# Reversing symmetries in dynamical systems

J.S.W. Lamb
Institute for Theoretical Physics, University of Amsterdam
Valckenierstraat 65, 1018 XE Amsterdam

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**Abstract:**

Dynamical systems may possess in addition to symmetries that leave the equations of motion invariant, reversing symmetries that invert the equations of motion. Such dynamical systems are called *(weakly) reversible*. Some consequences of the existence of reversing symmetries for dynamical systems with discrete time (mappings) are discussed. A reversing symmetry group is introduced and it is shown that every (weakly) reversible mapping $L$ can be decomposed into two mappings $K_0$ and $K_1$ of the same order $2^l$ (limit $l \to \infty$ included) such that $K_0^2 \circ K_1^2 = I$. Some applications are discussed briefly.

# 1 Introduction

Symmetries play an important role in physics. In a lot of applications they provide a simplification of calculations or even determine a true classification of phenomena. In dynamical systems one may distinguish next to symmetries that leave the equations of motion invariant, reversing symmetries that invert the equations of motion.

A well known family of dynamical systems with a reversing symmetry are evolutions in phase space that are governed by a Hamiltonian of the form

$$H = \sum_n \vec{p}_n^{\,2} + \Phi\left(\{\vec{q}_n\}\right). \tag{1}$$

These systems are said to be time reversal invariant, for the equations of motion are left invariant under time reversal $t \to -t$ and the transformation $\vec{p}_n \to -\vec{p}_n$. This transformation is an involution (i.e. its own inverse).

Devaney [1] generalized this allowing as the reversing transformation any involution. These dynamical systems are called *reversible*. If the reversing symmetry is not required to be an involution the dynamical system is called *weakly reversible* [2].

Much attention has been paid to reversible dynamical systems [3]. It has been shown that they may show a combination of conservative and dissipative features such as KAM-tori and strange attractors at the same time [2, 4, 5]. However, weakly reversible systems have not been studied so well until now.

The concept of reversing symmetries addresses to weakly reversible dynamical systems. Regarding them, the idea of a symmetry group can be extended to a *reversing symmetry group* that contains next to symmetries also reversing symmetries. In this paper I will focus on the importance of this for dynamical systems with discrete time. The latter can be regarded as stroboscopic pictures or as Poincaré-sections of dynamical systems with continuous time.

In section 2 the concept of reversing symmetries is introduced for both dynamical systems with continuous and discrete time. In section 3 reversing symmetry groups are considered and in section 4 some applications are discussed. Extended details have been put in the appendices.

# 2 Reversing symmetries

The concept of symmetries in dynamical systems is well known in physics. Symmetries leave the equations of motion invariant. If $M$ is a *symmetry* this implies for a dynamical system with continuous time

$$\frac{d}{dt}\vec{x} = F\vec{x} \tag{2}$$

that

$$\frac{d}{dt}(M\vec{x}) = F \circ M\vec{x}, \tag{3}$$

and for the dynamical system with discrete time

$$\vec{x}\,' = L\vec{x}, \tag{4}$$

that

$$\begin{aligned} & M\vec{x}\,' = L \circ M\vec{x}, \\ \Leftrightarrow \quad & M \circ L = L \circ M. \end{aligned} \tag{5}$$

We will define $S$ to be a *reversing symmetry* of a dynamical system such that, in the continuous case (2)

$$\frac{d}{dt}(S\vec{x}) = -F \circ S\vec{x}, \tag{6}$$

and in the discrete case (4)

$$\begin{aligned} & S\vec{x}\,' = L^{-1} \circ S\vec{x} \\ \Leftrightarrow \quad & S \circ L = L^{-1} \circ S. \end{aligned} \tag{7}$$

Definition (7) coincides with the definition of *weak reversibility* given by Sevryuk [2]. Hence dynamical systems with a reversing symmetry are called weakly reversible.

If $S$ is an involution, i.e. $S = S^{-1}$, we may write (7) as

$$S \circ L \circ S = L^{-1}. \tag{8}$$

Mappings $L$ with this property, i.e. that have an involution as a reversing symmetry, are called *reversible* (after Devaney [1]).

In the continuous case the same terminology is used. Thus a dynamical system (2) with a reversing symmetry $S$ is called weakly reversible in general, and reversible if the reversing symmetry $S$ is an involution.

If a dynamical system has certain (reversing) symmetries one may construct from them other (reversing) symmetries by composition:

- The composition of two symmetries is a symmetry.
- The composition of two reversing symmetries is a symmetry.
- The composition of a symmetry and a reversing symmetry is a reversing symmetry.

These properties follow directly from the definitions and hold both in the discrete and continuous case.

