Time-reversal symmetry in dynamical systems: a survey

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Abstract

In this paper we survey the topic of time-reversal symmetry in dynamical systems. We begin with a brief discussion of the position of time-reversal symmetry in physics. After defining time-reversal symmetry as it applies to dynamical systems, we then introduce a major theme of our survey, namely the relation of time-reversible dynamical sytems to equivariant and Hamiltonian dynamical systems. We follow with a survey of the state of the art on the theory of reversible dynamical systems, including results on symmetric periodic orbits, local bifurcation theory, homoclinic orbits, and renormalization and scaling. Some areas of physics and mathematics in which reversible dynamical systems arise are discussed. In an appendix, we provide an extensive bibliography on the topic of time-reversal symmetry in dynamical systems.

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1 Introduction

Time-reversal symmetry is one of the fundamental symmetries discussed in natural science. Consequently, it arises in many physically-motivated dynamical systems, in particular in classical and quantum mechanics.

The aim of this paper is to give a brief and compact survey of the state of the art with regards to time-reversal symmetry in dynamical systems theory. That is, we consider ordinary differential equations and diffeomorphisms possessing (what we call) *reversing* symmetries.

We are aware that the interest in time-reversal symmetry goes beyond this confined setting. For instance, there is extensive work on time-reversal symmetry in statistical and quantum mechanics that falls outside the scope of our survey. Our survey also does not include a discussion on reversible cellular automata. For further reading in these areas we recommend the books by Brush [6], Sachs [22] and Hawking [14], and the survey paper by Toffoli & Margolus [23].

Our survey is largely self-contained and accompanied by an extensive bibliography in Appendix A. However, in areas where other good recent surveys are available (most of them in this special volume: [Sevryuk, 1998; Champneys, 1998; Hoover, 1998]), our discussion will be brief and will refer to those papers for more details. We will focus here on surveying areas of research that are complimentary to those reviewed elsewhere.

Needless to say, we aim to give a balanced account of the work and interests in the field of time-reversal symmetry in dynamical systems. However, we realize very well that our survey is subjective and we would like to apologize to those colleagues who might find their work under- or misrepresented. Also, rather than trying to discuss results in this field in detail, we have aimed at giving the reader a taste of the state of the art. Our bibliography in Appendix A is inevitably incomplete and the process of compiling the bibliography is without end.[1] However, we hope that our bibliography will provide an encouragement and opportunity for the reader to explore further from there.

The paper is organized as follows. In Section 2 we briefly discuss the position of time-reversal symmetry in physics. In Section 3 we introduce the setting of our survey, defining time-reversal symmetry in dynamical systems and sketching its relation to equivariant and Hamiltonian dynamical systems. In Section 4 we survey the state of the art on the theory of reversible dynamical systems, including results on symmetric periodic orbits, local bifurcation theory, homoclinic orbits, and renormalization and scaling. In Section 5 we briefly discuss some areas of applications in physics and mathematics that have stimulated the research into reversible dynamical systems. Our concluding section is devoted to an outlook. Appendix A contains an extended bibliography on time-reversal symmetry in dynamical systems. References to this bibliography are separated by style ([author,year]) from other references ([number]).

As a guide to the reader, we note that the first three sections provide a nontechnical introduction, aimed at a non-specialized audience. In contrast, Sections 4 and 5 contain more details and references.

[1] Regular updates of this bibliography will be made available at
`http://www.maths.warwick.ac.uk/~lamb/revbib.html`.

2 Time-reversal symmetry in physics

Before addressing the topic of time-reversal symmetry in dynamical systems in §3, in this section we will briefly discuss the position of time-reversal symmetry in physics, i.e. in classical mechanics, thermodynamics and quantum mechnics.

2.1 Time-reversal symmetry in classical mechanics

The conventional notion of time-reversal symmetry relates to observations of physical phenomena.

To fix the discussion, consider the example of the dynamics of a classical *ideal* pendulum, that experiences no energy loss due to friction.

We now propose the following experiment: we let the pendulum swing, film it, and watch it using a projector that plays the film backward (in the reverse direction). So we see the pendulum moving backward in time. If we are not familiar with the original film, then as a viewer it would be impossible to tell that the film was actually played in reverse. This is because the motion on the reverse film also corresponds to a possible motion of the same pendulum. Namely, the reverse motion satisfies the same *laws of motion* as the forward motion. The only difference between the motion depicted on the forward and reverse versions of the film is the initial position and speed of the pendulum at the point where we start showing the movie.

If for a motion picture of a mechanical system one cannot decide whether it is shown in the forward or reverse direction, the system is said to have *time-reversal symmetry.*

When we consider the more realistic physical situation of a swinging pendulum in the presence of friction, we *can* tell the difference between a forward and a reverse film of this pendulum. Namely, the original (forward) film will show the swinging pendulum losing amplitude until it comes to a stand-still. However, the film in reverse direction shows a swinging pendulum whose amplitude is increasing in time. The latter film is clearly *unphysical* as it does not satisfy the natural laws of motion anymore (assuming there is no hidden source of energy feeding the pendulum). The presence of friction breaks the *time-reversal symmetry* of the ideal pendulum.

The time-reversal symmetry described in this example arises very frequently in classical mechanics. Although in nature we hardly ever encounter mechanical systems with perfect time-reversal symmetry, in *theory* a truly isolated pendulum has time-reversal symmetry. The friction and energy transfer are merely due to the coupling of the pendulum with its environment.

In the Hamiltonian formulation of classical mechanics, we describe the system with variables $(\mathbf{q}, \mathbf{p})$, where $\mathbf{q}$ is a vector describing the position of the system and $\mathbf{p}$ a vector describing its momentum.

In its simplest form, the Hamiltonian $H(\mathbf{q}, \mathbf{p})$ is a function which generates the equations of motion via:

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \ \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}. \tag{2.1}$$

The classical notion of time-reversal symmetry as discussed above is directly related to a symmetry property of the Hamiltonian,

$$H(\mathbf{q}, \mathbf{p}) = H(\mathbf{q}, -\mathbf{p}). \tag{2.2}$$

Namely, if the Hamiltonian satisfies (2.2), then the equations of motion (2.1) are invariant under the transformation

$$R_0 : \ (\mathbf{q}, \mathbf{p}, t) \mapsto (\mathbf{q}, -\mathbf{p}, -t). \tag{2.3}$$

In turn, this implies that when $(\mathbf{q}(t), \mathbf{p}(t))$ is a trajectory in phase space describing a possible motion of the system with initial position and momentum $(\mathbf{q}_0, \mathbf{p}_0)$, then so is $(\mathbf{q}(-t), -\mathbf{p}(-t))$ with initial condition $(\mathbf{q}_0, -\mathbf{p}_0)$.

In configuration (position) space this means that if we have a trajectory $\mathbf{q}(t)$, then we also have a trajectory $\mathbf{q}(-t)$. This is precisely what we see when we play a film of a time-reversible system in reverse.

2.2 Thermodynamics

Let us now consider a macroscopic number of classical particles and describe their collective behaviour. In fact, such a system is in principle described by Hamiltonian equations of motion where $\mathbf{q}$ and $\mathbf{p}$ describe the positions and momenta of N particles, where N is a very large number (e.g. in the order of Avogadro's number 10^{24}).

Despite the time-reversal symmetry property of the equations of motion, the collective behaviour of a macroscopic number of classical particles displays a clear direction of time, i.e. if $\mathbf{q}(t)$ is a likely trajectory of the system then $\mathbf{q}(-t)$ is not necessarily ! As an example, consider the motion of a macroscopic number of gas molecules in one of the two compartments of a tank. When we open up a connection between the two compartments, molecules from one compartment will flow to the second compartment and in the end they will distribute evenly over the two compartments. If one watches a film of this dynamical process in reverse, one observes the gas flowing from an evenly distributed state towards a state in which all the gas is in one of the two compartments of the tank. Despite the theoretical possibility of this happening, the realistic chance of this occurring is extremely small.

In statistical mechanics, when we want to describe the bulk dynamics of many particles ($N \to \infty$), there is a true sense of direction of time. The most well-known result in this direction is Boltzmann's second law of thermodynamics, saying that entropy is a monotonically increasing function of time [2].

This result needs careful interpretation, and it is not surprising that this result has led to a lot of confusion. Loschmidt challenged Boltzmann by pointing out that his result "violated" the time-reversal symmetry of the (microscopic) equations of motion of the particles concerned [18]. In recognition of his critique, the situation whereby an ensemble of particles with time-reversible dynamics displays irreversible behaviour is called *Loschmidt's paradox*.

A first solution to the paradox was proposed by Gibbs [11], who gave an explanation involving the course-grained structure of the phase-space. However, to the present day, many papers are written that provide an explanation of the paradoxical situation in which a system that has time-reversal symmetry on a microscopic scale breaks this symmetry in its collective macroscopic behaviour. A popular resolution of Loschmidt's paradox is to argue that despite the reversibility of the equations of motion, not all solutions need possess the full time-reversal symmetry. A different view is presented by Kumicak & De Hemptinne [17] in this volume. For an historical account, see Brush [6].

A general discussion of this area is outside the scope of our survey (but we refer the reader to the work of Prigogine and the Brussels school on reversibility/irreversibility and arrows of time in physics and chemistry [21; 20]). A recent approach in which irreversible dynamics of nonequilibrium thermodynamical systems is modelled by reversible dynamical systems has attracted a lot of interest, see e.g. § 5 and the papers [Dellago & Posch, 1998; Gallavotti, 1998; Hoover, 1998] in this volume.

2.3 Time-reversal symmetry in quantum mechanics

In the 1930's, Wigner [25] successfully introduced a quantum mechanical version of the classical conventional time-reversal operator. With it, he explained the two-fold degeneracy of energy levels that was reported by Kramers [16] in systems with an odd number of electrons in the absence of a magnetic field. In the presence of a magnetic field the time-reversal symmetry is broken and the degeneracy disappears yielding a splitting of energy bands, cf. also Herring [15]. Time-reversal symmetry is also important in quantum field theories for elementary particle physics, cf. [22].

With the growing interest in chaotic dynamics in the 1980's, there was also a growing interest in quantum mechanical systems whose classical limit displays chaotic behaviour. In this field (called *quantum chaology* by Berry [1]), time-reversal symmetry enters in the analysis of level statistics. Random matrix theory predicts that the rate of energy-level repulsion is classified by the presence or absence of time-reversal symmetry. These predictions have been discussed by many, and for an introduction, see Haake [13]. We note that, despite the success of group representation theory in quantum mechanics [26], the role of comparable symmetry methods in quantum-chaology appears to be relatively unexplored. For a recent exception see [8], who discuss the use of symmetries (but not time-reversal symmetries !) in the calculation of zeta-functions.

3 Time-reversal symmetry in dynamical systems

We will now give a more precise mathematical description of time-reversal symmetry in the setting of dynamical systems, as considered in this survey. We also give an historical account of its origin.

We consider two types of dynamical systems, with continuous time ($t \in \mathbb{R}$) and discrete time ($t \in \mathbb{Z}$) on some phase space Ω. Continuous time systems are taken to be flows of vector fields. Discrete time dynamical systems are taken to be generated by an invertible map f. In most applications of interest Ω will be a manifold, e.g. $\Omega = \mathbb{R}^n$.

In the continuous time context we consider autonomous ordinary differential equations of the form

$$\frac{d\mathbf{x}}{dt} = F(\mathbf{x}), \ \ (\mathbf{x} \in \Omega). \tag{3.1}$$

where $F : \Omega \mapsto T\Omega$ is a (smooth, continuous) vector field. The dynamics of (3.1) is given by a one-parameter family of evolution operators

$$\varphi_t : \Omega \mapsto \Omega, \ \varphi_t : \mathbf{x}(\tau) \mapsto \varphi_t(\mathbf{x}(\tau)) = \mathbf{x}(\tau + t), \tag{3.2}$$

such that[2]

$$\varphi_{t_1} \circ \varphi_{t_2} = \varphi_{t_1+t_2} \ \text{ for all } t_1, t_2 \in \mathbb{R}. \tag{3.3}$$

[2] $\circ$ denotes composition.

