Reversible Relative Periodic Orbits

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Abstract

We study the bundle structure near reversible relative periodic orbits in reversible equivariant systems. In particular we show that the vector field on the bundle forms a skew product system, by which the study of bifurcation from reversible relative periodic solutions reduces to the analysis of bifurcation from reversible discrete rotating waves. We also discuss possibilities for drifts along group orbits. Our results extend those recently obtained in the equivariant context by Sandstede *et al.* [24] and Wulff *et al.* [29].

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1 Introduction

Symmetry is a frequently occurring property of dynamical systems. In recent years, bifurcation in the presence of symmetries has been studied extensively, in particular for equivariant dynamical systems (i.e. systems with *time-preserving* symmetries).

In addition to equivariance, dynamical systems may also possess *time-reversal* symmetries. Such symmetries are well known to occur in many applications of interest, many of which – but not all! – are Hamiltonian. Examples include mechanical systems, and differential equations obtained by reduction from certain types of equivariant PDE's, see Lamb and Roberts [20] for a recent survey.

In the context of equivariant dynamical systems, group theoretical tools have been employed to develop a comprehensive understanding of bifurcation theory, see for instance [11]. The work presented in this paper is part of a program to extend the existing theory for equivariant systems to reversible equivariant systems.

The aim of this paper is to extend recently obtained results of Sandstede *et al.* [24] and Wulff *et al.* [29] on the bundle structure near relative periodic orbits in equivariant dynamical systems, to reversible equivariant dynamical systems, thereby reducing the problem of bifurcation from reversible periodic orbits to that of reversible discrete rotating waves. As a simple special case, bifurcation from reversible relative equilibria reduces to bifurcation from reversible equilibria.

A complicating fact at this point is that there is not yet a local bifurcation for general reversible equivariant systems. But some results are known to apply to special cases, most notably the *purely* reversible case (in the absense of equivariance).

For instance, reversible Hopf bifurcation has been studied by Arnol'd and Sevryuk [2], Vanderbauwhede [26], and – taking into account certain reversible equivariant situations – by Knobloch & Vanderbauwhede [14]. A Lyapunov center theorem for reversible equilibria is due to Devaney [6]. Generalizations to certain equivariant reversible systems are given in Golubitsky *et al.* [10].

Bifurcation from reversible discrete rotating waves (or even reversible equivariant equilibria) has upto now been studied only in certain special cases. Examples of bifurcation from reversible discrete rotating waves were discussed in [17].

But most results on local bifurcations have been derived in the purely reversible context. For instance, the papers [3, 25, 27, 28, 5] discuss various results on branching of reversible periodic orbits without additional symmetry, including period doubling and period preserving bifurcations as well as subharmonic bifurcations. See [20] for a more extended discussion and additional references (also on other aspects of reversible systems). For a survey on homoclinics in reversible systems, see Champneys [4].

Apart from their independent importance, the results of this paper form a first step towards understanding the bundle structure near relative periodic orbits in reversible equivariant Hamiltonian systems, which will be discussed in a forthcoming paper (with R.M. Roberts [22]).

2 Setting of the problem

We consider differential equations of the form

$$\dot{u} = f(u) \tag{2.1}$$

where f is a smooth vector field defined on a finite-dimensional smooth connected manifold M. We denote the flow of (2.1) by Φ_t.

Let Γ_ρ be a finite-dimensional (possibly noncompact) Lie group acting smoothly and properly on M, possessing a normal subgroup Γ of index two, i.e. $\Gamma_\rho/\Gamma \cong \mathbb{Z}_2$. Then, taking ρ to be any element of $\Gamma_\rho \setminus \Gamma$, we have

$$\Gamma_\rho = \langle \Gamma, \rho \rangle = \Gamma \cup \rho\Gamma. \tag{2.2}$$

We assume that our differential equation (2.1) is Γ-equivariant and ρ-reversible, i.e. the flow Φ_t satisfies $\gamma\Phi_t(u) = \Phi_t(\gamma u)$ and $\gamma\rho\Phi_{-t}(u) = \Phi_t(\gamma\rho u)$ for all $\gamma \in \Gamma$ and $u \in M$. In other words, whenever $u(t)$ is a solution of (2.1) then so are $\gamma u(t)$ and $\gamma\rho u(-t)$ for all $\gamma \in \Gamma$. In terms of the vectorfield f in (2.1) these equivariance and reversibility conditions read

$$\gamma f(u) = f(\gamma u) \quad \text{and} \quad \rho f(u) = -f(\rho u) \quad \text{for all} \quad \gamma \in \Gamma. \tag{2.3}$$

Note that from this definition it follows that (2.1) is not only reversible w.r.t. ρ, but also with respect to all elements in $\rho\Gamma$. We call the elements of Γ *symmetries* and the elements of $\rho\Gamma$ *reversing (or time-reversal) symmetries* of f. The above imposed structure on Γ_ρ naturally follows from the composition properties of symmetries and reversing symmetries. It should be noted that $\rho\Gamma$ need not contain an element of order two (a so-called *involution*), so that reversible equivariant systems need not possess an involutory reversing symmetry.

We say that a point u_0 lies on a *relative periodic orbit* (RPO) $\mathcal{P}$ if there is some (least) $T > 0$ (the relative period) such that $\Phi_T(u_0) = \sigma u_0$ for some $\sigma \in \Gamma$. The relative periodic orbit $\mathcal{P}$ is formed by the union of the Γ group orbit and the time orbit of u_0:

$$\mathcal{P} = \{\gamma\Phi_t(u_0) \mid \gamma \in \Gamma, t \in \mathbb{R}\}. \tag{2.4}$$

Definition 2.1 *A relative periodic orbit $\mathcal{P}$ is called* reversible *if there exists a reversing symmetry ρ such that $\rho\mathcal{P} = \mathcal{P}$.*

Note that indeed $\rho\mathcal{P}$ is always a relative periodic orbit if $\mathcal{P}$ is. Namely,

$$\Phi_T(\rho u_0) = \rho\Phi_{-T}(u_0) = \rho\sigma^{-1}\rho^{-1}(\rho u_0) = \tilde{\sigma}(\rho u_0). \tag{2.5}$$

for some $\tilde{\sigma} \in \Gamma$, since Γ is normal in Γ_ρ.

Similarly a relative equilibrium is a flow-invariant group orbit Γu_0. In this case $\Phi_t(u_0) = \mathrm{e}^{\xi t} u_0$ for some $\xi \in L\Gamma$, where $L\Gamma$ denotes the Lie algebra of Γ. A relative equilibrium Γu_0 is called reversible if $\rho\Gamma u_0 = \Gamma u_0$.

Before we discuss reversible relative periodic orbits we briefly recall the results by Wulff et al. [29] on relative periodic solutions in the (non-reversible) equivariant case.

Let $\mathcal{P}$ be a relative periodic orbit of relative period 1, i.e. there is some $u_0 \in \mathcal{P}$ and some $\sigma \in \Gamma$ so that $\sigma^{-1}\Phi_1(u_0) = u_0$. Let $\Delta \subset \Gamma$ denote the isotropy subgroup of the point u_0 and let

$$\Sigma = \langle \sigma, \Delta \rangle = \{\gamma \in \Gamma \mid \Phi_\theta(u_0) = \gamma u_0 \text{ for some } \theta \in \mathbb{R}\}$$

denote the spatio-temporal symmetry group of $\mathcal{P}$ w.r.t. u_0. Note that due to the proper group action of Γ, the isotropy subgroup Δ must be compact.

In [24, 29] it was shown that one can parametrize a neighborhood U of $\mathcal{P}$ as

$$U \equiv (\Gamma \times V \times \mathbb{R})/(\Delta \rtimes \mathbb{Z}). \tag{2.6}$$

Here V is a Δ-invariant section transversal to $\mathcal{P}$ in u_0 and 1 is the relative period of $\mathcal{P}$. Note that V is a Poincaré section which is transverse not only to the time orbit of u_0, but also to the group orbit of u_0. The variable $\theta \in \mathbb{R}$ plays the role of the phase along the relative periodic solution.

Let (γ, v, θ) be coordinates in $\Gamma \times V \times \mathbb{R}$. Then the quotient by $\Delta \rtimes \mathbb{Z}$ corresponds to identifications due to the isotropy Δ and spatiotemporal symmetry Σ (both defined w.r.t. some fixed $u_0 \in \mathcal{P}$):

$$(\gamma, v, \theta) = (\gamma\delta^{-1}, \delta v, \theta) = (\gamma\sigma^{-1}, Qv, \theta + 1). \tag{2.7}$$

Here $Q^{-1} \in \mathrm{O}(V)$ is an orthogonal *twisted equivariant* map of finite (even) order $2k_V$, i.e. for all $\delta \in \Delta$ we have

$$\delta Q = Q\phi(\delta), \tag{2.8}$$

where $\phi : \Delta \to \Delta$ is an automorphism of Δ (the *twist automorphism*) defined as

$$\phi(\delta) = \sigma^{-1}\delta\sigma.$$

For many groups, for example, algebraic groups, it is possible to choose σ within $\sigma\Delta$ such that ϕ has finite order k, see [29]. Actually even if the order k of the automorphism ϕ of Δ cannot be chosen to be finite then still ϕ^{k_V} effectively acts as the identity morphism on the representation of Δ on V for an appropriate choice of σ within $\sigma\Delta$, see [29] and section 5.3.2 below for more details. If k is finite we have $k_V|k$.

Due to the bundle structure one can lift the differential equations on the bundle to $(\Delta \rtimes \mathbb{Z})$-equivariant differential equations on $\Gamma \times V \times \mathbb{R}$. In terms of the coordinates (γ, v, θ) on $\Gamma \times V \times \mathbb{R}$ the resulting differential equations have the form

$$\dot{\gamma} = \gamma f_\Gamma(v,\theta), \quad \dot{v} = f_V(v,\theta), \quad \dot{\theta} = f_\Theta(v,\theta) \tag{2.9}$$

where

$$f_\Gamma(0,\theta) = f_V(0,\theta) = 0, \ f_\Theta(0,\theta) = 1, \tag{2.10}$$

and the $(\Delta \rtimes \mathbb{Z})$-equivariance reads as follows ($\delta \in \Delta$):

$$f_\Gamma(\delta v,\theta) = \mathrm{Ad}_\delta f_\Gamma(v,\theta), \ f_V(\delta v,\theta) = \delta f_V(v,\theta), \ f_\Theta(\delta v,\theta) = f_\Theta(v,\theta), \tag{2.11}$$

$$f_\Gamma(Qv,\theta+1) = \mathrm{Ad}_\sigma f_\Gamma(v,\theta), \ f_V(Qv,\theta+1) = Q f_V(v,\theta), \ f_\Theta(Qv,\theta+1) = f_\Theta(v,\theta). \tag{2.12}$$

Here Ad_γ denotes the adjoint action of Γ on $L\Gamma$, i.e., $\mathrm{Ad}_\gamma(\xi) = \gamma\xi\gamma^{-1}$, $\gamma \in \Gamma$.

Because Q has finite order $2k_V$, it follows that the (v,θ)-subsystem of (2.9) has the compact symmetry group $\Delta \rtimes \mathbb{Z}_{2k_V}$. Consequently a relative periodic orbit $\mathcal{P}$ corresponds to a discrete rotating wave of the $\Delta \rtimes \mathbb{Z}_{2k_V}$-equivariant (v,θ)-subsystem. Hence, the study of bifurcations from relative periodic orbits reduces to the study of bifurcations from discrete rotating waves. In particular, when the automorphism ϕ is of finite order k, the bifurcations from relative periodic orbits become fixed point bifurcations in systems with $\Delta \rtimes \mathbb{Z}_{2k}$ symmetry which were studied by Lamb and Melbourne [18].

The above result can be improved in two ways. First, by a reparametrization of time, one may set $f_\Theta(v,\theta) \equiv 1$, so that θ becomes the natural time-coordinate and (2.9) becomes a nonautonomous differential equation on $\Gamma \times V$. Second, subject to some mild conditions on Γ, we may go to a comoving frame in which the relative periodic orbit is observed as a periodic orbit. In particular, if Γ is an algebraic group then there is a comoving frame $\xi \in LZ(\Sigma)$ (where $Z(\Sigma)$ is the centralizer of Σ in Γ) satisfying the equations $\sigma = \exp(\xi)\sigma'$, so that $(\sigma')^n = 1$ for some positive $n \in \mathbb{N}$ in which the whole bundle becomes periodic of period $2n$. Moreover, the order k of the twist automorphism associated to σ' divides n, see [29]. In the comoving frame the identification (2.7) holds with σ replaced by the finite order element σ'.