## 3 Reversing symmetry groups

The symmetries of a dynamical system, together with its reversing symmetries form a group, if one requires the inverses of the (reversing) symmetries to exist.[1]

I will now proceed to reveal the structure of a *reversing symmetry group* by proving a number of propositions. As direct consequences of these propositions, two theorems are formulated in which statements are made about the structure of reversing symmetry groups and a related special decomposition property of mappings that have a reversing symmetry.

**Proposition 1:**
If $M$ is a symmetry then $M^{-1}$ is a symmetry and if $S$ is a reversing symmetry then $S^{-1}$ is a reversing symmetry.
**Proof:**

[1] If one does not require the inverses to exist, they form a semigroup.

In the discrete case it follows directly from (5) and (7) that if $M$ is a symmetry and $S$ is a reversing symmetry that

$$\begin{aligned} M^{-1} \circ L &= L \circ M^{-1} \\ \text{and} \quad S^{-1} \circ L &= L^{-1} \circ S^{-1}, \end{aligned}$$

i.e. $M^{-1}$ is a symmetry and $S^{-1}$ is a reversing symmetry.

In the continuous case we may write (6) as

$$dS|_{\vec{x}} \circ F\vec{x} = -F \circ S\vec{x}, \tag{9}$$

where $dS|_{\vec{x}}$ is the linearized form of $S$ in $\vec{x}$. Furthermore we may write (2) as

$$\begin{aligned} \frac{d}{dt}(S^{-1} \circ S\vec{x}) &= F \circ S^{-1} \circ S\vec{x} \\ \Leftrightarrow dS^{-1}|_{S\vec{x}} \circ dS|_{\vec{x}} \circ F\vec{x} &= F \circ S^{-1} \circ S\vec{x} \\ \text{(using (9) and writing } \vec{x}\,' = S\vec{x}) & \\ \Leftrightarrow -dS^{-1}|_{\vec{x}\,'} \circ F\vec{x}\,' &= F \circ S^{-1}\vec{x}\,' \end{aligned}$$

and hence $S^{-1}$ is a reversing symmetry. It can easily be shown, by a similar argument that $M^{-1}$ is a symmetry if $M$ is.

**Definition 1:**
If a dynamical system has a reversing symmetry, its *reversing symmetry group* is defined to be the group containing its symmetries and reversing symmetries.

The reversing symmetry group is constructed as follows:

- The unity operator $I$ is a symmetry and the unity element of the group.
- The group operation is the composition. The composition operation is associative and the reversing symmetry group is closed under the composition and hence it is a proper group operation.
- Because of *Proposition 1*, the inverses of all group elements are contained in the group.

In practice it is hard, if not impossible, to recognize all symmetries and reversing symmetries of a dynamical system. However, one may recognize certain (reversing) symmetries. To avoid problems I will distinguish between *the reversing symmetry group* and *a reversing symmetry group* of a dynamical system. The latter points to a subgroup of *the reversing symmetry group* that contains at least one reversing symmetry of the dynamical system that is considered.

**Proposition 2:**
The conjugation classes of a reversing symmetry group contain either symmetries or reversing symmetries.
**Proof:**
We know that:

- The composition of two symmetries or two reversing symmetries produces a symmetry.

- The composition of a symmetry with a reversing symmetry produces a reversing symmetry.
- If $T$ is a (reversing) symmetry, then $T^{-1}$ is too.

Because of the fact that the conjugation operation requires two compositions

$$T \mapsto A \circ T \circ A^{-1}$$

$T$ and its image after conjugation by an element of the reversing symmetry group are both (reversing) symmetries. Hence the elements in one conjugation class are either symmetries or reversing symmetries.

*Proposition 1*, *Proposition 2* and *Definition 1* hold for both dynamical systems with continuous and discrete time. In the case of continuous time the reversing symmetry group is a Lie group. This Lie group and its corresponding Lie algebra can provide an instrument of analysis as has been succesfully used recently by Golubitsky *et al.* [6]. However, from now on I will focus on dynamical systems with discrete time. The reversing symmetry groups of these systems are discrete groups.

**Proposition 3:**
$L$ is a symmetry of $L$.
**Proof:**
$L \circ L \circ L^{-1} = L$.

**Corollary 1:**
$L$ and $L^{-1}$ are symmetries of $L$ and hence elements of its reversing symmetry group.

**Proposition 4:**
$L$ and $L^{-1}$ form a conjugation class of the reversing symmetry group.
**Proof:**
Conjugation of $L$ by a symmetry $M$ gives $L$ and conjugation by a reversing symmetry $S$ results in $L^{-1}$. This follows directly from the definition of (reversing) symmetries. Hence $L$ and $L^{-1}$ form a conjugation class.

**Proposition 5:**
Let $S$ be a reversing symmetry of a mapping $L$ of finite order $n$ (i.e. $S^n = I$). If $n$ is odd then $L$ is an involution.
**Proof:**
By application of definition (7) $n$ times we find

$$S^n \circ L \circ S^{-n} = L^{(-)^n} \tag{10}$$

(with the latter notation is meant: $L$ if $n$ is even and $L^{-1}$ if $n$ is odd). For $S^n = I$ we find from (10) that $L = L^{-1}$ if $n$ is odd.

