but in addition, the center of rotation varies periodically. In the case V_-, $L = -1$, the approximate axis of reflection slowly propagates in a direction approximately normal to itself, but the direction of propagation rigidly rotates with nonuniform speed (and zero mean speed).

Example 3.11 We now generalize Example 3.9 to the case $\Gamma = \mathbf{O}(2)$, $\Sigma = \mathbb{D}_{2\ell}$, $\Delta = \mathbb{D}_\ell$ where ℓ is odd. Since ℓ is odd, we can choose $\sigma = R_\pi$ so that $k = 1$, $n = 2$.

We assume familiarity with the notation and results of [25, Example 5.1] where the only difference is that $\Gamma = \mathbb{D}_{2\ell}$. Recall that $V_{0,\pm}$ denotes the one dimensional irreducible representations of $\mathbb{D}_{2\ell}$, and V_j, $1 \le j < \ell/2$ denotes the two dimensional irreducible representations. Our results are almost identical to those given in [25, Table 1] and are tabulated in Table 5. We note that the first five columns and most of the eighth column can be read off directly from [25, Table 1]. The remaining entries are straightforward.

Space	L	$\Delta^{\rm bif}$	p	$\sigma^p\delta$	$Z(\Delta^{\rm bif})^0$	$\Sigma^{\rm bif}$	Remarks
$V_{0,+}$	1	$\mathbb{D}_\ell(R_{2\pi/\ell},\kappa)$	1	R_π	$\mathbf{1}$	$\mathbb{D}_{2\ell}$	saddle-node
$V_{0,+}$	-1	$\mathbb{D}_\ell(R_{2\pi/\ell},\kappa)$	2	id	$\mathbf{1}$	$\mathbb{D}_\ell$	
$V_{0,-}$	1	$\mathbb{Z}_\ell(R_{2\pi/\ell})$	1	R_π	$\mathbf{SO}(2)$	$\mathbf{SO}(2)$	MRW
$V_{0,-}$	-1	$\mathbb{Z}_\ell(R_{2\pi/\ell})$	1	$R_\pi\kappa$	$\mathbf{SO}(2)$	$\mathbb{D}_\ell$	
V_j $1 \le j < \ell/2$	I	$\mathbb{D}_{j'}(R_{2\pi/j'},\kappa)$	1	R_π	$\mathbf{1}$	$\mathbb{D}_{2j'}$	transcritical if $\ell/j' = 3$
V_j $1 \le j < \ell/2$	$-I$	$\mathbb{D}_{j'}(R_{2\pi/j'},\kappa)$	2	id	$\mathbf{1}$	$\mathbb{D}_{j'}$	
		$\mathbb{Z}_{j'}(R_{2\pi/j'})$	1	$R_\pi\kappa$	$\mathbf{SO}(2)$	$\mathbb{D}_{j'}$	

Table 5: Spatiotemporal symmetry of bifurcating solutions in nonHopf bifurcation from a discrete rotating wave with $\Gamma = \mathbf{O}(2)$, $\Sigma = \mathbb{D}_{2\ell}$, $\Delta = \mathbb{D}_\ell$, where ℓ is odd. The entries for $\Sigma^{\rm bif}$ are given only up to conjugacy. All bifurcations are period preserving. All bifurcations are pitchforks of discrete rotating waves unless stated otherwise. Notation: MRW = modulated rotating wave, $j' = \gcd(j, \ell)$.

The interpretations in physical space for the cases $V_{0,\pm}$ are identical to those in Example 3.9. The V_j representations, $1 \le j < \ell/2$ lead only to discrete rotating waves due to the presence of reflections in the spatial or spatiotemporal symmetries. However, in the case V_j, $L = -1$, there are two branches of bifurcating relative periodic solutions, and one of these has no purely spatial reflection symmetry. For this branch of solutions, the constant term of the $\dot\varphi$ equation vanishes, but $\dot\varphi \not\equiv 0$. In physical space, the bifurcating solutions have ℓ approximate axes of reflection symmetry and these axes slowly rigidly rotate with nonuniform speed and zero mean speed.

Remark 3.12 Lamb and Melbourne [25] consider in detail the possible nonHopf bifurcations in the cases $\Gamma = \Sigma = \mathbb{D}_{2\ell}$, $\Delta = \mathbb{D}_\ell$ (ℓ odd or even) and $\Gamma = \Sigma = \mathbb{D}_\ell$, $\Delta = \mathbb{Z}_\ell$. Proceeding as in Example 3.11, we can read off the corresponding results when Γ is replaced by $\mathbf{O}(2)$ or $\mathbf{E}(2)$. We omit the details.

Further examples of nonHopf bifurcation from nonisolated discrete rotating waves can be found in Rucklidge and Silber [32].

(b) NonHopf Bifurcation from modulated rotating waves

A modulated rotating wave is a relative periodic solution that is periodic in a corotating frame. In this subsection, we consider nonHopf bifurcation from modulated rotating waves. The main difference from the analysis in Subsection (a) is that in general $\alpha \neq \sigma$ and $\xi \neq 0$.

Example 3.13 We consider nonHopf bifurcation from a modulated rotating wave with $\Gamma = \mathbf{SO}(2)$, $\Sigma = \mathbf{SO}(2)$ and $\Delta = \mathbb{Z}_\ell$, where $\ell \geq 1$. Such modulated rotating waves arise in a variety of different forms in cellular flame experiments, see [19, 6].

It is immediate that $k = n = 1$. Moreover, the element σ (which is generically an irrational rotation) can be written as $\sigma = \exp \xi$ for some $\xi \in \mathbf{LSO}(2)$. In particular, $\alpha = \mathrm{id}$.

Passing to the skew product equations (1.2) in the corotating frame rotating with speed ξ, the modulated rotating wave becomes an ordinary periodic solution with spatial symmetry $\Delta = \mathbb{Z}_\ell$ and no further spatiotemporal symmetry.

Since $\Delta \rtimes \mathbb{Z}_{2k} = \mathbb{Z}_\ell \times \mathbb{Z}_2$ is abelian, the absolutely irreducible representations V are one dimensional. Let V_+ denote the trivial irreducible representation of $\Delta = \mathbb{Z}_\ell$. When ℓ is even, there is also a nontrivial irreducible representation V_- with kernel $\mathbb{Z}_{\ell/2}$.

Again, there is a unique axial isotropy subgroup $J \subset \Delta \rtimes \mathbb{Z}_{2k}$. Proceeding as in the examples of the previous subsection, we arrive at Table 6.

Space	L	Δ^{bif}	p	$\alpha^p\delta$	$Z(\Delta^{\mathrm{bif}})^0$	Σ^{bif}	Remarks
V_+	1	$\mathbb{Z}_\ell$	1	id	$\mathbf{SO}(2)$	$\mathbf{SO}(2)$	saddle-node
V_+	-1	$\mathbb{Z}_\ell$	2	id	$\mathbf{SO}(2)$	$\mathbf{SO}(2)$	period doubling
V_- (ℓ even)	1	$\mathbb{Z}_{\ell/2}$	1	id	$\mathbf{SO}(2)$	$\mathbf{SO}(2)$	
V_- (ℓ even)	-1	$\mathbb{Z}_{\ell/2}$	1	$R_{2\pi/\ell}$	$\mathbf{SO}(2)$	$\mathbf{SO}(2)$	period doubling

Table 6: Spatiotemporal symmetry of bifurcating solutions in nonHopf bifurcation from a modulated rotating wave with $\Gamma = \mathbf{O}(2)$, $\Sigma = \mathbf{SO}(2)$, $\Delta = \mathbb{Z}_\ell$, $\ell \geq 1$. All bifurcations are to modulated rotating waves. All bifurcations are period preserving pitchforks bifurcations unless stated otherwise. Period preserving and period doubling refers to periodicity in the corotating frame (ignoring new slow drifts).

As mentioned after Lemma 3.8, the notion of period preserving/doubling depends *a priori* on the choice of comoving frame. However, in this example, there is a natural choice of σ, namely $\sigma = \exp \xi$ with $\alpha = \mathrm{id}$. In the corresponding corotating frame, the underlying modulated rotating wave reduces to a periodic solution with a well-defined absolute period (in this case, absolute period 1).

Similarly, the bifurcating relative periodic solutions have well-defined absolute periods in the corotating frame modulo new drifts on $Z(\Delta^{\mathrm{bif}})^0$. As in Example 3.9, we define $q \geq 1$ to be the least integer such that $(\alpha^p\delta)^q \in \Delta^{\mathrm{bif}}$. Then the absolute period is given by pq. Note that the absolute period pq depends only on the choice of σ. Since we have a natural choice of σ in this example (and also in Examples 3.14 and 3.15 below), we may speak of period preserving and period doubling bifurcations.

We caution that even with the natural choice of σ, we are not claiming that there is a 'natural' comoving frame (since the equation $\exp \xi = \sigma$ does not determine ξ uniquely).

Example 3.14 We consider a modulated rotating wave with $\Gamma = \mathbf{SE}(2)$, $\Sigma = \mathbf{SO}(2)$ and $\Delta = \mathbb{Z}_\ell$, where $\ell \geq 1$. Such modulated rotating waves arise as "meandering spiral waves" in chemical reactions [35].

The analysis of nonHopf bifurcation is almost identical to the the analysis for $\Gamma = \mathbf{O}(2)$. In particular, we have $k = n = 1$, $\alpha = \mathrm{id}$, $\Delta \rtimes \mathbb{Z}_{2k} = \mathbb{Z}_\ell \times \mathbb{Z}_2$ as before. The entries in Table 6 are unchanged except that when $\Delta^{\mathrm{bif}} = 1$ we obtain $Z(\Delta^{\mathrm{bif}})^0 = \mathbf{SE}(2)$. This change occurs only when $\ell = 1$ and in the V_- cases when $\ell = 2$.

Under the assumption (valid generically) that $\xi \neq 0$, the only change in the conclusions when $Z(\Delta^{\mathrm{bif}})^0 = \mathbf{SE}(2)$ is that the center of rotation of the bifurcating modulated rotating waves varies periodically in time.

(c) NonHopf Bifurcation from modulated traveling waves

A modulated traveling wave is a relative periodic solution that is periodic in a cotraveling frame. In this subsection, we consider an example of nonHopf bifurcation from a modulated traveling wave.

Example 3.15 We consider nonHopf bifurcation from a modulated traveling wave with $\Gamma = \mathbf{SE}(2)$, $\Sigma = \mathbb{D}_1 \times \mathbb{Z}$ and $\Delta = \mathbb{D}_1$. Such modulated traveling waves arise as "bound pairs of spiral waves", see [10].

Observe that $Z(\Sigma)^0 \cong \mathbb{R}$. It is immediate that $k = n = 1$. Moreover, the element σ (which is generically a translation) can be written as $\sigma = \exp \xi$ for some $\xi \in \mathrm{L}\mathbb{R}$. In particular, $\alpha = \mathrm{id}$. Again, we pass to the skew product equations (1.2) in the cotraveling frame traveling with speed ξ. Our results are given in Table 7.

Space	L	Δ^{bif}	p	$\alpha^p\delta$	$Z(\Delta^{\mathrm{bif}})^0$	Σ^{bif}	Remarks
V_+	1	$\mathbb{D}_1(\kappa)$	1	id	$\mathbb{R}$	$\mathbb{D}_1 \times \mathbb{Z}$	saddle-node
V_+	-1	$\mathbb{D}_1(\kappa)$	2	id	$\mathbb{R}$	$\mathbb{D}_1 \times \mathbb{Z}$	period doubling
V_-	1	$\mathbf{1}$	1	id	$\mathbf{SE}(2)$	$\mathbf{SO}(2)$	MRW
V_-	-1	$\mathbf{1}$	1	κ	$\mathbf{SE}(2)$	$\mathbb{Z}$	period doubling

Table 7: Spatiotemporal symmetry of bifurcating solutions in nonHopf bifurcation from a modulated traveling wave with $\Gamma = \mathbf{SE}(2)$, $\Sigma = \mathbb{D}_1 \times \mathbb{Z}$, $\Delta = \mathbb{D}_1$. The entries for Σ^{bif} are given only up to conjugacy. All bifurcations are period preserving pitchforks of modulated traveling waves unless stated otherwise. Notation: MRW = modulated rotating wave

In the case V_-, $L = 1$, the approximate reflection axis of the bifurcating modulated rotating wave undergoes slow rigid rotation about a center of rotation far from the axis of reflection. In particular, the center of rotation 'comes in from infinity' in a manner similar to certain transitions studied in [5]. In the limit, as $\lambda \to 0$, we have finite translation drift parallel to the axis of reflection.