In this paper we show how to incorporate reversibility into the above approach towards relative periodic orbits. We obtain a parametrization of a neighborhood of the relative periodic solution as in (2.6), but with Γ replaced by Γ_ρ and with an additional identification due to reversibility of the relative periodic solution.

The paper is organized as follows. In section 3 we first treat reversible relative equilibria as a special (simple) case. In section 4 we discuss our results on the bundle structure near reversible relative periodic orbits in detail, including possible drifts of reversible RPOs and give some examples. Our main results on the bundle structure and the form of the differential equations on the bundle are stated in section 4.3 and section 4.4. Section 5 is devoted to the proofs of the main theorems.

3 Reversible relative equilibria

Before discussing reversible relative periodic orbits, we will first briefly treat the simpler case of reversible relative equilibria.

We consider a relative equilibrium Γu_0 and let $\Phi_t(u_0) = \exp(\xi t)u_0$. In the non-reversible case one can write a tubular neighborhood U of the relative equilibrium Γu_0 as $U = \Gamma \times_\Delta V$, see [23, 15, 7]. Here V is a Δ-invariant section, transversal to the group orbit Γu_0 in u_0.

We will now consider a ρ-reversible relative equilibrium Γu_0, where u_0 has isotropy $\Delta \subset \Gamma$. It is easily verified that there always exists a reversing symmetry (an element of the coset $\rho\Gamma$) that fixes the point u_0. Namely, since $\rho\Gamma u_0 = \Gamma u_0$ there is some $\gamma \in \Gamma$ so that $\rho u_0 = \gamma u_0$ or equivalently $\gamma^{-1}\rho u_0 = u_0$. Hence, we may assume without loss of generality that $u_0 \in \mathrm{Fix}(\rho)$ (for some reversing symmetry ρ).

Let Δ_ρ be the isotropy subgroup of Γ_ρ w.r.t. u_0 generated by Δ and ρ. Note that Δ_ρ is an index two extension of Δ because $\rho^2 \in \Delta$ and $\rho\Delta = \Delta\rho$. Further, we call the subgroup Σ_ρ of Γ_ρ generated by Δ, ρ and the Lie algebra element ξ the *spatiotemporal symmetry group* of the relative equilibrium in Γ_ρ.

The reversibility of the relative equilibrium leads to some natural reversibility conditions on the drift. Before stating these conditions, we introduce the notion of reversible maximal tori. Let $\Gamma = N(\Delta)/\Delta$ for a moment so that $\rho^2 = \mathrm{id}$ (here $N(\Delta)$ denotes the normalizer of Δ).

Definition 3.1 *An element $\xi \in L\Gamma$ is called ρ-reversible if $\mathrm{Ad}_\rho \xi = -\xi$. A torus T^ρ is called a* maximal ρ-reversible *torus if T^ρ is generated by a ρ-reversible $\xi \in L\Gamma$, i.e. $\mathrm{Ad}_\rho(\xi) = -\xi$, and is not contained in any ρ-reversible torus of higher dimension.*

Proposition 3.2 *Let Γu_0 be a ρ-reversible relative equilibrium, with $u_0 \in \mathrm{Fix}(\rho)$. Let $\Phi_t(u_0) = \exp(\xi t)u_0$. Then w.l.o.g. ξ may be taken to be Δ-equivariant and ρ-reversible, i.e. $\xi \in LZ(\Delta)$ and $\mathrm{Ad}_\rho(\xi) = -\xi$.*

Generically, if the drift is compact, the dimension of the closure of the time orbit of u_0 is equal to $\dim(T^\rho)$, where T^ρ is the maximal ρ-reversible torus in $N(\Delta)/\Delta$ containing ξ. In other words, generically ξ generates a maximal ρ-reversible torus T^ρ.

In the case of noncompact drift the time orbit of each u_0 on the relative equilibrium is diffeomorphic to a line.

Proof. Since $\Phi_t(u_0) = \mathrm{e}^{\eta t}u_0$ for some $\eta \in L\Gamma$ we have

$$\Phi_t(\rho u_0) = \rho\Phi_{-t}(u_0) = \mathrm{e}^{-\mathrm{Ad}_\rho \eta t}(\rho u_0)$$

similarly as for reversible relative periodic orbits, cf. (2.5). Since $u_0 = \rho u_0$ we conclude that

$$\eta + \mathrm{Ad}_\rho\eta \in L\Delta, \quad \rho^2 \in \Delta.$$

This is a restriction on the possible drift directions and the possible isotropies of reversible relative equilibria. Since $\mathrm{e}^{-\eta t}\delta\mathrm{e}^{\eta t}u_0 = \mathrm{e}^{-\eta t}\delta\Phi_t(u_0) = \mathrm{e}^{-\eta t}\Phi_t(u_0) = u_0$ for $\delta \in \Delta$, $t \in \mathbb{R}$, we always have $\eta \in LN(\Delta)$.

Since Δ_ρ is compact we may choose a Δ_ρ-equivariant scalar product on $L\Gamma$ and write $\eta = \chi + \xi$, where $\chi \in L\Delta$ and $\xi \in (L\Delta)^\perp$. Now, using that the scalar product is ρ-invariant, we find that $\xi + \mathrm{Ad}_\rho(\xi) \in (L\Delta)^\perp$, but on the other hand also $\xi + \mathrm{Ad}_\rho(\xi) = \eta + \mathrm{Ad}_\rho(\eta) - \chi - \mathrm{Ad}_\rho(\chi) \in L\Delta$. Hence, $\xi + \mathrm{Ad}_\rho(\xi) = 0$. Since $\chi \in L\Delta$ we may w.l.o.g. change η to ξ. As shown in [9] we have $(L\Delta)^\perp \cap LN(\Delta) = LZ(\Delta)$. Hence also $\xi \in LZ(\Delta)$.

Thus Φ_t generates a reversible torus if Σ_ρ is compact. Following Field [8] and Ashwin and Melbourne [1], we conclude that generically ξ will generate a reversible torus of maximal dimension, i.e. a maximal ρ-reversible torus in $N(\Delta)/\Delta$ provided that the drift is compact.

∎

In describing the bundle structure, we employ a local section V that is transversal to the relative equilibrium. In particular we may choose this section to be Δ_ρ-invariant. Namely, since $u_0 \in \mathrm{Fix}(\Delta_\rho)$ the group Δ_ρ acts on $T_{u_0}M$, and since Δ_ρ is a compact group there exists a Δ_ρ-invariant scalar product on $T_{u_0}M$. Since $\rho\Gamma\rho^{-1} = \Gamma$ we have $\mathrm{Ad}_\rho L\Gamma = L\Gamma$. Therefore $\rho\xi u_0 = \mathrm{Ad}_\rho\xi u_0 \in L\Gamma u_0$ for $\xi \in L\Gamma$. So the space $T_{u_0}\Gamma u_0 = L\Gamma u_0$ is Δ_ρ-invariant. We hence choose $V = (T_{u_0}\Gamma u_0)^\perp$ as Δ_ρ-invariant complement of $T_{u_0}\Gamma u_0$ in $T_{u_0}M$.

The bundle structure near the relative equilibrium and the form of the corresponding differential equations are discussed in the following theorem.

Theorem 3.3 *Let Γu_0 be a ρ-reversible relative equilibrium with $u_0 \in \mathrm{Fix}(\Delta_\rho)$. Then a neighbourhood U of Γu_0 has the form*

$$U \equiv (\Gamma_\rho \times V)/\Delta_\rho = \Gamma_\rho \times_{\Delta_\rho} V. \tag{3.1}$$

The differential equations (2.1) on U lift to $\Gamma \times V$. With coordinates $(\gamma, v) \in \Gamma \times V$ they take the form

$$\dot{\gamma} = \gamma f_\Gamma(v), \quad \dot{v} = f_V(v), \tag{3.2}$$

and are Δ-equivariant and reversible in the following sense:

$$\forall\ \delta \in \Delta \quad f_\Gamma(\delta v) = \mathrm{Ad}_\delta f_\Gamma(v), \quad f_V(\delta v) = \delta f_V(v), \tag{3.3}$$

$$f_\Gamma(\rho v) = -\mathrm{Ad}_\rho f_\Gamma(v), \quad f_V(\rho v) = -\rho f_V(v). \tag{3.4}$$

Proof. We focus on the consequences of the reversibility, as the equivariant part of the result is well known, see [15, 7].

Let $\psi : T_{u_0}M \to M$ be a Δ_ρ-equivariant smooth local chart of M near u_0. Due to reversibility we have

$$\begin{aligned}(\gamma, v) &= \gamma\psi(v) = \gamma\rho^{-1}\rho\psi(v) = \gamma\rho^{-1}\psi(\rho v)\\ &= (\gamma\rho^{-1}, \rho v), \quad v \in V.\end{aligned}$$

Moreover, Δ-equivariance implies

$$\forall \delta \in \Delta \ \ (\gamma, v) = (\gamma\delta^{-1}, \delta v).$$

By these identifications, a neighbourhood U of the relative equilibrium Γu_0 is of the form $U \equiv (\Gamma_\rho \times V)/\Delta_\rho = \Gamma_\rho \times_{\Delta_\rho} V$.

In the coordinates $(\gamma, v) \in \Gamma \times V$ the differential equations on U have the form

$$\dot{\gamma} = \gamma f_\Gamma(v), \quad \dot{v} = f_V(v)$$

where $f_\Gamma(v) \in (L\Delta)^\perp$ for some Δ_ρ-invariant scalar-product on $L\Gamma$. Reversibility implies that whenever $u(t) = (\gamma(t), v(t)) = \gamma(t)\psi(v(t))$ is a solution, then so is

$$\begin{aligned}\rho u(-t) &= \rho\gamma(-t)\psi(v(-t)) = \rho\gamma(-t)\rho^{-1}\psi(\rho v(-t))\\ &= (\rho\gamma(-t)\rho^{-1}, \rho v(-t)).\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}(-t)}(\rho v) = f_V(\rho v) \quad \Leftrightarrow \quad -\rho f_V(v) = f_V(\rho v), \tag{3.5}$$

and similarly

$$\frac{\mathrm{d}}{\mathrm{d}(-t)}(\rho\gamma\rho^{-1}) = f_\Gamma(\rho v) \quad \Leftrightarrow \quad -\mathrm{Ad}_\rho f_\Gamma(v) = f_\Gamma(\rho v). \tag{3.6}$$

■

We conclude this section with some simple examples of reversible relative equilibria. Just as in the non-reversible case they might be rotating waves, invariant tori or travelling waves.

Example 3.4 (Reversible rotating wave) Let $\Gamma_\rho = \mathrm{O}(2)$, $\Gamma = \mathrm{SO}(2)$. Then, a relative equilibrium Γu_0 with isotropy $\Delta = \mathbf{1}$ and $\Delta_\rho = \mathbb{Z}_2$ is typically a reversible rotating wave satisfying $\Phi_t(u_0) = R_{\omega t}u_0$, and thus an equilibrium in a rotating frame with angular velocity ω. By reversibility ρ reflects the rotating wave to a solution rotating with opposite orientation, i.e.

$$\rho\Phi_t(u_0) = \Phi_{-t}(u_0) = R_{-\omega t}u_0.$$

Example 3.5 (Reversible invariant torus) Let $\Gamma_\rho = \mathrm{O}(N)$, $\Gamma = \mathrm{SO}(N)$. Then, a relative equilibrium Γu_0 with $\Delta = \mathbf{1}$ and $\Delta_\rho = \mathbb{Z}_2$ is typically a reversible invariant torus satisfying $\Phi_t(u_0) = \exp(\xi t)u_0$, with $\xi \in \mathrm{so}(N)$. Again we have

$$\rho\Phi_t(u_0) = \Phi_{-t}(u_0) = \exp(-\xi t)u_0.$$

Example 3.6 (Reversible travelling wave) Let $\Gamma_\rho = \mathbb{R} \rtimes Z_2 = \mathrm{E}(1)$, $\Gamma = \mathbb{R}$. Let T_a denote a translation by $a \in \mathbb{R}$ so that $\rho T_a = T_{-a}\rho$ for all $a \in \mathbb{R}$. Then, a relative equilibrium Γu_0 with $\Delta = \mathbf{1}$ and $\Delta_\rho = \mathbb{Z}_2$ is typically a reversible travelling wave satisfying

$$\Phi_t(u_0) = T_{vt}u_0, \quad \rho\Phi_t(u_0) = \Phi_{-t}(u_0) = T_{-vt}u_0.$$

4 Reversible relative periodic orbits

Starting point for our results on reversible relative periodic orbits are the results on reversible relative equilibria presented in the previous section and the results in [24, 29] on the bundle structure near RPOs in non-reversible systems. First, in subsection 4.1, we make some general statements on reversible relative periodic orbits and give some examples. In subsection 4.2 we show what kinds of drifts are possible for reversible RPOs. In subsection 4.3 we state our three main theorems on the bundle structure near reversible RPOs. Detailed proofs of the main theorems are deferred to Section 5. In subsection 4.4 we study the differential equations in the bundle coordinates and show that reversible RPOs correspond to reversible discrete rotating waves on a slice $(V \times S^1)$.