If $L$ is an involution, its dynamics are trivial: all points are fixed points of order 2. From *Proposition 5* it follows that only when $L$ has no reversing symmetries of odd order, it is possible for its dynamics to be not trivial. Hence we will focus on the case that the reversing symmetries (of finite order) are of even order.

**Proposition 6:**
If $S$ is a reversing symmetry of order $2n$, then $S \circ L$ is a reversing symmetry of order $2n$ and $(S \circ L)^2 = S^2$.
**Proof:**

$$(S \circ L)^2 = (S \circ L \circ S^{-1})(S^2 \circ L \circ S^{-2})S^2 = L^{-1} \circ L \circ S^2 = S^2 \tag{11}$$

Because of (11) it follows directly from the fact that $S^{2n} = I$ that $(S \circ L)^{2n} = I$. $S \circ L$ is a reversing symmetry for it is a composition of a reversing symmetry and a symmetry.

From this point we can start building a reversing symmetry group. We will consider a very simple reversing symmetry group of a weakly reversible mapping, that is always present as a subgroup of its reversing symmetry group. The necessary ingredients are a mapping $L$ and a reversing symmetry $S$. We will require $S$ to be of even order $2k$, for we don't want $L$ to be an involution.

We might thus take $L$ and $S$ to be the generators of this group, but it makes more sense to use *Proposition 6* that states that there is always a second reversing symmetry of order $2k$, $T = S \circ L$. From *Proposition 6* we immediately find that $S^2 = T^2$. We can construct the mapping from $S$ and $T$:

$$L = S^{-1} \circ T. \tag{12}$$

We will now consider the reversing symmetry group that is generated by $S$ and $T$. If $L$ is of order $m$ (i.e. $L^m = I$), the group generated by $S$ and $T$ is isomorphic to $R^m_{2k}$.[2]

**Definition 2:**
The group $R^m_{2k}$ is generated by $a$ and $b$ and defined by the following relations:

**(i)** $a^{2k} = e$ ($e$ is the unity element of the group)

**(ii)** $a^2 = b^2$

**(iii)** $(a^{-1}b)^m = e$

However, if $L = S^{-1} \circ T$ is of finite order $m$, its dynamics are not interesting for all points are fixed points of order $m$. We will call a weakly reversible mapping *nontrivial* if $L^m \neq I$ for any nonzero integer $m$. A more interesting group is hence

**Definition 3:**
The group $R_{2k}$ is generated by two elements $a$ and $b$ and defined by the following relations

**(i)** $a^{2k} = e$

**(ii)** $a^2 = b^2$

[2] The group generated by $S$ and $L$ and defined by the relations $S \circ L \circ S^{-1} = L^{-1}$ and $L^m = I$ is also isomorphic to $R^m_{2k}$. The group homomorphism is surjective since any element of $R^m_{2k}$ can be written in terms of $S$ and $L$ and injective since the kernel of the homomorphism is $I$.

We may regard $R_{2k}$ as $R_{2k}^{\infty}$.

**Proposition 7:**
Let $L$ be a nontrivial weakly reversible mapping that has a reversing symmetry of finite order. Then $L$ is an element of a subgroup of its reversing symmetry group that is isomorphic to $R_{2k}$ (for some integer $k$).
**Proof:**
The reversing symmetry of finite order is of even order $2k$, because $L$ is nontrivial and hence not an involution (*Proposition 5*). The subgroup of the reversing symmetry group that is generated by a nontrivial $L$ and a reversing symmetry of order $2k$ is isomorphic to $R_{2k}$.

**Proposition 8:**
$R_{2^l}$ is isomorphic to a subgroup of $R_{2k}$ where $2^l$ is the largest power of 2 that is a divisor of $2k$. $2^l$ can be found, writing $2k$ uniquely as a product of primes:

$$2k = 2^l \cdot 3^{l'} \cdot 5^{l''} \cdot ....... \tag{13}$$

Moreover,[3]

$$R_{2k} \simeq H \times Z_{2^l} \simeq R_{2^l} \times C_{(2k/2^l)} \tag{14}$$

where $a$ and $b$ are the generators of $R_{2k}$,

$$H = \langle a^{(2k/2^l)}, b^{(2k/2^l)} \rangle, Z_{2^l} = \langle a^{(2^l)} \rangle$$

and $C_{(2k/2^l)}$ is the cyclic group of order $2k/2^l$.

**Proof:**

**(i)** $2^l$ is a divisor of $2k$ and $2k/2^l$ is odd (a product of odd primes).

**(ii)** $a^2$ commutes with every element of $R_{2k}$. $Z_2$ is a normal subgroup because of this, and hence $Z_{2^l}$ too.