We now say that an invertible (smooth, continuous) map $R : \Omega \mapsto \Omega$ is a *reversing symmetry* of (3.1) when

$$\frac{dR(\mathbf{x})}{dt} = -F(R(\mathbf{x})), \tag{3.4}$$

or equivalently, when

$$dR|_{\mathbf{x}} \cdot F(\mathbf{x}) = -F(R(\mathbf{x})), \tag{3.5}$$

where $dR|_{\mathbf{x}}$ denotes the (Frechet) derivative of R in $\mathbf{x}$. In terms of the evolution operator φ_t, (3.4)-(3.5) imply

$$R \circ \varphi_t = \varphi_{-t} \circ R = \varphi_t^{-1} \circ R \text{ for all } t \in \mathbb{R}. \tag{3.6}$$

In the context of classical mechanics, where the ordinary differential equations are derived from a Hamiltonian $H(\mathbf{q}, \mathbf{p})$, the conventional reversing symmetry is given by

$$R(\mathbf{q}, \mathbf{p}) = (\mathbf{q}, -\mathbf{p}). \tag{3.7}$$

Note that in this particular case R is an *involution* (i.e. $R^2 = \mathrm{id}$), and R is anti-symplectic.

By analogy to the definition (3.6) in the case of flows, we call an invertible map $R : \Omega \mapsto \Omega$ a *reversing symmetry* of an invertible map $f : \Omega \mapsto \Omega$, whenever[3]

$$R \circ f = f^{-1} \circ R. \tag{3.8}$$

The notion of reversing symmetries for autonomous flows extends in a natural way to nonautonomous flows,

$$\frac{d\mathbf{x}}{dt} = F(\mathbf{x}, t). \tag{3.9}$$

Namely, we call $R_a : (\mathbf{x}, t) \mapsto (R(\mathbf{x}), -t + a)$ a reversing symmetry of (3.9) whenever (3.9) is invariant under the transformation R_a (for some $a \in \mathbb{R}$), i.e.

$$\frac{dR(\mathbf{x})}{dt} = -F(R(\mathbf{x}), -t + a). \tag{3.10}$$

Note that by introducing a new variable $\tau = t - a/2$, the extended differential equation $d(\mathbf{x}, \tau)/dt = (F'(\mathbf{x}, \tau), 1)$ (with $F'(\mathbf{x}, \tau) := F(\mathbf{x}, \tau + a/2)$) is autonomous and has reversing symmetry $R_0 : (\mathbf{x}, \tau) \mapsto (R(\mathbf{x}), -\tau)$.

The presence and importance of time-reversal symmetry was recognized in the early days of dynamical systems by Birkhoff. He utilized it in his study of the restricted three body problem in classical mechanics [Birkhoff, 1915]. In particular, he noted that a map f with an involutory reversing symmetry R can always be written as the composition of two involutions

$$f = R \circ T, \text{ where } R^2 = T^2 = \mathrm{id}. \tag{3.11}$$

[3]Usually – but not always – one is interested in maps f and R that are not just invertible, but also homeomorphisms or diffeomorphisms.

In this context, note that when R is not an involution one readily verifies that the decomposition property (3.11) generalizes to

$$f = R \circ T, \text{ where } R^2 \circ T^2 = \mathrm{id}. \tag{3.12}$$

In flows of nonautonomous vector fields (3.9) when $F(\mathbf{x}, t)$ is periodic in time, i.e. $F(\mathbf{x}, t) = F(\mathbf{x}, t+1)$ (with period scaled to 1), then in a natural way the time-one return map of such a flow is autonomous. Moreover, it is readily checked that when the nonautonomous system is invariant under R_a, then the time-one return map with respect to the surface of section $t = a/2$ has reversing symmetry R. A similar result also applies to local return maps for R-symmetric periodic orbits of autonomous flows with reversing symmetry R.

After the work of Birkhoff, time-reversal symmetry did not receive much attention until the 1960s [DeVogelaere, 1958; Heinbockel & Struble, 1965; Moser, 1967; Bibikov & Pliss, 1967; Hale, 1969]. In particular, Hale described the property of time-reversal symmetry as *property E*.[4]

Devaney [Devaney, 1976] noted that many consequences of conventional time-reversal symmetry are shared by dynamical systems which have a different type of involutory reversing symmetry than the conventional anti-symplectic one (3.7). This led him to a definition of *reversible* systems in which the involutory nature of the time-reversal operator R was central, together with the fact that R should fix a subspace half the dimension of the phase space. Later, Arnol'd and Sevryuk [Arnol'd, 1984; Arnol'd & Sevryuk, 1986] relaxed this further to allow for any involutory reversing symmetry. The latter definition of *reversibility* was adopted by many, and hence for a map was taken to be synonymous with the decomposition property (3.11).

Arnol'd and Sevryuk [Arnol'd, 1984; Arnol'd & Sevryuk, 1986] remarked that systems with reversing symmetries need not have an involutory reversing symmetry (for some examples, see [Arnol'd & Sevryuk, 1986; Lamb, 1996a; Baake & Roberts, 1997]). In quantum mechanics, the importance of noninvolutory time-reversal symmetries was long before acknowledged by Wigner [26]. Arnol'd and Sevryuk proposed to call systems with only noninvolutory reversing symmetries *weakly reversible*. They found that many results for reversible systems actually also hold for weakly reversible systems, by showing that in many problems the reversing symmetry enters the analysis with an effectively involutory action.

In this respect it is interesting to note that if a linear system in $\mathbb{R}^n$ possesses a reversing symmetry it also possesses a linear involutory reversing symmetry [Sevryuk, 1986]. (This follows directly from Jordan normal form theory.) However one should be careful interpreting this result. For instance, when considering continuous families of matrices possessing a given noninvolutory reversing symmetry (such as in reversible linear normal form theory [Hoveijn, 1996; Lamb & Roberts, 1997]) this observation is not particularly useful as, for instance, the form of the involutory reversing symmetry may change discontinuously. Also, it should be stressed that this implication does not hold in general. For example it does not hold for nonlinear systems in $\mathbb{R}^n$ [Lamb, 1994a; Lamb, 1996a] and not even for linear diffeomorphisms of the 2-torus [Baake & Roberts, 1997] (cf. Example 3.6 below).

In the light of a more general approach towards symmetry properties of dynamical systems (see § 3.1), it turns out to be unnecessarily restrictive to explicitly mention the nature of

[4]Despite the fact that Hale did not publish many papers on time-reversal symmetry in dynamical systems, his interest has been a catalyst for further research, cf. e.g. the acknowledgements in [Kirchgässner, 1982a] and [Vanderbauwhede, 1990b].

the reversing symmetry in a definition of reversibility (and so distinguish between reversible and *weakly* reversible systems). Therefore, we define

Definition 3.1 (Reversible Dynamical System) A dynamical system is called reversible when it possesses a reversing symmetry R satisfying (3.4), (3.8), or (3.10) for autonomous flows, maps, or nonautonomous flows, respectively.

In the literature there is sometimes confusion about the use of terminology. Sometimes, a system is called *reversible* when its inverse exists. This notion of *invertibility* differs from the notion of *reversibility* adopted here. In particular, note that all reversible systems are invertible, but not all invertible systems are reversible (because not all invertible systems have a reversing symmetry).

We now list some examples of reversible dynamical systems:

Example 3.2 All Hamiltonian systems with Hamiltonian $H(\mathbf{q},\mathbf{p})$ satisfying (2.2) with an antisymplectic R of the form (3.7).

Example 3.3 All oscillation equations of the form

$$\frac{d^2}{dt^2}\mathbf{q} = F(\mathbf{q}), \quad F : \mathbb{R}^m \mapsto \mathbb{R}^m. \tag{3.13}$$

When (3.13) is rewritten as a first order system in the variables q_i and $\frac{d}{dt}q_i$ in $\mathbb{R}^{2m}$, R is the involution that changes the signs of the $\frac{d}{dt}q_i$.

Example 3.4 In many partial differential equations, the equations governing steady-state solutions are reversible.

For example, Malomed & Tribelski [Malomed & Tribelski, 1984] considered a class of partial differential equations, one of which

$$\tfrac{\partial}{\partial t}\xi + \tfrac{\partial^4}{\partial x^4}\xi + 2\alpha\tfrac{\partial^2}{\partial x^2}\xi + \xi + (\tfrac{\partial}{\partial x}\xi)^2 = 0, \tag{3.14}$$

describes the evolution of a gas flame under certain physical conditions. This equation is not reversible with respect to the time-variable. However, the steady-state solutions are described by (3.14) with $\frac{\partial}{\partial t}\xi = 0$. The resulting steady-state ordinary differential equation is reversible with respect to the space variable x. Namely, the fourth-order ordinary differential equation can be written as a system of four first order equations in the variables ξ, $\frac{\partial}{\partial x}\xi$, $\frac{\partial^2}{\partial x^2}\xi$, and $\frac{\partial^3}{\partial x^3}\xi$, and this dynamical system is reversible with respect to the involution

$$R : (\xi,\ \tfrac{\partial}{\partial x}\xi,\ \tfrac{\partial^2}{\partial x^2}\xi,\ \tfrac{\partial^3}{\partial x^3}\xi) \to (\xi, -\ \tfrac{\partial}{\partial x}\xi,\ \tfrac{\partial^2}{\partial x^2}\xi, -\ \tfrac{\partial^3}{\partial x^3}\xi).$$

In a similar way, one finds that all autonomous even-order (odd-order) ordinary differential equations in which the odd (even) derivatives occur only in even combinations are reversible when rewritten as a first order system. The reversing symmetry R is the transformation of the even-dimensional (odd-dimensional) phase space that corresponds to changing the sign of the variables corresponding to odd (even) derivatives.

Example 3.5 Symmetric difference equations of the form

$$x_{n+1} + x_{n-1} = f(x_n), \tag{3.15}$$

may arise as a discretization of $\frac{d^2}{dt^2}x = 2x - f(x)$, but may also arise due to spatial symmetry properties of physical models on a chain (n labels the position on the chain).

A well known example of a mapping of the form (3.15) arises in the study of stationary states of a chain of coupled oscillators with a convex nearest neighbour interaction potential. By Newton's *action=reaction* principle this interaction potential should be invariant with respect to interchanging x_n and x_{n+1}.

For instance, in the case of the Frenkel-Kontorova model, the total (interaction + background) potential is given by $\sum_n (\frac{1}{2}(x_n - x_{n+1})^2 + v\cos x_n)$, and the stationary states satisfy (3.15) with $f(x_n) = 2x_n - v\sin(x_n)$, which is equivalent to the area-preserving Chirikov-Taylor standard mapping.

Introducing new variables, $p_n := x_n$ and $q_n := x_{n-1}$ the system (3.15) can be written as a mapping of the plane

$$\begin{cases} p_{n+1} &= f(p_n) - q_n, \\ q_{n+1} &= p_n. \end{cases} \tag{3.16}$$

This mapping is reversible with respect to the involution $R(p,q) = (q,p)$. This is a direct consequence of the fact that (3.15) is invariant under the transformation $x_{n-1} \leftrightarrow x_{n+1}$. In the context of the Frenkel-Kontorova model this is in turn a direct consequence of the symmetry of the interaction potential.

Remarkably, (3.16) is not only reversible, but also area-preserving. Many area-preserving (symplectic) mappings studied in the literature are reversible (eg. the well-studied area-preserving Hénon mapping, cf. [Roberts & Quispel, 1992] and references therein).

Example 3.6 The hyperbolic toral automorphism given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 7 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pmod 1. \tag{3.17}$$

is reversible with the order four reversing symmetry R given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pmod 1. \tag{3.18}$$

In fact, the map (3.17) has *no* involutory reversing symmetry within the group of toral homeomorphisms.

We note that the reversibility of hyperbolic toral automorphisms has no obvious physical cause. The symmetry properties of hyperbolic toral automorphisms follow directly from the structure of the matrix group $Gl(2,\mathbb{Z})$ [Baake & Roberts, 1997].