In the case V_-, $L = -1$, the approximate reflection axis of the bifurcating modulated traveling wave slowly rotates with zero mean speed. This is in contrast to the case V_-, $L = -1$ in Example 3.10, where the approximate reflection axis propagates slowly in normal directions.

4 Hopf bifurcation from relative periodic solutions

In this section, we show how to analyze Hopf bifurcation from a relative periodic solution $\mathcal{P}$ with spatial symmetry Δ and spatiotemporal symmetry Σ. Again, the theory follows in principle from [25], but it is convenient as in Section 3 to repeat certain of the arguments in [25] to take account of additional structure in the (v,θ)-subsystem.

As in the nonHopf case, we have the twisted equivariant linear map $L : V \to V$ given by $L = P_0\sigma^{-1}(\mathrm{D}\Phi_1)_{u_0}|_V = Q^{-1}(\mathrm{D}g^{(1)})_0$. We may suppose that V coincides with the center subspace of L. Since we are concerned with Hopf bifurcation, we may suppose, generically, that L^k has no negative eigenvalues. It then follows from the proof of Lemma 2.2 that the orthogonal map $Q : V \to V$ can be chosen to have order k. Hence, as described in Remark 2.5, we may assume from the outset that the skew product equations (1.2) are n-periodic and that the (v,θ)-subsystem is $\Delta \rtimes \mathbb{Z}_k$-equivariant and k-periodic.

Again, we define Δ_L to be the closed group generated by the actions of Δ and L on V. Applying Theorem 3.2, at Hopf bifurcation we have that V is an irreducible representation of Δ_L of complex type. In particular, $L^k = \mu I$, where $\mu \in \mathbb{C}$ lies in general position on the unit circle.

Proposition 4.1 *Suppose that μ is an irrational angle on the unit circle. Then, we can choose a k'th root $\mu^{1/k}$ so that $Q_V = \mu^{1/k}L^{-1}$. Moreover, $\Delta_L \cong (\Delta \rtimes \mathbb{Z}_k) \times S^1$, where $\Delta \rtimes \mathbb{Z}_k$ is generated by Δ and Q_V.*

Proof. Since μ is irrational, Δ_L contains a copy of S^1 generated by L^k. We note that real scalar multiples of elements of S^1 form a two-dimensional space of equivariant linear maps (since S^1 lies in the center of Δ_L). Since V is irreducible of complex type, the commuting linear maps coincide with the real scalar multiples of elements of S^1.

Now, $Q_V L$ is equivariant and V is irreducible of complex type, so we may write $Q_V L = \alpha I$ where $\alpha \in \mathbb{C}$. By the above discussion, α is a scalar multiple of an element of S^1 and hence commutes with L. It follows that Q_V and L commute, and that

$$\alpha^k I = (Q_V L)^k = Q_V^k L^k = \mu I.$$

Hence, $Q_V = \alpha L^{-1}$ where $\alpha^k = \mu$.

Finally, Δ_L is the closed group generated by L^k and $\mu^{-1/k}L$, and hence has the given structure. ■

Remark 4.2 (a) It follows from Proposition 4.1 that Q_V coincides with the linear map $L_0 = \mu^{-1/k}L$ in [25].

(b) To obtain the identification $Q_V = \mu^{1/k}L^{-1}$ in Proposition 4.1, the assumption that μ is irrational may be weakened to the assumption that $\mu \neq \pm 1$. For then L^k generates a copy of $\mathbb{Z}_q$ where $q \geq 3$, so that the commuting linear maps coincide with the real vector space spanned by elements of $\mathbb{Z}_q$.

As in Section 3, we define a Δ-equivariant diffeomorphism $h : V \to V$ by writing $g^{(1)}(v,0) = (h(v),1)$. Suppose that μ is irrational. By passing to Birkhoff normal form, we may suppose that to arbitrarily high order, h is a general $\Delta_L = (\Delta \rtimes \mathbb{Z}_k) \times S^1$-equivariant diffeomorphism satisfying $h(0) = 0$ and such that $(dh)_0$ has eigenvalues in general position on the unit circle.

For many purposes, nothing changes if h is $(\Delta \rtimes \mathbb{Z}_k) \times S^1$-equivariant to "sufficiently high" order, see [25]. By Remark 4.2(b) combined with [25, Theorem 6.4], this can be achieved by excluding finitely many values of μ (the so-called *strong resonances*). ¿From now on, we assume that μ is not strongly resonant.

The bifurcation theory for h is given in [9]. In particular, suppose that $J \subset (\Delta \rtimes \mathbb{Z}_k) \times S^1$ is a $\mathbb{C}$-axial isotropy subgroup (that is, the fixed-point subspace of J is two dimensional). By the equivariant Hopf theorem for maps [9], there exists a branch of invariant circles C with spatial symmetry J for the diffeomorphism h. This invariant circle C corresponds to an invariant torus T of the (v,θ)-subsystem of the skew product equations (1.2) (in the comoving frame) and hence to a relative invariant torus for the underlying ODE (1.1). Both the invariant torus T for the subsystem and the relative invariant torus for the underlying ODE have spatial symmetry $\Delta^{\rm bif} = J \cap \Delta$.

Finally, we consider the setwise symmetries Σ_T of the invariant torus T. Let $J_C \subset (\Delta \rtimes \mathbb{Z}_k) \times S^1$ be the symmetry which fixes the invariant circle as a set. Let $p \geq 1$ be least with $Q_V^p\delta \in J_C$ for some $\delta \in \Delta$. Then the symmetry group $\Sigma_T \subset (\Delta \rtimes \mathbb{Z}_k) \times S^1$ which fixes as a set the invariant torus T of the (v,θ)-system is generated by $J_C \cap \Delta$ together with $Q^p\delta$, see [25].

Example 4.3 As in Example 3.14, we consider a modulated rotating wave with $\Gamma = \mathbf{SE}(2)$, $\Sigma = \mathbf{SO}(2)$ and $\Delta = \mathbb{Z}_\ell$, $\ell \geq 1$. However, this time we analyze Hopf bifurcation from such a modulated rotating wave. The case $\ell = 1$ was analyzed previously by [34].

Again, we pass to a corotating frame with $\alpha = 1$. Note that $k = n = 1$, so that $(\Delta \rtimes \mathbb{Z}_k) \times S^1 = \mathbb{Z}_\ell \times S^1$. We concentrate on the faithful irreducible representations of $\mathbb{Z}_\ell$. These are two dimensional, and $\mathbb{Z}_\ell$ acts on $V \cong \mathbb{C}$ by

$$R_{2\pi/\ell} \cdot v = e^{2\pi i j/\ell} v,$$

for some $j = 0, 1, \ldots, [\ell/2]$.

Since V is two dimensional, there is a unique $\mathbb{C}$-axial isotropy subgroup $J \subset \mathbb{Z}_\ell \times S^1$ given by $J = \mathbb{Z}_\ell(R_{2\pi/\ell}, -2\pi j/\ell)$. By the equivariant Hopf theorem for maps, there is a branch of invariant circles C for the diffeomorphism h with spatial symmetry J and setwise symmetry $J_C = \mathbb{Z}_\ell \times S^1$. Moreover, $Q_V = 1$ (since $k = 1$) so that the corresponding invariant tori for the (v,θ)-subsystem have spatial symmetry $\Delta^{\rm bif} = \mathbb{Z}_{\gcd(\ell,j)}$ and setwise symmetry $\Sigma_T = \mathbb{Z}_\ell$.

Phaselocking in the (v,θ)-subsystem Before describing the dynamics in the (v,θ)-subsystem, we introduce some notation. Since $k = n = 1$, we have $Q_V = 1$ and the Poincaré map for the (v,θ)-subsystem is given by $h = g^{(1)} : V \to V$. The eigenvalues of $(dh)_0$ are complex conjugate eigenvalues on the unit circle and we write these eigenvalues as $e^{\pm 2\pi i\omega}$. Previously, we have normalized so that the underlying periodic solution at $v = 0$ has relative period 1. However, in this example it is instructive not to normalize, and we denote the relative period of this periodic solution by $T_{\rm per}$. (Note that $T_{\rm per}$ is also the absolute period of the underlying periodic solution in this example.)

The dynamics on the (v,θ)-subsystem is governed by $\mathbb{Z}_\ell$-equivariant Hopf bifurcation from a periodic solution. The relevant results for the case $\ell = 1$ can be found for example in Arnold [2] and in Arrowsmith and Place [3]. We assume familiarity with the case $\ell = 1$ (referring to [2, 3] for details), and concentrate instead on the differences that arise when $\ell > 1$.

Two parameters are required to describe the bifurcation: the bifurcation parameter λ and the frequency ω defined above. Parameter space is divided up according to the rotation number of the flow on the invariant torus. For each rational p/q (written in its lowest terms), there is an *Arnold tongue* $A_{p/q}$ in parameter space emanating from the point $(\lambda,\omega) = (0, p/q)$. Generically, the torus flow for $(\lambda,\omega) \in A_{p/q}$ is Morse-Smale and consists of periodic sinks, periodic sources and trajectories connecting these periodic solutions. The Arnold tongues are open and the corresponding dynamics is said to be *p/q-phaselocked*. The periodic solutions have period $T_{\rm bif}(\lambda)$, where $T_{\rm bif}(\lambda) \to qT_{\rm per}$ as $\lambda \to 0$.

The union of Arnold tongues $\cup A_{p/q}$ forms an open and dense subset of parameter space. The complement of the union $\cup A_{p/q}$ corresponds to torus flows with irrational rotation number. By Denjoy's Theorem, such flows are topologically conjugate to rigid irrational rotations. In fact, the complement of $\cup A_{p/q}$ has positive measure, converging to full measure as $\lambda \to 0$.

In systems without symmetry ($\ell = 1$), we can say more about the phaselocked flows. Consider an Arnold tongue $A_{p/q}$, where $q \geq 5$. (We do not consider the strongly resonant cases $q \leq 4$ here.) Generically in systems without symmetry, it is the case that for each $(\lambda, \omega) \in A_{p/q}$ (with λ close enough to 0), there is a single sink-source pair of periodic solutions on the invariant torus (see [3, Theorem 5.4.1]). We now formulate the corresponding result for $\mathbb{Z}_\ell$-equivariant systems.

Proposition 4.4 *Suppose that the $\mathbb{Z}_\ell$-equivariant (v, θ)-subsystem undergoes Hopf bifurcation from the periodic solution $v = 0$, where the action of $\mathbb{Z}_\ell$ on the critical eigenspace is given by $v \mapsto e^{2\pi ij/\ell}v$ for some $j = 0, 1, \ldots, [\ell/2]$. Define $\tilde{j} \geq 1$ to be least such that $j\tilde{j} \equiv \gcd(\ell, j) \bmod \ell$.*

Suppose that $(\lambda, \omega) \in A_{p/q}$ (λ small). Let $s = \gcd(\ell, q)$ and write $\ell = \ell' s$ and $q = q' s$. Suppose (to avoid strong resonance) that $\ell' q/\gcd(\ell', j) \geq 5$. Then generically there are precisely $\ell'/\gcd(\ell', j)$ sink-source pairs of periodic solutions (of absolute period $T_{\mathrm{bif}}(\lambda) \sim qT_{\mathrm{per}}$) on the invariant torus. Moreover, each periodic solution $y(t, \lambda)$ has spatial symmetry $\Delta^{\mathrm{bif}} = \mathbb{Z}_{\gcd(\ell,j)}$ and satisfies the spatiotemporal symmetry

$$y\left(\frac{q' \gcd(\ell, j)}{q} T_{\mathrm{bif}}(\lambda), \lambda\right) = R_{2\pi\tilde{j}p\ell'/\ell} \cdot y(0, \lambda).$$

Proof. First, we compute the number of sink-source pairs of periodic solutions. We sketch the proof, relying on familiarity with the corresponding arguments in the nonequivariant context $\ell = 1$ (again, we refer to [2, 3] for details). The main difference for $\ell \geq 2$ is the Birkhoff normal form symmetry of h. The linearization $(dh)_0 = e^{2\pi ip/q}$ gives rise to $\mathbb{Z}_q$ Birkhoff normal form symmetry (for any ℓ). We claim that this symmetry, when coupled with $\mathbb{Z}_\ell$-equivariance, leads to $\mathbb{Z}_Q$ Birkhoff normal form symmetry where $Q = \ell' q/\gcd(\ell', j)$.