4.1 Some general properties of reversible RPOs

Let $\mathcal{P}$ be a reversible relative periodic orbit with relative period 1, i.e. fixing $u_0 \in \mathcal{P}$ we have $\Phi_1(u_0) = \sigma u_0$ for some $\sigma \in \Gamma$ and $\rho\mathcal{P} = \mathcal{P}$.

As in the case of relative equilibria we can always choose a point $u_0 \in \mathcal{P}$ such that $u_0 \in \mathrm{Fix}(\rho)$ for some reversing symmetry ρ. Namely, since $\rho u_0 \in \mathcal{P}$ there exists a $\gamma_0 \in \Gamma$ and some $t_0 \in \mathbb{R}$ such that $\rho u_0 = \gamma_0 \Phi_{t_0}(u_0)$, and we readily verify that $\gamma_0^{-1}\rho\Phi_{t_0/2}(u_0) = \gamma_0^{-1}\Phi_{-t_0/2}(\rho u_0) = \Phi_{t_0/2}(u_0)$. So we change u_0 to $\Phi_{t_0/2}(u_0)$ and ρ to $\gamma_0^{-1}\rho$. From now on we assume that u_0 and ρ are such that $u_0 \in \mathrm{Fix}(\rho)$. We denote the isotropies of u_0 by $\Delta \subset \Gamma$ and $\Delta_\rho = \langle \Delta, \rho \rangle \subset \Gamma_\rho$. As mentioned in the case of reversible relative equilibria, Δ is a normal subgroup of Δ_ρ of index 2.

In reversible systems a point $u_0 \in \mathrm{Fix}(\rho)$ lies on a ρ-reversible T-periodic solution if and only if $\Phi_{T/2}(u_0) \in \mathrm{Fix}(\rho)$. An analogous observation applies to reversible RPOs.

Lemma 4.1 *Let $u_0 \in \mathrm{Fix}(\rho)$. Then u_0 lies on a ρ-reversible relative periodic orbit $\mathcal{P}$ with relative (minimal) period T, if and only if $\Phi_{T/2}(u_0) \in \mathrm{Fix}(\rho')$ where ρ' is some reversing symmetry in $\rho\Gamma$.*

Proof. We readily verify that, with $\Phi_T(u_0) = \sigma u_0$ for some $\sigma \in \Gamma$, we obtain

$$\sigma\rho\Phi_{T/2}(u_0) = \sigma\Phi_{-T/2}(\rho u_0) = \sigma\Phi_{-T/2}(u_0) = \Phi_{T/2}(u_0),$$

so that indeed $\Phi_{T/2}(u_0) \in \mathrm{Fix}(\rho')$ with $\rho' = \sigma\rho$.

Vice-versa, if $u_0 \in \mathrm{Fix}(\rho)$ and $\Phi_{T/2}(u_0) \in \mathrm{Fix}(\sigma\rho)$ for some $T > 0$, $\sigma \in \Gamma$ then u_0 lies on a relative periodic solution with relative period T. ∎

In the following proposition we collect some easy observations:

Proposition 4.2 *Let u_0 be a point on a reversible relative periodic orbit $\mathcal{P}$ with (minimal) relative period $T > 0$, such that $\Phi_T(u_0) = \sigma u_0$, and with isotropy $\Delta_\rho \subset \Gamma_\rho$ and let $\Sigma = \langle \Delta, \sigma \rangle$. Then,*

1. $\dim \mathrm{Fix}(\Delta_\rho) \leq \dim \mathrm{Fix}(\Delta) - 1$.
2. *If* $\dim \mathrm{Fix}(\Delta_\rho) = \dim \mathrm{Fix}(\Delta) - 1$ *and Σ is compact then $\mathcal{P}$ is periodic.*

Proof. The time orbit of u_0 is contained in $\mathrm{Fix}(\Delta)$, so we can restrict to the case $M = \mathrm{Fix}(\Delta)$, and $\Gamma = N(\Delta)/\Delta$. Now assume that

$$\dim \mathrm{Fix}(\rho) = \dim(M) - 1.$$

In case Σ is compact, σ^ℓ can be made arbitrarily close to identity by choosing some appropriate $\ell \in \mathbb{N}$ large enough. Since

$$\Phi_\ell(u_0) = \sigma^\ell u_0 \in \mathrm{Fix}(\sigma^\ell \rho \sigma^{-\ell})$$

and since $\mathrm{Fix}(\sigma^\ell\rho\sigma^{-\ell})$ is close to $\mathrm{Fix}(\rho)$ we conclude that due to the transversal crossing of $\Phi_t(u_0)$ through $\mathrm{Fix}(\sigma^\ell\rho\sigma^{-\ell})$ and since $\mathrm{Fix}(\rho)$ has codimension 1 the orbit must cross transversely through $\mathrm{Fix}(\rho)$, i.e. $\Phi_T(u_0) \in \mathrm{Fix}(\rho)$ for some $T > 0$. Then u_0 lies on a periodic orbit with period $2T$.

If $\dim M = \dim \mathrm{Fix}(\rho)$ then $M = \mathrm{Fix}(\rho)$ and since $f(u) \in \mathrm{Fix}(-\rho)$ we have $f(u) = 0$ on M. Hence $\mathcal{P}$ must be a steady state, which contradicts our assumptions.

∎

We conclude this subsection with some examples of reversible relative periodic orbits.

Example 4.3 (Reversible discrete rotating wave) One of the simplest examples of a reversible relative periodic orbit is a reversible discrete rotating wave as studied in [16], i.e. a periodic orbit with discrete spatiotemporal symmetry. For example, let $\Gamma_\rho = \mathbb{D}_n$, $\Gamma = \mathbb{Z}_n$, let σ generate $\mathbb{Z}_n$ and let $\rho \in \Gamma_\rho \setminus \Gamma$. Then we may have a reversible periodic solution of period n passing through $u_0 \in \mathrm{Fix}(\rho)$ and satisfying $\Phi_1(u_0) = \sigma u_0$.

Example 4.4 (Reversible modulated travelling wave) Another example of a reversible relative periodic orbit is a reversible modulated travelling wave. Let $\Gamma_\rho = \mathrm{E}(1)$, $\Gamma = \mathbb{R}$, and let σ generate $\mathbb{Z} \subset \mathbb{R}$. Then we may have a relative periodic orbit $\mathcal{P}$ passing through $u_0 \in \mathrm{Fix}(\rho)$ satisfying $\Phi_1(u_0) = \sigma u_0$. Such a solution is called a modulated travelling wave and may have bifurcated from a reversible travelling wave, see Example 3.6.

Example 4.5 (Reversible modulated rotating wave) A reversible modulated rotating wave is a relative periodic orbit that is nonperiodic, but whose drift is compact. For instance, let $\Gamma_\rho = \mathrm{O}(2)$ and $\Gamma = \mathrm{SO}(2)$ act on a two-torus $M = S^1 \times S^1$, so that Γ acts naturally on the first copy of S^1 and trivial on the second copy. Let ρ act as a reflection on both copies of S^1. Then $\dim \mathrm{Fix}(\rho) = 0$, and typically a relative periodic orbit $\mathcal{P}$ passing through $u_0 \in \mathrm{Fix}(\rho)$ satisfies $\Phi_T(u_0) = \sigma u_0$ for some $T > 0$ and some $\sigma \in \mathrm{SO}(2)$ generating $\mathrm{SO}(2)$. Such a solution is quasiperiodic, and a reversible modulated rotating wave.

4.2 Drift of reversible RPOs

In this subsection we investigate what kind of drifts may occur for reversible RPOs. As we saw above, without loss of generality we may assume that there exists a point u_0 on the reversible RPO $\mathcal{P}$ in $\mathrm{Fix}(\rho)$.

We find that the drift element σ with $\Phi_1(u_0) = \sigma u_0$ must satisfy certain conditions due to the isotropy $\Delta_\rho \subset \Gamma_\rho$ of u_0.

Lemma 4.6 *Let $\mathcal{P}$ be a ρ-reversible relative periodic orbit with relative period 1, containing a point $u_0 \in \mathrm{Fix}(\Delta_\rho)$, such that $\Phi_1(u_0) = \sigma u_0$ for some $\sigma \in \Gamma$. Then σ must be inside $N(\Delta)$ and satisfies*

$$\sigma\rho\sigma\rho^{-1} \in \Delta. \tag{4.1}$$

Proof. If $\mathcal{P}$ has isotropy Δ, then σ has to lie in the normalizer $N(\Delta)$ of the isotropy group Δ due to the fact that for all $\delta \in \Delta$

$$\sigma^{-1}\delta\sigma u_0 = \sigma^{-1}\delta\Phi_1(u_0) = u_0.$$

If $\mathcal{P}$ is ρ-reversible, with $u_0 \in \mathrm{Fix}(\rho) \cap \mathcal{P}$, we have

$$\rho\sigma\rho^{-1}u_0 = \rho\Phi_1(u_0) = \Phi_{-1}(\rho u_0) = \sigma^{-1}u_0,$$

so that $\sigma\rho\sigma\rho^{-1} \in \Delta$. ∎

In contrast to the situation of drift of reversible relative equilibria, note that in the case of reversible RPOs σ cannot always be chosen to be Δ-equivariant and ρ-reversible (here ρ-reversible means $\rho\sigma\rho^{-1} = \sigma^{-1}$), see [18, 29, 17] for counter-examples.

Let $\Sigma = \langle \Delta, \sigma \rangle$ denote the spatio-temporal symmetry of $\mathcal{P}$ w.r.t u_0 in Γ. Because of Lemma 4.6 the group $\Sigma_\rho = \langle \Sigma, \rho \rangle$ is an index-2-extension of Σ. The spatial and spatio-temporal symmetry groups Δ, Δ_ρ, Σ and Σ_ρ of $\mathcal{P}$ w.r.t. u_0 in Γ rsp. Γ_ρ satisfy the following relations

$$\Delta \trianglelefteq \Sigma \trianglelefteq \Sigma_\rho, \quad \Delta \trianglelefteq \Delta_\rho, \quad \Sigma_\rho/\Sigma \cong \Delta_\rho/\Delta \cong \mathbb{Z}_2 \tag{4.2}$$

(here $A \trianglelefteq B$ denotes that A is a normal subgroup of B). Analogous statements for reversible discrete rotating waves can be found in [17].

In the following consideration let $\Gamma = N(\Delta)/\Delta$. Adapting the results of [8] and [1] to the reversible context, we obtain the following result on generic drift of reversible RPOs.

Proposition 4.7 *Let $\mathcal{P}$ be a ρ-reversible relative periodic orbit, as in Lemma 4.6. Then, if Σ is compact, we generically have the equality*

$$\Sigma = C^\rho = \{\gamma \in C \mid \rho\gamma = \gamma^{-1}\rho\} \tag{4.3}$$

where C is the Cartan subgroup of $N(\Delta)/\Delta$ containing σ. This implies that in the case of compact drift generically the dimension of the closure of the time-orbit of u_0 is $\dim(C^\rho) + 1$.

If Σ is non-compact, the time-orbit of $u_0 \in \mathcal{P}$ is a line.

We call C^ρ a *reversible Cartan subgroup.*

Proof. If the spatio-temporal symmetry Σ_ρ of the reversible RPO $\mathcal{P}$ is compact, that is, if $\mathcal{P}$ is a modulated rotating or a discrete rotating wave, then there is a Cartan subgroup C, $\sigma \in C$, containing σ [1]. In the non-reversible case, generically, $\Sigma = C$ in $N(\Delta)/\Delta$ and therefore the dimension of the time orbit of each $u_0 \in \mathcal{P}$ is equal to $\dim(C) + 1$, see [9, 15, 1].

Due to the reversibility condition (4.1) we conclude that if $\mathcal{P}$ is a generic reversible RPO with compact spatio-temporal symmetry Σ_ρ, the dimension of the closure of the time-orbit of $u_0 \in \mathcal{P}$ equals $\dim(C^\rho) + 1$ where C^ρ is defined as in (4.3).