The above examples illustrate the fact that reversible dynamical systems arising in the literature obtain their reversibility due to a variety of reasons. In particular, we observe:

- *Reversibility in time* arising due to a natural assumption of time-reversibility of the equations of motion (cf. Example 3.2 and Example 3.3)

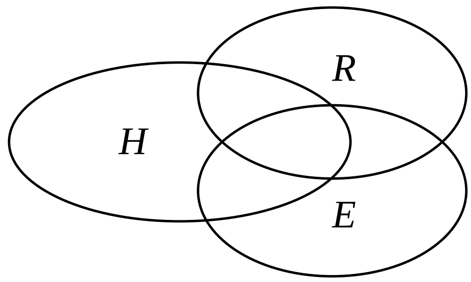

Figure 1: Schematic diagram indicating the intersections of reversible dynamical systems (R), equivariant dynamical systems (E), and Hamiltonian dynamical systems (H).

- *Reversibility in space* arising due to natural assumptions of spatial symmetries of a physical model (cf. Example 3.4 and Example 3.5)
- *Reversibility* arising due to the specific structure of a mathematical problem under consideration (cf. Example 3.6).

For more examples in the above three categories, see § 5.

There are generally two perspectives from which reversible dynamical systems are considered. On the one hand they can be treated from a symmetry perspective, as reversible systems are defined in terms of a symmetry property. On the other hand, historically the interest in reversible systems has often been in the context of Hamiltonian systems. Firstly this is because many examples of reversible dynamical systems in applications are actually also Hamiltonian. Secondly there is the remarkable fact that reversible and Hamiltonian systems have many interesting dynamical features in common. This duality may be the reason why so few systematic results on reversible dynamical systems have been derived, compared to the overwhelming machinery developed for symmetric (equivariant[5]) and Hamiltonian dynamical systems. The dual approach towards reversible systems will be the dominant theme in this paper. A schematic view of the situation is depicted in Figure 1, where it is shown how the Hamiltonian, reversible, and equivariant systems overlap.

3.1 Reversible versus equivariant dynamics

When a reversible system possesses more than one reversing symmetry, one finds that the composition of an odd number of reversing symmetries yields again a reversing symmetry, but that the composition of an even number of reversing symmetries yields a *symmetry*. S is called a *symmetry* of the equations of motion if, in the case of an autonomous or nonautonomous flow (3.1) or (3.9), we find

$$\frac{dS(\mathbf{x})}{dt} = dS|_{\mathbf{x}} \cdot F(\mathbf{x}, t) = F(S(\mathbf{x}), t), \tag{3.19}$$

or, in the case of a map f, we have

$$S \circ f = f \circ S. \tag{3.20}$$

Dynamical systems with a symmetry S are also called *S-equivariant*, and have attracted lots of attention in recent years, cf. for instance [9; 10; 12].

[5]A definition of equivariance will be given in §3.1.

In describing the symmetry properties of flows and maps, it is natural to discuss symmetries and reversing symmetries on an equal footing as they form a group under composition. We call a group of symmetries and reversing symmetries of a dynamical system a *reversing symmetry group* G [Lamb, 1992] and note that the symmetries (equivariances) form a normal subgroup H of G, i.e. $H \trianglelefteq G$. Moreover, when $H \neq G$ then H is a subgroup of index 2,

$$G/H \simeq \mathbb{Z}_2. \tag{3.21}$$

Note that G can be written as the semi-direct product $G \simeq H \rtimes \mathbb{Z}_2$ if and only if $G \setminus H$ contains an involution.

When a dynamical system possesses a reversing symmetry but no symmetries, it follows that $H = \{\mathrm{id}\}$ and $G \simeq \mathbb{Z}_2$, so that the dynamical system possesses precisely one involutory reversing symmetry. We will call such a dynamical system *purely reversible*.

The dynamical consequences of symmetries (equivariance) and reversing symmetries differ substantially. Symmetries map trajectories to other trajectories preserving the direction in which they are traversed in time. Reversing symmetries also map trajectories to trajectories, but now the time-direction of the two trajectories is reversed.

A very obvious difference resulting from this is the role of fixed point subspaces. The fixed point subspace of a map $U : \Omega \mapsto \Omega$ is defined as $\mathrm{Fix}(U) := \{\mathbf{x} \in \Omega \mid U(\mathbf{x}) = \mathbf{x}\}$. Fixed point subspaces of symmetries are setwise invariant under the dynamics. However, fixed point subspaces of reversing symmetries are usually not setwise invariant under the dynamics, but give rise to symmetric periodic orbits, homoclinics and heteroclinics (see § 4.1 and § 4.5).

As a simple contrasting example, two phase portraits of flows of planar vector fields are sketched in Figure 2. Both flows are symmetric with respect to a reflection in a horizontal line, but in one portrait (a) the reflection is a symmetry and in the other (b) it is a reversing symmetry.

Despite the dynamical differences between reversible and equivariant dynamical systems, techniques developed for the equivariant context sometimes carry over to the reversible one. For instance, local bifurcation problems in reversible systems can often be studied via equivariant singularity theory after performing a *Liapunov-Schmidt* reduction [Vanderbauwhede, 1982; Golubitsky *et al.*, 1995], cf. also §4.3 for more references.

k-Symmetry and space-time symmetry

It may happen that a map f possesses less symmetries and/or reversing symmetries than its kth iterate f^k. If k is the smallest positive integer for which a transformation U is a (reversing) symmetry of f^k, then U is called a (reversing) k-symmetry of f [Lamb & Quispel, 1994].

It turns out that k-symmetry naturally arises in the study of return maps of flows of time-periodic vector fields with mixed space-time symmetries. In the context of such systems we consider reversing symmetries $R_a : (\mathbf{x}, t) \mapsto (R(\mathbf{x}), -t + a)$ and (time-shift) symmetries of the form $S_a : (\mathbf{x}, t) \mapsto (S(\mathbf{x}), t + a)$, that leave the equations of motion invariant. In a natural way these space-time symmetries form a group under composition.

Let us consider a nonautonomous system that is invariant with respect to the time-shift $t \to t + 1$. The symmetry properties of the time-one return map of such a system are related to the space-time symmetries of the flow under consideration. For instance, the time-one return map with surface of section $t = 0$ inherits the space-time symmetries that

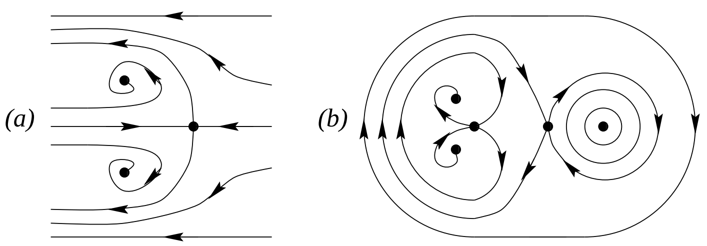

Figure 2: Sketches of the phase portraits of planar flows that are symmetric with respect to a horizontal reflection, where (a) the reflection is a symmetry (equivariance), and (b) the reflection is a reversing symmetry. In (b) some typical reversible dynamical phenomena can be observed: the right-most equilibrium is a center surrounded by a Liapunov centre family of periodic orbits (§4.1). Going outward the periods of these orbits start tending to infinity (blue sky catastrophe) as they come close to a symmetric homoclinic orbit connecting the symmetric saddle to itself (§4.5). In the left side of the flow one observes an asymmetric attractor-repeller pair.

fix the surface of section setwise, i.e. S_0 gives rise to a symmetry S and R_0 to a reversing symmetry R for such a return map. The remaining time-shift symmetry properties arise in a less obvious way. Namely, when a time-periodic flow admits a symmetry of the form $S_{-1/q}(\mathbf{x}, t) = (S(\mathbf{x}), t - 1/q)$ (for some $q \in \mathbb{N}$), then the time-one return map with surface of section $t = 0$ can be written as being decomposed into q pieces that are related to each other by time-shift symmetries. Consequently, the time-one return map can be written as $S^{-q} \circ f^q$ where $f := S \circ \varphi_{[0,1/q]}$ and $\varphi_{[0,1/q]}$ denotes the first hit map between surfaces of section at $t = 0$ and $t = 1/q$ [Lamb, 1995; Lamb, 1998] . The map f conveniently characterizes the dynamics of the flow. Interestingly, the space-time symmetry properties of the flow arise as (reversing) k-symmetries of the map f. k-Symmetry arises in a similar way in the study of local return maps of symmetric periodic orbits in autonomous flows. For a more detailed discussion, we refer to [Lamb, 1998].

Interestingly, many results for reversible maps have (nontrivial) extensions to the domain of k-reversible maps. We will give detailed references in relevant sections below. For a more extended introduction to k-symmetric dynamical systems see [Lamb & Brands, 1994; Lamb & Quispel, 1994; Lamb, 1994a; Lamb, 1996b; Lamb, 1998].

3.2 Reversible versus Hamiltonian dynamics

Reversibility is a symmetry property that most prominently arises in Hamiltonian dynamical systems, in particular in the context of mechanical systems. In physics, the terms *reversible* and *Hamiltonian* might sometimes be thought to be nearly synonymous, since in practice the vast majority of reversible dynamical systems arising in applications so far appear to be Hamiltonian.

In studies of classical mechanical systems reversibility has been gratefully welcomed as a tool in studying periodic orbits, homoclinics and heteroclinics, cf. e.g. [Devaney, 1976; Devaney, 1977; Churchill & Rod, 1980; Churchill *et al.*, 1983; Churchill & Rod, 1986; Meyer, 1981]. Reversibility began to be taken more seriously as a symmetry property in Hamiltonian dynamical systems after it turned out that many results originally established for Hamiltonian dynamical systems can also be obtained by assuming that the dynamical system is reversible (without taking into account the Hamiltonian structure).

In two seminal papers, Devaney [Devaney, 1976; Devaney, 1977] established the reversible Liapunov centre theorem (§4.1) and the reversible blue sky catastrophe theorem (§ 4.5) for periodic orbits of reversible systems and noted the close analogy to comparable results for Hamiltonian systems, noting that "reversible systems, near symmetric periodic orbits, behave qualitatively just like Hamiltonian systems."

Along the same lines, various other "Hamiltonian" results have been extended to the reversible domain. Most notably, the Kolmogorov-Arnol'd-Moser (KAM) theory has a reversible analogue, as do some results in local bifurcation theory, cf. §4.2 and §4.3. In the same spirit, the Aubry-Mather theory for area-preserving monotone twist maps has recently been extended to the domain of reversible monotone twist maps of the plane [Chow & Pei, 1995]. The origin of these coincidences is still an area of investigation and not very well understood.

Many reversible systems in applications happen to be Hamiltonian at the same time. When studying reversible Hamiltonian systems, it may be more convenient to prove results using the reversibility rather than the Hamiltonian structure. In dynamical systems obtained as reductions from partial differential equations, reversibility is often more easily recognized than an underlying symplectic structure.[6]

Unfortunately, the fact that most reversible systems in applications are Hamiltonian and most Hamiltonian systems in applications are reversible, seems to have obscured a bit which properties of reversible Hamiltonian systems are due to the reversibility and which are due to the Hamiltonian structure.

Because of the overwhelming number of reversible Hamiltonian systems of interest, it is somewhat surprising that a systematic theory on reversible Hamiltonian systems has not reallly been developed. Also a systematic comparison between dynamical features of Hamiltonian systems, reversible systems, and reversible Hamiltonian systems is far from established.

In the literature there are not so many examples of Hamiltonian dynamical systems that are not reversible. Arnol'd & Sevryuk [Arnol'd & Sevryuk, 1986] and Roberts & Capel [Roberts & Capel, 1992; Roberts & Capel, 1997] constructed examples of non-reversible Hamiltonian systems using local obstructions to reversibility. However, the non-reversibility in these examples is not persistent under small (Hamiltonian) perturbations. Examples of persistently non-reversible Hamiltonian systems in $\mathbb{R}^2$ were given by Mather [MacKay, 1993] (global topological obstruction) and Lamb [Lamb, 1996a] (local topological obstructions). Recently, it was shown that the reversibility of hyperbolic toral automorphisms can always be decided [Baake & Roberts, 1997]. Since such mappings are structurally stable, this yields persistent examples of both reversible and non-reversible area-preserving diffeomorphisms of the torus.