To prove the claim, observe that the combined effect of the $\mathbb{Z}_q$ Birkhoff normal form symmetry together with $\mathbb{Z}_\ell$-equivariance yields symmetries of the form

$$v \mapsto e^{2\pi i\{\frac{a}{q} + \frac{bj}{\ell}\}}v = e^{2\pi i(a\ell' + bjq')/\ell' q}v$$

where $a, b \in \mathbb{Z}$. But $a\ell' + bjq'$ is a general multiple of $\gcd(\ell', jq')$ and hence $\gcd(\ell', j)$ (since ℓ' and q' are coprime). Hence, we have symmetries of the form

$$v \mapsto e^{2\pi ic \gcd(\ell', j)/\ell' q}v = e^{2\pi ic/Q}v,$$

where $c \in \mathbb{Z}$, proving the claim.

As in the case $\ell = 1$, we write h (in Birkhoff normal form) as $h(v) = e^{2\pi ip/q}\chi_1(v)$, where $\chi_1(v)$ is the time 1 map of a $\mathbb{Z}_Q$-equivariant vector field [2, 3]. Since $Q \geq 5$, it can be shown (just as in the case $\ell = 1$, $q \geq 5$, see [3, pages 282–286]) that generically (in the region of parameter space corresponding to $A_{p/q}$) the vector field has precisely Q sink-source pairs of equilibria and hence that h has $Q/q = \ell'/\gcd(\ell', j)$ sink-source pairs of q-periodic orbits as required.

Next, we compute the spatiotemporal symmetry of these periodic orbits. Observe that in the $\mathbb{Z}_Q$-equivariant Birkhoff normal form, the periodic sinks (and similarly the periodic sources) for h lie on a single group orbit and hence are equally spaced on the invariant circle. Suppose that v_0 is a periodic point for h. Then $h(v_0) \sim (dh)_0 v_0 = e^{2\pi ip/q}v_0 \in \mathbb{Z}_Q \cdot v_0$. Hence $h^m(v_0) \sim e^{2\pi ipm/q}v_0 \in \mathbb{Z}_Q \cdot v_0$ for all $m \geq 1$. Moreover, we have exact equality if (and generically, only if) $e^{2\pi ipm/q} \in \mathbb{Z}_\ell$. This means that m satisfies

$$\frac{pm}{q} = \alpha\frac{j}{\ell} + \beta,$$

for integers $\alpha, \beta \in \mathbb{Z}$. Equivalently,

$$\frac{pm\ell'}{q'} = \alpha j + \beta \ell \equiv 0 \bmod \gcd(\ell, j).$$

But $p\ell'$ and q' are coprime, so m is a multiple of $q' \gcd(\ell, j)$. Taking $m = q' \gcd(\ell, j)$, we obtain the spatiotemporal symmetry

$$h^{q' \gcd(\ell,j)}(v_0) = e^{2\pi i p q' \gcd(\ell,j)/q} v_0 = e^{2\pi i p \ell' \gcd(\ell,j)/\ell} v_0 = e^{2\pi i p \ell' j \tilde{j}/\ell} v_0 = R_{2\pi p \tilde{j} \ell'/\ell} \cdot v_0,$$

as required. ∎

Noncompact drift on the phaselocked relative periodic solutions Next, we consider the dynamics on the corresponding relative invariant torus for the underlying ODE on M. Define the frequency $\omega_{\rm per} = 2\pi/T_{\rm per}$. Let $\omega_{\rm rot} = \xi$, the speed of the corotating frame.

It follows from Proposition 1.4 that the p/q-phaselocked periodic solution in Proposition 4.4 corresponds to a relative periodic solution $u(t, \lambda)$ for the underlying ODE satisfying

$$u\left(\frac{q' \gcd(\ell,j)}{q} T_{\rm bif}(\lambda), \lambda\right) = \gamma\left(\frac{q' \gcd(\ell,j)}{q} T_{\rm bif}(\lambda), \lambda\right) R_{2\pi \tilde{j} p \ell'/\ell} \cdot u(0, \lambda), \tag{4.1}$$

where $\gamma(\frac{q' \gcd(\ell,j)}{q} T_{\rm bif}(\lambda), \lambda)$ is a general element of $Z(\Delta^{\rm bif})$ close to the rotation $R_{\frac{q' \gcd(\ell,j)}{q} T_{\rm bif}(\lambda)\omega_{\rm rot}}$ (by the same argument as in Proposition 3.6).

There are two distinct cases. When $\gcd(\ell, j) \geq 2$, we have that $Z(\Delta^{\rm bif}) = \mathbf{SO}(2)$. Generically, $\gamma(\frac{q' \gcd(\ell,j)}{q} T_{\rm bif}(\lambda), \lambda)$ is an irrational rotation and we have a modulated rotating wave.

We now concentrate on the case $\gcd(\ell, j) = 1$ so that $\Delta^{\rm bif} = \mathbf{SE}(2)$, and there is the possibility of noncompact drift.

Proposition 4.5 *Suppose that* $\gcd(\ell, j) = 1$. *Let* $(\lambda, \omega) \in A_{p/q}$, *so that we have* p/q*-phaselocking in the* (v, θ)*-subsystem. A necessary condition for noncompact drift on the corresponding phaselocked relative periodic solutions near* $\lambda = 0$ *is that*

$$\frac{q\omega_{\rm rot}}{\omega_{\rm per}} + \tilde{j} p \equiv 0 \bmod s.$$

Proof. Equation (4.1) reduces to

$$u\left(\tfrac{q'}{q} T_{\rm bif}(\lambda), \lambda\right) = \gamma\left(\tfrac{q'}{q} T_{\rm bif}(\lambda), \lambda\right) R_{2\pi \tilde{j} p \ell'/\ell} \cdot u(0, \lambda).$$

where $\gamma(\frac{q'}{q} T_{\rm bif}(\lambda), \lambda)$ is a general element of $\mathbf{SE}(2)$ close to the rotation $R_{\frac{q'}{q} T_{\rm bif}(\lambda)\omega_{\rm rot}}$. In particular,

$$\gamma(\tfrac{q'}{q} T_{\rm bif}(\lambda), \lambda) R_{2\pi \tilde{j} p \ell'/\ell} \sim R_{\frac{q'}{q} T_{\rm bif}(\lambda)\omega_{\rm rot} + 2\pi \tilde{j} p \ell'/\ell} \sim R_{q' T_{\rm per}\omega_{\rm rot} + 2\pi \tilde{j} p \ell'/\ell}.$$

Noncompact drift occurs only when $\gamma(\frac{q'}{q} T_{\rm bif}(\lambda), \lambda) R_{2\pi \tilde{j} p \ell'/\ell}$ is a translation. In particular, a necessary condition for noncompact drift at $\lambda \sim 0$ is that $R_{q' T_{\rm per}\omega_{\rm rot} + 2\pi \tilde{j} p \ell'/\ell} = \mathrm{id}$. In other words

$$q' \frac{2\pi}{\omega_{\rm per}} \omega_{\rm rot} + \frac{2\pi \tilde{j} p \ell'}{\ell} \equiv 0 \bmod 2\pi,$$

yielding the required result. ∎

A more precise condition is obtained by writing the $\dot{\phi}$ equation (which is $\frac{q'}{q}T_{\rm bif}(\lambda)$-periodic) in a Fourier series and defining $\omega_{\rm rot}(\lambda)$ to be the time-independent term of the Fourier series. Define $\omega_{\rm bif}(\lambda) = 2\pi/T_{\rm bif}(\lambda)$. In particular, $\omega_{\rm rot}(0) = \omega_{\rm rot}$ and $\omega_{\rm bif}(0) = \omega_{\rm per}/q$. Noncompact drift at the value λ of the bifurcation parameter occurs generically if and only if

$$\frac{\omega_{\rm rot}(\lambda)}{\omega_{\rm bif}(\lambda)} + \tilde{j}p \equiv 0 \bmod s.$$

5 Relative periodic solutions and comoving frames

In this section we study the structure of the relative periodic solution itself.

In Subsection (a), we recall some basic results concerning topologically cyclic subgroups and Cartan subgroups. As shown in [23, 16, 4], these concepts drive the dynamics on relative periodic solutions. These results are described and extended slightly in Subsection (b).

In Subsection (c), we introduce the index m of a relative periodic solution and we show that this index is finite if and only if Hypothesis (H) is satisfied (in which case m divides n). Moreover, we show that m is stable if Γ is compact and also if Γ is the Euclidean group $\mathbf{E}(N)$ for some N.

In Subsection (d), we show that, assuming Hypothesis (H), any relative periodic solution can be reduced to a group orbit of periodic solutions of period m by passing to a suitable moving frame.

In Subsection (e), we show that, again assuming Hypothesis (H), a neighborhood of a relative periodic solution can be written as a $2m$-periodic bundle. This is the optimal periodicity that may be obtained in general.

(a) Topologically cyclic subgroups and Cartan subgroups

We begin by recalling the main definitions and results concerning Cartan subgroups [7]. Let G be a finite dimensional Lie group. We denote the connected component of the identity by G^0, and define the projection $\pi : G \to G/G^0$. Let $g \in G$ and define $H(g)$ to be the closure of the subgroup of G generated by g. Such a subgroup $H(g)$ is said to be *topologically cyclic* and either $H(g) \cong \mathbb{Z}$ or $H(g) \cong T^p \times \mathbb{Z}_q$ where $p \geq 0$, $q \geq 1$ are integers. Note that $\pi(H(g)) = \langle \pi(g) \rangle$, where $\langle \pi(g) \rangle \subset G/G^0$ is the cyclic subgroup generated by $\pi(g)$.

We concentrate on the case when $H(g)$ is compact. In particular, the cyclic group $\pi(H(g))$ is finite. Define $\widetilde{G} = \pi^{-1}\pi(H(g))$. Then $\widetilde{G}$ is a finite dimensional Lie group consisting of finitely many connected components of G. Hence $\widetilde{G}$ is diffeomorphic to $K \times \mathbb{R}^a$ where $a \geq 0$ and K is the (unique up to conjugacy) maximal compact subgroup of $\widetilde{G}$, see for example [8]. Observe that $\langle \pi(g) \rangle = \pi(H(g)) = \pi(K) = \pi(\widetilde{G})$. It follows from the theory of Cartan subgroups [7] that there is a maximal (with respect to inclusion) topologically cyclic subgroup $H \subset K$ containing g such that $\pi(H) = \langle \pi(g) \rangle$. This subgroup H is called the *Cartan subgroup corresponding to* g and is unique up to conjugacy. In fact, it is sufficient to require maximality with respect to dimension in defining H (instead of maximality with respect to inclusion).

We note that this construction depends only on $\pi(g)$. More precisely, if g_1 and g_2 lie in the same connected component of G, and $H(g_1)$ and $H(g_2)$ are compact, then (up to conjugacy) we obtain the same group $\widetilde{G} \cong K \times \mathbb{R}^a$ and the same Cartan subgroup H corresponding to g_1 and g_2. Moreover, for generic g in this connected component of G, either $H(g) \cong \mathbb{Z}$ or $H(g) \cong H$ [7].

Definition 5.1 Let $g \in G$ and suppose that $H(g)$ is compact. Let H be the Cartan subgroup corresponding to g and write $H \cong T^d \times \mathbb{Z}_m$. We say that g has *rank* d and *index* m.

Example 5.2 Let G be the nonsplit extension of $\mathbf{SO}(2)$ by an element g such that $g^2 = R_\pi \in \mathbf{SO}(2)$. A matrix representation of this group is

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The Cartan subgroup corresponding to g is isomorphic to $\mathbb{Z}_4$, although $G = \widetilde{G}$ has only two connected components.

Thus, even when G is compact, the index m of an element $g \in G$ may be larger than the number of connected components of G in contrast to what is claimed in [16, Lemmas 4.1,4.2], [4, Theorem 4.1, Proposition 4.3]. (This error is of no consequence for the results in [16, 4].)

(b) Dynamics on relative periodic solutions

We now return to the set up of vector fields with relative periodic solutions. Let Γ be a finite dimensional Lie group acting smoothly and properly on a finite dimensional manifold M. Let $\mathcal{F} : M \to TM$ be a smooth Γ-equivariant vector field with flow Φ_t, and suppose that $\mathcal{P}$ is a relative periodic solution with relative period 1. Choose $u_0 \in \mathcal{P}$ and write $\Phi_1(u_0) = \sigma u_0$ where $\sigma \in \Gamma$.

As usual, we have the (compact) spatial symmetry group Δ and the spatiotemporal symmetry group Σ. Recall that $\sigma \in N(\Delta)$ and that Σ is the closed subgroup of $N(\Delta)$ generated by Δ and σ. Observe that Σ/Δ is the topologically cyclic subgroup of $N(\Delta)/\Delta$ generated by the coset $\sigma\Delta$.

