# $k$-Symmetry and return maps of space-time symmetric flows

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February 10, 1998

**Abstract**

A diffeomorphism $U : \Omega \mapsto \Omega$ is called a (reversing) $k$-symmetry of a dynamical system in $\Omega$ represented by the diffeomorphism $f : \Omega \mapsto \Omega$ if $k$ is the smallest positive integer for which $U$ is a (reversing) symmetry of $f^k$ (the $k$-times iterate of $f$), i.e. $U \circ f^k = f^{\pm k} \circ U$.

In this paper we show how $k$-symmetry naturally arises in the context of return maps of flows with space-time symmetries. We discuss the connection between periodic orbits of $k$-symmetric maps and symmetric periodic orbits of the flows they represent, and illustrate the application of our results in local bifurcation theory. We also provide a geometric interpretation of formal (Birkhoff) normal form symmetries for diffeomorphisms as a time-shift symmetry of a locally constructed space-time symmetric suspension flow.

Finally, we explain the occurrence of dual (representations of) reversing $k$-symmetry groups in $k$-symmetric maps in relation to different choices for the position of the surface of section for return maps of space-time symmetric flows.

**MSC**: 58Fxx
**Keywords**: dynamical systems, symmetry, reversibility, return map, normal form

## Contents

# 1 Introduction

In dynamical systems, symmetry properties have received an increasing interest over the last decade. In particular, many studies have been devoted to equivariant dynamical systems in $\mathbb{R}^n$ of which the equations of motion commute with the linear representation of a compact Lie group [8, 9]. Apart from equivariant dynamical systems, there has been an independent interest in time-reversal symmetry, mainly – but not only – in Hamiltonian dynamical systems [22, 24, 27].

Relatively recently, investigations in yet another type of symmetry property – $k$*-symmetry* – was initiated in [20]. $k$-Symmetries are symmetries of dynamical systems with discrete time (maps) arising when the $k$th iterate of a map possesses more (reversing) symmetries than the map itself. $k$-Symmetries were originally observed to occur in a variety of examples from physics, in particular in time-one maps of kicked systems [14, 18].

One of the main questions arising in the context of this observation is why iterates of maps of certain models possess more symmetry properties than the map itself, and in which type of models such maps arise. In [16] a mechanism was presented by which $k$-symmetry occurs in the context of return maps of space-time symmetric flows, the $k$-symmetry properties of the map $f$ being directly related to the spatiotemporal symmetry group of the related flow.

In this paper, motivated by the result of [16], we further explore the occurrence of $k$-symmetry in return maps. Our approach systematically generalizes and extends previous observations in treatises on return maps of space-time symmetric flows and symmetric periodic orbits by Swift and Wiesenfeld [28], Fiedler [7], Vanderbauwhede [31], Lamb [16] and Nikolaev [23] (cf. Remark 8.1 for a more detailed comparison).

The main aim of this paper is to explain and illustrate the occurrence and the use of $k$-symmetry in the study of return maps of space-time symmetric periodic orbits and flows. In particular, we will discuss the relationship between $k$-symmetric periodic orbits of $k$-symmetric maps and space-time symmetric periodic orbits of flows, and the application of this relationship in local bifurcation theory.

Up to now, most studies in equivariant bifurcation theory have been concerned with the study of local bifurcations from a symmetric equilibrium point or periodic orbit that is pointwise invariant under the action of a symmetry group [9]. After [12], also a lot of attention has been devoted to local bifurcations from flow-invariant group orbits (relative equilibria). The situation in which none of the above applies, and a periodic orbit is setwise but not pointwise invariant under the action of a compact Lie group, has not received much attention yet. In this paper we will show that when we study local bifurcations of such period orbits using return maps, we are naturally led to a description in terms of $k$-symmetric maps. The Birkhoff normal form theory for $k$-symmetric maps as developed in [17], can then be used to study generic local bifurcations of space-time symmetric periodic orbits in detail (including their stability properties).

In order to give a flavour of the latter application of the results in this paper, we will now present two examples of how $k$-symmetric maps may be used in the study of local bifurcations of space-time symmetric periodic orbits. Example 1.1 involves equivariance and time-shift symmetries, whereas Example 1.2 also contains time-reversal symmetries. We note that both these examples – like most of this paper – discusses the context of return maps for nonautonomous flows. However, our results also apply to local return maps for periodic orbits of autonomous flows with spatiotemporal symmetry properties (*twisted* subgroups of $\Gamma \times S^1$ in the terminology of [9]), cf. Section 8.