**(iii)** Any element of $R_{2k}$ can be decomposed uniquely as a product of an element of $Z_{2^l}$ and an element of $H$. In fact $c \in R_{2k}$, $h \in H$, $z \in Z_{2^l}$ can be written without loss of generality as $c = a^{2n} s_p$, $h = a^{2m(2k/2^l)} s_q$ and $z = a^{r2^l}$ (where $n$, $m$, $p$, $q$ and $r$ are nonnegative integers such that $2n < 2k$, $r < 2k/2^l$ and $2m < 2^l$, and $s_p$ (resp. $s_q$) is an alternating sequence of $a$'s and $b$'s of length $p$ (resp. $q$)). If $c = hz$ it follows directly that $s_p = s_q$ and

$$\left[2m(2k/2^l) + r2^l\right] \bmod 2k = 2n.$$

From the latter expression it can easily be checked that $m$ and $r$ are uniquely determined by $n$.

Since all elements of $Z_{2^l}$ commute with all elements of $H$, $H$ is a normal subgroup of $R_{2k}$ too. Hence $R_{2k} \simeq H \times Z_{2^l}$.

**(iv)** $Z_{2^l} \simeq C_{(2k/2^l)}$, this follows directly from the fact that $a$ is of order $2k$.

[3] $\times$ is a direct product and $\langle a_1, ...., a_n \rangle$ denotes a group generated by $a_1, ..., a_n$.

**(v)** $H \simeq R_{2^l}$, it is easy to check that the generators of $H$: $a^{(2k/2^l)}$ and $b^{(2k/2^l)}$ satisfy the defining relations of $R_{2^l}$.

**Corollary 2:**
A nontrivial mapping $L$ that has a reversing symmetry of finite order has a reversing symmetry group with a subgroup isomorphic to $R_{2^l}$ that contains $L$ as an element.

As an illustration of *Proposition 8* and *Corollary 2*, an explicit example of a mapping with a reversing symmetry of order $2k \neq 2^l$ is presented in Appendix A.

**Proposition 9:**
$R_{2^l}$ has no subgroup isomorphic to $R_{2m}$, where $2m < 2^l$.
**Proof:**
$R_{2^l}$ is generated by $a$ and $b$, and each element of $R_{2^l}$ can be written as

$$(a^2)^k s_p$$

where $s_p$ is an alternating sequence of $a$'s and $b$'s of length $p$, and $k$ some integer. If there exists a subgroup of $R_{2^l}$ that is isomorphic to $R_{2m}$ then it has a generator

$$c = (a^2)^k s_p$$
$$\text{and} \quad c^2 = \begin{cases} (a^2)^{2k}(a^2)^p & \text{if } p \text{ is odd} \\ (a^2)^{2k} s_{2p} & \text{if } p \text{ is even} \end{cases}.$$

For $c$ to be a generator of $R_{2m}$ we must have

$$c^{2m} = e.$$

Hence $p$ has to be odd and

$$(2k + p)m = 2^l$$

so $m$ must be even.
We can write any integer as a product of primes

$$\begin{aligned} 2k + p \quad & = 2^{n_1} \cdot 3^{n_2} \cdot \ldots\ldots \text{odd} \;\Leftrightarrow n_1 = 0 \\ m \quad & = 2^{q_1} \cdot 3^{q_2} \cdot \ldots\ldots \text{even} \;\Leftrightarrow q_1 \neq 0 \\ \text{and } (2k+p)m \quad & = 2^{q_1} \cdot 3^{q_2+n_2} \cdot \ldots\ldots \\ & \neq 2^l \text{ unless } q_1 = l,\, n_i = 0 \text{ for all } i \\ & \qquad \text{and } q_j = 0 \text{ for all } j \neq 1. \end{aligned}$$

If a mapping $L$ has a reversing symmetry $S$ of infinite order then $T = S \circ L$ is also a reversing symmetry of infinite order and still $T^2 = S^2$.
**Definition 4:**
The group $R_\infty$ is generated by two elements of infinite order $a$ and $b$ and defined by the relation

**(i)** $a^2 = b^2$.

A group-theoretical discussion on the identification of the groups $R_{2k}^m$, $R_{2k}$ and $R_\infty$ in *Definition 2, 3,* and *4* is presented in Appendix B.

**Proposition 10:**
A nontrivial mapping $L$ that has a reversing symmetry of infinite order has a reversing symmetry group that has a subgroup isomorphic to $R_\infty$.
**Proof:**
The subgroup is generated by $L$ and its reversing symmetry of infinite order.