Now, Σ is compact if and only if Σ/Δ is compact, in which case $\dim \Sigma/\Delta \le d$ where d is the rank of $\sigma\Delta \in N(\Delta)/\Delta$. Moreover, when Σ is compact, it is generically the case that $\dim \Sigma/\Delta = d$, see [23, 16, 4]. (Here, genericity is within the class of smooth Γ-equivariant vector fields $\mathcal{F} : M \to TM$ with relative periodic solution $\mathcal{P}$.)

Altogether, we have the following result [23, 16, 4]. (In contrast to [4], we do not require that $N(\Delta)/\Delta$ has finitely many connected components.)

Proposition 5.3 *Let Γ be a finite dimensional Lie group, and let $\mathcal{P}$ be a relative periodic solution. If Σ is noncompact, then $\mathcal{P}$ is foliated by unbounded trajectories. If Σ is compact, then $\mathcal{P}$ is foliated by $(p+1)$ dimensional tori with $(p+1)$-frequency quasiperiodic flows for some $p \le d$, where d is the rank of $\sigma\Delta \in N(\Delta)/\Delta$. Moreover, generically $p = d$.*

(c) The index of a relative periodic solution

We continue to assume the set up of Subsection (b).

Definition 5.4 Suppose that Γ is a finite dimensional Lie group and that $\mathcal{P}$ is a relative periodic solution. Then the *index* of $\mathcal{P}$ is the least positive integer m such that $\sigma^m \in \exp \mathrm{L}Z(\Sigma) \cdot \Delta$. If no such integer m exists, we say that $\mathcal{P}$ has *infinite index.*

In the remainder of this subsection, we show how the index m in Definition 5.4 is related to the integer m in Definition 5.1 and to the integer n in Hypothesis (H).

Proposition 5.5 *Suppose that Γ is a compact Lie group and that $\mathcal{P}$ is a relative periodic solution. Then the index of $\mathcal{P}$ is finite and coincides with the index of the element $\sigma\Delta \in N(\Delta)/\Delta$.*

Proof. Without loss, we may suppose that $\Delta = \mathbf{1}$. In particular, $Z(\Sigma) = Z(\sigma)$.

Let m be the index of $\mathcal{P}$ and let q be the index of σ, so that the Cartan subgroup corresponding to σ has the form $H = T^d \times \mathbb{Z}_q$. We show that $q = m$.

Since $\sigma \in H = T^d \times \mathbb{Z}_q$, it follows that $\sigma^q \in T^d \subset Z(\sigma)^0$ and hence m divides q. In particular, m is finite.

It remains to show that q is the least such positive integer. Since $\sigma^m \in T^d$, we can form a subgroup $\widehat{H} = T^d \times \mathbb{Z}_m \subset H$ generated by T^d and σ. But $\widehat{H}$ is a topologically cyclic subgroup containing σ and satisfying $\pi(\widehat{H}) = \langle \pi(\sigma) \rangle$. Since $\widehat{H}$ has the same (hence maximal) dimension as H it follows that $\widehat{H}$ is the Cartan subgroup corresponding to σ. Hence $q = m$. ∎

Corollary 5.6 *Suppose that Γ is a compact Lie group. Then the index m of a relative periodic solution is stable under perturbations (of the element $\sigma \in N(\Delta)$).*

Proof. By Proposition 5.5, the index m is determined by the Cartan subgroup H corresponding to $\sigma\Delta$. But H depends only on the connected component of $N(\Delta)/\Delta$ containing $\sigma\Delta$ and hence is stable under perturbations. ∎

Corollary 5.7 *Let $\Gamma = G \ltimes \mathbb{R}^N$ be the semidirect product of a compact subgroup $G \subset \mathbf{O}(N)$ and a normal vector subgroup $\mathbb{R}^N$, $N \geq 1$. Suppose that $\Delta = \mathbf{1}$. Then the index m of a relative periodic solution is finite and stable under perturbations (of the element $\sigma \in \Gamma$).*

Proof. Write $\sigma \in \Gamma$ in the form $\sigma = (R, w)$, where $R \in G$, $w \in \mathbb{R}^N$. We show that the index m of σ in Γ coincides with the index m_R of $R \in G$. The result then follows immediately from Corollary 5.6.

First, we verify that $m_R \leq m$. Suppose that $\sigma^m = \exp \xi$ where $\xi \in \mathrm{L}Z(\sigma)$, and write $\xi = (\xi_R, \xi_w)$ where $\xi_R \in \mathrm{L}G$, $\xi_w \in \mathrm{L}\mathbb{R}^N$. It is an immediate consequence of the semidirect product structure that $R^m = \exp \xi_R$ and $\xi_R \in \mathrm{L}Z(R)$. Hence R has index $m_R \leq m$.

Next, we verify that $m \leq m_R$. Conjugating $\sigma = (R, w)$ by an element of the form (I, y) where $y \in \mathbb{R}^N$ is chosen appropriately, we can transform σ into an element $\sigma = (R, w)$ where $Rw = w$. Since G is compact, R has finite index m_R. Hence there is an element $\xi_R \in \mathrm{L}G$ such that $R^{m_R} = \exp \xi_R$ and $\mathrm{Ad}_R \xi_R = \xi_R$. Let $\xi = (\xi_R, m_R w) \in \mathrm{L}\Gamma$. A calculation shows that $\sigma^{m_R} = \exp \xi$ and $\mathrm{Ad}_\sigma \xi = \xi$. Hence σ has index $m \leq m_R$. ∎

Remark 5.8 Corollaries 5.6 and 5.7 show that the index m is stable for compact groups and for groups that are 'Euclidean-like' (taking $G = \mathbf{O}(N)$ in Corollary 5.7 yields the Euclidean group $\Gamma = \mathbf{E}(N)$).

However, in general, the index m need not be stable to perturbations of the element $\sigma \in N(\Delta)$. For example, consider the case $\Gamma = \mathbf{SL}(2, \mathbb{R})$, $\Delta = \mathbf{1}$. The image of the exponential map $\exp : \mathrm{L}\mathbf{SL}(2, \mathbb{R}) \to \mathbf{SL}(2, \mathbb{R})$ consists of those 2×2 matrices $\gamma \in \Gamma$ for which $\mathrm{tr}\, \gamma > -2$ together with $\gamma = -I$, see for example [8, p. 74]. If σ lies in the image of the exponential map, then $m = 1$. Otherwise, $m = 2$. It follows that the index m of $\sigma \in \Gamma$ is stable to perturbations if and only if $\mathrm{tr}\, \sigma \neq -2$.

Next, we explore the relationship between the index m and the integer n in Hypothesis (H).

Proposition 5.9 *The relative periodic solution $\mathcal{P}$ has finite index m if and only if σ can be chosen (within $\sigma\Delta$) to satisfy Hypothesis (H), that is, so that $\sigma^n \in \exp \mathrm{L}Z(\Sigma)$ for some positive integer n.*

Proof. It is immediate that Hypothesis (H) implies finite index. Conversely, suppose that $\sigma^m = \exp(\zeta)\delta_0$ where $\zeta \in \mathrm{L}Z(\Sigma)$ and $\delta_0 \in \Delta$. Define the compact Lie group $\widetilde{\Delta} = \Delta \cap Z(\sigma)$ and observe that $\delta_0 \in \widetilde{\Delta}$. Let $H \subset \widetilde{\Delta}$ be the Cartan subgroup containing δ_0. Then $\delta_0^q \in H^0$ for some $q \geq 1$, and we can write $\delta_0^q = \epsilon^{mq}$ where $\epsilon \in \widetilde{\Delta}$. Let $\widetilde{\sigma} = \sigma\epsilon^{-1} \in \sigma\Delta$. Then

$$\widetilde{\sigma}^{mq} = \sigma^{mq}\epsilon^{-mq} = \exp(q\zeta)\delta_0^q\,\epsilon^{-mq} \in \exp \mathrm{L}Z(\Sigma).$$

Hence $\widetilde{\sigma}$ satisfies Hypothesis (H), with $n = mq$. ∎

The index m divides the integer n. Note also that m depends only on the coset $\sigma\Delta$, whereas n depends on the choice of σ within this coset.

(d) Structure of a relative periodic solution

It is well known that a relative equilibrium with spatial symmetry Δ is diffeomorphic to Γ/Δ. Similarly, for a relative periodic solution $\mathcal{P}$, we have

$$\mathcal{P} \cong \frac{(\Gamma/\Delta) \times \mathbb{R}}{\mathbb{Z}} \cong \frac{\Gamma \times \mathbb{R}}{\Delta \rtimes \mathbb{Z}}$$

where the action of $\Delta \rtimes \mathbb{Z}$ on $\Gamma \times \mathbb{R}$ is given by

$$(\gamma, \theta) \mapsto (\gamma\delta^{-1}, \theta),\ \delta \in \Delta, \qquad (\gamma, \theta) \mapsto (\gamma\sigma^{-1}, \theta + 1).$$

Under Hypothesis (H), we can simplify this representation of $\mathcal{P}$ by passing to a suitable comoving frame. In particular, we show that in a comoving frame, $\mathcal{P}$ is the quotient by a compact Lie group of an m-periodic trivial bundle $\Gamma \times S^1$, where m is the index of $\mathcal{P}$. Here, m-periodicity is optimal.

Let m be the index of $\mathcal{P}$ and write $\sigma^m = \delta_0 \exp m\zeta$, where $\zeta \in \mathrm{L}Z(\Sigma)$ and $\delta_0 \in \Delta$. Let Σ_m denote the subgroup of Γ generated by Δ and $\beta = \exp(-\zeta)\sigma$. Note that $\beta \in N(\Delta)$ and that $\beta^m \in \Delta$. Hence Σ_m is a cyclic extension of Δ of order m, so $\Sigma_m/\Delta \cong \mathbb{Z}_m$. We call Σ_m the *discrete spatiotemporal symmetry group* of the relative periodic solution.

Lemma 5.10 *Suppose that $\mathcal{P}$ has index m. Write $\sigma^m = \delta_0 \exp m\zeta$ where $\zeta \in \mathrm{L}Z(\Sigma)$ and $\delta_0 \in \Delta$. Define β and Σ_m as above. Then, in a comoving frame moving with velocity ζ,*

$$\mathcal{P} \cong \frac{(\Gamma/\Delta) \times S^1}{\mathbb{Z}_m} \cong \frac{\Gamma \times S^1}{\Sigma_m}$$

where $S^1 = \mathbb{R}/m\mathbb{Z}$.

The action of Σ_m on $\Gamma \times S^1$ is given by

$$(\gamma, \theta) \mapsto (\gamma\delta^{-1}, \theta),\ \delta \in \Delta, \qquad (\gamma, \theta) \mapsto (\gamma\beta^{-1}, \theta + 1).$$

Proof. Define the smooth parameterization $\tau : \Gamma \times \mathbb{R} \to \mathcal{P}$ given by

$$\tau(\gamma, \theta) = \gamma \exp(-\zeta\theta)\Phi_\theta(u_0).$$

Since $\zeta \in \mathrm{L}Z(\Delta)$, we have that $\tau(\gamma\delta^{-1}, \theta) = \tau(\gamma, \theta)$ for all $\delta \in \Delta$. Moreover, since $\zeta \in \mathrm{L}Z(\sigma)$, we have that $\tau(\gamma, \theta + 1) = \gamma \exp(-\zeta) \exp(-\zeta\theta)\sigma\Phi_\theta(u_0) = \tau(\gamma\beta, \theta)$. In particular, $\tau(\gamma, \theta + m) = \tau(\gamma\beta^m, \theta) = \tau(\gamma\delta_0, \theta) = \tau(\gamma, \theta)$. Hence τ induces a smooth map $\tau : \Gamma \times S^1 \to \mathcal{P}$ with the required properties. ∎

In the comoving frame in Lemma 5.10, the relative periodic solution $\mathcal{P}$ becomes a group orbit of ordinary periodic solutions with spatial symmetry Δ and spatiotemporal symmetry Σ_m. Moreover, the integer m is the least possible. When $\mathcal{P}$ is a discrete rotating wave, m is the absolute period of $\mathcal{P}$, and $\Sigma_m = \Sigma$. Hence the index m corresponds to the integer m in [25].

(e) A $2m$-periodic bundle

In Subsection (d), we passed to a comoving frame in which $\mathcal{P}$ could be represented as an m-periodic bundle; more precisely, the quotient of an m-periodic trivial bundle by a free compact Lie group action. Here, the integer m is the least possible. It is natural to ask whether a neighborhood of $\mathcal{P}$ can be represented as a $2m$-periodic bundle. (A factor 2 is to be expected to take account of orientability issues.) In this subsection, we answer this question positively.