**Example 1.1** We consider the flow of a time-periodic vector field in $\mathbb{R}^2$,

$$\frac{d\mathbf{x}}{dt} = F(\mathbf{x}, t), \tag{1.1}$$

with $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and $F(\mathbf{x}, t) = F(\mathbf{x}, t+1)$. We furthermore assume that the vector field possesses the space-time symmetries

$$(\mathbf{x}, t) \mapsto (M_x(\mathbf{x}), t) \quad \text{and} \quad (\mathbf{x}, t) \mapsto (R_{\frac{\pi}{2}}(\mathbf{x}), t + \frac{1}{2}), \tag{1.2}$$

where $M_x(x, y) = (x, -y)$, and $R_{\frac{\pi}{2}}(x, y) = (y, -x)$. By this, we mean that $F(M_x(\mathbf{x}), t) = M_x F(\mathbf{x}, t)$ and $F(R_{\frac{\pi}{2}}(\mathbf{x}), t + \frac{1}{2}) = R_{\frac{\pi}{2}} F(\mathbf{x}, t)$.

Because the vector field has periodicity in time, the time-one return map $\mathcal{X}_{[0,1]}$ of the flow of this vector field is autonomous (independent of time). Moreover, due to the time-shift symmetry, the time-one return map possesses a decomposition property:[1]

$$\mathcal{X}_{[0,1]} = f^2, \quad \text{where} \quad f := A \circ \mathcal{X}_{[0,\frac{1}{2}]} \quad \text{and} \quad A := R_{\frac{\pi}{2}} \circ M_x. \tag{1.3}$$

Note that $A$ is (the spatial) part of a time-shift symmetry.

Rather than using the return map $\mathcal{X}_{[0,1]}$, we use the map $f$ to study generic local bifurcations of a periodic orbit $\mathbf{x}(t)$ with period one and full symmetry, i.e. $\mathbf{x}(t)$ satisfies $\mathbf{x}(t) = \mathbf{x}(t+1)$, $\mathbf{x}(t) = M_x(\mathbf{x}(t))$ and $\mathbf{x}(t) = R_{\frac{\pi}{2}}(\mathbf{x}(t + \frac{1}{2}))$. Such a periodic orbit is represented by a fixed point $\mathbf{x}_0 = \mathbf{x}(0)$ of the map $f$ (which is at the same time a fixed point of the return map $\mathcal{X}_{[0,1]}$).

An important observation is that due to the space-time symmetries of the flow, the map $f$ is 2-symmetric with respect to $M_x$ and $R^2_{\frac{\pi}{2}} \circ M_x$, i.e. $f$ does not commute with these transformations, but $f^2$ does. In particular,

$$f \circ M_x = R^2_{\frac{\pi}{2}} \circ M_x \circ f \quad \text{and} \quad f \circ R^2_{\frac{\pi}{2}} \circ M_x = M_x \circ f. \tag{1.4}$$

It should be noted that the spatiotemporal symmetries involving a time-shift do not arise as $k$-symmetries of $f$. However, their presence is encoded in the the relations (1.4). Importantly, the spatiotemporal symmetry properties of the underlying flow induce no other structure on the map $f$ than the $k$-symmetry properties described by (1.4), cf. Section 3.

We can now use Birkhoff normal forms to study the generic local bifurcations of the periodic orbit $\mathbf{x}(t)$. Generic local bifurcations and Birkhoff normal forms of maps $f$ satisfying (1.4) above were discussed in a previous paper [17, Section 3.1]. It turns out that the bifurcation problem of the fixed point $\mathbf{x}_0$ is closely related to that of a $\mathbb{D}_4$ equivariant steady state bifurcation. At this point it is interesting to note that the linear symmetries of the time-one return map $\mathcal{X}_{[0,1]}$ only form the group $\mathbb{D}_2$. That our bifurcation problem is $\mathbb{D}_4$ rather than $\mathbb{D}_2$ equivariant is due to the presence of the time-shift symmetries of the flow, which lead us to study the map $f$ that is 2-symmetric with respect to $\mathbb{D}_2$, rather than $\mathcal{X}_{[0,1]}$ that is $\mathbb{D}_2$ equivariant.