**Theorem 1:**
All nontrivial weakly reversible mappings $L$, i.e. mappings that are not of finite order and have a reversing symmetry $S$ such that

$$S \circ L \circ S^{-1} = L^{-1}, \tag{15}$$

have a reversing symmetry group that has a subgroup isomorphic to $R_{2^l}$ (limit $l \to \infty$ included).
**Proof:**
*Corollary 2* and *Proposition 10.*

**Theorem 2:**
Every nontrivial weakly reversible mapping $L$ is decomposable into two mappings of order $2^l$ (limit $l \to \infty$ included), i.e. $L = K_1 \circ K_0$, such that $K_0^2 \circ K_1^2 = I$.
**Proof:**
$L$ is the element of a group isomorphic to $R_{2^l}$ (*Theorem 1*) and hence can be written as $L = S^{-1} \circ T$ were $S$ and $T$ are of order $2^l$ (limit $l \to \infty$ included). $S^{-1}$ is of the same order as $S$. Identifying $K_1$ as $S^{-1}$ and $K_0$ as $T$ we find from $S^2 = T^2$ that $K_0^2 \circ K_1^2 = I$.

A weakly reversible mapping is decomposable in the sense of *Theorem 2*. However, this decomposition does not have to be unique. A weakly reversible mapping may for instance be decomposable into two involutions as well as in two elements of order four (such that $K_0^2 \circ K_1^2 = I$). These decompositions are totally independent. Each decomposition into mappings $K_0$ and $K_1$ of the same order $2^l$ such that $K_0^2 \circ K_1^2 = I$ is irreducible, e.g. a fourfold reversing symmetry never implies reversibility. Several types of (weak) reversibility can coexist. This suggests a characterization of (weakly) reversible mappings by their irreducible decompositions.

In general, reversing symmetry groups will be isomorphic to extensions of $R_{2^l}$. For instance, if there is, in addition to a reversing symmetry $S$ of order $2^l$, a second reversing symmetry $U$ that is independent[4] of $S$. In the special case that $U$ commutes with $S$, the reversing symmetry group generated by $S$, $T = S \circ L$ and $U$ is isomorphic to $R_{2^l} \times \langle U \rangle$. A reversing symmetry group generated by $S$, $T$ and $U$ has a subgroup generated by $U$ and $L$ that is reducible to $R_{2^{l'}}$, for some integer $l'$ (limit $l' \to \infty$ included). $l'$ may equal $l$, but it does not have to. These independent reversing symmetries give rise to independent decompositions in the sense of *Theorem 2.*

[4] If $S$ is a reversing symmetry, it follows directly that $S^{2k+1} \circ L^n$ (for any integer $k$ and $n$) is a reversing symmetry too. These reversing symmetries are regarded to be dependent.

## 4 Applications

- It has been shown for reversible mappings that one can use the reversibility to find periodic orbits [7]. The search for symmetric periodic orbits using the symmetry lines can be extended easily to the weakly reversible case. This idea is worked out in more detail in Appendix C.

- For the construction of weakly reversible mappings of a certain kind we can make use of a method, exploited already for reversible mappings [5]. One may construct a weakly reversible mapping with a $2^l$-fold reversing symmetry from one element $A$ of order $2^l$ (e.g. a rotation over $2\pi/2^l$) and a transformation $T$. From $A$ we can construct $B = T \circ A \circ T^{-1}$ that is automatically of the same order as $A$. The requirement that $A^2 = B^2$ restricts $T$ to transformations that commute with $A^2$. Constructing nontrivial reversible mappings, there is no restriction on $T$ because it always commutes with $A^2 = I$. Using this method one is able to produce nontrivial weakly reversible mappings of various kinds. In Appendix D this method is used to construct a family of weakly reversible mappings with a four-fold reversing symmetry.

## 5 Concluding remarks

Dynamical systems with a reversing symmetry and discrete time, (weakly) reversible mappings, are shown to have a special structure. They are always decomposable into two mappings of order $2^l$ (limit $l \to \infty$ included) such that the squares of these mappings are each other's inverse (*Theorem 2*). This property has been shown to be understood entirely from a group-theoretical point of view.

As symmetry groups do, reversing symmetry groups provide much information about the dynamics. I have shown that certain periodic orbits are directly related to the reversing symmetries (see Appendix C). A more profound study of reversing symmetry groups, i.e. extensions of $R_{2^l}$, may lead to a better insight in the structure of the dynamics. As has been recognized by MacKay [8], symmetries may give rise to anomalous behaviours of the dynamics. Since weakly reversible mappings inherently possess symmetries, this notion is very relevant in further studies of the dynamics of weakly reversible mappings.

To study the explicit difference between reversible and weakly reversible mappings that are not reversible the family of mappings that is presented in Appendix D may serve to provide an example. Using the concept of local reversibility [9, 10] one may construct a nontrivial weakly reversibile mapping that is definitely not reversible. We plan to study such mappings in the near future.