Non-Hamiltonian reversible systems also do not appear frequently in the literature. A few examples are a laser model [Politi *et al.*, 1986], the Stokeslet model describing sedimenting spheres [Caflisch *et al.*, 1988], and a model of coupled Josephson junctions [Tsang *et al.*, 1991a; Tsang *et al.*, 1991b]. Politi *et al.* and Tsang *et al.* observed that their reversible systems may possess attractors and repellers (pair-wise), and at the same time display Hamiltonian-like behaviour. The Stokeslet model has led to various interesting papers on reversible equivariant systems, cf. [Golubitsky *et al.*, 1991; Lim & McComb, 1995; Lim &

[6]For instance, the authors of [Eckmann & Procaccia, 1991] did not realize that their dynamical system obtained by PDE reduction is not only reversible but also Hamiltonian, until this was pointed out by R.S. MacKay (Woudtschoten Conference, 1992).

McComb, 1998; McComb & Lim, 1993; McComb & Lim, 1995]. Roberts & Quispel [Roberts & Quispel, 1992] studied scalings in non area-preserving reversible mappings of the plane, and also found a mixture of dissipative and conservative (Hamiltonian) behaviour, see also § 4.4.

In the following sections we will discuss in more detail certain aspects of the theory of reversible dynamical systems. We will make more precise comments on the reversible versus Hamiltonian dichotomy at various points.

4 Aspects of reversible dynamics

In this section we will discuss some dynamical consequences of reversibility in more detail. It is organized as follows. In § 4.1-4.5 we review results on symmetric periodic orbits (in particular in relation to their natural occurrence in families), KAM theory, local bifurcations (including Birkhoff normal form theory), scaling properties and renormalization, and homoclinic and heteroclinic behaviour. § 4.6 contains a brief discussion of some other topics. It should be noted that the length of our discussion of the different topics in this section has been largely influenced by the existence or absense of other recent relevant surveys.

4.1 Symmetric periodic orbits

Understanding a dynamical system on the basis of its periodic orbits has been a predominant theme in dynamical systems theory ever since the studies of Poincaré.

It is therefore not surprising that a result on periodic orbits is by far the most well-known and used result in reversible dynamical systems. In 1915, Birkhoff [Birkhoff, 1915] described the use of reversibility to find periodic orbits of the restricted three body problem. In 1958 DeVogelaere [DeVogelaere, 1958] described the method again, but now as a tool for searching for symmetric periodic orbits of reversible systems (by computer).

Definition 4.1 (Symmetric orbits) Let $o(\mathbf{x})$ be an orbit of a dynamical system, i.e. $o(\mathbf{x}) = \{\varphi_t(\mathbf{x}) \mid t \in \mathbb{R}\}$ in the case of flows and $o(\mathbf{x}) = \{f^n(\mathbf{x}) \mid n \in \mathbb{Z}\}$ in the case of maps. Then $o(\mathbf{x})$ is *R-symmetric* or *symmetric with respect to R* when the orbit is setwise invariant under R, i.e. $R(o(\mathbf{x})) = o(\mathbf{x})$.

The results on finding symmetric periodic orbits in reversible systems are folklore and many have derived (and are still deriving !) these results apparently independently. We present here a general version of the results on periodic orbits. Note that the results nowhere require that R is an involution [Lamb, 1992].

Theorem 4.2 (Symmetric orbits for flows) *Let $o(\mathbf{x})$ be an orbit of the flow of an autonomous vector field with time-reversal symmetry R. Then,*

- *an orbit $o(\mathbf{x})$ is symmetric with respect to R if and only if $o(\mathbf{x})$ intersects* $\mathrm{Fix}(R)$*, in which case the orbit intersects* $\mathrm{Fix}(R)$ *in no more than two points and is fully contained in* $\mathrm{Fix}(R^2)$.
- *an orbit $o(\mathbf{x})$ intersects* $\mathrm{Fix}(R)$ *in precisely two points if and only if the orbit is periodic (and not a fixed point) and symmetric with respect to R.*

Theorem 4.3 (Symmetric orbits for maps) *Let $o(\mathbf{x})$ be an orbit of an invertible map f with reversing symmetry R. Then,*

- *an orbit $o(\mathbf{x})$ is symmetric with respect to R if and only if $o(\mathbf{x})$ intersects* $\mathrm{Fix}(R) \cup \mathrm{Fix}(f \circ R)$*, in which case the orbit intersects* $\mathrm{Fix}(R) \cup \mathrm{Fix}(f \circ R)$ *in no more than two points and is fully contained in* $\mathrm{Fix}(R^2)$.

- *an orbit $o(\mathbf{x})$ intersects* $\mathrm{Fix}(R) \cup \mathrm{Fix}(f \circ R)$ *in precisely two points if and only if the orbit is symmetric with respect to R and periodic (but not a fixed point). Such an orbit intersects both* $\mathrm{Fix}(R)$ *and* $\mathrm{Fix}(f \circ R)$ *if and only if it has odd period. In particular:*

  - *$o(\mathbf{x})$ is a periodic orbit of f with period $2p$ if and only if there exists a $\mathbf{y} \in o(\mathbf{x})$ such that $\mathbf{y} \in \mathrm{Fix}(R) \cap f^p \mathrm{Fix}(R)$ or $\mathbf{y} \in \mathrm{Fix}(f \circ R) \cap f^p \mathrm{Fix}(f \circ R)$.*
  - *$o(\mathbf{x})$ is a periodic orbit of f with period $2p+1$ if and only if there exists a $\mathbf{y} \in o(\mathbf{x})$ such that $\mathbf{y} \in \mathrm{Fix}(R) \cap f^p \mathrm{Fix}(f \circ R)$.*

Theorem 4.2 or Theorem 4.3 are used in almost every paper discussing reversible dynamical systems. In particular, these theorems imply efficient techniques for tracking down R-symmetric periodic orbits, as it justifies searching for them in only a subset of the full phase space, cf. e.g. [Greene *et al.*, 1981; Kook & Meiss, 1989], § 4.4 and Appendix A for more examples. In the special case of reversible maps in $\mathbb{R}^2$ with an involutory reversing symmetry R fixing a one-dimensional subspace, $\mathrm{Fix}(R)$, $\mathrm{Fix}(f \circ R)$ and their iterates have been often referred to as the *symmetry lines* of the map, cf. e.g. [MacKay, 1993; Roberts & Quispel, 1992].

From the results on periodic orbits, we find that the fixed point subspaces of reversing symmetries are important. The fixed sets of reversing symmetries can take various forms. For instance, when a reversing symmetry acts freely then its fixed point subspace is empty. Examples of involutions with free actions include rotations on a two-torus or the involutory action on a unit two-sphere embedded in $\mathbb{R}^3$ induced by the transformation $-\mathrm{id}$ (the latter example arose recently in a study of relative equilibria of molecules [Montaldi & Roberts, 1997]). On more exotic manifolds, involutions may even exist whose fixed point subspace consists of several connected components of different dimension, cf. [Quispel & Sevryuk, 1993].

Generalizations of Theorem 4.2 and Theorem 4.3 that apply to flows of time-periodic vector fields with space-time symmetries were recently described in [Lamb, 1998]. Generalizations that apply to maps with reversing k-symmetries can be found in [Lamb & Quispel, 1994; Brands *et al.*, 1995; Lamb, 1998].

The above theorems imply that in reversible systems periodic orbits generically arise in families. Under some smoothness assumptions, the fixed point subspaces mentioned in the above theorems are manifolds. Their intersection will again be a manifold and its generic dimension follows from elementary considerations.

Theorem 4.4 (Families of symmetric periodic orbits)

(i) In a continuous m-parameter family of diffeomorphisms f_a with reversing symmetry R, R-symmetric periodic orbits of a given even period generically come in $(2\dim\mathrm{Fix}(R) - \dim\mathrm{Fix}(R^2) + m)$*-parameter families and* $(2\dim\mathrm{Fix}(f_a \circ R) - \dim\mathrm{Fix}(R^2) + m)$*-parameter families. R-symmetric periodic orbits of a given odd period generically form* $(\dim\mathrm{Fix}(R) + \dim\mathrm{Fix}(f_a \circ R) - \dim\mathrm{Fix}(R^2) + m)$*-parameter families.*

(ii) In a continuous m-parameter family of autonomous flows with reversing symmetry R, R-symmetric periodic orbits of a given period typically arise as $(2 \dim \mathrm{Fix}(R) - \dim \mathrm{Fix}(R^2) + m)$*-parameter families.*

(iii) In a continuous m-parameter family of autonomous flows with reversing symmetry R, R-symmetric periodic orbits typically arise as $(2 \dim \mathrm{Fix}(R) - \dim \mathrm{Fix}(R^2) + m + 1)$*-parameter families with smoothly varying period, in which the families with constant period mentioned in (ii) are embedded.*[7]

Theorem 4.4 implies that under suitable conditions, in reversible systems one may find n-parameter families ($n > 0$) of R-symmetric periodic orbits (whenever one of the formulas with $m = 0$ gives a positive number).

Stability properties of symmetric orbits

A well known property of linear reversible systems is that their eigenvalue structure is similar to that of Hamiltonian systems.

Theorem 4.5 (Eigenvalues of linear reversible systems)

- *Let λ be an eigenvalue of a linear reversible vector field. Then so is $-\lambda$ and $\bar{\lambda}$ (complex conjugate of λ).*
- *Let λ be an eigenvalue of a linear reversible diffeomorphism. Then so is λ^{-1} and $\bar{\lambda}$.*

Hence, for linear flows the eigenvalues come in singlets $\{0\}$, doublets $\{\lambda, -\lambda\}$ with $\lambda \in \mathbb{R}$ or $\lambda \in i\mathbb{R}$, or quadruplets $\{\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}\}$. And for linear reversible maps the eigenvalues come in singlets $\{\pm 1\}$, doublets $\{\lambda, \lambda^{-1}\}$ with $\lambda \in \mathbb{R}$ or $\lambda \in S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, or quadruplets $\{\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}\}$. Note that when R is an involutory reversing symmetry and $\dim \mathrm{Fix}(R) \neq \frac{1}{2} \dim \mathrm{Fix}(R^2)$, then necessarily the linearized vector field, respectively diffeomorphism, at an R-symmetric fixed point is forced to have eigenvalues equal to 0, respectively ± 1. In case $\dim \mathrm{Fix}(R) > \frac{1}{2} \dim \mathrm{Fix}(R^2)$ these eigenvalues precisely support the families of periodic orbits described in Theorem 4.4.

From the characterization of eigenvalues of reversible linear systems it can be seen that a range of interesting phenomena might arise. For instance, note that the stability properties of fixed points and periodic orbits are decided by linearized vector fields and return maps (via Floquet theory). The eigenvalue properties confirm the fact that R-symmetric periodic solutions cannot be asymptotically (un)stable. (In fact, this comment applies to any R-symmetric ω-limit set, cf. e.g. [Lamb & Nicol, 1998]). Indeed, precisely this generic occurrence of "balanced" stability characterizes reversible dynamical systems. It gives rise to complicated (and interesting) dynamical behaviour which is partly similar to dynamical features of volume preserving and Hamiltonian dynamical systems.

[7] The observation in Theorem 4.4(iii) can be derived from considering a local return map for a symmetric periodic orbit of a flow. The dimension of the surface of section S is one lower than the dimension of the phase space and the periodic orbit is a fixed point of the return map. Now, importantly $\dim \mathrm{Fix}(R) = \dim \mathrm{Fix}(R)|_S$. Hence, by Theorem 4.4(i) the symmetric fixed point is typically embedded in a $(2 \dim \mathrm{Fix}(R) - \dim \mathrm{Fix}(R^2) + m + 1)$-parameter family of fixed points representing periodic orbits of the flow with nearby period.