It is well known [9] that generic $\mathbb{D}_4$ equivariant steady state bifurcations involve two equivalence classes of $\mathbb{D}_1$-symmetric branches. In this example, one $\mathbb{D}_1$-symmetric branch

[1] It should be noted that we could have decomposed $\mathcal{X}_{[0,1]}$ also in the following way: $\mathcal{X}_{[0,1]} = A^2 \circ f^2$ where $f := A \circ \mathcal{X}_{[0,\frac{1}{2}]}$ and $A := R_{\frac{\pi}{2}}$. However, the results in terms of the flow $\mathcal{X}$ are independent of the particular choice for $A$ and $f$.

consists of solutions fixed under the reflections $M_x$ or $R^2_{\frac{\pi}{2}} \circ M_x$ and the other branch consists of solutions invariant under $R_{\frac{\pi}{2}} \circ M_x$ or $R^{-1}_{\frac{\pi}{2}} \circ M_x$.

In turn, these bifurcating solutions represent periodic orbits of the map $f$ and periodic orbits of the flow $\mathcal{X}$. (The precise connection will be discussed in Section 5). In the present example, one $\mathbb{D}_1$-branch represents periodic solutions with stationary symmetry

$$(\mathbf{x},t) \mapsto (M_x(\mathbf{x}),t) \quad \text{or} \quad (\mathbf{x},t) \mapsto (R^2_{\frac{\pi}{2}} \circ M_x(\mathbf{x}),t), \tag{1.5}$$

and the second branch represents periodic solutions with time-shift symmetry

$$(\mathbf{x},t) \mapsto (R_{\frac{\pi}{2}} \circ M_x(\mathbf{x}),t+\frac{1}{2}) \quad \text{or} \quad (\mathbf{x},t) \mapsto (R^{-1}_{\frac{\pi}{2}} \circ M_x(\mathbf{x}),t+\frac{1}{2}). \tag{1.6}$$

It turns out (using Proposition 7.4) that all these bifurcating solutions have period one.

From the Birkhoff normal forms we can also deduce the stability properties of the bifurcating periodic solutions. For instance, from the analysis in [17] it can be seen that one of the possible generic local bifurcations is a supercritical bifurcation, in which an attracting symmetric periodic orbit loses stability (the symmetric orbit turns into a repeller) and bifurcates into two pairs of solutions of saddle type, with respective symmetries (1.5), and another two pairs of attracting solutions, with respective symmetries (1.6).

**Example 1.2** As a second example, we consider a time-periodic vector field (1.1) in $\mathbb{R}^2$ with the space-time symmetries

$$(\mathbf{x},t) \mapsto (M_x(\mathbf{x}),-t) \quad \text{and} \quad (\mathbf{x},t) \mapsto (R_{\frac{\pi}{2}}(\mathbf{x}),\frac{1}{2}-t). \tag{1.7}$$

We suppose the flow has a periodic orbit $\mathbf{x}(t)$ with full symmetry, i.e. $\mathbf{x}(t) = M_x(\mathbf{x}(-t)) = R_{\frac{\pi}{2}}(\mathbf{x}(\frac{1}{2}-t))$, so that $\mathbf{x}(0) = \mathbf{x}_0$ is a fixed point of the time-one return map $\mathcal{X}_{[0,1]}$. The difference with Example 1.1 is that now certain space-time symmetries involve a time-reversal ($t \mapsto c-t$, $c \in \mathbb{R}$), whereas in the previous example no time-reversals arose.

We may write the time-one return map as (1.3), but now $f$ satisfies (cf. Section 3)

$$f \circ M_x = R^2_{\frac{\pi}{2}} \circ M_x \circ f^{-1} \quad \text{and} \quad f \circ R^2_{\frac{\pi}{2}} \circ M_x = M_x \circ f^{-1}. \tag{1.8}$$

Hence $M_x$ and $R^2_{\frac{\pi}{2}} \circ M_{\mathbf{x}}$ do not conjugate $f$ to its inverse $f^{-1}$, but they do conjugate $f^2$ to its inverse $f^{-2}$. We call them reversing 2-symmetries of $f$. Again, the relations (1.8) describe all the structure on $f$ imposed by the spatiotemporal symmetries of the underlying vector field.

Generic local bifurcations of a map $f$ satisfying (1.8) were discussed in [17, Section 3.2], and were shown to be related to generic steady state bifurcations with reversing symmetry group $\mathbb{D}_4$ (and equivariance subgroup $\mathbb{D}_2$). As in Example 1.1, $\mathbb{D}_1$-symmetric solution branches are involved in generic local bifurcations of such a symmetric steady state.