In microscopic crystal models such as the DIFFOUR model [11, 12], reversing symmetries in a mapping that is related to the search for ground states may occur as a direct consequence of symmetries in the local potential. Phase transitions can be related to bifurcations of this mapping. The connection between possible kinds of phase transitions and the reversing symmetries is obvious, e.g. para-ferro phase transitions in the DIFFOUR model are related to symmetry breaking (Rimmer) bifurcations. We plan to study explicit examples in the near future.

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# Appendices

## A Modified Townsville mappings

As an example I will present in this appendix a family of modified Townsville mappings (for the original Townsville mapping see Post *et al.* [5]). These mappings have a reversing symmetry group isomorphic to $R_{4n+2}$. They are decomposable into two elements of order $(4n+2)$ and so are additionally decomposable into two involutions in the spirit of *Theorem 2*.

To find a nontrivial example of a reversible mapping with a reversing symmetry element of order $(4n+2)$, we may construct it from two mappings $A$ and $B$ with the following properties:

$$
\begin{aligned}
A^{4n+2} &= I && (16)\\
B^2 &= I && (17)\\
A^2 \circ B &= B \circ A^2 && (18)\\
A^{2n+1} \circ B &\neq B \circ A^{2n+1} && (19)
\end{aligned}
$$

Condition (19) ensures that our reversible mapping $L = B \circ A^{2n+1}$ is not an involution. This to prevent trivial dynamics. I will now present a modification of the Townsville mapping that has the proposed symmetries. $A$ and $B$ are given by

$$
A \;:\; \left\{ \begin{array}{lcl} x' &=& x + \frac{1}{4n+2} + \omega(y) \pmod 1, \\ y' &=& -y, \end{array} \right. \tag{20}
$$

$$
B \;:\; \left\{ \begin{array}{lcl} x' &=& x, \\ y' &=& \frac{-y+g(x)}{1+yh(x)}. \end{array} \right. \tag{21}
$$

It is furthermore required that $\omega(y) = -\omega(-y)$, $g(x+\frac{1}{2n+1}) = g(x)$ and $h(x+\frac{1}{2n+1}) = h(x)$. $B^2 = I$ [5] and we have

$$
A^2 \;:\; \left\{ \begin{array}{lcl} x' &=& x + \frac{1}{2n+1} \pmod 1, \\ y' &=& y, \end{array} \right. \tag{22}
$$

$$
A^{2n+1} \;:\; \left\{ \begin{array}{lcl} x' &=& x + \frac{1}{2} + \omega(y) \pmod 1, \\ y' &=& -y. \end{array} \right. \tag{23}
$$

Hence it is immediately clear that (18) is satisfied and that for almost all $\omega(y)$ (19) is satisfied. The resulting modified Townsville mapping $L = B \circ A^{2n+1}$ has the following form

$$L : \left\{ \begin{array}{rcl} x' & = & x + \frac{1}{2} + \omega(y) \pmod 1, \\ y' & = & \frac{y+g(x')}{1-yh(x')}, \end{array} \right. \tag{24}$$

with $g(x+\frac{1}{2n+1}) = g(x)$, $h(x+\frac{1}{2n+1}) = h(x)$ and $\omega(-y) = -\omega(y)$.
$L$ is decomposable into two involutions

$$\begin{array}{rrcl} & L & = & I_1 \circ I_0 \\ \text{where} & I_1 & = & B \\ \text{and} & I_0 & = & A^{2n+1} \end{array} \tag{25--27}$$

as well as in two mappings of order $4n+2$

$$\begin{array}{rrcl} & L & = & K_1 \circ K_0 \\ \text{where} & K_1 & = & A \\ \text{and} & K_0 & : & \left\{ \begin{array}{rcl} x' & = & x + \omega(y) - \omega(y') + \frac{n}{2n+1} \pmod 1, \\ y' & = & \frac{y+g(x+\omega(y)+\frac{1}{2})}{yh(x+\omega(y)+\frac{1}{2})-1}. \end{array} \right. \end{array} \tag{28--30}$$

It is straightforward to check that $K_0^2 \circ K_1^2 = I$ and $K_0^{4n+2} = I$.

# B Identification of $R_{2k}^m$, $R_{2k}$ and $R_\infty$

In this appendix I present a more detailed discussion on the identification of the groups that have been defined in this paper. As a standard reference I use the book of Coxeter and Moser [13].