Reversible Liapunov centre theorem

Devaney [Devaney, 1976] showed that the Liapunov centre theorem for Hamiltonian systems has a reversible analogue. The theorem describes the existence of families of symmetric periodic orbits in the neighbourhood of an (partially) elliptic symmetric fixed point $\mathbf{0}$ of a vector field F in $\mathbb{R}^{2n}$ with reversing symmetry R.

Suppose that $\pm i\omega$ are simple eigenvalues of the linearized vector field $dF|_{\mathbf{0}}$, and that $\pm ik\omega$ are not eigenvalues for all $k = 0, 2, 3, 4, \ldots$. To avoid resonances, it then follows that $R^2 = \mathrm{id}$ and $\dim \mathrm{Fix}(R) = n$. The reversible Liapunov centre theorem asserts that there exists a two-dimensional invariant manifold containing $\mathbf{0}$ that, in a neighbourhood of $\mathbf{0}$, contains a nested one-parameter family of R-symmetric periodic orbits. Moreover, the periods of these periodic orbits tend to $2\pi/\omega$ as the initial conditions of these orbits tend to $\mathbf{0}$. To fix the idea, note that the centre theorem precisely describes the familiar planar picture of a centre type equilibrium surrounded by a family of periodic orbits (cf. the rightmost equilibrium point in Figure 2(b)).[8]

Golubitsky, Krupa & Lim [Golubitsky *et al.*, 1991] extended this result by allowing certain types of symmetry-induced resonances to occur, namely zero eigenvalues due to reversibility and 1:1 resonances due to equivariance. Contrasting Devaney's geometric approach, they used a Liapunov-Schmidt reduction, adapting an alternative proof of the reversible Liapunov centre theorem given by Vanderbauwhede [Vanderbauwhede, 1982]. (For a recent application of this result, see [Chang & Kazarinoff, 1996].)

When resonances occur that are not a simple consequence of symmetry properties, then the families of periodic orbits of two pairs of purely imaginary eigenvalues interact. For a discussion, see §4.3.

While considering the Liapunov centre family in the right hand side of Figure 2(b), it is interesting to note that when we follow the one-parameter family of periodic orbits going away from the centre point, the periods of the closed orbits tend to infinity as they approach a reversible homoclinic orbit (a closed orbit starting and ending at a symmetric saddle point). Devaney called this a *blue sky catastrophe* [Devaney, 1977], and proved that such families of symmetric periodic orbits always arise around reversible homoclinic orbits, cf. § 4.5 for more details.

4.2 KAM-theory

The interest in reversible dynamical systems has been boosted not simply because of the results on periodic orbits. Another important line of investigation has been that of Kolmogorov-Arnol'd-Moser (KAM) theory. KAM-theory deals with the persistence of invariant tori constituting quasiperiodic motion in nearly integrable dynamical systems.

Originally, KAM-theory was developed in a Hamiltonian setting, i.e. only smooth perturbations were considered that preserve the symplectic structure of an integrable dynamical system. However, early on it was acknowledged that KAM-theory can also be applied to the setting of reversible systems, cf. Moser [Moser, 1967] and Bibikov & Pliss [Bibikov & Pliss, 1967]. Interestingly, in his expository paper [Moser, 1973], Moser chooses to present KAM-theory for reversible systems as he finds it "technically somewhat simpler" than for the Hamiltonian context.

[8] Note that the Liapunov centre families of periodic orbits in flows often appear in the associated return maps as one-parameter families of periodic orbits (with a fixed period).

Although integrable systems need not be reversible, many integrable systems in the literature happen to be reversible. For examples of a large class of integrable reversible mappings of the plane that are the composition of two involutions that also preserve the integrals, see [Quispel *et al.*, 1989; Roberts & Quispel, 1992]. For other examples of integrable reversible mappings, cf. [Boukraa *et al.*, 1994; Rerikh, 1995; Rerikh, 1996].

Sevryuk has carried through a thorough program of studying reversible KAM-theory, starting with his lecture notes [Sevryuk, 1986] (for further references see Appendix A). For a survey on the reversible KAM-theory see Sevryuk [Sevryuk, 1998] (in this volume) and Broer, Huitema & Sevryuk [Broer *et al.*, 1996a]. The latter notes treat KAM-theory from a general perspective, presenting reversible systems as just one of the contexts to which the KAM techniques apply.

With the recognition that KAM-theory applies to reversible systems, the natural question has arisen whether there is a reversible analogue of (the problem of) Arnol'd-diffusion. Matveyev [Matveyev, 1995a; Matveyev, 1995b; Matveyev, 1996] obtained some results that show that indeed there is diffusion related to the break-up of invariant tori. However, the mechanism of diffusion is not identical to the mechanism of diffusion observed in Hamiltonian systems.

4.3 Local bifurcation theory

It is well known that symmetry properties of a system may influence the genericity of the occurrence of local bifurcations. That is, bifurcations that typically occur in certain symmetric systems might only be rarely observed in non-symmetric systems. The influence of symmetry on local bifurcations (steady-state and Hopf) has been extensively studied in the context of equivariant dynamical systems, cf. Golubitsky *et al.* [12].

Bifurcation theory for reversible systems has been developed in a less systematic way than for equivariant systems. First of all, in most papers the analysis is restricted to purely reversible systems (i.e. those with no equivariance properties). Often, the further assumption is made that the reversing symmetry R acts in $\mathbb{R}^{2n}$ as a linear involution with an n-dimensional fixed point subspace.[9]

In papers on reversible equivariant systems there are often additional hypotheses: e.g. the existence of an involutory reversing symmetry or some explicit assumption on how a reversing symmetry R acts with respect to the action of the additional equivariances. These hypotheses are usually motivated by properties of the particular models under consideration, but unfortunately obscure the general applicability of some of these results.

Nevertheless, many interesting results on local bifurcations in reversible systems have been obtained and their embedding in a systematic theory that applies to more general space-time symmetric systems is an interesting open problem.

Before we discuss reversible local bifurcation theory in more detail, we note that most results on reversible systems are indeed of a local nature. Interestingly, however, Fiedler & Heinze [Fiedler & Heinze, 1996a; Fiedler & Heinze, 1996b] recently developed a topological index theory for reversible periodic orbits. This might well be a starting point for the use of global techniques in the study of reversible systems. Also Fiedler & Turaev [Fiedler & Turaev, 1996] illustrate how topological arguments can be used to prove that elliptic periodic

[9] As many authors note, the assumption of a locally linear action of an involution is justified by the Montgomery-Bochner Theorem [19]. Actually this comment applies more generally to local actions of compact Lie groups cf. [3].

orbits arise at certain elementary homoclinic bifurcations.

In discussing local bifurcations of reversible systems, it is important to note from Theorem 4.5 that instabilities may arise when eigenvalues are on the imaginary axis or unit circle in the complex plane. In particular, in analogy to the situation in Hamiltonian systems, bifurcations may arise when such eigenvalues pass through resonances.

This section is organized as follows. First, we briefly discuss the Birkhoff normal form theory for reversible systems. Thereafter we survey the literature on several types of local bifurcations: steady-state bifurcations, bifurcations at resonant centres, subharmonic branching and reversible Krein crunch.

Steady-state bifurcations involve the collision at zero of eigenvalues of the linearized vector field at an equilibrium point of a flow, or a collision at $+1$ of eigenvalues of the linearized map at a fixed point of a diffeomorphism. More complicated bifurcations involve the occurrence of resonances (rational relationships between the eigenvalues of a linearized vector field or map on the imaginary axis or unit circle in the complex plane). We distinguish between bifurcations at resonant equilibrium points of flows (when eigenvalues on the imaginary axis pass through a resonance), subharmonic branching (when the return map of a symmetric periodic orbit of a flow has (a pair of) eigenvalues passing through a root of unity), and the reversible Krein crunch (when two pairs of eigenvalues of a map collide on the unit circle).

Birkhoff normal forms

Various authors have studied reversible bifurcation problems using, in one way or another, a Birkhoff normal form analysis. The starting point of Birkhoff normal form theory is to consider the Taylor expansion of a diffeomorphism or vector field at a fixed point and to find a local coordinate frame in which the Taylor expansion looks “simple”, i.e. in normal form (with respect to a certain convention). The aim is then to study certain aspects of the local dynamics around a fixed point (local bifurcations, stability properties) using the truncated normalised expansion of the diffeomorphism or vector field. However, it is important to keep in mind that this strategy should be applied with care as some dynamical features of the truncated system may not arise in the original system.

The Birkhoff normal form procedure about a fixed point starts with bringing the linear part of the flow or map into normal form, and then studying its (versal) unfoldings.

In the context of symmetric systems, the derivation of Birkhoff normal forms is naturally done in a structure preserving (symmetry respecting) framework [5]. In the presence of a reversing symmetry group G this means that only G-equivariant coordinate transformations are to be considered.

For purely reversible systems, several papers on reversible linear normal forms have appeared [Palmer, 1977; Wan, 1991; Sevryuk, 1992; Shih, 1993; Hoveijn, 1996]. The paper by Hoveijn [Hoveijn, 1996] is the most recent and complete reference,and includes a discussion of the Hamiltonian and reversible Hamiltonian cases as well. For technical details on the symmetry-respecting linear normal form theory, see also Knobloch & Vanderbauwhede [Knobloch & Vanderbauwhede, 1995]. Recently, Lamb & Roberts [Lamb & Roberts, 1997] discussed the normal form theory for linear reversible equivariant systems on the basis of the representation theory of reversing symmetry groups.

After bringing the linear part of a system into normal form, one can subsequently normalize higher order terms in the Taylor expansion of a map or vector field. It turns out that most basic results for nonsymmetric systems (cf. [24; 7]) carry over to the reversible (and equivariant) context without further complications, cf. e.g. [Iooss & Adelmeyer, 1992; Lamb, 1996b; Shih, 1997; Vanderbauwhede, 1990a; van der Meer *et al.*, 1994].

The most common method for characterizing Birkhoff normal forms uses the unique decomposition of a matrix into its nilpotent and semisimple part.[10] We first consider flows. Let $dF|_{\mathbf{0}}$ be the (Frechet) derivative of a vector field F at a fixed point $\mathbf{0}$, and $dF|_{\mathbf{0}} = A_s + A_n$ be the decomposition into its semi-simple and nilpotent part. Then, when the flow is sufficiently differentiable, a normal form to any desired order can be obtained that is equivariant with respect to $\exp(tA_s)$ for all $t \in \mathbb{R}$, while preserving the (reversing) symmetry properties of the flow.[11] The additional symmetry properties of the normal form are usually referred to as the *formal* normal form symmetry.

For smooth diffeomorphisms $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ an analogous scheme applies. After decomposing the (Frechet) derivative $df|_{\mathbf{0}}$ of f at $\mathbf{0}$ as $df|_{\mathbf{0}} = A_s + A_n$, up to any desired order of the Taylor expansion a normal form can be obtained that is (formally) A_s-equivariant.

Above we described normalizations with respect to the semi-simple part A_s. Further normalization with respect to the nilpotent part A_n can also be implemented. Note however, that in the case of maps, when $df|_{\mathbf{0}}$ is not semi-simple the explicit calculations for obtaining reversible normal forms for diffeomorphisms are potentially cumbersome, cf. [Lamb *et al.*, 1993; Lamb, 1996b].

Jacquemard & Teixeira [Jacquemard & Teixeira, 1996b; Jacquemard & Teixeira, 1996c; Jacquemard & Teixeira, 1997] have developed an alternative method for calculating reversible normal forms of diffeomorphisms. They successfully implement this method within a computer algebra program. Their method might be particularly useful, as an alternative to the method described above, when calculating normal forms for maps with non semi-simple linear parts.

The normal form results for flows and diffeomorphisms extend in a natural way to the reversible Hamiltonian setting under natural additional assumptions on the (anti-)symplecticity of the representation of G.

The structure preserving Birkhoff normal form strategy also extends naturally to diffeomorphisms with a reversing k-symmetry group G in which case the structure preserving transformations are again G-equivariant [Lamb, 1996b].