In turn, the periodic orbits of $f$ represent periodic orbits of the flow $\mathcal{X}$ (cf. Section 5). One $\mathbb{D}_1$-branch represents periodic solutions with reversing symmetry

$$(\mathbf{x},t) \mapsto (M_x(\mathbf{x}),-t) \quad \text{or} \quad (\mathbf{x},t) \mapsto (R^2_{\frac{\pi}{2}} \circ M_x(\mathbf{x}),-t), \tag{1.9}$$

and the second branch represents periodic solutions with time-shift symmetry

$$(\mathbf{x},t) \mapsto (R_{\frac{\pi}{2}} \circ M_x(\mathbf{x}),\frac{1}{2}+t) \quad \text{or} \quad (\mathbf{x},t) \mapsto (R^{-1}_{\frac{\pi}{2}} \circ M_x(\mathbf{x}),\frac{1}{2}+t). \tag{1.10}$$

The periods of these solutions are again all equal to one (cf. Proposition 7.4).

As a consequence of the fourfold reversing symmetry, the fully symmetric periodic solution $\mathbf{x}(t)$ is generically of saddle type. In the case of a supercritical bifurcation one either finds four orbits of saddle type and four elliptic ones (the saddles symmetric with respect to a symmetry (1.9), and the elliptic orbits symmetric with respect to a reversing symmetry (1.10)) or four orbits of saddle type, two attractors and two repellers (the saddles now symmetric with respect to a reversing symmetry (1.10) and the attractors and repellers symmetric with respect to a symmetry (1.9)).

In Example 1.1 all the symmetries of the orbit that do not involve a time-shift appeared as $k$-symmetries of the map $f$. In the case of reversing symmetries a new phenomenon arises. Namely, the spatial part $R_{\frac{\pi}{2}}$ of the reversing symmetry $(\mathbf{x}, t) \mapsto (R_{\frac{\pi}{2}}(\mathbf{x}), \frac{1}{2} - t)$ does not appear as a reversing $k$-symmetry of $f$. However, something else happens. Namely, $R_{\frac{\pi}{2}}$ arises as a reversing 2-symmetry of the map $g = A \circ \mathcal{X}_{[-\frac{1}{4},\frac{1}{4}]}$, since $g \circ R_{\frac{\pi}{2}} = R_{\frac{\pi}{2}}^{-1} \circ g^{-1}$ and $g \circ R_{\frac{\pi}{2}}^{-1} = R_{\frac{\pi}{2}} \circ g^{-1}$. In this respect, we notice that $f = \mathcal{X}_{[-\frac{1}{4},0]} \circ g \circ \mathcal{X}_{[-\frac{1}{4},0]}^{-1}$, and $g^2 = \mathcal{X}_{[-\frac{1}{4},\frac{3}{4}]}$. Thus, by a (nonlinear) coordinate transformation we can change the surface of section and thereby the group of linear reversing symmetries of the return map in an interesting way. Importantly, these different choices may lead to reversing 2-symmetry groups with nonequivalent representations and/or group structures. In this example, we find that the linear reversing symmetries of $\mathcal{X}_{[0,1]}$ are $M_x$ and $R_{\frac{\pi}{2}}^2 \circ M_x$ generating a *reversing symmetry group* isomorphic to the dihedral group $\mathbb{D}_2$, whereas the linear reversing symmetries of $\mathcal{X}_{[-\frac{1}{4},\frac{3}{4}]}$ are $R_{\frac{\pi}{2}}$ and $R_{\frac{\pi}{2}}^{-1}$, generating a reversing symmetry group of $\mathcal{X}_{[-\frac{1}{4},\frac{3}{4}]}$ isomorphic to the cyclic group $\mathbb{Z}_4$. These occurrence of different symmetry properties in return maps of the *same* system is illustrated in Figure 1. This phenomenon will be discussed in more detail in Section 4.