The group $R_{2k}^m$ that is defined in *Definition 2* can be identified as one of the families of groups known as *Miller's generalization of polyhedral groups* ([13] §6.6):

$$\langle l, m \mid n; q \rangle \tag{31}$$

generated by $R$, $S$ and $Z$ and defined by ([13] eq. (6.62))

$$R^l = S^m = Z, \ \ (RS)^n = Z^q = E, \tag{32}$$

where $E$ is the unity element of the group. Comparing this with *Definition 2* we find immediately that

$$R_{2k}^m \simeq \langle -2, 2 \mid m; k \rangle \tag{33}$$

and from *Definition 3* and *Definition 4* that

$$\begin{array}{rcl} R_{2k} & = & R_{2k}^\infty \simeq \langle -2, 2 \mid \infty; k \rangle, \\ R_\infty & = & R_\infty^\infty \simeq \langle -2, 2 \mid \infty; \infty \rangle. \end{array} \tag{34--35}$$

For $\langle l, m \mid n; q \rangle$ it has already been recognized ([13] eq. (6.624)) that

$$\begin{array}{rcl} \langle l, m, \mid n; q \rangle & \simeq & \langle l, m \mid n; 2^c \rangle \times C_r \\ \text{where} & & q = 2^c r \ \ , r \text{ is odd} \\ \text{and} & C_r & \text{is the cyclic group of order } r. \end{array} \tag{36}$$

As a corollary of this, *Proposition 8* and (14) follow immediately.

Instead of building $R^m_{2k}$ from $a$ and $b$ we may also generate it from $a$ and $\lambda = a^{-1}b$ (with $a$ of order $2k$, $\lambda$ of order $m$ and $a\lambda a^{-1} = \lambda^{-1}$). It is easily checked that this is entirely equivalent. Looking at $R^m_{2k}$ from this point of view its structure can be recognized as a cyclic expansion of a cyclic group.

$$\begin{aligned} \langle\lambda\rangle &\simeq C_m, && (37)\\ \langle a\rangle &\simeq C_{2k}, && (38)\\ \text{and} \quad R^m_{2k} &\simeq \langle\lambda\rangle \wedge \langle a\rangle && (39)\end{aligned}$$

where $\wedge$ is a semidirect product and $\langle\lambda\rangle$ a normal subgroup of $R^m_{2k}$. Hence we find that

$$R^m_{2k} \simeq C_m \wedge C_{2k} \simeq (C_m \wedge C_{2^l}) \times C_{(2k/2^l)} \tag{40}$$

where the last isomorphism follows from (36).

$Z_2 = \langle a^2\rangle$ is a normal subgroup of $R^m_{2k}$ (the elements of $Z_2$ commute with all elements of $R^m_{2k}$) and we find that

$$R^m_{2k}/Z_2 = \langle aZ_2, bZ_2\rangle \simeq R^m_2 \simeq D_m. \tag{41}$$

$D_m$ is the dihedral group ([13] eq. (1.52)). The dihedral group typically acts as a reversing symmetry group for reversible mappings: $D_m$ for trivial and $D_\infty$ for nontrivial reversible mappings.

# C Symmetric periodic orbits

For reversible mappings it has been recognized that some periodic orbits follow directly from the symmetry properties of the mapping. These periodic orbits are called *symmetric periodic orbits*. However, this idea can be extended easily to weakly reversible mappings. The following is a straightforward generalization of the work of DeVogelaere [7] who considered explicitly reversible mappings.

Define the family of sets:

$$\mathcal{R}_n := \{\vec{x} \in \mathcal{C} : \ L^n \circ K_0\vec{x} = \vec{x}\}, \tag{42}$$

where $K_0$ is a reversing symmetry of $L$ that is an automorphism on the configuration space $\mathcal{C}$. We find that

$$\vec{x} \in \mathcal{R}_n \cap \mathcal{R}_m \quad \Rightarrow \quad L^{n-m}\vec{x} = \vec{x}. \tag{43}$$

Proof:
Note that

$$L^n \circ K_0\vec{x} = \vec{x} \quad \Leftrightarrow \quad K_0 \circ L^{-n}\vec{x} = \vec{x} \quad \Leftrightarrow \quad \vec{x} = L^n \circ K_0^{-1}\vec{x} \tag{44}$$

and we find that for any $\vec{x} \in \mathcal{R}_n \cap \mathcal{R}_m$

$$L^{n-m}\vec{x} = L^n \circ K_0 \circ L^m \circ K_0^{-1}\vec{x} = L^n \circ K_0 \circ L^m \circ K_0\vec{x} = \vec{x}. \tag{45}$$

We will call these periodic orbits *symmetric periodic orbits.*
All $\mathcal{R}_n$ can be found from $\mathcal{R}_0$ and $\mathcal{R}_1$:

$$\mathcal{R}_{2n} = L^n\mathcal{R}_0 \qquad \mathcal{R}_{2n+1} = L^n\mathcal{R}_1. \tag{46}$$

$$\underline{\text{Proof:}} \quad \begin{array}{rcl} \vec{x} \in \mathcal{R}_{2n} & \Leftrightarrow & \vec{x} = L^{2n} \circ K_0 \vec{x} = L^n \circ K_0 \circ L^{-n} \vec{x} \\ & & \Leftrightarrow K_0 \circ L^{-n} \vec{x} = L^{-n} \vec{x} \Rightarrow L^{-n} \vec{x} \in \mathcal{R}_0 \\ \vec{x} \in \mathcal{R}_{2n+1} & \Leftrightarrow & \vec{x} = L^{2n+1} \circ K_0 \vec{x} = L^n \circ L \circ K_0 \circ L^{-n} \vec{x} \\ & & \Leftrightarrow L \circ K_0 \circ L^{-n} \vec{x} = L^{-n} \vec{x} \Rightarrow L^{-n} \vec{x} \in \mathcal{R}_1 \end{array}$$