In relation to the reversible-Hamiltonian duality discussed in § 3.2, it is interesting to note that Hamiltonian normal forms are almost always *formally* reversible. For example, an equilibrium of a Hamiltonian vector field with purely imaginary eigenvalues $\pm i\omega$ has a normal form that is, up to any desired order, rotationally ($SO(2)$) equivariant. Consequently, it can be written in polar coordinates as [Birkhoff, 1927]

$$\frac{dr}{dt} = 0, \ \frac{d\vartheta}{dt} = g(r^2), \ \text{with } g(0) = \omega, \tag{4.1}$$

which is reversible with respect to the involution $(r, \vartheta) \mapsto (r, -\vartheta)$. For counter examples of such coincidences, see [Roberts & Capel, 1992; Lamb *et al.*, 1993; Roberts & Capel, 1997]. The reason for the formal reversibility of Hamiltonian normal forms is not well understood, but it foreshadows some of the similarities in the (local) dynamics of reversible and Hamiltonian systems.

[10] A linear operator is semi-simple whenever it is $\mathbb{C}$-diagonalizable. A linear operator N is nilpotent if there exists an integer n such that N^n is zero.

[11] For local actions of compact Lie groups, we may assume without loss of generality that a reversing symmetry group G acts locally linear and orthogonal. However, for a normal form approach allowing for nonlinear symmetry actions, see Gaeta [Gaeta, 1994].

Despite the inherent problem of the convergence properties of Birkhoff normal forms, they are very helpful in bifurcation analysis and in understanding certain aspects of local dynamics, cf. e.g. [Iooss & Kirchgässner, 1992; Iooss & Pérouème, 1993; Iooss, 1995a; Iooss, 1997] and [Broer *et al.*, 1998b; Hanßmann, 1998] in this volume.

Steady-state bifurcations in reversible systems

To our knowledge, Rimmer [Rimmer, 1978; Rimmer, 1983], was one of the first to discuss a reversible bifurcation problem. He considered reversible symplectic diffeomorphisms of the plane with a reversing reflection symmetry R. He showed that symmetry-breaking pitchfork bifurcations from an R-symmetric fixed point are generic (codimension one). He also showed that such pitchfork bifurcations cease to be generic when the fixed point under consideration is not R-symmetric (e.g. when the symplectic diffeomorphism is not reversible). This result nicely illustrates the importance of acknowledging the presence of a reversing symmetry, also in Hamiltonian systems.

More recently, steady-state bifurcations in reversible Hamiltonian systems in $\mathbb{R}^2$ have been studied by [Broer *et al.*, 1998b; Broer *et al.*, 1998a; Hanßmann, 1998] (in this volume). It is interesting to note that (anti)symplectic (reversing) symmetries of Hamiltonian vector fields arise as invariance (or *anti*-invariance) properties of the Hamiltonian, posing the problem of steady-state bifurcations automatically as a singularity theory problem. We note, in this respect, that in the classification of planar polynomial vector fields reversibility is a recurrent theme, either in connection with the study of centres [Zoladek, 1994; Berthier & Moussu, 1994; Teixeira, 1997b] or as a symmetry assumption restricting the class of vector fields under consideration [Guimond & Rousseau, 1996].

Surprisingly, steady-state bifurcations in reversible systems have not been studied very intensively. Motivated by a laser model, Politi *et al.* [Politi *et al.*, 1986] observed a symmetry breaking pitchfork bifurcation in non-area preserving planar maps and flows of the plane and noted the birth of an attractor-repeller pair in such a bifurcation, cf. also [Post *et al.*, 1990; Roberts & Quispel, 1992]. Local steady-state bifurcations of certain planar reversible equivariant vector fields were discussed by Lamb & Capel [Lamb & Capel, 1995]. Most of this analysis was done on the basis of a Birkhoff normal form approach.

A different approach towards steady-state bifurcations of reversible flows has been pursued by Teixeira [Teixeira, 1997a; da Rocha Medrado & Teixeira, 1998]: purely reversible flows in $\mathbb{R}^n$ where $\dim \mathrm{Fix}(R) = n-1$ are treated as a half-infinite system with a boundary $\mathrm{Fix}(R)$. In this approach, symmetric local bifurcations (and indeed the local flow around $\mathrm{Fix}(R)$) are characterized by the contact of the vector field with $\mathrm{Fix}(R)$.

Jacquemard & Teixiera [Jacquemard & Teixeira, 1996a; Jacquemard & Teixeira, 1996c; Jacquemard & Teixeira, 1997] study local bifurcations of fixed points of reversible diffeomorphisms in $\mathbb{R}^2$ and $\mathbb{R}^3$ using normal forms to describe the contact between the fixed sets of the two involutions that constitute a (purely) reversible map.

Yet another approach has been pursued by Lim & McComb [Lim & McComb, 1998] (in this volume). They use a Liapunov-Schmidt reduction to prove the genericity of symmetry breaking pitchfork bifurcations in reversible systems. Their technique allows for the occurrence of resonances and zero eigenvalues that arise due to equivariance and reversibility.

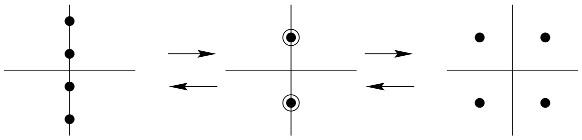

Figure 3: Schematic diagram illustrating the eigenvalue movements at the reversible Hopf bifurcation (1:1 resonance).

Bifurcations at resonant centres

The linearized flow at an R-symmetric equilibrium point of a reversible autonomous flow may have pairs of purely imaginary eigenvalues on the imaginary axis. In the absence of equivariance, these eigenvalues are typically simple (no two eigenvalues are the same). Also, typically, they are nonresonant (no eigenvalues are positive integer multiples of others). However, such situations may arise in generic one-parameter families for suitable values of the parameter.

We consider the situation where two pairs of purely imaginary (nonzero) eigenvalues $\pm i\omega_1$, $\pm i\omega_2$ collide when a parameter is varied. Then, typically, after such a collision they branch off as a quadruplet in the complex plane, cf. Figure 3. In Hamiltonian systems, such a bifurcation is known as a Hamiltonian Hopf bifurcation; in reversible systems, by analogy, it is called a reversible Hopf bifurcation or reversible 1:1 resonance. By the centre theorem, before the collision each pair of purely imaginary eigenvalues has a one-parameter family of periodic orbits associated with it. However, after the collision the quadruplet of eigenvalues is in the complex plane and does not give rise anymore to families of periodic solutions around the origin.

Arnol'd and Sevryuk [Arnol'd & Sevryuk, 1986; Sevryuk, 1986] studied this bifurcation and found that either both families simultaneously disappear, shrinking as a unit at the origin (the so-called elliptic regime), or they disappear only around the origin but persist outside (the hyperbolic regime). The approach by Arnol'd & Sevryuk is geometrical and based on an analysis of the curves along which the families of periodic orbits intersect the fixed point subspace of the reversing involution. Vanderbauwhede [Vanderbauwhede, 1990a] also studied this bifurcation but from a different (analytical) perspective, using Liapunov-Schmidt reduction. This allows him to include generalizations to some situations in which the system is not only reversible but also equivariant. Recently, Vanderbauwhede & Knobloch [Knobloch & Vanderbauwhede, 1995] extended the analysis to reversible Hopf bifurcations at k-fold resonances (where k pairs of purely imaginary eigenvalues collide, involving the merging of k one-parameter families of symmetric periodic orbits), using general results on periodic solutions obtained by the same authors in [Knobloch & Vanderbauwhede, 1996]. Iooss & Pérouème [Iooss & Pérouème, 1993] used a normal form approach to analyze homoclinic solutions in the reversible 1:1 resonance. For peculiarities about the normal form at the reversible 1:1 resonance, see Van der Meer *et al.* [van der Meer *et al.*, 1994] and Bridges [Bridges, 1998] (in this volume).

Higher $1 : N$ resonances (where $i\omega_1 = Ni\omega_2$) were also studied by Arnol'd & Sevryuk [Arnol'd & Sevryuk, 1986; Sevryuk, 1986]. An important difference with the reversible Hopf bifurcation is that now, before and after passing through the resonance, the purely imaginary eigenvalues remain on the imaginary axis. The two families of (short and long) periodic orbits interact at the bifurcation points. There are again two regimes (elliptic and hyperbolic) and the cases $N = 2$, $N = 3$, and $N \geq 4$ are treated separately. McComb & Lim [McComb & Lim, 1995] extended these results by allowing for zero eigenvalues and resonances due to reversibility and equivariance along the lines of [Golubitsky *et al.*, 1991]. Sevryuk [Sevryuk, 1986] further describes $p : q$ resonances (with $\gcd(p, q) = 1$) in which case the scenarios involve short and long periodic orbits (associated with p and q) and very long periodic orbits (associated with $\gcd(p, q)$). Recently, Shih [Shih, 1997] extended the analysis to case studies of resonances involving three frequencies.

Subharmonic branching

The dynamics of periodic orbits in flows can be studied by a (local) Poincaré return map such that the periodic orbit corresponds to a fixed point of this map. When the flow is reversible and the periodic orbit is symmetric with respect to a reversing symmetry, the return map is reversible too and the fixed point is symmetric. Hence, among the eigenvalues of the linearized map at the fixed point (Floquet multipliers), we may have a pair of eigenvalues on the unit circle in the complex plane. When the eigenvalues pass through a resonance ($\lambda_{\pm} = \exp(\pm 2i\pi p/q)$, with $\gcd(p, q) = 1$) then, in the absence of other resonances, Vanderbauwhede [Vanderbauwhede, 1986; Vanderbauwhede, 1990b; Vanderbauwhede, 1992a] showed that at such resonances generically Liapunov-centre families of periodic solutions meet, cf. also Gervais [Gervais, 1988] and Sevryuk [Sevryuk, 1986]. The cases $q = 1$, $q = 2$ (period doubling) and $q \geq 3$ (subharmonic bifurcations) require separate discussions.

For a detailed bifurcation analysis in the strongly resonant case $q = 1$, see also Pérouème [Pérouème, 1993]. The situation where there are additional resonances due to reversibility in the form of eigenvalues locked at $+1$ is discussed by Furter [Furter, 1991] on the basis of a singularity theoretic approach of the reduced bifurcation equations.

Reversible subharmonic branching is similar to the corresponding phenomenon of subharmonic branching in Hamiltonian systems, and gives rise to a *subharmonic branching tree* in its phase space [Vanderbauwhede, 1990b]. In the Hamiltonian case, this tree is normally foliated by the level sets of the Hamiltonian. However, in the non-Hamiltonian reversible case there need not be any conserved quantities. The tree is embedded in phase space and longitudinal drifting motion parallel to such a tree may arise [Arnol'd, 1984]. For a picture of part of a branching tree in $\mathbb{R}^3$ see [Roberts & Lamb, 1995]. Associated scaling behaviour is discussed in §4.4.

Reversible Krein crunch

In the study of diffeomorphisms in $\mathbb{R}^n$ $(n \geq 4)$, resonances may arise when a quadruplet of eigenvalues on the unit circle collides at $\exp(\pm i\omega)$ and the four eigenvalues branch off the unit circle as a quadruplet into the complex plane (the eigenvalues thus behaving as the exponents of eigenvalues in the reversible Hopf bifurcation depicted in Figure 3). By analogy to a similar bifurcation in Hamiltonian systems, this is sometimes called the reversible *Krein crunch.* The normal form theory is formally similar to the reversible $1 : 1$ resonance, which is used in Sevryuk & Lahiri [Sevryuk & Lahiri, 1991] to conjecture a description in case $\omega/(2\pi)$ is sufficiently irrational, cf. also [Bridges *et al.*, 1995]. In a number of papers, Lahiri *et al.* have further numerically studied bifurcations near resonant reversible Krein crunches in four-dimensional reversible maps [Bhowal *et al.*, 1993b; Bhowal *et al.*, 1993a; Lahiri *et al.*, 1993; Lahiri *et al.*, 1995], see also [Lahiri *et al.*, 1998] (in this volume) for a recent account.