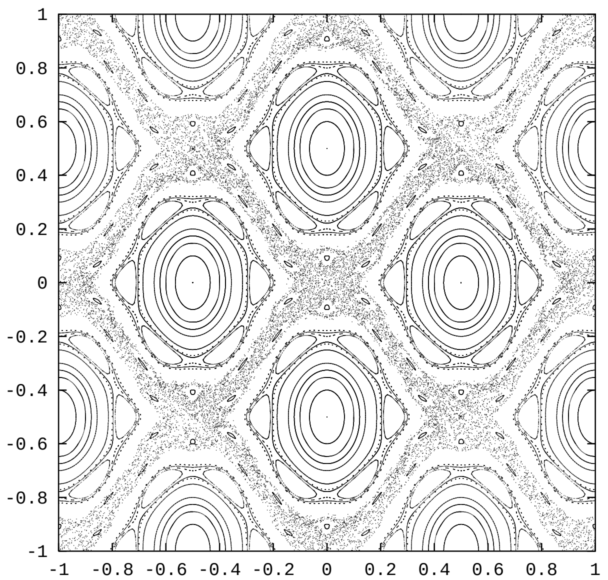
The above examples illustrate the importance of recognizing the decomposition properties of the time-one return map for understanding the dynamics. For instance, generic local bifurcations of diffeomorphisms with the linear symmetry properties of the time-one return map $\mathcal{X}_{[0,1]}$ differ significantly from the generic bifurcations of the diffeomorphisms with these symmetry properties which also possess special decomposition properties (as the ones in the above examples). Remarks along this line were made previously also by Swift and Wiesenfeld [28] and Fiedler [7].

The decomposition properties of the time-one return map leads us to study $k$-symmetric maps $f$. The $k$-symmetry properties of these maps precisely describe the interaction between the time-shift symmetries and the other spatiotemporal symmetry properties (cf. the relations (1.4) and (1.8) above). In Section 3 we will see that in the more abstract theory of $k$-symmetric dynamical systems these relations are described in the form of so-called $\phi_f$ orbits.

One could also study local bifurcations of space-time symmetric periodic orbits with other methods than return maps (e.g. using a Liapunov-Schmidt reduction directly on the flow). However, the advantage of the use of return maps is that one can obtain information not only on the singularities describing the local bifurcations but also about the stability properties of the bifurcating solutions and other aspects of the local dynamics (information that gets lost in a Liapunov-Schmidt reduction on the flow).

In this paper, we will also see how certain aspects of the dynamics of $k$-symmetric maps – some of which observed previously in [17, 18, 20, 21] – can be understood in the

(a)




(b)

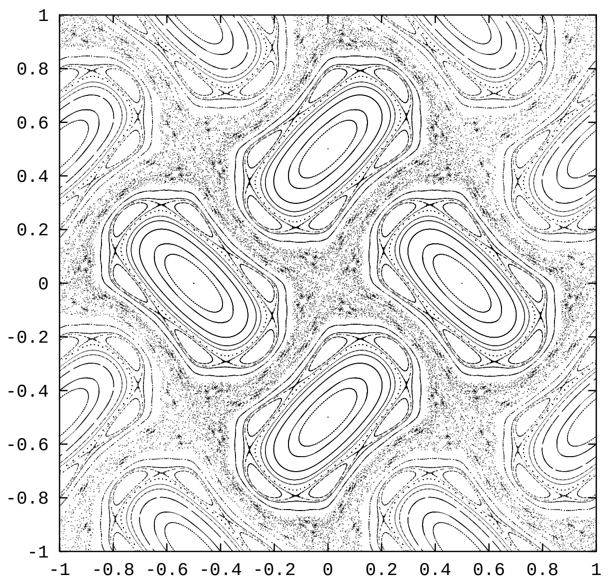



Figure 1: Illustration of the occurrence of different symmetry properties for return maps of systems with time-reversal symmetries. The phase portraits present return maps of a flow possessing the spatiotemporal symmetries discussed in Example 1.2. In (a) a phase portrait is shown of a return map $\mathcal{X}_{[0,1]} = f^2$ with reversing symmetry group $\mathbb{D}_2$ generated by the reflections in the $x$-axis and $y$-axis. In (b) a phase portrait representing the *same* flow is shown. However now we have a different surface of section and our return map is $\mathcal{X}_{[-\frac{1}{4},\frac{3}{4}]} = g^2$. It is obvious from the phase portraits that the linear symmetry properties have changed: $\mathcal{X}_{[-\frac{1}{4},\frac{3}{4}]}$ possesses a reversing symmetry group $\mathbb{Z}_4$ generated by the rotation around the origin $(0,0)$ over an angle $\frac{\pi}{2}$. Despite their different linear symmetry properties $\mathcal{X}_{[0,1]}$ and $\mathcal{X}_{[-\frac{1}{4},\frac{3}{4}]}$ are conjugate because they represent the same flow, i.e. $\mathcal{X}_{[0,1]} = T^{-1} \circ \mathcal{X}_{[-\frac{1}{4},\frac{3}{4}]} \circ T$. The transformation $T = \mathcal{X}_{[0,-\frac{1}{4}]}$ is nonlinear. See Example 4.3 for more details.

framework of return maps of space-time symmetric flows. For instance, in Theorem 5.6 we present an efficient characterization of periodic orbits that are *$k$-symmetric* with respect to reversing $k$-symmetries (improving upon results in [3, 20]), and in Theorem 4.1 we explain the simultaneous occurrence of *dual* (representations of) reversing $k$-symmetry groups (first observed in an example in [21]).