For practical use one should note that every orbit that intersects $\mathcal{R}_n \cap \mathcal{R}_m$ also intersects $\mathcal{R}_0$ if $n$ is even, and $\mathcal{R}_1$ if $n$ is odd. It is hence sufficient to search for symmetric periodic orbits at

$$\left\{ \begin{array}{ll} \mathcal{R}_0 \cap \mathcal{R}_m & \text{(orbits of period } m) \\ \mathcal{R}_1 \cap \mathcal{R}_m & \text{(orbits of period } m-1) \end{array} \right. \tag{47}$$

(43) and (46) have been found already by DeVogelaere [7] for reversible mappings. (Note that in general not all periodic orbits can be found in this way. Asymmetric periodic orbits may exist as well.)

# D 2D mappings with a four-fold reversing symmetry

In this appendix a family of two-dimensional mappings is constructed with a reversing symmetry of order 4. We start with a mapping of order 4 (the rotation over $\frac{\pi}{2}$):

$$A : \left\{ \begin{array}{rcl} x' & = & -y, \\ y' & = & x, \end{array} \right. \tag{48}$$

and a transformation $T$

$$T : \left\{ \begin{array}{rcl} x' & = & xp(y) + q(y) \\ y' & = & y + r(x') \end{array} \right. , \qquad T^{-1} : \left\{ \begin{array}{rcl} x' & = & \frac{x - q(y')}{p(y')} \\ y' & = & y - r(x) \end{array} \right. . \tag{49}$$

A second mapping of order 4 is constructed from $A$ and $T$: $B = T \circ A \circ T^{-1}$. $A$ and $B$ can serve as generators of the subgroup of the reversing symmetry group isomorphic to $R_4$ of $L = A^{-1} \circ B$ if $A^2 = B^2$. The latter implies that $T \circ A^2 = A^2 \circ T$ and hence that $p$ is an even function and $q$ and $r$ are odd functions.

$$B : \left\{ \begin{array}{rcl} x' & = & (r(x) - y)\; p\,(y' - r(x')) + q\,(y' - r(x')), \\ y' & = & \frac{x - q(y - r(x))}{p(y - r(x))} + r(x'), \end{array} \right. \tag{50}$$

and $L = A^{-1} \circ B$

$$L : \left\{ \begin{array}{rcl} x' & = & \frac{x - q(y - r(x))}{p(y - r(x))} - r(y'), \\ y' & = & -\,(r(x) - y)\; p\,(x' + r(y')) - q\,(x' + r(y')). \end{array} \right. \tag{51}$$

$L$ is weakly reversible with $A$ as a reversing symmetry of order 4 if $p$ is an even function and $q$ and $r$ are odd functions.

## References


[1] Devaney R L 1976 *Trans.Am.Math.Soc.* **218** 89

[2] Sevryuk M B 1986 *Reversible systems*, Lecture Notes in Mathematics vol. 1211 (Berlin: Springer)

[3] Roberts J A G and Quispel G R W 1991 *Chaos and Time-reversal Symmetry* to appear in *Phys.Rep.*

[4] Quispel G R W and Roberts J A G 1989 *Phys.Lett.A* **135** 337

[5] Post T, Capel H W, Quispel G R W and Van der Weele J P 1990 *Physica A* **164** 625

[6] Golubitsky M, Krupa M and Lim C 1991 *SIAM J.Appl.Math.* **51**(1) 49

[7] DeVogelaere R 1958 in: *Contribution to the theory of nonlinear oscillations* vol. 4 (Princeton: Princeton Univ. Press) 53

[8] MacKay R S 1984 *Phys.Lett.* **106A**(3) 99

[9] Quispel G R W and Capel H W 1989 *Phys.Lett.A* **142** 112

[10] Brown A 1991 *Physica A* **173** 267

[11] Janssen T and Tjon J A 1983 *J.Phys.A: Math.Gen.* **16** 673

[12] Lamb J S W 1991 *The ferro-antiferro bifurcation transition ...* to appear in *J.Phys.A: Math.Gen.*

[13] Coxeter H S M and Moser W O J 1965 *Generators and Relations for Discrete Groups* (2nd ed.), Ergebnisse der Mathematik und ihrer Grenzgebiete - Neue Folge - Band 14 (Berlin: Springer)