4.4 Renormalisation and scaling

Universal scaling in dissipative dynamical systems, and accompanying explanations using renormalization group theory, were introduced by Feigenbaum. Such investigations have also been made in low-dimensional conservative/reversible mappings in two main areas : (i) break-up of KAM-tori; and (ii) period-multifurcation cascades. Historically, both areas were first explored numerically in area-preserving mappings. As both investigations require finding many long periodic orbits, it was natural to study reversible area-preserving mappings in which Theorem 4.3 above could be used to find symmetric periodic orbits. In fact, it seems prohibitive to conduct such studies without the benefits of reversibility. This means that the universal results obtained pertain to mappings that are both area-preserving and reversible. Various authors have investigated whether the results are different if one of the properties is relaxed. This seems to be another area where the similarities induced by symplecticity and by reversibility are quite striking. Although a complete explanation is yet to be given, it seems analysis should focus on how dependent the spectrum of appropriate renormalization operators is to the properties of the space of maps on which the operators act.

Break-up of KAM-tori

Greene's residue criterion [Greene, 1979] uses the stability of a sequence of nearby symmetric periodic orbits to suggest the existence or non-existence of a given KAM-torus. The sequence of periodic orbits analyzed has rotation number converging to the irrational winding number of the torus. Since its illustration for the standard mapping [Greene, 1979], it has been employed widely to study break-up of tori in area-preserving reversible mappings and universal scalings associated with the break-up have been identified [MacKay, 1983b; MacKay, 1988; MacKay, 1993]. Renormalization group explanations of the scalings have been advanced by MacKay [MacKay, 1988; MacKay, 1992]. Numerical results suggest that the same scalings characterise reversible mappings that are not area-preserving [Roberts & Quispel, 1992]. [Khanin & Sinai, 1986] have given a renormalization group theory that works in the space of reversible (not necessarily area-preserving) maps.

Period-multifurcation cascades

In the early 1980's, various authors discovered universal scalings in parameter and phase space in period-doubling trees in area-preserving reversible mappings [Greene *et al.*, 1981; Bountis, 1981; Benettin *et al.*, 1980b; Benettin *et al.*, 1980a]. Only the paper [Greene *et al.*, 1981] identified two phase-space scalings, associating them with scaling along and scaling across the so-called dominant symmetry line containing two points of each even cycle (cf. Theorem 3.2). Renormalization group explanations in the space of area-preserving reversible mappings were advanced by various authors [Greene *et al.*, 1981; Collet *et al.*, 1981; MacKay, 1993], the fixed point of the doubling operator being assumed reversible. There was limited discussion as to how important it was to have both properties : area-preservation and reversibility. Again, [Roberts & Quispel, 1992] show that symmetric period-doubling cascades in reversible mappings that are not necessarily area-preserving appear to be governed by the scalings found earlier. Meanwhile, the analysis in [Davie & Dutta, 1993] would seem to explain this by highlighting the significance of the spectrum of the doubling operator when restricted to area-preserving maps.

[Roberts & Lamb, 1995] showed that the self-same scalings found in 2D (reversible) period-doubling describe self-similarity of period-doubling branching trees in 3D reversible mappings (cf. also [Komineas *et al.*, 1994]). Here the symmetric periodic orbits in the tree form one-parameter families (because of Theorem 4.4) and branch in phase space (rather than parameter space) according to the subharmonic branching theory of Vanderbauwhede described in the previous section.

Self-similar universal scalings also govern multifurcations in (area-preserving) reversible maps [Meiss, 1986; Turner & Quispel, 1994].

4.5 Reversible homoclinics and heteroclinics

Homoclinics and heteroclinics form connections between saddle points and thereby usually constitute recurrent transport through a dynamical system.

Let $\mathbf{x}_0$ be an equilibrium point of a dynamical systems with reversing symmetry R. We denote the stable and unstable manifolds of $\mathbf{x}_0$ by $W^s_{\mathbf{x}_0}$ and $W^u_{\mathbf{x}_0}$. Recall that stable, respectively unstable manifolds, contain all the points that tend to $\mathbf{x}_0$ for $t \to -\infty$, respectively $t \to +\infty$. A point $\mathbf{y}$ is a *homoclinic point* of $\mathbf{x}_0$ if $\mathbf{y}$ lies in the intersection of $W^s_{\mathbf{x}_0}$ and $W^u_{\mathbf{x}_0}$. A point $\mathbf{y}$ is a *heteroclinic point* of two points $\mathbf{x}_1$ and $\mathbf{x}_2$ when $\mathbf{y} \in W^s_{\mathbf{x}_1} \cap W^u_{\mathbf{x}_2}$.

In general, it is not easy to locate homoclinic or heteroclinic points and orbits. However, in reversible systems, symmetric homoclinic and heteroclinic orbits can be found relatively easily, because of the characterizations of symmetric orbits in Theorem 4.2 and Theorem 4.3. In fact, if a dynamical system has a reversing symmetry R, it follows immediately that the the intersection of $\mathrm{Fix}(R)$ and the stable or unstable manifold of a hyperbolic point yields homoclinic or heteroclinic points. More precisely, let $\mathbf{y} \neq \mathbf{x}_0$, then $\mathbf{y} \in \mathrm{Fix}(R) \cap W^{s,u}_{\mathbf{x}_0}$ is a homoclinic point of $\mathbf{x}_0$ if and only if $\mathbf{x}_0$ is R-symmetric. Alternatively, $\mathbf{y} \in \mathrm{Fix}(R) \cap W^{s,u}_{\mathbf{x}_0}$ is a heteroclinic point of $\mathbf{x}_0$ if and only if $\mathbf{x}_0$ is not R-symmetric.

For flows, this gives a full description of R-symmetric homoclinics and heteroclinics. For maps f, one should also consider the above statements not only with $\mathrm{Fix}(R)$ but also with $\mathrm{Fix}(f \circ R)$, by analogy to the characterization of R-symmetric orbits in § 4.1. Note in this respect that R-symmetric homoclinic and heteroclinic orbits are always contained in $\mathrm{Fix}(R^2)$.

Homoclinic (and heteroclinic) orbits are of great interest in dynamical systems because in their neighbourhood one usually finds *chaotic* behaviour. Importantly, the above characterization yields the persistent occurrence of symmetric homoclinic and heteroclinic solutions in many reversible systems of interest. In this respect it is interesting to note that in the prototype model for homoclinic dynamics, the Smale-horseshoe map, the symbolic dynamics on the nonwandering set is reversible in a very natural way, with the reversing symmetry interchanging the stable and unstable leaves [Devaney, 1989].

Devaney [Devaney, 1984; Devaney, 1988] used reversibility to prove the existence of transversal homoclinics (horse-shoes) in the area-preserving reversible Hénon map (cf. also [Brown, 1995]). Others have used normal forms to prove the existence of homoclinic and/or heteroclinic solutions, cf. e.g. Churchill & Rod [Churchill & Rod, 1986] in the context of the Hénon-Heiles system and Iooss & Pérouème [Iooss & Pérouème, 1993] in the context of the reversible 1:1 resonance.

The dynamics around reversible homoclinics enjoys special properties. For instance, Devaney[Devaney, 1977] showed that an R-symmetric homoclinic orbit in a reversible vector field invokes a "blue sky catastrophe", i.e. a family of periodic orbits with periods tending to infinity. More precisely, he found that in the neighbourhood of a nondegenerate R-symmetric homoclinic orbit of an R-reversible vector field there exists a one-parameter family of R-symmetric periodic orbits whose periods tend to infinity as the periodic orbits approach the homoclinic orbit. Devaney's approach is mainly geometrical and uses the classical properties of symmetric periodic orbits. Vanderbauwhede & Fiedler [Vanderbauwhede & Fiedler, 1992] proved this theorem with a different (analytical) method which works for both reversible and Hamiltonian systems. Interestingly, the latter paper extends a result from the reversible category into the Hamiltonian setting, rather than vice-versa.

In case the equilibrium of the homoclinic orbit is of saddle-focus type, the dynamics around the homoclinic orbit tends to become very intricate, cf. [Devaney, 1977] and [Härterich, 1998] (in this volume).

The interest in reversible homoclinics arises not only from their relation to complicated dynamics, but also for their practical relevance. Namely, in the context of traveling waves of certain partial differential equations, homoclinic solutions represent solitary waves. Such waves are of interest in various applications, e.g. in optical communication systems [Sandstede *et al.*, 1997]. For a survey on the theory and applications of reversible homoclinics (in particular in the context of partial differential equations), see Champneys [Champneys, 1998] in this volume.

Reversible heteroclinics have attracted considerably less attention than reversible homoclinics. A few exceptions are [Churchill & Rod, 1986; Vanderbauwhede, 1992b; Rabinowitz, 1994a; Rabinowitz, 1994b; Maxwell, 1997]. Following up the recent interest in heteroclinic cycles in equivariant systems, reversible heteroclinic cycles are also beginning to be studied in more detail, cf. [Reißner, 1998].

4.6 Miscellaneous topics

Admissible symmetry properties of periodic orbits and attractors

One may address the following question: what symmetry properties of periodic orbits, attractors, or other types of ω-limit sets are admissible in dynamical systems with a given reversing symmetry group ? In the equivariant setting (without reversibility) there is a fairly complete understanding [10]. However, in reversible equivariant systems many elementary questions are still open. For a discussion, see Lamb & Nicol [Lamb & Nicol, 1998] in this volume.

Ergodic theory

It is somewhat surprising that in the field of ergodic theory, reversibility has, until recently, received very little attention. This is despite the success of ergodic theory in many areas of dynamical systems theory and its obvious relevance to thermodynamics. In a recent series of publications, Goodson *et al.* have begun a study of reversible dynamical systems from an ergodic theory point of view. We refer the reader to [Goodson *et al.*, 1996; Goodson & Lemańczyk, 1996; Goodson, 1996c; Goodson, 1996b] for further details.

Reversible integrators

The numerical study of a dynamical system with continuous time (i.e. a flow) often involves a discretization method to *integrate* the equations of motion. In recent years there has been much interest in studying the properties of such *integrators*. In particular, when a dynamical system possesses certain structures (symmetry, reversibility, Hamiltonian, gradient) one would like to find a discretized approximant that preserves such a structure. This is done to prevent the numerical calculation of evidently erroneous global phenomena, such as asymptotically stable attractors in Hamiltonian systems.

Especially in the context of reversible Hamiltonian systems, it turns out that preserving reversibility in integrators has lots of advantages (in the context of such systems it is sometimes argued that preserving reversibility in an integration method is more important than preserving the symplectic structure), cf. [Scovel, 1991; Stoffer, 1995; McLachlan *et al.*, 1995; Hairer & Stoffer, 1997; Leimkuhler, 1997], and McLachlan & Quispel [McLachlan & Quispel, 1998] in this volume.

5 Reversible dynamical systems in physics and mathematics

In this section we will briefly describe some areas of physics and mathematics in the context of which reversible dynamical systems have occurred in the literature. In particular, we follow up our observation in § 3 that reversible systems have appeared in the literature not only in relation to *conventional* reversibility with respect to time, but also due to spatial symmetries in the context of partial difference and differential equations, or due to the (group) structure in certain abstract mathematical problems.

5.1 Reversibility in time: mechanics and nonequilibrium thermodynamics

Conventional time-reversibility in mechanical systems

Needless to say, mechanical (Hamiltonian) systems contain a large class of examples of reversible dynamical systems with the conventional antisymplectic reversing symmetry (3.7). The bibliography in Appendix A contains only a limited number of references in this direction, trying to include those that use reversibility in a systematic, rather than ad-hoc, way.

Reversibility is certainly an important symmetry property, even (or especially) in the context of Hamiltonian systems. A nice illustration of this point was made by Montaldi [Montaldi, 1991]. He showed that in configuration ($\mathbf{q}$-) space, projections of tori of reversible Hamiltonian systems have different caustics than projections of tori in non-reversible systems. The configuration-space point of view was also taken by Golubitsky *et al.* [Golubitsky *et al.*, 1996] in a recent study of the admissible types of symmetric periodic orbits in configuration space for reversible equivariant potential systems.