If the spatio-temporal symmetry Σ_ρ of the reversible RPO $\mathcal{P}$ is noncompact, for example, if $\mathcal{P}$ is a modulated travelling wave, then the time-orbit of $u_0 \in \mathcal{P}$ is always a line, i.e. one-dimensional, see [1]. ∎

In analogy to [29] we define the *index* m of a reversible RPO as the least number such that

$$\sigma^m = \delta_0 \exp(m\zeta) \quad \text{for some} \quad \zeta \in LZ(\Sigma), \quad \mathrm{Ad}_\rho \zeta = -\zeta, \quad \delta_0 \in \Delta.$$

This index may be infinity, e.g. if $\Sigma = \mathbb{Z}$, $\Sigma_\rho = \mathbb{D}_\infty$. If Σ is compact then m equals the number $\tilde{m}$ of connected components of the reversible Cartan subgroup C^ρ containing σ. This can be seen as follows: obviously $\tilde{m}|m$. On the other hand we know that $\sigma^{\tilde{m}} = \exp(\tilde{m}\tilde{\zeta})$ with $\mathrm{Ad}_\sigma\tilde{\zeta} = \tilde{\zeta}$ and $\mathrm{Ad}_\rho\tilde{\zeta} = -\tilde{\zeta}$ in $N(\Delta)/\Delta$. Then $\sigma^{\tilde{m}} = \delta_0 \exp(\tilde{m}\zeta)$ for some $\delta_0 \in \Delta$ and some representative ζ of $\tilde{\zeta} = \zeta + L\Delta$. Choose a Σ_ρ-invariant scalar-product on $LN(\Delta)$ and choose the representative $\zeta \in (L\Delta)^\perp \cap LN(\Delta) = LZ(\Delta)$. Then $(\mathrm{Ad}_\sigma\zeta - \zeta) \in (L\Delta)^\perp \cap L\Delta$, and therefore $\mathrm{Ad}_\sigma\zeta = \zeta$. Similarly, $(\mathrm{Ad}_\rho\zeta + \zeta) \in (L\Delta)^\perp \cap L\Delta$ and henceforth $\mathrm{Ad}_\rho\zeta = -\zeta$ and $\tilde{m} = m$.

We will see in subsection 5.3.1 and Theorem 4.12 below that m is finite if Γ is an algebraic group. If m is finite denote the group generated by Δ and the element $\beta = \sigma\exp(-\zeta)$ by Σ^m, and let Σ^m_ρ be the group generated by Σ^m and ρ. We have

$$\Sigma^m/\Delta \cong \mathbb{Z}_m, \quad \Sigma^m_\rho/\Delta \cong \mathbb{D}_m, \quad \Sigma^m_\rho/\Sigma^m \cong \Delta_\rho/\Delta \cong \mathbb{Z}_2. \tag{4.4}$$

Assume that the relative period of the reversible RPO $\mathcal{P}$ is scaled to one. In a frame moving with velocity ζ the reversible relative periodic orbit $\mathcal{P}$ is foliated by periodic orbits of period m as the following lemma shows:

Lemma 4.8 *Assume that the reversible RPO $\mathcal{P}$ has finite index m. Then*

$$\mathcal{P} = \frac{\Gamma_\rho \times S^1}{\Sigma^m_\rho}, \quad S^1 = \mathbb{R}/m\mathbb{Z}.$$

Proof. Write $\gamma \exp(-\theta\zeta)\Phi_\theta(u_0) = (\gamma, \theta)$ where $\gamma \in \Gamma_\rho$, $\theta \in \mathbb{R}$. One readily verifies that

$$(\gamma\delta^{-1}, \theta) = (\gamma\beta^{-1}, \theta + 1) = (\gamma\rho^{-1}, -\theta)$$

where $\beta = \sigma \exp(-\zeta)$. In particular $(\gamma, \theta + m) = (\gamma\delta_0, \theta) = (\gamma, \theta)$, with $\delta_0 \in \Delta$. ∎

4.3 Bundle structure near reversible RPOs

In this section we describe the bundle structure in the neighborhood of a reversible RPO. Before we state our main theorems we need to discuss some preliminaries.

We let $\mathcal{P}$ be a reversible relative periodic solution of (2.1) with relative period 1, and let ρ be a reversing symmetry fixing a point $u_0 \in \mathcal{P}$ (such a reversing symmetry ρ and point u_0 always exist, see section 4.1). Let $\Delta \subset \Gamma$ be the isotropy subgroup of u_0, and $\Phi_1(u_0) = \sigma u_0$. We define again $\Delta_\rho = \langle \Delta, \rho \rangle$.

We now extend the definition of the *twist morphism* ϕ of section 2 to Δ_ρ by

$$\forall \delta \in \Delta \quad \phi(\delta\rho) = \sigma^{-1}\delta\rho\sigma^{-1}. \tag{4.5}$$

Due to Lemma 4.6 we have $\phi(\delta\rho) \in \rho\Delta$ for all $\delta \in \Delta$ so that ϕ maps Δ_ρ into itself. Note that in general ϕ is a morphism but not necessarily an automorphism or homomorphism of Δ_ρ: one readily verifies that [17]

$$\phi(\delta\rho) = \phi(\delta)\phi(\rho), \quad \text{but} \quad \phi(\rho\delta) = \phi(\rho)\phi^{-1}(\delta). \tag{4.6}$$

Definition 4.9 *Let Δ_ρ act smoothly on a manifold X. We say that a diffeomorphism $F : X \to X$ is* twisted reversible equivariant *if*

$$F\delta = \phi(\delta)F, \quad F\delta\rho = \phi(\delta\rho)F^{-1}$$

for all $\delta \in \Delta$.

Twisted (reversible) equivariant diffeomorphisms and the twist morphism ϕ were introduced in Lamb and Quispel [19] and referred to as *k-symmetric* maps where k is least such that ϕ^k is the identity morphism on Δ_ρ. In the study of RPOs linear twisted reversible equivariant maps play an important role, see below.

When Γ_ρ is algebraic, there always exists a choice of σ for which k is finite, see Theorem 4.12 below and section 5.3 for more details. As in section 2 we will let k_V denote the effective order of ϕ as acting on a representation of Δ_ρ on V.

In describing the bundle structure we will mention a group Ξ_ρ that is related to the spatiotemporal symmetries of the RPO. We define

$$\Xi = \Delta \rtimes \mathbb{Z} \tag{4.7}$$

as the group which was factored out in the bundle structure near non-reversible RPOs, see section 2. We now write the elements of Ξ as (δ, i) where $\delta \in \Delta$ and $i \in \mathbb{Z}$. The group multiplication of the semidirect product is defined via the twist morphism ϕ

$$(0,i)(\delta,0)(0,-i) = (\phi^{-i}(\delta),0), \quad \delta \in \Delta, i \in \mathbb{Z}. \tag{4.8}$$

We define $\Xi_\rho = (\Delta \rtimes \mathbb{Z})_\rho$ as index-2-extension of Ξ by an element ρ such that

$$\rho(\delta,i)\rho^{-1} = (\rho\delta\rho^{-1}, -i), \quad \delta \in \Delta, \ i \in \mathbb{Z}, \tag{4.9}$$

and $\rho^2 \in \Delta$. Evidently,

$$\Xi \trianglelefteq \Xi_\rho, \quad \Xi_\rho/\Xi \cong \mathbb{Z}_2, \quad \text{and} \quad \Delta \trianglelefteq \Xi_\rho, \quad \Xi_\rho/\Delta \simeq \mathbb{D}_\infty. \tag{4.10}$$

Before we state our main results on the bundle, we verify the existence of a Δ_ρ-invariant section V that is locally transversal to $\mathcal{P}$.

Lemma 4.10 *Let $\mathcal{P}$ be a ρ-reversible relative periodic orbit through u_0 with isotropy Δ_ρ. Then there exists a Δ_ρ-invariant complement V to $T_{u_0}\mathcal{P}$ in $T_{u_0}M$.*

Proof. Because $u_0 \in \mathrm{Fix}(\Delta_\rho)$, we have $\rho f(u_0) = -f(u_0)$. Therefore the space $\langle f(u_0)\rangle$ is Δ_ρ-invariant. Since $\rho\xi u_0 = \mathrm{Ad}_\rho \xi u_0$, $\xi \in L\Gamma$, we conclude that $L\Gamma u_0$ is Δ_ρ-invariant. So the tangent space $T_{u_0}\mathcal{P}$ to the relative periodic orbit in u_0 is invariant under the group Δ_ρ.

Since $u_0 \in \mathrm{Fix}(\Delta_\rho)$ and Δ_ρ is compact we can assume w.l.o.g. that Δ_ρ acts orthogonally on $T_{u_0}M$. Then the space $V = (T_{u_0}\mathcal{P})^\perp$ is a Δ_ρ-invariant complement to $T_{u_0}\mathcal{P}$. ∎

Finally, we now state our main result:

Theorem 4.11 *Let $\mathcal{P}$ be a reversible RPO as discussed above. Then a neighborhood U of $\mathcal{P}$ is of the form*

$$U \equiv \frac{\Gamma_\rho \times V \times \mathbb{R}}{\Xi_\rho}.$$

Here V is a Δ_ρ-invariant section transversal to $\mathcal{P}$. Let (γ, v, θ) be coordinates on $\Gamma_\rho \times V \times \mathbb{R}$. Then the quotient by Ξ_ρ corresponds to the identifications ($\gamma \in \Gamma_\rho$, $v \in V$, $\theta \in \mathbb{R}$)

$$(\gamma, v, \theta) = (\gamma\rho^{-1}, \rho v, -\theta) = (\gamma\sigma^{-1}, Qv, \theta+1) = (\gamma\delta^{-1}, \delta v, \theta) \quad \forall \delta \in \Delta. \tag{4.11}$$

The linear map $Q^{-1} \in \mathrm{O}(V)$ occurring in these identifications is twisted reversible equivariant (w.r.t. the twist morphism ϕ on Δ_ρ) and of finite order $2k_V$.

By definition $(\Delta \rtimes \mathbb{Z}) \trianglelefteq \Xi_\rho$ and $\Xi_\rho/(\Delta \rtimes \mathbb{Z}) \cong \mathbb{Z}_2$, so that indeed the bundle structure here is a natural extension of the result obtained in [29]. The proofs of this theorem and the two following theorems can be found in section 5.

Now assume that Γ is an algebraic group, i.e. defined by algebraic equations. This assumption is often satisfied in applications: examples of algebraic groups include compact groups, Euclidean groups and the classical Lie groups. In this case the bundle structure near a reversible relative periodic orbit becomes periodic in a comoving frame, more precisely:

Theorem 4.12 *Let $\mathcal{P}$ be a reversible RPO as in Theorem 4.11, and assume that Γ is algebraic. Then there exists a $\xi \in LZ(\Sigma)$, satisfying $\mathrm{Ad}_\rho \xi = -\xi$, such that $\sigma^n = \exp(n\xi)$ for some $n \in \mathbb{N}$ and an appropriate choice of σ in the coset $\sigma\Delta$. As a consequence, k and m are finite and there exists a comoving frame (moving with velocity ξ) in which the neighborhood U of $\mathcal{P}$ has the form*

$$U \equiv \frac{\Gamma_\rho \times V \times \mathbb{R}/2n\mathbb{Z}}{\Xi'_\rho}.$$

Here $\Xi' = \Delta \rtimes \mathbb{Z}_{2n}$ is defined by the group multiplication (4.8) where $i \in \mathbb{Z}_{2n}$ instead of $i \in \mathbb{Z}$ and Ξ'_ρ is defined as index-2-extension of Ξ' with group multiplication (4.9) so that Ξ'_ρ satisfies $\Delta \trianglelefteq \Xi'_\rho$ and $\Xi'_\rho/\Delta \cong \mathbb{D}_{2n}$.

The quotient corresponds to the identifications (4.11) where σ is replaced by σ'.

Analogously as for the group Ξ_ρ we have $(\Delta \rtimes \mathbb{Z}_{2n}) \trianglelefteq \Xi'_\rho$ and $\Xi'_\rho/(\Delta \rtimes \mathbb{Z}_{2n}) \cong \mathbb{Z}_2$.