Although the $2n$-periodic bundle obtained in Section 2 is adequate for applications, there are at least three reasons for looking for an $2m$-periodic bundle. First, the integer m is intrinsically defined, independent of any choices. Second, m is optimally small. (Recall that in general m divides n.) Third, this bundle makes transparent how the action of the discrete spatiotemporal symmetry group Σ_m on the cross-section V comes about. Overall, the $2m$-periodic bundle seems more natural.

We continue to assume that $\mathcal{P}$ is a relative periodic solution satisfying Hypothesis (H). Let m be the index of $\mathcal{P}$ and write $\sigma^m = \delta_0 \exp m\zeta$ where $\zeta \in \mathrm{L}Z(\Sigma)$ and $\delta_0 \in \Delta$. In Section 5(c), we introduced the discrete spatiotemporal symmetry group Σ_m, namely the subgroup of Γ generated by Δ and $\beta = \exp(-\zeta)\sigma$.

The group Σ_m is a cyclic extension of Δ of order m. To take account of orientability problems, we define a related cyclic extension Σ_{2m} of Δ of order $2m$.

Proposition 5.11 *Let Σ_{2m} be the group generated by Δ and an element R such that $R\delta R^{-1} = \sigma\delta\sigma^{-1}$, for $\delta \in \Delta$, and $R^{2m} = \delta_0^2$. Then Σ_{2m} is a cyclic extension of Δ.*

Proof. The group Σ_{2m} can be realized as a matrix group:

$$\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}, \; \delta \in \Delta, \qquad \begin{pmatrix} \beta & 0 \\ 0 & e^{\pi i/m} \end{pmatrix}.$$

■

Theorem 5.12 *Let $\mathcal{P}$ be a relative periodic solution with index m, and write $\sigma^m = \delta_0 \exp m\zeta$, where $\zeta \in \mathrm{L}Z(\Sigma)$ and $\delta_0 \in \Delta$. Form the group Σ_{2m} generated by Δ and R as in Proposition 5.11. Then, there is a neighborhood U of the relative periodic solution $\mathcal{P}$ such that, in a comoving frame, moving uniformly with velocity ζ,*

$$U \cong \frac{(\Gamma \times_\Delta V) \times S^1}{\mathbb{Z}_{2m}} \cong \frac{\Gamma \times V \times S^1}{\Sigma_{2m}},$$

where V is a representation of the group Σ_{2m}, and $S^1 = \mathbb{R}/2m\mathbb{Z}$.

The action of Σ_{2m} on $\Gamma \times V \times S^1$ is given by

$$\delta \cdot (\gamma, v, \theta) = (\gamma\delta^{-1}, \delta v, \theta), \; \delta \in \Delta, \quad R \cdot (\gamma, v, \theta) = (\gamma\beta^{-1}, Rv, \theta + 1),$$

where $\beta = \exp(-\zeta)\sigma$.

Proof. Let $A : V \to V$ be the orthogonal twisted equivariant linear map in equation (2.2). Let G be the closed subgroup of $\mathbf{O}(V)$ acting on V generated by the actions of Δ and A and let Z denote the centralizer of G in $\mathbf{O}(V)$.

We claim that $\delta_0 A^m \in Z$. Note that $\sigma^{-m}\delta_0 \in Z(\Delta)$ so that $\delta_0\delta = \sigma^m\delta\sigma^{-m}\delta_0$, for all $\delta \in \Delta$. In other words, $\delta_0\delta = \phi^{-m}(\delta)\delta_0$, for all $\delta \in \Delta$. By twisted equivariance, $A^m\delta = \phi^m(\delta)A^m$ for all $\delta \in \Delta$. Hence the composition $\delta_0 A^m$ commutes with elements of Δ. At the same time, $A\delta_0 = \phi(\delta_0)A = \delta_0 A$ since δ_0 commutes with σ. It follows that $\delta_0 A^m$ commutes with A proving the claim.

Now, Z is a space of equivariant linear maps, and it follows from general arguments, see Appendix A, that $B^2 \in Z^0$ for all $B \in Z$. Hence $\delta_0^2 A^{2m} \in Z^0$. Since Z is compact, we can write

$\delta_0^2 A^{2m} = \exp(2m\eta)$ where $\eta \in \mathrm{L}Z$. In particular, $\exp(t\eta)$ is Δ-equivariant and commutes with A for all $t \in \mathbb{R}$. Set $R = Q\exp(\eta)$, where $Q = A^{-1}$ so that $R^{2m} = \delta_0^2$.

Let $\tau : \Gamma \times V \times \mathbb{R} \to M$ be the submersion defined in Section 2(b). As in the proof of Theorem 1.6, we consider a modified submersion τ^{new}. This time, we define

$$\tau^{\mathrm{new}}(\gamma, v, \theta) = \tau(\gamma\exp(-\theta\zeta), \exp(-\theta\eta)v, \theta).$$

As before, $\tau^{\mathrm{new}}(\gamma\delta, v, \theta) = \tau^{\mathrm{new}}(\gamma, \delta v, \theta)$ for all $\delta \in \Delta$, and we compute that

$$\begin{aligned}\tau^{\mathrm{new}}(\gamma, v, \theta+1) &= \tau(\gamma\exp(-\zeta)\exp(-\theta\zeta), \exp(-\eta)\exp(-\theta\eta)v, \theta+1)\\ &= \tau(\gamma\exp(-\zeta)\exp(-\theta\zeta)\sigma, Q^{-1}\exp(-\eta)\exp(-\theta\eta)v, \theta)\\ &= \tau(\gamma\beta\exp(-\theta\zeta), \exp(-\theta\eta)R^{-1}v, \theta)\\ &= \tau^{\mathrm{new}}(\gamma\beta, R^{-1}v, \theta).\end{aligned}$$

Since $\beta^m = \delta_0$ and $R^{2m} = \delta_0^2$, it follows that

$$\tau^{\mathrm{new}}(\gamma, v, \theta + 2m) = \tau^{\mathrm{new}}(\gamma\delta_0^2, \delta_0^{-2}v, \theta) = \tau^{\mathrm{new}}(\gamma, v, \theta).$$

Hence τ^{new} induces a Γ-equivariant submersion

$$\tau^{\mathrm{new}} : \Gamma \times V \times S^1 \to M,$$

where $S^1 = \mathbb{R}/2m\mathbb{Z}$. Moreover, we have a fixed-point free action of the compact group Σ_{2m} given by

$$(\gamma, v, \theta) \mapsto (\gamma\delta^{-1}, \delta v, \theta), \quad \delta \in \Delta, \qquad (\gamma, v, \theta) \mapsto (\gamma\beta^{-1}, Rv, \theta+1).$$

It follows that τ induces a Γ-equivariant diffeomorphism

$$\tau : \frac{(\Gamma \times_\Delta V) \times S^1}{\mathbb{Z}_{2m}} \cong U,$$

where U is a neighborhood of $\mathcal{P}$ and $\Gamma \times_\Delta V = (\Gamma \times V)/\Delta$. ∎

Remark 5.13 (a) As was the case for the $2n$-periodic bundle in Theorem 1.6, the cross-section V may be an arbitrary representation of the group Σ_{2m}. In particular, the symbol R is used to denote the abstract group element $R \in \Sigma_{2m}$, the action of R on the representation space V, and the induced action of R on $\Gamma \times V \times S^1$.

(b) Although the index m is well-defined, the elements $\delta_0 \in \Delta$ and $\zeta \in \mathrm{L}Z(\Sigma)$ are not in general well-defined. It is natural to demand that δ_0 has the least order possible. In many examples, this enables a well-defined choice of σ and hence of δ_0, $\exp\zeta$ and so on (but ζ itself is still not well-defined).

(c) In general, we cannot hope to obtain m-periodicity in Theorem 5.12. However, let δ_0, Q and Z be as in the proof of Theorem 5.12. If $\delta_0 Q^m \in Z^0$, then we obtain an m-periodic bundle

$$U \cong \frac{(\Gamma \times_\Delta V) \times S^1}{\mathbb{Z}_m} \cong \frac{\Gamma \times V \times S^1}{\Sigma_m},$$

where $S^1 = \mathbb{R}/m\mathbb{Z}$.

(d) The derivation of the bundle in Theorem 5.12 involves passing to a comoving frame in the physical variables ($\gamma \mapsto \gamma\exp(-\theta\zeta)$) and simultaneously passing to a comoving frame in the phase space variables ($v \mapsto \exp(-\theta\eta)v$).

If the relative periodic solution $\mathcal{P}$ is a discrete rotating wave then there is no need to go into a comoving frame, and Theorem 5.12 applies with $\zeta = 0$, $\beta = \sigma$, and $\Sigma_m = \Sigma$. In particular, Σ_{2m} is precisely the group Σ_0 introduced in Vanderbauwhede [37, 36] for studying period doubling bifurcation (see also [30, 26]). Hence Theorem 5.12 clarifies the role of Σ and Σ_0 in the approach of [37, 36, 30, 32] to bifurcation from discrete rotating waves. A discussion of these approaches can be found in [26].

When $\mathcal{P}$ is a modulated traveling wave, it follows easily from Theorem 5.12 that the action of Δ on the cross-section V extends to an (unfaithful) action of Σ: let σ act as R^{-1}.

When $\mathcal{P}$ is a modulated rotating wave, it is not necessarily the case that the action of Δ on V extends to an action of Σ. For example, suppose that $\Sigma = \mathbf{SO}(2)$, $\Delta = \mathbb{Z}_2$, and let V be the one dimensional nontrivial representation of Δ.

In general, the $2k$-periodicity of the equations on the slice, achieved in Theorem 1.2, is not preserved. This is due to the fact that k and m are unrelated in general [25].

Example 5.14 Consider a relative periodic solution $\mathcal{P}$ with $\Gamma = \mathbf{O}(2)$, $\Sigma = \mathbb{D}_\ell$ and $\Delta = \mathbb{Z}_\ell$, $\ell \geq 3$. A concrete example is given by "pulsating waves" in two dimensional convection, see [32]. Since Σ contains reflections, $\mathcal{P}$ is a discrete rotating wave (so $\zeta = 0$, $\beta = \sigma$ in Theorem 5.12). Observe that $k = m = n = 2$. In addition, σ is necessarily of order two, so that $\delta_0 = \mathrm{id}$.

Assume that the cross-section V is two dimensional and that we have a faithful action of $\Delta = \mathbb{Z}_\ell$ on V. We show that the bundle in Theorem 5.12 can be chosen to be m-periodic and not only $2m$-periodic.

Let Q, G and Z be as in the proof of Theorem 5.12. Since Q is orthogonal, Q is either a rotation or reflection on V. If Q is a reflection, then $\delta_0 Q^m = Q^2 = I$. If Q is a rotation, then $Z^0 = Z = \mathbf{SO}(2)$, so that $\delta_0 Q^m = Q^2 \in Z^0$. Either way, it follows that Remark 5.13(c) applies, and the 4-periodic bundle in Theorem 5.12 reduces to a 2-periodic bundle.

6 Validity of Hypotheses (H) and (H′)

In the introduction, we stated the two hypotheses

(H) There is a positive integer n such that $\sigma^n \in \exp \mathrm{L}Z(\Sigma)$, and

(H′) There is a positive integer k such that $\sigma^k \in Z(\Delta)$.

Clearly, Hypothesis (H) implies Hypothesis (H′). The main results of this paper assume Hypothesis (H), though some results go through under the weaker Hypothesis (H′).

In this section, we derive algebraic results that substantiate Hypotheses (H) and (H′). In particular, as promised in Remark 1.1, we show in Theorem 6.11 below that Hypothesis (H) can always be satisfied when Γ is an algebraic group, whereas Hypothesis (H′) requires only that Γ is a matrix group (Theorem 6.4 below).

Throughout this section, Γ is a finite dimensional Lie group, $\Delta \subset \Gamma$ is a compact subgroup, and σ lies in the normalizer $N(\Delta)$ of Δ. The centralizer of Δ is denoted by $Z(\Delta)$. Each element $\sigma \in N(\Delta)$ defines an automorphism $\phi \in \mathrm{Aut}(\Delta)$ given by $\phi(\delta) = \sigma^{-1}\delta\sigma$.

In Subsection (a), we prove a result about the relationship between $N(\Delta)$ and $Z(\Delta)$. In Subsection (b), we prove that Hypothesis (H′) can be satisfied for matrix groups. In Subsection (c), we prove that Hypothesis (H) can be satisfied for algebraic groups.

(a) Normalizers and centralizers

Suppose that Γ is a finite dimensional Lie group with compact subgroup Δ. Field [16, Proposition 3.2] showed that $N(\Delta)^0 = \Delta^0\, Z(\Delta)^0$. We give a different proof of this result. (The ideas in this proof are required in Lemma 6.2 below.)