As hardly any mechanical system in engineering is perfectly reversible, it is interesting to study how a reversible symmetric system behaves when small additional non-reversible perturbations are taken into account. O'Reilly *et al.* [O'Reilly *et al.*, 1995; O'Reilly *et al.*, 1996] have considered the consequences of non-reversible dissipative perturbations to reversible mechanical systems. We emphasize that this approach is different to the approach in which generic phenomena of symmetric systems are studied. In the latter studies, from a practical point of view the underlying idea is that in many physically relevant situations idealized symmetric models represent less ideally symmetric experiments quite well (thus quietly assuming that small symmetry breaking perturbations do not cause drastic changes in the dynamical behaviour).

Reversible models for nonequilibrium thermodynamics

The study of the behaviour of the dynamics of many particles is naturally the domain of statistical physics. Recently, in the study of nonequilibrium systems, a *reversible* dynamical systems point of view towards such problems has received a lot of interest.

In 1984, Nosé showed that a molecular system connected to a heat reservoir can be described as an an isolated system after the introduction of additional *bath* variables. These bath variables can be chosen in such a way that the thermodynamics properties of the system can be derived using *microcanonical* rather *canonical* ensembles. Importantly, after adopting the *ergodic hypothesis*, in the latter formulation the relevant thermodynamic variables of a system can be found by averaging over an ergodic trajectory of the system (computationally this is very advantageous). Interestingly, Nosé's additional bath variables keep the dynamical system reversible. The reversibility in the extended nonequilibrium system is such that the reversing symmetry maps sources to sinks and vice-versa. After Nosé's discovery, various modifications to his initial ideas have been made. We refer to the review paper of Hoover [Hoover, 1998] in this volume for a discussion and more references. Despite the large number of experimental (numerical) studies of the Nosé-Hoover type dynamical systems (cf. e.g. Dellago & Posch [Dellago & Posch, 1998] in this volume), as yet not many theoretical studies have been devoted to revealing their properties.

Gallavotti & Cohen [Gallavotti & Cohen, 1995] propose a point of view in which the *ergodic hypothesis* common in equilibrium statistical mechanics is replaced by a similar *chaotic hypothesis* for *reversible* nonequilibrium systems (eg. of the Nosé-Hoover type). The latter approach assumes the existence of a transitive reversible Anosov system. For some studies of the properties of such systems see e.g. [Gallavotti, 1995; Tasaki & Gaspard, 1995] and [Gallavotti, 1998] in this volume. The property of reversibility in the theory is important as it ensures the pairing of negative and positive Liapunov exponents.

5.2 Reversibility in space: reductions of partial differential and difference equations

Many physically-significant examples of reversible flows arise by considering partial differential equations (PDE's) involving space and time, and looking for their steady-state or travelling wave solutions. The resulting ordinary differential equations can then be reversible with the independent "time" variable now being played by a spatial coordinate (the reversibility here is thus equivalent to a spatial symmetry). An example was presented earlier in Example 3.4 (in fact, this example was an important motivation for the studies by Arnol'd & Sevryuk [Arnol'd & Sevryuk, 1986]).

Once the spatial reversibility is noted, the full force of the reversible theory can be applied to yield information about the steady-states and/or travelling waves, e.g. their appearance in one-parameter families, their stability and their bifurcations. Also, reversible homoclinics describe physically relevant solutions such as defects or solitary waves, cf. § 4.5.

There is a wide range of applications in which reversibility arises in the study of steady-states and travelling waves in partial differential equations. Examples range from steady-states of reaction diffusion equations [Kazarinoff & Yan, 1991; Yan, 1992; Yan, 1993b; Yan, 1995; Yan & Hwang, 1996b; Yan *et al.*, 1995], to water-waves (see [Iooss, 1995a] for a recent survey). Sleeman [Sleeman, 1996a; Sleeman, 1996b] recently stressed the importance of reversibility in these kind of models arising in the context of mathematical biology. We note, though, that in many of these applications the reversible differential equations obtained by reduction from a PDE are not only reversible, but also Hamiltonian.

5.3 Reversibility in abstract mathematical settings

Arnol'd [Arnol'd, 1984] already noted that there are several interesting mathematical contexts in which reversibility surprisingly appears. Examples are found in the work of Moser & Webster [Moser & Webster, 1983] who study reversible maps in the context of normal forms for real surfaces in $\mathbb{C}^2$, or in the work of Teixeira [Teixeira, 1981] on discontinuous ordinary differential equations. In both these cases, in a natural way the problem comes down to the study of compositions of two involutions (i.e. a reversible map). These are not isolated examples. A more recent example is provided in the work of [Boukraa *et al.*, 1994; Maillard & Rollet, 1994; Meyer *et al.*, 1994] which studies birational representations of discrete groups generated by involutions. This work has connections with hyperbolic Coxeter groups (but, interestingly enough, also has physical connections to statistical mechanical models). Another example is the study of holomorphic correspondences, which we mention next. We follow this with some algebraic aspects of the study of reversible dynamical systems.

Holomorphic correspondences

Bullett and co-workers have studied the dynamics of complex polynomial correspondences $z \mapsto z'$ defined implicitly by $g(z, z') = 0$ with g polynomial in both arguments and having complex coefficients [Bullett, 1988; Bullett, 1991; Bullett, 1992; Bullett *et al.*, 1986; Bullett & Penrose, 1994b; Bullett & Penrose, 1994a] (cf. also [Webster, 1996] for related work). In particular, they have studied the case with g quadratic in both variables, whereby we have a 2-valued map of the Riemann sphere with 2-valued inverse. Under suitable conditions on g, involutory reversing symmetries arise naturally in the correspondence dynamics. For example, if $g(z, z') = 0$ if and only if $g(\bar{z}', \bar{z}) = 0$, then complex conjugation reverses time so that $z \mapsto z'$ if and only if $\bar{z}' \mapsto \bar{z}$. In the ensuing dynamics induced by the correspondence, one observes Hamiltonian-like behaviour in the form of Siegel discs around symmetric periodic orbits, together with attracting and repelling asymmetric periodic orbits (we remark that reversible polynomial mappings of $\mathbb{C}^n$ to itself are necessarily volume-preserving and cannot have attractors/repellers [Roberts, 1997]).

Reversing symmetry groups and algebraic structures

The *reversing symmetry group* G obtained from combining the symmetries and reversing symmetries of a map f has been previously mentioned in Section 3. From the algebraic structure of reversing symmetry groups, interesting consequences can be drawn. For instance, Goodson [Goodson, 1996a] shows that if $f^2 \neq \mathrm{id}$, f has a reversing symmetry R and the group of symmetries H of f is precisely the trivial centralizer $\{f^n : n \in \mathbb{Z}\}$, then it follows from algebraic considerations that $R^4 = \mathrm{id}$. The is an example where the nature of f and its symmetry group impose the nature of any reversing symmetry R, cf. also [Goodson, 1996a; Goodson, 1997].

It might also be the case that the dynamical systems under consideration form a group with a known structure. Then it may be possible to deduce the structure of possible reversing symmetry groups within this group (and so decide, for instance, if a particular system has any reversing symmetry). An example in which this can be done is the group of hyperbolic toral automorphisms [Baake & Roberts, 1997], which belong to the integer matrix group

$Gl(2,\mathbb{Z})$ (in a related problem, the reversing symmetry group can be calculated for a group of 3-dimensional polynomial maps arising in solid state physics [Baake & Roberts, 1997; Roberts & Baake, 1994]). More generally, since, for example, the set of invertible polynomial maps of $\mathbb{C}^n$ also form a group, there may be possibilities to also understand the prevalence of reversibility in this situation.

Another example in which algebraic considerations arise is in the work of McLachlan *et al.* [McLachlan *et al.*, 1995]. They recently pointed out, in the context of reversible integration methods, that large classes of reversible maps can be viewed as fixed points of anti-automorphisms.

In k-symmetric systems analogous algebraic considerations arise very naturally and non-trivially in the description of the interaction between a map and its (reversing) k-symmetry group, cf. e.g. [Lamb & Quispel, 1995]. The algebraic structures arising in the context of reversing symmetry groups generalizes in a nontrivial way to the context of reversing k-symmetry groups.

6 Discussion

In this paper we have presented a compact survey of the literature on reversible dynamical systems. Dynamical systems with time-reversal symmetry have certainly received a lot of interest in recent years, cf. the bibliography in Appendix A, and a lot of interesting results have been obtained. However, given the importance and relevance of reversible dynamical systems, there is still a range of problems to be tackled.

A theme throughout this survey has been the relation of reversible dynamical systems to equivariant dynamical systems, on the one hand, and Hamiltonian dynamical systems on the other hand. The main task for the future seems to be bringing the theory of reversible systems to a similar maturity as that of equivariant and Hamiltonian systems (for example, many results on reversible systems are obtained in specific problem-related contexts). In so doing, the interconnections between the three classes of systems will also be better understood. We conclude by making some further remarks along these lines.

Because reversibility is a symmetry property, and the present theory for equivariant dynamical systems is powerful and successful, it seems most desirable to adopt an approach that smoothly connects to the theory for equivariant dynamical systems. In particular a theory for reversible systems could be developed as an extension of the equivariant one, in a similar way as reversing symmetry groups are extensions of symmetry groups. In this way equivariance and reversibility can be studied on an equal footing, as particular cases of space-time symmetry properties. In order to achieve this, the introduction of a more systematic use of group (representation) theory for reversing symmetry groups would be useful. (A first step in this direction has recently been made in [Lamb & Roberts, 1997].) As we mentioned in Section 3, a historical distinction in reversible dynamical systems has been among systems with involutory reversing symmetries and ones with non-involutory reversing symmetries. It seems, from a unified symmetry approach, more natural to distinguish between purely reversible systems (with only one involutory reversing symmetry) and reversible equivariant systems.

The relationship between reversible and Hamiltonian dynamical systems is a very intriguing one and deserves further attention. To now, most Hamiltonian systems of interest in the literature are reversible and most reversible systems of interest are Hamiltonian. Reversibility is often used as a tool in reversible Hamiltonian systems to study a particular dynamical phenomenon. Although it is often noted that reversible systems have many features in common with Hamiltonian systems, this is by no means a guarantee of no differences, cf. e.g. [Rimmer, 1978; Champneys, 1994; Matveyev, 1995a; Matveyev, 1996]. Hence, in order to understand the dynamics of reversible Hamiltonian systems, it will be essential to take the full structure (reversibility as well as the Hamiltonian properties) of such systems into account. For a deeper understanding of the similarities of reversible, Hamiltonian and reversible Hamiltonian dynamical systems, more comparative studies of these three categories will be needed.

From the literature it appears that KAM-theory, local bifurcations and homoclinics have been focus points for the research in reversible dynamical systems. However, many basic problems in these fields of research are still open and deserve prompt attention. In particular, we think of the embedding of results for purely reversible systems (e.g. on local bifurcations) into the context of reversible equivariant (Hamiltonian) systems. Also, investigations into reversible homoclinic bifurcations and reversible heteroclinic networks have began only recently [Fiedler & Turaev, 1996; Knobloch, 1997; Reißner, 1998].

Other future directions of research might include the study of more general space-time symmetries of ordinary differential equations, and symmetry properties of partial differential and difference equations that involve transformations of both the dependent and independent variables. We note in this respect that the possibility for PDE's to be reversible in both space and time seems an interesting starting point for such investigations [4].

We feel that our survey illustrates that the field of reversible dynamical systems is still in its adolescense, but enjoying growing interest. We hope that this paper – and indeed this entire special volume of Physica D – will provide encouragement for further studies into reversible dynamical systems.

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We are very grateful to all colleagues and friends who supplied us with valuable remarks on the survey and bibliography. Appendix A has its roots in the bibliographies of Sevryuk [Sevryuk, 1991b] and Roberts & Quispel [Roberts & Quispel, 1992], and was updated and extended with the use of Mathematical Reviews (MathSciNet: `http://www.ams.org/`) and Science Citation Index (Bath Information Services (BIDS): `http://www.bids.ac.uk/`) "online". Without the electronic availability and search tools of these databases, our bibliography would certainly be more incomplete than it is in its present form.

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Appendix A: A Bibliography on time-reversal symmetry in dynamical systems

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