Similarly as in the non-reversible case [29] we can obtain a $2m$-periodic bundle if the index m of the reversible RPO is finite, i.e. if $\sigma^m = \delta_0 \exp(m\zeta)$ for some $\zeta \in LZ(\Sigma)$, $\mathrm{Ad}_\rho\zeta = -\zeta$, $\delta_0 \in \Delta$. For the bundle construction we need the group Σ_ρ^{2m} which is a cyclic extension of Δ_ρ of order $2m$. The group Σ_ρ^{2m} is generated by Δ, ρ and some element R satisfying

$$R^{-1}\delta R = \phi(\delta),\ R^{-1}\delta\rho R^{-1} = \phi(\delta\rho) \text{ for } \delta \in \Delta, \quad R^{2m} = \delta_0^2.$$

Theorem 4.13 *Let $\mathcal{P}$ be are reversible RPO of index m and write $\sigma^m = \delta_0 \exp(m\zeta)$ where $\zeta \in LZ(\Sigma)$, $\mathrm{Ad}_\rho\zeta = -\zeta$ and $\delta_0 \in \Delta$. Form the group Σ_ρ^{2m} as described above and denote $\beta = \sigma\exp(-\zeta)$. Then, there is a neighborhood U of the RPO $\mathcal{P}$ such that, in a comoving frame, moving with velocity ζ*

$$U \equiv \frac{\Gamma_\rho \times V \times S^1}{\Sigma_\rho^{2m}}, \quad S^1 = \mathbb{R}/(2m\mathbb{Z}).$$

Here V is an orthogonal representation of the group Σ_ρ^{2m}. The action of Σ_ρ^{2m} on $\Gamma_\rho \times V \times S^1$ is given as

$$\begin{aligned}
\delta \cdot (\gamma, v, \theta) &= (\gamma\delta^{-1}, \delta v, \theta),\\
R \cdot (\gamma, v, \theta) &= (\gamma\beta^{-1}, Rv, \theta + 1),\\
\rho \cdot (\gamma, v, \theta) &= (\gamma\rho^{-1}, \rho v, -\theta).
\end{aligned}$$

4.4 Differential equations on the bundle

In the following theorem we state the form of the differential equations in the bundle coordinates on $\Gamma_\rho \times V \times \mathbb{R}$. We will see that the analysis of the bundle structure reduces the study of bifurcations from reversible RPOs to the study of bifurcations from reversible discrete rotating waves in the slice $V \times S^1$. Whereas the overall symmetry group Γ may be noncompact the symmetry group of the slice which determines bifurcations is compact.

Theorem 4.14 *In terms of the bundle coordinates $(\gamma, v, \theta) \in \Gamma_\rho \times V \times \mathbb{R}$, the differential equations on the lifted bundle have the form (2.9):*

$$\dot{\gamma} = \gamma f_\Gamma(v, \theta), \quad \dot{v} = f_V(v, \theta), \quad \dot{\theta} = f_\Theta(v, \theta).$$

They are ρ-reversible, i.e.,

$$f_\Gamma(\rho v, -\theta) = -\mathrm{Ad}_\rho f_\Gamma(v, \theta), \;\; f_V(\rho v, -\theta) = -\rho f_V(v, \theta), \;\; f_\Theta(\rho v, -\theta) = f_\Theta(v, \theta) \quad (4.12)$$

and $(\Delta \rtimes \mathbb{Z})$-equivariant in the sense of (2.11), (2.12). Further $f_\Gamma(0, \theta) = 0$, $f_V(0, \theta) = 0$, $f_\Theta(0, \theta) = 1$.

If there exists a comoving frame $\xi \in LZ(\Sigma)$ in which the bundle becomes periodic as in Theorem 4.12, then (2.12) holds with σ replaced by σ', and $f_\Gamma(0, \theta) = \xi$.

Proof. Due to reversibility, whenever $u(t) = (\gamma(t), v(t), \theta(t))$ is a solution of (2.9), then also

$$\rho u(-t) = (\rho\gamma(-t), v(-t), \theta(-t)) = (\rho\gamma(-t)\rho^{-1}, \rho v(-t), -\theta(-t))$$

is a solution of (2.9). Here we used identification (4.11). In analogy to the case of reversible relative equilibria, see section 3, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}(-t)}(\rho v) = f_V(\rho v, -\theta) \quad \Leftrightarrow \quad -\rho f_V(v, \theta) = f_V(\rho v, -\theta)$$

and that

$$\frac{\mathrm{d}}{\mathrm{d}(-t)}(\rho\gamma\rho^{-1}) = \rho\gamma\rho^{-1} f_\Gamma(\rho v, -\theta) \quad \Leftrightarrow \quad -\mathrm{Ad}_\rho f_\Gamma(v, \theta) = f_\Gamma(\rho v, -\theta)$$

and finally, that

$$\frac{\mathrm{d}}{\mathrm{d}(-t)}(-\theta) = f_\Theta(\rho v, -\theta) \quad \Leftrightarrow f_\Theta(v,\theta) = f_\Theta(\rho v, -\theta)$$

which yields the conditions (4.12). The equivariance conditions (2.11), (2.12) are known from [29]. In subsection 5.4 below we will show how the vectorfields f_Γ, f_V and f_Θ are related to the vectorfield f from (2.1). ■

Again, we may reparametrize time to set $f_\Theta(v,\theta) = 1$. In particular, whenever Γ_ρ is algebraic, we can proceed to a comoving frame in which the relative periodic orbit is periodic, and the equations take the form of a periodically driven reversible equivariant system on $\Gamma \times V$.

Since Q has finite order $2k$ (rsp. $2k_V$) we note that relative periodic solutions correspond to reversible discrete rotating waves (periodic solutions with discrete spatiotemporal symmetry) of the (v,θ)-subsystem. Hence, modulo drifts, the problem of bifurcation from reversible relative periodic orbits reduces to the problem of bifurcation from reversible discrete rotating waves. More precisely, reversible RPOs become fixed points of the twisted reversible equivariant diffeomorphism $F : V \to V$ defined as $F = Q^{-1}g^{(1)}$ where $g^{(1)}$ is the time-one map of the nonautonomous v-equation in (2.9).

There is no general bifurcation theory available yet for reversible discrete rotating waves, although it is clear that the techniques of [18] for discrete rotating waves in equivariant systems extend to the context of reversible equivariant systems: By employing Birkhoff normal form techniques [16] the problem of bifurcation from discrete rotating waves may be reduced to a problem of fixed point bifurcation of a $(\Delta \rtimes \mathbb{Z}_{2k})$-equivariant and ρ-reversible diffeomorphism (up to higher order terms), see e.g. [17] for some examples.

As mentioned before, even a general local steady-state bifurcation theory for reversible equivariant systems is not yet in place, however see [21, 13] for the linear theory.

5 Proofs of the main theorems

In this section we prove our main Theorems 4.11, 4.12, 4.13 on the bundle structure near a reversible relative periodic solution.

First, in subsection 5.1, we briefly review the bundle construcion of [24, 29] and sketch what modifications need to be made in the reversible case. Subsection 5.2 deals with decompositions of twisted reversible equivariant linear maps which occur as linearizations of relative periodic solutions. In subsection 5.3 we show that the order of the twist morphism ϕ is finite for appropriate choices of σ in the coset of $\sigma\Delta$ if Γ is algebraic. Moreover we show that for any twisted reversible equivariant linear map $L \in \mathrm{Mat}(V)$ there is some k_V and $\delta_0 \in \Delta$ such that $(\delta_0 L)^{k_V}$ is equivariant and reversible. Finally, in subsection 5.4 we prove Theorems 4.11, 4.12 and 4.13 and discuss the relationship between the equations (2.9) in bundle coordinates and the original vectorfield (2.1) in more detail.

5.1 The bundle construction of [24, 29].

Let $\mathcal{P}$ be a relative periodic orbit of (2.1) of relative period 1. Fixing $u_0 \in \mathcal{P}$ there is a $\sigma \in \Gamma$ such that $\sigma^{-1}\Phi_1(u_0) = u_0$. We let $\Delta \subset \Gamma$ denote the isotropy of u_0 and $\Sigma \subset \Gamma$ denote the group generated by σ and Δ. As mentioned in Section 2, in [24, 29] it is shown that a neighborhood U of $\mathcal{P}$ can be parametrized as (2.6) with identifications (2.7).

To prove this result, in [24] a family of cross-sections V_θ around the relative periodic solution is constructed and the linearized flow $(\mathrm{D}\Phi_\theta)_{u_0}$ is used to define coordinates on V_θ. First, write

$$T_{\Phi_\theta(u_0)}M = T_{\Phi_\theta(u_0)}\mathcal{P} \oplus V_\theta, \tag{5.1}$$

where $V_0 = V$ and the cross-sections V_θ are Δ-invariant and depend smoothly on θ. Let $P_\theta : T_{\Phi_\theta(u_0)}M \to V_\theta$ denote the associated family of Δ-equivariant projections. The spaces V_θ are chosen such that the linearized flow $(\mathrm{D}\Phi_\theta)_{u_0}$ restricts to a Δ-equivariant map $P_\theta(\mathrm{D}\Phi_\theta)_{u_0} : V \to V_\theta$ and that

$$P_{\theta+1} = \sigma P_\theta \sigma^{-1}. \tag{5.2}$$

The neighborhood U of the relative periodic solution is parametrized by the submersion $\tau : U \to \Gamma \times V \times \mathbb{R}$ where

$$u = \tau(\gamma, v, \theta) = \gamma\psi(\Phi_\theta(u_0), P_\theta \mathrm{D}\Phi_\theta(u_0)J_\theta v). \tag{5.3}$$

Here $\psi(u, w) \in M$, $u \in M$, $w \in T_uM$, is a local smooth Γ-equivariant chart for M near u and $J_\theta \in \mathrm{GL}(V)$ is a Δ-equivariant homotopy such that

$$J_0 = \mathrm{id}, \quad P_0\sigma^{-1}\mathrm{D}\Phi_1(u_0)J_{\theta+1} = J_\theta Q^{-1} \tag{5.4}$$

and $Q^{-1} \in \mathrm{O}(V)$ is an orthogonal twisted equivariant matrix of finite order $2k_V$ (rsp. $2k$ if k is finite, e.g. in the case of algebraic groups Γ). Using the parametrization (5.3) one verifies the identifications (2.7).

In the following two subsections we show how to adapt this approach to reversible systems.

5.2 Twisted reversible equivariant linear maps

In this subsection we show that twisted reversible equivariant linear maps can be decomposed into a product of two commuting matrices one of them being reversibly and equivariantly homotopic to identity and the other one being orthogonal and of twice the order k of the twist morphism ϕ. We need this result to find reversible homotopies J_θ which are used later in defining the parametrization of a neighborhood of a reversible RPO, cf. (5.3), (5.4).

Lemma 5.1 *Let Δ_ρ be a compact Lie group acting orthogonally on some finite-dimensional vectorspace V and suppose that ϕ is a morphism of finite order k, such that $\phi|_\Delta \in \mathrm{Aut}(\Delta)$ and for all $\delta \in \Delta$ $\phi(\delta\rho) = \phi(\delta)\phi(\rho)$ and $\phi(\rho\delta) = \phi(\rho)\phi^{-1}(\delta)$. Let $L : V \to V$ be a twisted reversible equivariant nonsingular linear map. Then there exists a twisted reversible equivariant orthogonal map $A : V \to V$ such that $A^{2k} = I$ and $A^{-1}L$ is homotopic to the identity with a homotopy $J_\theta = \exp(\theta\eta)$ which is Δ-equivariant, commutes with A and L and is reversible*

$$\rho J_\theta \rho^{-1} = J_\theta^{-1} = J_{-\theta}.$$

Proof. We decompose $L = L_s L_n$ where L_s is the semisimple part of L, L_n is unipotent and $[L_s, L_n] = 0$. As shown in [16, Lemma 2.2] the map L_s is twisted reversible equivariant as L. Therefore the matrix L_n is Δ-equivariant and ρ-reversible. Note that $L_n - \mathrm{id}$ is nilpotent. Therefore $\ln(L_n)$ is a polynomial in $L_n - \mathrm{id}$ and consequently Δ-equivariant. Moreover

$$\begin{aligned} \rho \ln(L_n)\rho^{-1} &= \rho \sum_i \frac{(-1)^i}{i!}(L_n - \mathrm{id})^i \rho^{-1} = \sum_i \frac{(-1)^i}{i!}(L_n^{-1} - \mathrm{id})^i \\ &= \ln(L_n^{-1}) = -\ln(L_n). \end{aligned}$$

Therefore the unipotent part L_n of L is Δ-equivariantly and reversibly homotopic to id and the homotopy $\mathrm{e}^{\ln(L_n)\theta}$ commutes with L and L_s. From now on we can assume without loss of generality that L is semisimple.

The map L^k is equivariant in the usual sense. Let μ denote an eigenvalue of L^k and denote the corresponding eigenspace by E_μ. Note that E_μ is Δ-invariant and is also invariant under L. However, E_μ is not necessarily invariant under ρ: If μ is an eigenvector of L^k with eigenvector v_μ then μ^{-1} is also an eigenvalue of L^k, with eigenvector $v_{\mu^{-1}} = \rho v_\mu$. Hence a Δ_ρ-invariant subspace for ρ is given by $\hat{E}_\mu = E_\mu + E_{\mu^{-1}}$. To prove the lemma, it is sufficient to restrict to $\hat{E}_\mu$, or, if $\mu \in \mathbb{C}$, to $\hat{E}_\mu + \hat{E}_{\bar{\mu}}$.