Lemma 6.1 *Suppose that Γ is a finite dimensional Lie group and that $\Delta \subset \Gamma$ is a compact subgroup. Then*

(a) $\mathrm{L}N(\Delta) = \mathrm{L}\Delta + \mathrm{L}Z(\Delta)$.

(b) $N(\Delta)^0 = \Delta^0\, Z(\Delta)^0$.

Proof. The adjoint action of Δ on $\mathrm{L}\Gamma$ restricts to a representation of Δ on $\mathrm{L}N(\Delta)$. Moreover, the subalgebra $\mathrm{L}\Delta \subset \mathrm{L}N(\Delta)$ is a Δ-invariant subspace. Since Δ is compact, we may choose a Δ-invariant inner product on $\mathrm{L}N(\Delta)$. We then have the Δ-invariant splitting

$$\mathrm{L}N(\Delta) = \mathrm{L}\Delta \oplus (\mathrm{L}\Delta)^\perp.$$

Let $\eta \in (\mathrm{L}\Delta)^\perp$. To prove part (a), it is sufficient to prove that $\eta \in \mathrm{L}Z(\Delta)$. Let $\delta \in \Delta$. We show that $\mathrm{Ad}_\delta\, \eta = \eta$. By Δ-invariance of $(\mathrm{L}\Delta)^\perp$, we have that $\mathrm{Ad}_\delta\, \eta - \eta \in (\mathrm{L}\Delta)^\perp$, so it remains to show that $\mathrm{Ad}_\delta\, \eta - \eta \in \mathrm{L}\Delta$. But $\eta \in \mathrm{L}N(\Delta)$ so that $\exp(t\eta)\delta^{-1}\exp(-t\eta) \in \Delta$. It follows that $\delta\exp(t\eta)\delta^{-1} \cdot \exp(-t\eta) \in \Delta$. Differentiating with respect to t and setting $t = 0$, we obtain $\mathrm{Ad}_\delta\, \eta - \eta \in \mathrm{L}\Delta$ as required.

Next, we prove part (b). Let G consist of those elements $\gamma \in N(\Delta)^0$ such that $\gamma \in \Delta^0\, Z(\Delta)^0$. We show that G is a nonempty open and closed subset of the connected component $N(\Delta)^0$ so that $G = N(\Delta)^0$.

Openness of G is immediate from part (a), and it is an elementary argument to show that G is nonempty and closed. Indeed, G contains the identity and so is nonempty. To see that G is closed, let $\{\gamma_n\}$ be a sequence in G with $\gamma_n \to \gamma \in N(\Delta)^0$. Write $\gamma_n = \delta_n\tau_n$ where $\delta_n \in \Delta^0$, $\tau_n \in Z(\Delta)^0$. By compactness of Δ, we may pass to a subsequence so that $\delta_n \to \delta \in \Delta^0$. Then $\tau_n = \delta_n^{-1}\gamma_n \to \delta^{-1}\gamma \in N(\Delta)^0$. Write $\tau = \lim_{n\to\infty}\tau_n = \delta^{-1}\gamma$. Since $Z(\Delta)^0$ is closed, $\tau \in Z(\Delta)^0$ so that $\gamma = \delta\tau \in \Delta^0\, Z(\Delta)^0$. Hence $\gamma \in G$ and G is closed. ∎

We require the following refinement of Lemma 6.1(a).

Lemma 6.2 *Suppose that Γ is a finite dimensional Lie group and that $\Delta \subset \Gamma$ is a compact subgroup. Let $\sigma \in N(\Delta)$ and suppose that there is an integer $k \geq 1$ such that $\sigma^k = \exp\xi$, where $\xi \in \mathrm{L}N(\Delta)$ and $\mathrm{Ad}_\sigma\, \xi = \xi$. Then $\xi = \chi + \eta$ where $\chi \in \mathrm{L}\Delta$, $\eta \in \mathrm{L}Z(\Delta)$ and $\mathrm{Ad}_\sigma\, \chi = \chi$, $\mathrm{Ad}_\sigma\, \eta = \eta$.*

Proof. As in the proof of Lemma 6.1, we have the unique decomposition $\xi = \chi + \eta$ where $\chi \in \mathrm{L}\Delta$, $\eta \in (\mathrm{L}\Delta)^\perp \subset \mathrm{L}Z(\Delta)$.

Let Σ be the group generated by σ and Δ, and observe that $\mathrm{L}\Delta$ is Σ-invariant. If Σ is compact, then the scalar product on $\mathrm{L}\Gamma$ can be chosen to be Σ-invariant. It then follows that $(\mathrm{L}\Delta)^\perp$ is Σ-invariant. In particular, we have the decomposition $\xi = \mathrm{Ad}_\sigma\, \xi = \mathrm{Ad}_\sigma\, \chi + \mathrm{Ad}_\sigma\, \eta$ where $\mathrm{Ad}_\sigma\, \chi \in \mathrm{L}\Delta$, $\mathrm{Ad}_\sigma\, \eta \in (\mathrm{L}\Delta)^\perp$. By uniqueness of the decomposition, we have $\mathrm{Ad}_\sigma\, \chi = \chi$, $\mathrm{Ad}_\sigma\, \eta = \eta$ as required.

When Σ is noncompact, the proof is more complicated. By Lemma 6.1 we have a preliminary decomposition $\xi = \tilde{\chi} + \tilde{\eta}$ where $\tilde{\chi} \in \mathrm{L}\Delta$, $\tilde{\eta} \in \mathrm{L}Z(\Delta)$. Consider the subspace $X = \mathrm{L}\Delta \oplus \mathbb{R}(\tilde{\eta}) = \mathrm{L}\Delta + \mathbb{R}(\xi) \subset \mathrm{L}N(\Delta)$. Then X is invariant under the adjoint actions of Δ and σ and hence is a Σ-invariant subspace.

Let $(\cdot,\cdot)$ be any Δ-invariant scalar product on X and set

$$\langle u,v\rangle=\sum_{j=0}^{k-1}(\mathrm{Ad}_{\sigma^j}\,u,\mathrm{Ad}_{\sigma^j}\,v).$$

We claim that $\langle u,v\rangle$ is Σ-invariant. We can then proceed as in the case when Σ is compact to obtain a new decomposition $\xi=\chi+\eta$, this time with the required properties.

Since $\sigma\in N(\Delta)$, it follows that $\langle u,v\rangle$ is Δ-invariant. It remains to show σ-invariance, specifically to show that $(\mathrm{Ad}_{\sigma^k}\,u,\mathrm{Ad}_{\sigma^k}\,v)=(u,v)$. Let $\delta=\exp\tilde\chi\in\Delta$, $\epsilon=\exp\tilde\eta\in Z(\Delta)$. Then it is sufficient, by Δ-invariance of the inner product, to show that $\mathrm{Ad}_{\sigma^k}=\mathrm{Ad}_\delta$ on X.

Let $u\in \mathrm{L}\Delta$. Then $\mathrm{Ad}_{\sigma^k}u=\mathrm{Ad}_\delta\mathrm{Ad}_\epsilon\,u=\mathrm{Ad}_\delta\,u$ since $\epsilon\in Z(\Delta)$, and so $\mathrm{Ad}_{\sigma^k}=\mathrm{Ad}_\delta$ on $\mathrm{L}\Delta$. Since $\tilde\eta\in \mathrm{L}Z(\Delta)$ we have that $\mathrm{Ad}_\delta\,\tilde\eta=\tilde\eta$. At the same time, $[\xi,\tilde\eta]=[\tilde\chi,\tilde\eta]+[\tilde\eta,\tilde\eta]=0$ so that $\mathrm{Ad}_{\sigma^k}\tilde\eta=\mathrm{Ad}_{\exp\xi}\tilde\eta=\tilde\eta$. Therefore, Ad_{σ^k} and Ad_δ are the identity on $\mathbb{R}(\tilde\eta)$. This completes the proof that $\mathrm{Ad}_{\sigma^k}=\mathrm{Ad}_\delta$ on X. ∎

(b) Hypothesis (H′)

We continue to assume that Γ is a finite dimensional Lie group with compact subgroup Δ and $\sigma\in N(\Delta)$. Let $\phi\in\mathrm{Aut}(\Delta)$ be the automorphism induced by σ, so $\phi(\delta)=\sigma^{-1}\delta\sigma$.

Hypothesis (H′) in the introduction asks that the automorphism ϕ has finite order. That is, ϕ^k is the identity automorphism of Δ for some $k\geq 1$. In general, σ need not satisfy Hypothesis (H′). For example, let $\Gamma=\Sigma=\Delta=\mathbf{SO}(3)$, and let $\sigma\in\mathbf{SO}(3)$ be an irrational rotation. Then the corresponding (inner) automorphism $\phi\in\mathrm{Aut}(\mathbf{SO}(3))$ has infinite order.

In our study of relative periodic solutions, the element σ is defined up to multiplication by elements of Δ. Hence the following question arises:

- Is there an element $\tilde\sigma$ in the coset $\sigma\Delta$ such that $\tilde\sigma^k\in Z(\Delta)$ for some $k\geq 1$?

Equivalently, can ϕ be modified by an inner automorphism so as to have finite order? Clearly, the answer is yes for the $\mathbf{SO}(3)$ example above, since we can choose $\tilde\sigma=1$.

We answer this question affirmatively when Γ is a matrix group, and give a counterexample for more general groups.

Proposition 6.3 *Suppose that Γ is compact and $\sigma\in N(\Delta)$. Then there is an element $\tilde\sigma\in\sigma\Delta$ such that $\tilde\sigma^k\in Z(\Delta)$ for some $k\geq 1$.*

Proof. Compact Lie groups have finitely many connected components, so $\sigma^k\in(N(\Delta)\cap Z(\sigma))^0$ for some $k\geq 1$. By surjectivity of the exponential map, $\sigma^k=\exp\xi$ where $\xi\in \mathrm{L}N(\Delta)$ and $\mathrm{Ad}_\sigma\,\xi=\xi$.

Let $\xi=\chi+\eta$ be the decomposition in Lemma 6.2 and define $\delta=\exp(\chi/k)\in\Delta$. Then $\sigma^k=\delta^k\exp\eta$, where $\exp\eta\in Z(\Delta)$. By Lemma 6.2, $\mathrm{Ad}_\sigma\chi=\chi$ and so σ commutes with δ. Hence $\tilde\sigma=\sigma\delta^{-1}$ satisfies $\tilde\sigma^k=\exp\eta\in Z(\Delta)$ as required. ∎

Theorem 6.4 *Suppose that Γ is a matrix group, Δ is a compact subgroup, and $\sigma\in N(\Delta)$. Then there is an element $\tilde\sigma\in\sigma\Delta$ such that $\tilde\sigma^k\in Z(\Delta)$ for some $k\geq 1$.*

Proof. Since Γ is a matrix group, Γ acts faithfully on some finite dimensional vector space V. In particular, Δ acts faithfully on V. Let $\phi\in\mathrm{Aut}(\Delta)$ be the automorphism induced by σ, so that $\sigma^{-1}\delta\sigma=\phi(\delta)$. Equivalently, $\sigma^{-1}:V\to V$ is a twisted equivariant nonsingular linear map

$(\sigma^{-1}\delta = \phi(\delta)\sigma^{-1})$. Choose a Δ-invariant inner product on V. It follows from Proposition 2.1 that there is a twisted equivariant orthogonal map $Q : V \to V$. In particular,

$$R^{-1}\delta R = \sigma^{-1}\delta\sigma = \phi(\delta),$$

where $R = Q^{-1}$.

Let $G \subset \mathbf{O}(V)$ be the closed group generated by Δ and R. Then G is a compact Lie group. It follows from Proposition 6.3 that there is an element $\delta_0 \in \Delta$ such that $\widetilde{R} = R\delta_0$ satisfies $\widetilde{R}^k \in Z(\Delta)$ for some $k \geq 1$.

Now, define $\tilde{\sigma} = \sigma\delta_0$. Since σ and R induce the same automorphism $\phi \in \mathrm{Aut}(\Delta)$, it follows that $\tilde{\sigma}$ and $\widetilde{R}$ induce the same automorphism. Hence $\tilde{\sigma}^k \in Z(\Delta)$. ∎

Remark 6.5 The hypothesis that Γ is a matrix group can be weakened further to the assumption that Γ has a finite dimensional representation in which Δ acts faithfully.

Similarly, it is sufficient to assume that Δ has a finite dimensional faithful representation V for which there is a nonsingular linear map $L : V \to V$ that is twisted equivariant with respect to the automorphism ϕ induced by σ.

Finally, we give a counterexample when Γ is not a matrix group.

Example 6.6 Let $\Delta = T^2$. Then $\mathrm{Aut}(\Delta)$ is isomorphic to the group of 2×2 matrices with integer entries and determinant ± 1. In particular, $\mathbf{SL}(2,\mathbb{Z}) \subset \mathrm{Aut}(\Delta)$. Hence we can form the semidirect product $\Gamma = T^2 \rtimes \mathbf{SL}(2,\mathbb{Z})$. Note that Γ is a two dimensional Lie group, $\Delta = T^2$ is a compact subgroup and $N(\Delta) = \Gamma$. Moreover $Z(\Delta) = \Delta$. Now let σ be any element of $\mathbf{SL}(2,\mathbb{Z})$ of infinite order. (For example, $\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.) Then $\sigma^k \notin Z(\Delta)$ for any $k \geq 1$. Since $(\sigma\Delta)^k = \sigma^k\Delta$, it follows that $\tilde{\sigma}^k \notin Z(\Delta)$ for any $\tilde{\sigma} \in \sigma\Delta$ and $k \geq 1$.

(c) Hypothesis (H)

Let Σ denote the closed subgroup of Γ generated by Δ and σ. We consider the following refinement of the question posed in Subsection (b):

- Is there an element $\tilde{\sigma}$ in the coset $\sigma\Delta$ such that $\tilde{\sigma}^n \in \exp \mathrm{L}Z(\Sigma)$ for some $n \geq 1$?

Proposition 6.7 *Suppose that Γ is compact and $\sigma \in N(\Delta)$. Then there is an element $\tilde{\sigma} \in \sigma\Delta$ such that $\tilde{\sigma}^n \in \exp \mathrm{L}Z(\Sigma)$ for some $n \geq 1$.*

Proof. The proof is already implicit in the proof of Proposition 6.3 (with $n = k$). We showed there that $\tilde{\sigma}$ can be chosen so that $\tilde{\sigma}^n = \exp\eta \in \exp \mathrm{L}Z(\Delta)$. Furthermore, η was constructed using Lemma 6.2, so that $\mathrm{Ad}_\sigma\,\eta = \eta$. Hence $\tilde{\sigma}^n \in \exp \mathrm{L}Z(\Sigma)$. ∎

Example 6.8 In contrast to the case for the integer k, Proposition 6.7 cannot be generalized to arbitrary matrix groups. For example, the group $\Gamma = \mathbb{Z}$ is a matrix group. Let $\Delta = \mathbf{1}$ and $\sigma \neq 0$. Then we are forced to take $\tilde{\sigma} = \sigma$, and no power lies in the connected component of the identity. In this example, $k = 1$ but there is no finite integer n.

There are two ways in which compactness is utilized in the proof of Propositions 6.3 and 6.7. The first, which is highlighted in Examples 6.6 and 6.8, is that compact Lie groups have finitely many connected components. The second is that the exponential map $\exp : \mathrm{L}G \to G$ is surjective for connected compact Lie groups. This property may fail for connected but noncompact Lie groups, an example being $G = \mathbf{SL}(2,\mathbb{R})$, see Remark 5.8. The proof of Proposition 6.7 can be revised to take account of elements for which some sufficiently high power lies in the image of the exponential map. However, even this property fails in general:

Example 6.9 Let G be the universal cover of $\mathbf{SL}(2,\mathbb{R})$. (Topologically, $G \cong \mathbb{R}^3$.) Again, G is a connected semisimple Lie group. However, it can be shown that there exist elements $g \in G$ for which g^n is not in the image of the exponential map for all $n \geq 1$, see [31, p.164].

A similar, but more computable example, is the universal cover G of $\mathbf{SE}(2)$. Again G is homeomorphic to $\mathbb{R}^3$, but now G is a connected solvable Lie group with a semidirect product structure $G = \mathbb{R} \ltimes \mathbb{R}^2$ where multiplication is defined by

$$(t_1, v_1)\,(t_2, v_2) = (t_1 + t_2, e^{2\pi i t_1} v_2 + v_1).$$

An elementary computation shows that (t, v) lies in the image of the exponential map if and only if $v = 0$ or $t \notin \mathbb{Z}$. Let $g = (1, v)$ where $v \neq 0$. Then $g^m = (m, mv)$, so g^m is not in the image of the exponential map for all $m \geq 1$.

It turns out that Proposition 6.7 can be generalized to the class of algebraic groups. For background material on algebraic groups, we refer to [31]. Such groups are characterized as being those subgroups of $\mathbf{GL}(n,\mathbb{R})$, $n \geq 1$, that are defined by polynomial equalities in the coefficients of the matrices. Thus, every algebraic group is a finite dimensional Lie group. We note that all compact Lie groups are algebraic, as are the Euclidean groups $\mathbf{E}(N)$. The classical Lie groups are algebraic, and more generally a semisimple Lie group is an algebraic group if and only if it is a matrix group.

We require the following lemma of Goto [21].

Lemma 6.10 *Suppose that G is an algebraic group and that $g \in G$. Then there is an integer n such that $g^n \in \exp \mathrm{L}G$.*

Proof. The centralizer $Z(g)$ is a subgroup of G defined by linear equalities and hence is algebraic. Similarly, the center C of $Z(g)$ is algebraic and hence has finitely many connected components. Of course, g lies in C, and so $g^n \in C^0$ for some $n \geq 1$. Moreover, C^0 is a finite dimensional connected abelian Lie group and hence $\exp : \mathrm{L}C \to C^0$ is surjective. It follows that $g^n \in \exp \mathrm{L}C \subset \exp \mathrm{L}G$. ∎

In fact, Goto [21] proves that when G is algebraic, there is an integer n (depending only on G) such that $g^n \in \exp \mathrm{L}G$ for all $g \in G$. We do not require this stronger result in this paper.

Theorem 6.11 *Suppose that Γ is an algebraic group with compact subgroup Δ and $\sigma \in N(\Delta)$. Then there is an element $\tilde{\sigma} \in \sigma\Delta$ such that $\tilde{\sigma}^n \in \exp \mathrm{L}Z(\Sigma)$ for some $n \geq 1$.*

Proof. The subgroup $N(\Delta) \cap Z(\sigma)$ is defined by linear equalities and hence is algebraic. It follows from Lemma 6.10 that there is an integer $n \geq 1$ such that $\sigma^n = \exp \xi$ where $\xi \in \mathrm{L}(N(\Delta) \cap Z(\sigma))$. Now proceed as in the proof of Proposition 6.7. ∎

Remark 6.12 In general, we define two integers k and n, where k divides n, associated with the element $\sigma \in N(\Delta)$.

An integer k such that $\tilde{\sigma}^k \in Z(\Delta)$ for some $\tilde{\sigma} \in \sigma\Delta$.

An integer n such that $\tilde{\sigma}^n \in \exp \mathrm{L}Z(\Sigma)$ for some $\tilde{\sigma} \in \sigma\Delta$.

When Γ is algebraic, we have shown that $\tilde{\sigma}$ can be chosen so that n (and hence k) is finite.

In applications, it is sufficient to make any choice of $\tilde{\sigma}$ for which n is finite. Obviously, calculations are simplest when $\tilde{\sigma}$ is chosen so that k and n are as small as possible. However, we note that in general it is not possible to choose a single representative of σ that simultaneously minimizes k and n. For example, let $\Delta = \mathbf{SO}(2) \rtimes \mathbb{Z}_4$ where $\mathbb{Z}_4$ is generated by an element τ of order 4 that induces the automorphism $\theta \mapsto -\theta$ on elements $\theta \in \mathbf{SO}(2)$. Let $\Gamma = \Sigma = \Delta \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ is generated by an element σ of order 2 such that $\sigma\tau$ lies in the center of Γ. (In other words, σ commutes with τ and induces the same automorphism as τ on $\mathbf{SO}(2)$.) Then the minimum values of k and n are $k = 1$ (achieved by $\sigma\tau$) and $n = 2$ (achieved by σ), but there is no element of $\sigma\Delta$ that simultaneously achieves $k = 1$ and $n = 2$.

7 A generalization of Theorem 1.2

In this section, we show that many aspects of bifurcation from a relative periodic solution are captured by the methods in this paper, even when Hypothesis (H) is not satisfied. The simplest example where Hypothesis (H) fails is when $\Gamma = \Sigma = \mathbb{Z}$ and $\Delta = \mathbf{1}$. We note that Hypothesis (H) fails whenever a traveling wave is discretized in space. Such "discrete traveling waves" occur in discrete models of spatially periodic media and in numerical simulations. We begin by assuming only the weaker Hypothesis (H$'$) (which is typically valid in the situations described above), and then we dispense with this hypothesis altogether.

We continue to suppose that Γ is a finite dimensional Lie group acting smoothly and properly on a finite dimensional manifold M, and that $\mathcal{P}$ is a relative periodic solution for a Γ-equivariant vector field on M. As before, the spatiotemporal symmetry Σ is the closed subgroup of Γ generated by the spatial symmetry Δ together with an element σ.

Suppose that Hypothesis (H$'$) is satisfied. Thus, σ can be shown to satisfy $\sigma^k \in Z(\Delta)$ for some $k = 1$. We form the semidirect product $\Delta \rtimes \mathbb{Z}_{2k}$ by adjoining to Δ an element Q of order $2k$ as described in the introduction. We also form a semidirect product $\Delta \rtimes \mathbb{Z}$ by adjoining an element of infinite order (with the same multiplication).

Theorem 7.1 *Suppose that Hypothesis (H$'$) is satisfied and define $\Delta \rtimes \mathbb{Z}_{2k}$ and $\Delta \rtimes \mathbb{Z}$ as above.*
(i) *There is a neighborhood U of the relative periodic solution $\mathcal{P}$ such that*

$$U \cong \frac{(\Gamma \times_\Delta V) \times \mathbb{R}}{\mathbb{Z}} \cong \frac{\Gamma \times V \times \mathbb{R}}{\Delta \rtimes \mathbb{Z}},$$

where V is a representation of the group $\Delta \rtimes \mathbb{Z}_{2k}$.

The action of $\Delta \rtimes \mathbb{Z}$ on $\Gamma \times V \times \mathbb{R}$ is given by

$$(\gamma, v, \theta) \mapsto (\gamma\delta^{-1}, \delta v, \theta),\ \delta \in \Delta, \quad (\gamma, v, \theta) \mapsto (\gamma\sigma^{-1}, Qv, \theta + 1),$$

(ii) *The equations on U lift to skew product equations on $\Gamma \times V \times \mathbb{R}$ of the form (after reparameterizing time)*

$$\dot{\gamma} = \gamma f_\Gamma(v, \theta), \qquad \dot{v} = f_V(v, \theta), \qquad \dot{\theta} = 1, \tag{7.1}$$

where $f_\Gamma : V \times \mathbb{R} \to \mathrm{L}\Gamma$ and $f_V : V \times \mathbb{R} \to V$ are smooth vector fields satisfying $f_\Gamma(0, \theta) = 0$, $f_V(0, \theta) = 0$, for all $\theta \in \mathbb{R}$.

In addition, the skew product equations (7.1) are $\Delta \rtimes \mathbb{Z}$-equivariant with respect to the above action of $\Delta \rtimes \mathbb{Z}$.

Finally, f_Γ and f_V are general vector fields satisfying the above conditions.

Proof. We proceed as in Subsections (b) and (c) of Section 2, except that we do not pass to a comoving frame (so $\xi = 0$). In particular, the twisted equivariant linear map $Q : V \to V$ is chosen to have order $2k$. ∎

Corollary 7.2 *Assume the setup of Theorem 7.1, and let $S^1 = \mathbb{R}/2k\mathbb{Z}$. Then the skew product equations restricted to $V \times \mathbb{R}$ define a general smooth $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant vector field on $V \times S^1$ of the form*

$$\dot{v} = f_V(v, \theta), \qquad \dot{\theta} = 1,$$

where $f_V(0, \theta) = 0$ for all $\theta \in S^1$. The action of $\Delta \rtimes \mathbb{Z}_{2k}$ is given by

$$(v, \theta) \mapsto (\delta v, \theta),\ \delta \in \Delta, \quad (v, \theta) \mapsto (Qv, \theta + 1),$$

It follows that provided Hypothesis (H′) is satisfied, modulo drifts along group orbits, bifurcation from a relative periodic solution $\mathcal{P}$ reduces to bifurcation from an isolated periodic solution with spatial symmetry Δ and spatiotemporal symmetry group $\Delta \rtimes \mathbb{Z}_{2k}$.

Finally, we show that a similar result is true even when Hypothesis (H′) fails, except that now the integer k depends on the specific representation of Δ on the cross-section V.