If $\mu \in \mathbb{R}^+$ then let λ be the positive real k-th root of μ and define

$$\eta|_{E_\mu} := \ln(\lambda), \quad \eta|_{E_{\mu^{-1}}} := -\ln(\lambda).$$

Then $J_\theta = \mathrm{e}^{\eta\theta}$ defines a Δ-equivariant reversible homotopy commuting with L such that $J_0 = \mathrm{id}$ and $J_k = L^k$.

If L^k has an eigenvalue $\mu \in \mathbb{R}^-$, then we homotopy $L^k|_{\hat{E}_\mu}$ to $-\mathrm{id}|_{\hat{E}_\mu}$ using the matrix η defined above.

If the eigenvalue $\mu \in \mathbb{C} \setminus \mathbb{R}$ of L^k lies on the unit circle then again we choose a k-th root λ of μ and define

$$\eta|_{E_\mu} = \mathrm{i}\arg(\lambda), \quad \eta|_{E_{\bar{\mu}}} = -\mathrm{i}\arg(\lambda)$$

and obtain a real-valued reversible homotopy $\mathrm{e}^{\eta\theta}$ between id and L^k on the real subspace corresponding to $\hat{E}_\mu$.

If the eigenvalue $\mu \in \mathbb{C} \setminus \mathbb{R}$ of L^k has module different from 1 then we get a quadruple of eigenvalues

$$\{\mu, \bar{\mu}, \mu^{-1}, \bar{\mu}^{-1}\}.$$

Again we choose a k-th root λ of μ and define the homotopy $J_\theta = e^{\eta\theta}$ where

$$\eta|_{E_\mu} = \ln(|\lambda|) + \mathrm{i}\ln(\arg(\lambda)), \quad \eta|_{E_{\mu^{-1}}} = -\ln(|\lambda|) - \mathrm{i}\ln(\arg(\lambda))$$

and the complex conjugated equations hold on $E_{\bar{\mu}}, E_{\bar{\mu}^{-1}}$. Again the matrix η anti-commutes with ρ and is Δ-equivariant.

In all cases we obtain a reversible Δ-equivariant path J_θ with

$$\rho J_\theta \rho^{-1} = J_{-\theta} = J_\theta^{-1}, \; \delta J_\theta = J_\theta \delta, \quad \delta \in \Delta, \theta \in \mathbb{R}$$

which is smooth in θ, commutes with L and satisfies

$$J(0) = \mathrm{id}, \quad J_{\theta+1} = A^{-1} L J_\theta, \quad A^{2k} = \mathrm{id}.$$

The matrix A is orthogonal, twisted reversible equivariant and commutes with J_θ and L.

∎

Remark 5.2 The order of A in the above proof is not optimally chosen concerning orientability issues. In fact it is often possible to choose A of order k even if L^k has eigenvalues in $\mathbb{R}^-$. Whether or not this is possible depends on the type of representation of Ξ_ρ on V.

5.3 Finiteness of k and k_V

In this subsection we study the order of the twist morphism $\phi : \Delta_\rho \to \Delta_\rho$ and the corresponding twisted reversible equivariant matrices L defined on some Δ_ρ-invariant finite-dimensional vectorspace V. Remember that the morphism ϕ is defined by some $\sigma \in N(\Delta)$ with $\rho\sigma\rho^{-1}\sigma \in \Delta$. As before, Δ is a compact subgroup of Γ with $\rho^2 \in \Delta$ and Δ_ρ is the index-2-extension of Δ by ρ. In subsection 5.3.1 we show that the order of the twist morphism ϕ is finite for an appropriate choice of σ in the coset of $\sigma\Delta$ if the overall symmetry group Γ is algebraic. In subsection 5.3.2 we show that there is always a $k_V \in \mathbb{N}$ and a $\delta_0 \in \Delta$ such that $(\delta_0 L)^{k_V}$ is reversible and Δ-equivariant for any twisted reversible equivariant matrix $L \in \mathrm{GL}(V)$ even if Γ is not algebraic.

We will need these results in subsection 5.4 below to prove that the matrix Q occuring in the identification (4.11) can always be chosen to have finite order $2k$ or $2k_V$.

To illustrate the problem, we will now present an example in which $\phi^m(\rho) \neq \rho$, $m \in \mathbb{Z}$, and therefore $k = \infty$, for bad choices of σ in $\sigma\Delta$.

Example 5.3 Let $\Gamma = \mathrm{SO}(3) \times \mathbb{Z}_2$, $\Delta = \mathrm{SO}(3)$ and $\tilde{\sigma}$ generate $\mathbb{Z}_2$. Define $\Gamma_\rho = \mathrm{O}(3) \times \mathbb{Z}_2$ so that ρ is an involution ($\rho^2 = \mathrm{id}$) in $\Gamma_\rho \setminus \Gamma$. Choose $\sigma = \tilde{\sigma}\delta_\sigma$, where $\delta_\sigma \in \Delta$ is taken as $\delta_\sigma = \delta_1\delta_2$, $\rho\delta_1 = \delta_1\rho$ and $\rho\delta_2 = \delta_2^{-1}\rho$.

More specifically, let ρ be the reflection about the (x_2, x_3)-plane. Then δ_1 is a rotation around the x_1-axis and δ_2 is a rotation around a vector in the (x_2, x_3)-plane, say the x_2-axis. We compute that

$$\phi(\rho) = \sigma^{-1}\rho\sigma^{-1} = (\sigma^{-1}\rho)^2\rho.$$

Similarly $\phi^m(\rho) = (\sigma^{-m}\rho)^2\rho$ and

$$(\sigma^{-m}\rho)^2 = (\delta_2^{-1}\delta_1^{-1})^m(\delta_2\delta_1^{-1})^m.$$

Obviously $\delta_2^{-1}\delta_1^{-1}$ and $\delta_2\delta_1^{-1}$ are rotations around different rotation axes. If we choose the rotation angles of δ_1, δ_2 such that $\delta_2^{-1}\delta_1^{-1}$ and $\delta_2\delta_1^{-1}$ are irrational rotations then $\phi^m(\rho) \neq \rho$ for all $m \in \mathbb{N}$. This implies that $k = \infty$. Of course, choosing by $\sigma = \tilde{\sigma}$ we get $k = 1$.

5.3.1 Algebraic groups

We now assume that Γ is an algebraic group. Compact and Euclidean groups are algebraic groups whereas $\Gamma = \mathbb{Z}$ is an example of a non-algebraic group.

Lemma 5.4 *Let Γ be an algebraic group, let $\Delta \subseteq \Gamma$ be compact with $\rho^2 \in \Delta$ and assume that $\sigma \in N(\Delta)$ satisfies $\sigma\rho\sigma\rho^{-1} \in \Delta$. Further let Σ be the group generated* *by σ and Δ. Then there is some $n \in \mathbb{Z}$ and some choice of σ in the coset $\sigma\Delta$ such that $\sigma^n = \mathrm{e}^{n\xi}$ where $\xi \in LZ(\Sigma)$ and $\mathrm{Ad}_\rho \xi = -\xi$. In particular, the morphism ϕ of Δ_ρ defined by σ has finite order k.*

Proof. Let

$$C = C(Z(\sigma)) \cap N(\Delta)$$

where $C(G) \subset G$ denotes the center of the group G. We have $\sigma \in C$. The group C is abelian and so the set

$$C^\rho = \{\gamma \in C \mid \gamma\rho\gamma\rho^{-1} \in \Delta\}$$

is a group: let $\gamma_1, \gamma_2 \in C$. Then there are $\delta_1, \delta_2 \in \Delta$ such that

$$\gamma_i = \delta_i \rho \gamma_i^{-1} \rho^{-1}, \quad i = 1, 2.$$

Therefore

$$\gamma_1\gamma_2 = \delta_1\rho\gamma_1^{-1}\rho^{-1}\delta_2\rho\gamma_2^{-1}\rho^{-1} = \delta_1\rho\gamma_1^{-1}\tilde{\delta}_2\gamma_2^{-1}\rho^{-1} = \delta_1\hat{\delta}_2\rho(\gamma_1\gamma_2)^{-1}\rho^{-1}.$$

where $\tilde{\delta}_2, \hat{\delta}_2 \in \Delta$. Here we used that ρ and γ_1 are normal in Δ and that C^ρ is abelian.

The group C^ρ is algebraic and $\sigma \in C^\rho$. Hence by [12] there is some $n \in \mathbb{N}$ and some $\eta \in LC^\rho$ such that $\sigma^n = \mathrm{e}^{n\eta}$.

Now we first treat the case that Σ is compact. Then Σ_ρ is compact. So we can choose a Σ_ρ-invariant scalar-product on $L\Gamma$. As in [29, Lemmata 6.1, 6.2] we decompose orthogonally

$$\eta = \xi + \chi$$

where $\chi \in L\Delta$ and $\xi \in (L\Delta)^\perp$. As shown in [29, Lemmata 6.1, 6.2] we have $\xi \in LZ(\Sigma)$. The argument there is as follows: since the scalar product on $L\Gamma$ is Σ-invariant we know that $\mathrm{Ad}_\gamma\xi - \xi \in (L\Delta)^\perp$ for $\gamma \in \Sigma$. Further $\mathrm{Ad}_\gamma\xi = \mathrm{Ad}_\gamma(\eta - \chi) \in L\Delta$ for $\gamma \in \Sigma$ since $\sigma \in N(\Delta)$ and $\eta \in LN(\Delta) \cap LZ(\sigma)$. Consequently $\mathrm{Ad}_\gamma\xi \in L\Delta \cap (L\Delta)^\perp = \{0\}$. In particular letting $\sigma = \mathrm{id}$ so that $\Sigma = \Delta$ the above argument shows that for any compact group Δ and any Δ-invariant scalar-product on $L\Gamma$ the equality $LN(\Delta) \cap (L\Delta)^\perp = LZ(\Delta)$ holds, cf. [9].

With a similar argument we can show that $\mathrm{Ad}_\rho\xi = -\xi$: Since the scalar product on $L\Gamma$ is ρ-invariant we conclude that $\mathrm{Ad}_\rho\xi + \xi \in (L\Delta)^\perp$. Further, since $\eta \in LC^\rho$ we have $\mathrm{Ad}_\rho\eta + \eta \in L\Delta$. Therefore $\mathrm{Ad}_\rho\xi = \mathrm{Ad}_\rho(\eta - \chi) \in -\xi + L\Delta$ and consequently $\mathrm{Ad}_\rho\xi + \xi = 0$. Changing σ to $\sigma\exp(-\chi)$ we see that $k < \infty$.

If Σ is non-compact then the relative periodic orbit has to be a modulated travelling wave. We proceed as in Lemma 6.2 of [29]: We choose a Δ_ρ-invariant scalar-product $(\cdot,\cdot)$ on $L\Gamma$. We decompose $\eta = \tilde{\xi} + \tilde{\chi}$ orthogonally w.r.t. this scalar product. i.e. $\tilde{\xi} \in LN(\Delta) \cap (L\Delta)^\perp = LZ(\Delta)$, $\tilde{\chi} \in L\Delta$.

Now we define a new scalar product on $X := \mathbb{R}\eta + L\Delta = \mathbb{R}\tilde{\xi} \oplus L\Delta$ as

$$\langle u, v\rangle = \sum_{i=0}^{n-1} (\mathrm{Ad}_{\sigma^i} u, \mathrm{Ad}_{\sigma^i} v), \quad u, v \in X.$$

Since $\sigma \in N(\Delta)$ this new scalar product is Δ-invariant. Since $\tilde{\xi} \in LZ(\Delta)$ we have $\mathrm{Ad}_{\sigma^n} = \mathrm{Ad}_{\exp(n\tilde{\chi})}$ on $L\Delta$ and $\mathrm{Ad}_{\sigma^n}\tilde{\xi} = \mathrm{Ad}_{\exp(n\tilde{\chi})\exp(n\tilde{\xi})}\tilde{\xi} = \mathrm{Ad}_{\exp(n\tilde{\chi})}\tilde{\xi} = \tilde{\xi}$. Therefore $(\mathrm{Ad}_{\sigma^n} u, \mathrm{Ad}_{\sigma^n} v) = (u, v)$ for $u, v \in X$ and the new scalar product $\langle\cdot,\cdot\rangle$ is σ-invariant. Finally $(\cdot,\cdot)$ is Δ_ρ-invariant and $\sigma\rho = \phi^{-1}(\rho)\sigma^{-1} \in \Delta\rho\sigma^{-1}$. Therefore

$$\begin{aligned}
\langle \mathrm{Ad}_\rho u, \mathrm{Ad}_\rho v\rangle &= \sum_{i=0}^{n-1} (\mathrm{Ad}_{\sigma^i}\mathrm{Ad}_\rho u, \mathrm{Ad}_{\sigma^i}\mathrm{Ad}_\rho v) \\
&= \sum_{i=0}^{n-1} (\mathrm{Ad}_{\phi^{-i}(\rho)}\mathrm{Ad}_{\sigma^{-i}} u, \mathrm{Ad}_{\phi^{-i}(\rho)}\mathrm{Ad}_{\sigma^{-i}} v) = \sum_{i=0}^{n-1} (\mathrm{Ad}_{\sigma^{-i}} u, \mathrm{Ad}_{\sigma^{-i}} v) \\
&= \langle u, v\rangle.
\end{aligned}$$

Now we decompose $\eta = \xi + \chi$ w.r.t. this new scalar product. Proceeding as in the case of compact spatio-temporal symmetry Σ_ρ we see that again $\mathrm{Ad}_\rho \xi = -\xi$. Changing σ to $\sigma\exp(-\chi)$ we achieve that k becomes finite. ∎

If the spatiotemporal symmetry group Σ is algebraic then we can define $\Gamma := \Sigma$ and the above lemma applies. This means that in particular for discrete rotating waves and modulated rotating waves the order of the automorphism ϕ is always finite for appropriate choices of σ.