Theorem 7.3 *Let V be a Δ-invariant cross-section to the relative periodic solution at u_0. Let $L : V \to V$ be the linear map defined, as in Section 2(b), by $L = P_0\sigma^{-1}(\mathrm{D}\Phi_1)_{u_0}$.*

Then the element σ can be chosen (in $\sigma\Delta$) so that for some $k \geq 1$,

- *(a) $\sigma^k \delta v = \delta \sigma^k v$ for all $\delta \in \Delta$, $v \in V$, and*
- *(b) There is a linear map $A : V \to V$ satisfying $A^{2k} = I$ such that $L^{-1}A$ is equivariantly isotopic to the identity.*

With these choices of σ, k and $Q = A^{-1}$, the conclusions of Theorem 7.1 hold.

Proof. The actions of Δ and L on V generate a group G acting linearly on V. In general, the action of G is not faithful. However, the quotient of G by the kernel of the action is a matrix group and Theorem 6.4 applies to this quotient. Hence σ can be chosen so that part (a) holds for some k. (In other words, σ^k lies in the centralizer of Δ modulo the kernel of the action of Δ on V.) Part (b) follows from Lemma 2.2. ∎

We stress that, in contrast to the case when Hypothesis (H′) is satisfied, the choices of σ, k and Q in Theorem 7.3 depend on the specific representation of Δ on V.

Remark 7.4 (a) This theorem enables us to study bifurcation from a relative periodic solution for any finite dimensional Lie group Γ. In particular, modulo drifts along continuous group orbits, the entire bifurcation is reduced to bifurcation from an isolated periodic solution for the group $\Delta \rtimes \mathbb{Z}_{2k}$ generated by the actions of Δ and Q on V.

(b) It is a consequence of Theorem 7.3 that situations such as that in Example 6.6 do not arise in the context of relative periodic solutions. That is, the existence of the twisted equivariant nonsingular linear map L arising from the flow precludes certain representations of Δ. In particular, in the case of the group $\Gamma = T^2 \rtimes \mathbf{SL}(2, \mathbb{Z})$ the subgroup $\Delta = T^2$ is forced to act trivially on the subspace V.

A Appendix: Equivariant linear maps

In the proof of Theorem 5.12, we exploited a standard result about nonsingular equivariant linear maps, namely that the square of a nonsingular equivariant linear map is connected to the identity by a path of such maps. Presumably this result is well known, but we give a proof in this appendix since a complete proof is hard to find elsewhere.

Suppose that G is a compact Lie group acting orthogonally on a real vector space X and let $\mathrm{Hom}(X)$ denote the space of equivariant linear maps. Let $Z(X) \subset \mathrm{Hom}(X)$ denote the group of equivariant nonsingular linear maps, and let $Z(X)^0$ denote the connected component of the identity in $Z(X)$. In this appendix, we prove that $B^2 \in Z(X)^0$ for all $B \in Z(X)$.

Let V be a G-irreducible subspace of X. Then $\mathrm{Hom}(V)$ is a real division ring $\mathcal{D}$ isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The irreducible subspace V is accordingly said to be of real, complex or quaternionic type.

The space X can be decomposed (nonuniquely) as a direct sum of irreducible subspaces. There is also a unique *isotypic decomposition* $X = \oplus W_j$, where each *isotypic component* W_j is a direct sum of isomorphic irreducible subspaces and distinct isotypic components consist of distinct irreducible subspaces. We say that an isotypic component W is of real, complex or quaternionic type if W consists of irreducible subspaces of real, complex or quaternionic type.

An important property of the isotypic decomposition is that the isotypic components are preserved by any equivariant linear map. In other words, $\mathrm{Hom}(X) = \oplus \mathrm{Hom}(W_j)$. Next, let W be a fixed isotypic component. If we write $W = \oplus_1^m V$, then the isomorphism $\mathrm{Hom}(V) \cong \mathcal{D}$ induces an isomorphism $\mathrm{Hom}(W) \cong M_m(\mathcal{D})$ where $M_m(\mathcal{D})$ is the space of $m \times m$ matrices with entries in $\mathcal{D}$. In particular, the eigenvalues of $B \in \mathrm{Hom}(W)$ are given by the eigenvalues of the corresponding matrix $a \in M_m(\mathcal{D})$ but with multiplicity $\dim_{\mathcal{D}} V$. Altogether, we have a complete description of $\mathrm{Hom}(X)$.

Proposition A.1 *Suppose that W is an isotypic component.*

(a) If W is of complex or quaternionic type, then $Z(W)$ is connected.

(b) If W is of real type, then $Z(W)$ has two connected components.

Proof. The isomorphism $\mathrm{Hom}(W) \cong M_m(\mathcal{D})$ induces an isomorphism $Z(W) \cong \mathbf{GL}(m, \mathcal{D})$. Of course, $\mathbf{GL}(m, \mathbb{R})$ has two connected components and $\mathbf{GL}(m, \mathbb{C})$ has one connected component, as can be seen easily from the Jordan normal forms for real and complex matrices. The corresponding result in the quaternionic case follows similarly from the Jordan normal form for quaternionic matrices which can be found in [38]. ∎

Corollary A.2 *Suppose that G acts orthogonally on X and that $B \in Z(X)$. Then $B^2 \in Z(X)^0$.*

Proof. Let $X = \oplus W_j$ be the isotypic decomposition of X, and write $B = \oplus B_j$ where $B_j \in Z(W_j)$. By Proposition A.1, $B_j^2 \in Z(W_j)^0$ and hence $B^2 \in Z(X)^0$. ∎

Finally, we note that the results in this appendix go through without change if we restrict to spaces of orthogonal equivariant linear maps.

References

[1] C.D. Andereck, S.S. Liu, and H.L. Swinney, *Flow regimes in a circular Couette system with independently rotating cylinders*, J. Fluid Mech. **164** (1986), 155–183.

[2] V.I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer, New York, 1988.

[3] D.K. Arrowsmith and C.M. Place, *An Introduction to Dynamical Systems*, Cambridge University Press, Cambridge, 1990.

[4] P. Ashwin and I. Melbourne, *Noncompact drift for relative equilibria and relative periodic orbits*, Nonlinearity **10** (1997), 595–616.

[5] P. Ashwin, I. Melbourne, and M. Nicol, *Drift bifurcations of relative equilibria and transitions of spiral waves*, preprint UH/MD-251, University of Houston, 1998.

[6] A. Bayliss, B.J. Matkowsky, and H. Riecke, *Symmetries in modulated traveling waves in combustion: jumping ponies on a merry-go-round*, Pattern Formation: Symmetry Methods and Applications (J. Chadam and W. Langford, eds.), Fields Institute Communications, vol. 5, AMS, 1995, pp. 19–43.

[7] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Graduate Texts in Mathematics, vol. 98, Springer, New York, Berlin, Heidelberg, Tokyo, 1985.

[8] R. Carter, G. Segal, and I. MacDonald, *Lectures on Lie Groups and Lie Algebras*, London Mathematical Society, Student Texts, vol. 32, Cambridge University Press, 1995.

[9] P. Chossat and M. Golubitsky, *Iterates of maps with symmetry*, SIAM J. Math. Anal. **19** (1988), 1259–1270.

[10] E.A. Ermakova, A.M. Pertsov, and E.E. Shnol, *On the interaction of vortices in two-dimensional active media*, Physica D **40** (1989), 185–190.

[11] B. Fiedler, *Global Bifurcation of Periodic Solutions with Symmetry*, Lecture Notes in Math., vol. 1309, Springer, Berlin, 1988.

[12] B. Fiedler, B. Sandstede, A. Scheel, and C. Wulff, *Bifurcation from relative equilibria to noncompact group actions: Skew products, meanders, and drifts*, Doc. Math. J. DMV **1** (1996), 479–505.

[13] B. Fiedler and D. Turaev, *Normal forms, resonances, and meandering tip motions near relative equilibria of Euclidean group actions*, Arch. Rat. Mech. Anal. (1998), to appear.

[14] M. J. Field, *Symmetry Breaking for Compact Lie Groups*, Memoirs of the AMS, vol. 574, Amer. Math. Soc., Providence, RI, 1996.

[15] M.J. Field, *Equivariant dynamical systems*, Trans. Amer. Math. Soc. **259** (1980), 185–205.

[16] M.J. Field, *Local structure for equivariant dynamics*, Singularity Theory and its Applications, Part II (M. Roberts and I. Stewart, eds.), Lecture Notes in Math., vol. 1463, Springer, New York, 1991, pp. 142–166.

[17] M. Golubitsky, V. LeBlanc, and I. Melbourne, *Hopf bifurcation from rotating waves and patterns in physical space*, preprint UH/MD-242, University of Houston, 1998.

[18] M. Golubitsky, I. Stewart, and D. Schaeffer, *Singularities and Groups in Bifurcation Theory, Vol. II*, Applied Mathematical Sciences, vol. 69, Springer, Berlin, 1988.

[19] M. Gorman, C.F. Hamill, M. el Hamdi, and K.A. Robbins, *Rotating and modulated rotating states of cellular flames*, Combustion Sci. and Technology. **98** (1994), 25–35.

[20] M. Gorman and H.L. Swinney, *Spatial and temporal characteristics of modulated waves in the circular Couette system*, J. Fluid Mech. **117** (1982), 123–142.

[21] M. Goto, *Index of the exponential map of a semialgebraic group*, Amer. J. Math. **100** (1978), 837–843.

[22] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Applied Mathematical Sciences, vol. 42, Springer, New York, Heidelberg, Berlin, 1990.

[23] M. Krupa, *Bifurcations of relative equilibria*, SIAM J. Math. Anal. **21** (1990), 1453–1486.

[24] J.S.W. Lamb, *Local bifurcations in k-symmetric dynamical systems*, Nonlinearity **9** (1996), 537–557.

[25] J.S.W. Lamb and I. Melbourne, *Bifurcation from discrete rotating waves*, preprint UH/MD-252, University of Houston, 1998.

[26] J.S.W. Lamb and I. Melbourne, *Bifurcation from periodic solutions with spatiotemporal symmetry*, Pattern Formation in Continuous and Coupled Systems (M. Golubitsky, D. Luss, and S. Strogatz, eds.), IMA, Springer, 1998, to appear.

[27] J.S.W. Lamb and G.R.W. Quispel, *Reversing k-symmetries in dynamical systems*, Physica D **73** (1994), 277–304.

[28] A.S. Landsberg and E. Knobloch, *Direction-reversing travelling waves*, Phys. Lett. A **159** (1991), 17–20.

[29] G. Li, Q. Ouyang, V. Petrov, and H.L. Swinney, *Transition from simple rotating chemical spirals to meandering and traveling spirals*, Phys. Rev. Lett. **77** (1996), 2105–2108.

[30] N. Nicolaisen and B. Werner, *Some remarks on period doubling in systems with symmetry*, ZAMP **46** (1995), 566–579.

[31] A.L. Onishchik and E.B. Vinberg, *Lie Groups and Lie Algebras III*, Encyclopaedia of Mathematical Sciences, vol. 42, Springer, 1991.

[32] A.M. Rucklidge and M. Silber, *Bifurcations of periodic orbits with spatio-temporal symmetries*, Nonlinearity **11** (1998), 1435–1455.

[33] B. Sandstede, A. Scheel, and C. Wulff, *Dynamics of spiral waves on unbounded domains using center-manifold reductions*, *J. Diff. Eq.* **141** (1997), 122–149.

[34] B. Sandstede, A. Scheel, and C. Wulff, *Bifurcations and dynamics of spiral waves*, J. Nonlin. Sci. **9** (1998), to appear.

[35] G.S. Skinner and H.L. Swinney, *Periodic to quasiperiodic transition of chemical spiral rotation*, Physica D **48** (1991), 1–16.

[36] A. Vanderbauwhede, *Equivariant period doubling*, Advanced Topics in the Theory of Dynamical Systems (G. Fusco et al, ed.), Notes Rep. Math. Sci. Engrg., vol. 6, Academic Press, Boston, 1989, pp. 235–246.

[37] A. Vanderbauwhede, *Period-doublings and orbit-bifurcations in symmetric systems*, Dynamical Systems and Ergodic Theory, Banach Center Publ, vol. 23, Polish Scientific Publishers, Warsaw, 1989, pp. 197–208.

[38] N.A. Wiegmann, *Some theorems on matrices with real quaternionic elements*, Canad. J. Math. **7** (1955), 191–201.

[39] A.T. Winfree, *The Geometry of Biological Time*, Biomathematics, vol. 8, Springer, New York, 1980.

[40] V.S. Zykov, *Cycloidal circulation of spiral waves in an excitable medium*, Biofizika **31** (1986), 862–865.