5.3.2 Non-algebraic groups

In the case of discrete travelling waves k may be ∞ for any choice of σ in the coset $\sigma\Delta$, see [29, Example 6.6]. Let $L = P_0\sigma^{-1}\mathrm{D}\Phi_1(u_0)$ be the linearization along the relative periodic orbit. In [29], for the twisted equivariant case, we proved that there is still a finite k_V such that L^{k_V} is Δ-equivariant for an appropriate choice of $\sigma \in \sigma\Delta$. Similarly in the twisted reversible equivariant case we can always choose a representative σ in $\sigma\Delta$ such that L^{k_V} is Δ-equivariant and ρ-reversible as we will now show.

First we prove that it suffices to treat the case that L is orthogonal.

Proposition 5.5 *Assume that the compact group Δ_ρ acts orthogonally on the finite-dimensional vectorspace V. Suppose that $L : V \to V$ is a twisted reversible equivariant* *nonsingular linear map. Then there exists a twisted reversible equivariant orthogonal map $A : V \to V$ such that $A^{-1}L$ is Δ-equivariant and (ρA)-reversible.*

Proof. By polar decomposition, we can write L uniquely as $L = AB$ where A is orthogonal and B is symmetric and positive definite. One easily computes, see [29, Proposition 1.1], that B is equivariant and A is twisted equivariant.

Recall that B is defined as the unique symmetric positive definite square root of $B^2 = L^T L$. We show that B satisfies $(\rho A)B = B^{-1}(\rho A)$. Using orthogonality of the Δ_ρ-action on V, we find

$$\begin{aligned} B^2(\rho A) &= L^T L \rho A = L^T \phi(\rho) L^{-1} A = ((\phi(\rho))^{-1} L)^T L^{-1} A = (L^{-1}\rho^{-1})^T L^{-1} A \\ &= \rho (L^{-1})^T L^{-1} A = \rho (A^{-1})^T (B^{-1})^T B^{-1} A^{-1} A = \rho A B^{-2}. \end{aligned}$$

Thus, the positive definite matrix $B' = (\rho A)^{-1} B(\rho A)$ satisfies

$$(B')^2 = (\rho A)^{-1} L^T L(\rho A) = B^{-2}.$$

Since B is the unique positive definite square root of $L^T L$, it follows that $B^{-1} = B' = (\rho A)^{-1} B(\rho A)$. The matrix A is twisted reversible equivariant. Namely,

$$A\rho = LB^{-1}\rho = LB^{-1}\rho A A^{-1} = L\rho A B A^{-1} = \phi(\rho) L^{-1} L A^{-1} = \phi(\rho) A^{-1}.$$

∎

Now consider the subgroup G of $\mathrm{O}(V)$ generated by the representation Δ_V of Δ on V and the orthogonal, twisted reversible equivariant matrix A. The group Δ_V is normal in G. Let G_ρ denote the index two extension of G by the orthogonal representation ρ_V of ρ on V and let ϕ_V be the morphism of Δ_V defined by $\phi_V(\delta) = A\delta A^{-1}$, $\delta \in \Delta_V$, and $\phi_V(\delta\rho_V) = A\delta\rho_V A$. Since G is compact and therefore algebraic we can apply Lemma 5.4 and conclude that there is some $\delta_0 \in \Delta$ and some $k_V \in \mathbb{N}$ such that $(\delta_0 A)^{k_V}$ is Δ-equivariant and ρ-reversible. Changing σ to $\sigma_{\mathrm{new}} = \sigma\delta_0^{-1}$ and A to $A_{\mathrm{new}} = \delta_0 A$ we see that now $A_{\mathrm{new}}^{k_V}$ is Δ-equivariant and ρ-reversible. Since A_{new} and $L_{\mathrm{new}} = \delta_0 L = P_0 \sigma_{\mathrm{new}}^{-1} \mathrm{D}\Phi_1(u_0)$ define the same morphism ϕ_V of Δ_V we conclude that also $L_{\mathrm{new}}^{k_V}$ is Δ-equivariant and reversible. We summarize this result in the following lemma.

Lemma 5.6 *Suppose that $L : V \to V$ is a twisted reversible equivariant nonsingular linear map. Then there exists a finite number $k_V \in \mathbb{N}$ and some $\delta_0 \in \Delta$ such that $(\delta_0 L)^{k_V}$ is reversible and Δ-equivariant.*

5.4 Parametrization and equations on the slice

In this section we come towards the proofs of Theorems 4.11, 4.12, 4.13. Let $\mathcal{P}$ be a reversible relative periodic solution with minimal relative period one. Choose some $u_0 \in \mathcal{P}$. As we saw in subsection 4.1 we may assume without loss of generality that $u_0 \in \mathrm{Fix}(\rho)$. We have $\Phi_1(u_0) = \sigma u_0$ for some $\sigma \in \Gamma$ and $\Phi_t(u_0) \notin \Gamma u_0$ for $0 < t < 1$. Let Δ and Σ denote the spatial and spatiotemporal symmetry groups of $\mathcal{P}$ in Γ. Remember that the isotropy subgroup $\Delta \subset \Gamma$ is compact and that Σ is the closed subgroup of Γ generated by Δ and σ.

Since Φ_1 is ρ-reversible and Δ-equivariant, i.e. $\Phi_1 \rho = \rho \Phi_{-1}$ and $\delta \Phi_1 = \Phi_1 \delta$ for all $\delta \in \Delta$, it follows that the diffeomorphism $\sigma^{-1}\Phi_1$ is twisted reversible equivariant: since $\sigma \Phi_{-1} \sigma^{-1} = \Phi_1^{-1}$ we obtain [17]

$$(\sigma^{-1}\Phi_1)\delta\rho = \sigma^{-1}\delta\rho\sigma^{-1}\sigma\Phi_{-1} = \phi(\delta\rho)\sigma\Phi_{-1}\sigma^{-1}\sigma = \phi(\delta\rho)\Phi_1^{-1}\sigma, \tag{5.5}$$

where ϕ is the twist morphism as defined before. Accordingly, the linearization $\sigma^{-1}\mathrm{D}\Phi_1$ is also twisted reversible equivariant.

In [24], it was first shown how the flow in a neighborhood U of the relative periodic solution $\mathcal{P}$ can be written as a skew product flow on a space of the form $\Gamma \times V \times \mathbb{R}$, and we shortly described this approach in section 2 and subsection 5.1. We will use the parametrization (5.3) from [24], but need to take reversibility into account. First of all we choose V as a Δ_ρ-invariant cross-section to $\mathcal{P}$. In the following lemma we show how the projections a P_θ and the homotopy J_θ mentioned in subsection 5.1 can be chosen to be ρ-reversible.

Lemma 5.7 *Let $L = P_0\sigma^{-1}(\mathrm{D}\Phi_1)_{u_0} : V \to V$.*
(a) There is an orthogonal twisted reversible equivariant map $A : V \to V$ of finite order $2k_V$ and a smooth family of Δ-equivariant nonsingular linear maps $J_\theta : V \to V$, $\theta \in \mathbb{R}$ commuting with A, such that

$$J_0 = I, \qquad \rho J_\theta \rho^{-1} = J_{-\theta} = J_\theta^{-1}, \qquad L J_{\theta+1} = J_\theta A, \qquad \theta \in \mathbb{R}. \tag{5.6}$$

(b) The Δ-equivariant projections P_θ can be chosen so that

$$P_\theta \rho = \rho P_{-\theta}, \qquad P_{\theta+1}\sigma = \sigma P_\theta, \qquad \theta \in \mathbb{R}.$$

Proof. Let $\langle\, , \rangle$ be a Δ_ρ-invariant inner product on $T_{u_0}M$ and define P_0 to be the orthogonal projection onto V such that $T_{u_0}\mathcal{P} = (\mathrm{id} - P_0)T_{u_0}M$ and that P_0 is Δ_ρ-equivariant. Since $\sigma^{-1}(\mathrm{D}\Phi_1)_{u_0}$ is twisted reversible equivariant as we saw above and $\sigma^{-1}(\mathrm{D}\Phi_1)_{u_0}$ maps $T_{u_0}\mathcal{P}$ into itself also $L = P_0\sigma^{-1}(\mathrm{D}\Phi_1)_{u_0} \in \mathrm{GL}(V)$ is twisted reversible equivariant, i.e. $L\delta = \phi(\delta)L$ and $L\delta\rho = \phi(\delta\rho)L^{-1}$ for all $\delta \in \Delta$, where ϕ is the twist morphism on Δ_ρ. By Lemma 5.6 there is some $k_V \in \mathbb{N}$ and some

$\sigma \in \sigma\Delta$ such that L^{k_V} is Δ-equivariant and reversible. Now we can apply Lemma 5.1 from section 5.2 with ϕ replaced by ϕ_V and conclude that there is some orthogonal twisted reversible matrix A of order $2k_V$ such that $A^{-1}L$ is homotopic to identity by a Δ-equivariant reversible homotopy J_θ which commutes with A.

Next, we prove part (b). We repeat the argument above, but applied to the twisted equivariant map $\widetilde{L} = \sigma^{-1}(\mathrm{D}\Phi_1)_{u_0} : T_{u_0}M \to T_{u_0}M$ to obtain an orthogonal map $\widetilde{A} : T_{u_0}M \to T_{u_0}M$ and a smooth family of Δ-equivariant nonsingular linear maps $\widetilde{J}_\theta : T_{u_0}M \to T_{u_0}M$ such that

$$\widetilde{J}_0 = I, \qquad \rho\widetilde{J}_\theta = \widetilde{J}_{-\theta}\rho, \qquad \widetilde{L}\widetilde{J}_{\theta+1} = \widetilde{J}_\theta\widetilde{A}, \qquad \theta \in \mathbb{R}.$$

Let $\langle\, , \rangle_\theta$ be the Δ-invariant inner product on $T_{\Phi_\theta(u_0)}M$ defined by

$$\langle v, w\rangle_\theta = \langle \widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}v, \widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}w\rangle, \quad v, w \in T_{\Phi_\theta(u_0)}M.$$

Following [29], V_θ is defined to be the orthogonal complement to $T_{\Phi_\theta(u_0)}\mathcal{P}$ in $T_{\Phi_\theta(u_0)}M$ with respect to the inner product $\langle\, , \rangle_\theta$. We let $P_\theta : T_{\Phi_\theta(u_0)}M \to V_\theta$ be the orthogonal projection with respect to this inner product. One easily computes, see [29], that $\langle v, w\rangle_{\theta+1} = \langle \sigma^{-1}v, \sigma^{-1}w\rangle_\theta$. Therefore we see that indeed $P_{\theta+1}\sigma = \sigma P_\theta$ as required. It remains to be shown that also $P_\theta\rho = \rho P_{-\theta}$. We compute that

$$(\mathrm{D}\Phi_{-\theta})_{u_0} = \mathrm{D}(\rho^{-1}\Phi_\theta\rho)_{u_0} = \rho^{-1}(\mathrm{D}\Phi_\theta)_{\rho u_0}\rho = \rho^{-1}(\mathrm{D}\Phi_\theta)_{u_0}\rho. \tag{5.7}$$

Here we used that $\rho u_0 = u_0$. Hence

$$\widetilde{J}_{-\theta}^{-1}(\mathrm{D}\Phi_{-\theta})_{u_0}^{-1} = \widetilde{J}_{-\theta}^{-1}\rho^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\rho = \rho^{-1}\widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\rho.$$

Using the orthogonality of ρ on $T_{u_0}M$, it follows that

$$\begin{aligned}
\langle v, w\rangle_{-\theta} &= \langle \widetilde{J}_{-\theta}^{-1}(\mathrm{D}\Phi_{-\theta})_{u_0}^{-1}v, \widetilde{J}_{-\theta}^{-1}(\mathrm{D}\Phi_{-\theta})_{u_0}^{-1}w\rangle \\
&= \langle \rho^{-1}\widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\rho v, \rho^{-1}\widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\rho w\rangle \\
&= \langle \widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\rho v, \widetilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\rho w\rangle \\
&= \langle \rho v, \rho w\rangle_\theta.
\end{aligned}$$

Since $P_\theta : T_{\Phi_\theta(u_0)}M \to V_\theta$ is the orthogonal projection w.r.t. the inner product $\langle\, , \rangle_\theta$ we see that $P_\theta\rho = \rho P_{-\theta}$ as required. ∎

Now, consider the Γ-equivariant submersion $\tau : \Gamma_\rho \times V \times \mathbb{R} \to M$ defined by

$$\tau(\gamma, v, \theta) = \gamma\psi\big(\Phi_\theta(u_0)\,,\, P_\theta(\mathrm{D}\Phi_\theta)_{u_0}J_\theta v\big). \tag{5.8}$$

Here $\psi(u,w) \in M$, $u \in M$, $w \in T_uM$, is a local smooth Γ_ρ-equivariant chart for M near u. It follows from Δ-equivariance of P_θ, $(\mathrm{D}\Phi_\theta)_{u_0}$ and J_θ, and Δ-invariance of the points $\Phi_\theta(u_0) \in \mathcal{P}$, that $\tau(\gamma\delta, v, \theta) = \tau(\gamma, \delta v, \theta)$ for all $\delta \in \Delta$. Also as in [29] we compute that

$$\tau(\gamma, v, \theta+1) = \tau(\gamma\sigma, Q^{-1}v, \theta) \tag{5.9}$$

where $Q = A^{-1}$. In the presence of reversibility, we moreover have

$$\begin{aligned}
\tau(\gamma, v, -\theta) &= \gamma\psi\big(\Phi_{-\theta}(u_0)\,,\; P_{-\theta}(\mathrm{D}\Phi_{-\theta})_{u_0}J_{-\theta}v\big)\\
&= \gamma\psi\big(\rho^{-1}\Phi_\theta(u_0)\,,\; \rho^{-1}P_\theta\rho(\mathrm{D}\Phi_{-\theta})_{u_0}J_{-\theta}v\big)\\
&= \gamma\psi\big(\rho^{-1}\Phi_\theta(u_0)\,,\; \rho^{-1}P_\theta(\mathrm{D}\Phi_\theta)_{u_0}J_\theta\rho v\big)\\
&= \tau(\gamma\rho^{-1}, \rho v, \theta).
\end{aligned}$$

Here we used the relations

$$\rho\Phi_\theta(u_0) = \Phi_{-\theta}(u_0), \quad \mathrm{D}\Phi_\theta(u_0) = \rho\mathrm{D}\Phi_{-\theta}(u_0)\rho^{-1}.$$

This proves Theorem 4.11.

Proof of Theorem 4.12

If Γ is algebraic then we can simplify the bundle structure by passing to a convenient comoving frame. Due to Lemma 5.4 we can write $\sigma^n = \exp(n\xi)$ where $\xi \in LZ(\Sigma)$ and $\mathrm{Ad}_\rho\xi = -\xi$. Let $\sigma' = \exp(-\xi)\sigma$, so that σ' has order n. We define the new submersion

$$\tau_{\mathrm{new}}(\gamma, v, \theta) = \tau(\gamma\exp(-\theta\xi), v, \theta). \tag{5.10}$$

Note that τ_{new} remains Γ-equivariant, since Γ acts on the left. Since $\tau(\gamma\delta, v, \theta) = \tau(\gamma, \delta v, \theta)$ for all $\delta \in \Delta$, and $\xi \in LZ(\Delta)$, it is immediate that $\tau_{\mathrm{new}}(\gamma\delta, v, \theta) = \tau_{\mathrm{new}}(\gamma, \delta v, \theta)$ for all $\delta \in \Delta$. Similarly, see [29], we compute that

$$\begin{aligned}
\tau_{\mathrm{new}}(\gamma, v, \theta+1) &= \tau(\gamma\exp(-\xi)\exp(-\theta\xi), v, \theta+1)\\
&= \tau(\gamma\exp(-\xi)\exp(-\theta\xi)\sigma, Q^{-1}v, \theta)\\
&= \tau(\gamma\sigma'\exp(-\theta\xi), Q^{-1}v, \theta)\\
&= \tau_{\mathrm{new}}(\gamma\sigma', Q^{-1}v, \theta).
\end{aligned}$$

Now we need to check the reversibility condition in (4.11). We get

$$\begin{aligned}
\tau_{\mathrm{new}}(\gamma\rho^{-1}, \rho v, -\theta) &= \tau(\gamma\rho^{-1}\exp(-\theta\xi), \rho v, -\theta) = \tau(\gamma\exp(\theta\xi)\rho^{-1}, \rho v, -\theta)\\
&= \tau(\gamma\exp(\theta\xi), v, \theta) = \tau_{\mathrm{new}}(\gamma, v, \theta).
\end{aligned} \tag{5.11}$$

Here we used that $\mathrm{Ad}_\rho\xi = -\xi$.

Since σ' has order n and $Q^{2n} = \mathrm{id}$, it follows that $\tau_{\mathrm{new}}(\gamma, v, \theta+2n) = \tau_{\mathrm{new}}(\gamma, v, \theta)$. Hence τ_{new} induces a Γ-equivariant and $(\Delta \rtimes \mathbb{Z}_{2n})$-equivariant submersion

$$\tau_{\mathrm{new}} : \Gamma_\rho \times V \times S^1 \to M, \quad S^1 = \mathbb{R}/2n\mathbb{Z},$$

onto a uniform neighborhood U of the reversible RPO $\mathcal{P}$ which respects reversibility in the sense of (5.11). This proves Theorem 4.12.

Proof of Theorem 4.13

Now let $\mathcal{P}$ be a reversible RPO with finite index m. Write $\sigma^m = \delta_0 \exp(m\zeta)$ where $\zeta \in LZ(\Sigma)$, $\mathrm{Ad}_\rho\zeta = -\zeta$ and $\delta_0 \in \Delta$ and let $\beta = \sigma\exp(-\zeta)$. The matrix $\delta_0^{-1}Q^m$ with Q as before is Δ-equivariant and ρ-reversible. Moreover, since $\mathrm{Ad}_\sigma\delta_0 = \delta_0$ also $Q\delta_0Q^{-1} = \delta_0$. Let $G \in \mathrm{O}(V)$ be the compact group generated by Q and the representation Δ_V of Δ on V. Since Q^{-1} is twisted reversible equivariant $\rho Q\rho^{-1} \in Q^{-1}\Delta_V \subset G$ and therefore $G_\rho = \langle G, \rho_V\rangle$ is an index-2-extension of G (here ρ_V is the representation of ρ on V). We saw in Lemma 5.1 that the square of a G-equivariant and ρ-reversible matrix is G-equivariantly and ρ-reversibly homotopic to identity by a homotopy $J_\theta = \exp(\eta\theta)$. This implies that $(\delta_0^{-1}Q^m)^2 = \exp(-2m\eta)$ where $\eta \in \mathrm{so}(V)$ is Δ-equivariant, commutes with Q and satisfies $\rho_V\eta\rho_V^{-1} = -\eta$. Set $R = Q\exp(\eta) \in \mathrm{O}(V)$ and define similarly as in the proof of Theorem 4.12

$$\tilde{\tau}(\gamma, v, \theta) = \tau(\gamma\exp(-\theta\zeta), \exp(-\theta\eta)v, \theta). \tag{5.12}$$

One easily verifies (see [29]) that

$$\tilde{\tau}(\gamma\beta^{-1}, Rv, \theta+1) = \tilde{\tau}(\gamma\delta^{-1}, \delta v, \theta) = \tilde{\tau}(\gamma, v, \theta), \quad \delta \in \Delta$$

Since $\mathrm{Ad}_\rho\eta = -\eta$ and $\mathrm{Ad}_\rho\zeta = -\zeta$ we moreover see that

$$\tilde{\tau}(\gamma\rho^{-1}, \rho v, -\theta) = \tilde{\tau}(\gamma, v, \theta). \tag{5.13}$$

Since $\beta^m = \delta_0$ and $R^{2m} = \delta_0^2$ it follows that

$$\tilde{\tau}(\gamma, v, \theta+2m) = \tilde{\tau}(\gamma\delta_0^2, \delta_0^{-2}v, \theta) = \tilde{\tau}(\gamma, v, \theta)$$

so that the bundle is indeed $2m$-periodic in θ. Hence $\tilde{\tau}$ induces a Γ-equivariant diffeomorphism $\tilde{\tau} : (\Gamma_\rho \times V \times S^1)/\Xi'_\rho \cong U$ where $S^1 = \mathbb{R}/(2m\mathbb{Z})$ and U is a uniform neighborhood of the reversible RPO. This proves the theorem.

Appendix on the Proof of Theorem 4.14

The differential equations (2.9) are defined by the equation

$$\mathrm{D}\tau(\gamma, v, \theta)(\gamma f_\Gamma(v,\theta), f_V(v,\theta), f_\Theta(v,\theta)) = f(\tau(\gamma, v, \theta)).$$

If Δ is continuous then $\mathrm{D}\tau$ has a kernel due to the identification (2.11). Therefore f_Γ is only determined within $f_\Gamma + L\Delta$. In the following we will show how to choose f_Γ within $f_\Gamma + L\Delta$ in such a way that $f_\Gamma(v,\theta)$ is smooth in v, θ and that the equivariance and reversibility conditions (2.11), (2.12) and (4.12) are satisfied. We choose $f_\Gamma(v,\theta) \in \mathbf{m}_\theta$ where $\mathbf{m}_\theta \subset L\Gamma$ is a complement to $L\Delta$ which is invariant under the adjoint action of Δ_ρ on $L\Gamma$. More precisely we set $\mathbf{m}_\theta = \exp(\eta\theta)\mathbf{m}_0$ where $\eta \in \mathrm{Mat}(L\Gamma)$ is defined as follows.

We first deal with the setting of Theorem 4.11. We apply Lemma 5.6 on the space $X = L\Gamma \times V$ and the twisted reversible equivariant matrix

$$L = \begin{pmatrix} \mathrm{Ad}_\sigma^{-1} & 0 \\ 0 & P_0\sigma^{-1}\mathrm{D}\Phi_1(u_0) \end{pmatrix}$$

and conclude that there is some finite k_X such that L^{k_X} is Δ-equivariant and ρ-reversible for an appropriate choice of $\sigma \in \sigma\Delta$ (this may enlarge the original k_V a bit). Applying Lemma 5.1 on the space $L\Gamma$ we see that there is some twisted reversible equivariant matrix A of finite order $2k_X$ and a reversible Δ-equivariant homotopy $\exp(\eta\theta)$ commuting with A such that $\mathrm{Ad}_\sigma^{-1} = A\exp(-\eta)$. We choose an inner product on $L\Gamma$ which is invariant under the compact matrix group generated by $A \in \mathrm{Mat}(L\Gamma)$, $\mathrm{Ad}_\delta \in \mathrm{Mat}(L\Gamma)$, $\delta \in \Delta$, and $\mathrm{Ad}_\rho \in \mathrm{Mat}(L\Gamma)$ and choose $\mathbf{m}_0$ as orthogonal complement to $L\Delta$ w.r.t. this scalar product on $L\Gamma$. Since $\mathbf{m}_\theta = \exp(\eta\theta)\mathbf{m}_0$ we get $\mathbf{m}_{\theta+1} = \mathrm{Ad}_\sigma \mathbf{m}_\theta$ so that $f_\Gamma(v,\theta)$ is smooth in v, θ and so that the equivariance and reversibility conditions (2.11), (2.12) and (4.12) for the equations on the group make sense.

In the case of algebraic groups (Theorem 4.12) we can choose $\eta = [\xi, \cdot]$ so that $\mathbf{m}_\theta = \mathrm{Ad}_{\exp(\theta\xi)}\mathbf{m}_0$. In this case the scalar product on $L\Gamma$ is chosen to be invariant under the compact group generated by Δ_ρ and σ'. In the comoving frame ξ we have $f_\Gamma(v,\theta) \in \mathbf{m}_0$ independent of θ.

In the setting of Theorem 4.13 we likewise define $\mathbf{m}_\theta = \mathrm{Ad}_{\exp(\theta\zeta)}\mathbf{m}_0$ and choose the scalar product on $L\Gamma$ to be Σ_ρ^m-invariant so that again in the comoving frame ζ we have $f_\Gamma(v,\theta) \in \mathbf{m}_0$ independent of θ.

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