Although we mainly focus on $k$-symmetric maps that arise in the context of return maps of space-time symmetric flows, it should be noted that not every $k$-symmetric diffeomorphism can be related to the return map of a space-time symmetric vector field. However, in Theorem 6.2 we find that an intimate relationship does exist in a local context. In fact, we will show in Section 7, that this result provides a geometrical interpretation of formal symmetry properties of Birkhoff normal forms at fixed points of diffeomorphisms in terms of the the possibility of a local construction of a space-time symmetric suspension flow. This construction is such that the formal symmetry property of the Birkhoff normal form corresponds to a time-shift symmetry of the locally constructed suspension flow.

This paper is organized as follows. In Section 2 we present a brief introduction into $k$-symmetry. In Section 3 we then show how $k$-symmetric maps naturally arise in the study of space-time symmetric flows. In Section 4 we discuss the simultaneous occurrence of dual (representations of) reversing $k$-symmetry groups. In Section 5 we focus on periodic orbits, and discuss the relation between $k$-symmetric periodic orbits of $k$-symmetric maps and symmetric periodic orbits of space-time symmetric flows. In Section 6 we change our perspective and discuss in what way $k$-symmetric maps are related to return maps of space-time symmetric flows, both globally and locally (around fixed points). In Section 7 we discuss in more detail the local correspondence in the context of Birkhoff normal forms, and its application in the context of local bifurcation problems as sketched in Examples 1.1 and 1.2 above.

Finally, in Section 8 we point out that $k$-symmetric maps arise not only in the context of return maps of space-time symmetric flows, but also in the context of local return maps for symmetric periodic orbits of autonomous flows. This recognition opens up a wide class of applications in symmetric dynamical systems in which $k$-symmetric mappings naturally arise. Interestingly, in the spirit of this remark, very recently Rucklidge and Silber [26] used 2-symmetric maps in a study of local bifurcations of convection flows.

### Acknowledgements

It is great pleasure to acknowledge valuable discussions with Anja Braathen, Henk Broer, Michael Dellnitz and Mark Roberts. I am also grateful to Melbourne University (and in particular Prof. C.J. Thompson) for the hospitality during a visit in which part of this paper was written. This research has been supported by the European Union through a Human Capital & Mobility Research Fellowship (ERBCHBICT941533).

## 2 $k$-Symmetry in dynamical systems

We consider dynamical systems with discrete time, defined by a map $f$ of a state space $\Omega$ to itself. A transformation $S : \Omega \mapsto \Omega$ is called a *symmetry* when $S$ maps orbits of $f$ to itself, i.e. when $f$ commutes with $S$:

$$f \circ S = S \circ f. \tag{2.1}$$

Alternatively, $f$ is also called to be *equivariant* with respect to $S$. When $f$ is invertible, $f$ may also possess *reversing symmetries* $R : \Omega \mapsto \Omega$ that conjugate $f$ to its inverse:

$$f \circ R = R \circ f^{-1}. \tag{2.2}$$

A reversing symmetry $R$ maps orbits of $f$ to orbits of $f^{-1}$. Reversing symmetries naturally arise in time-reversible dynamical systems, e.g. in autonomous Hamiltonian dynamical systems of which the Hamiltonian $H(\mathbf{p}, \mathbf{q})$ is invariant under the transformation $R : (\mathbf{p}, \mathbf{q}) \mapsto (-\mathbf{p}, \mathbf{q})$.

We will focus only on symmetries and reversing symmetries that are invertible transformations of $\Omega$ to itself. In this context, the symmetries of a map $f$ form a group under composition, and so do the union of the sets of symmetries and reversing symmetries. A group of symmetries of $f$ is called a *symmetry group* and a group of symmetries and reversing symmetries is called a *reversing symmetry group* of $f$ [13].

Any reversing symmetry group $\Sigma$ that contains a reversing symmetry has a normal subgroup $\tilde{\Sigma}$ of index two, containing only symmetries, i.e.

$$\tilde{\Sigma} \trianglelefteq \Sigma, \text{ and if } \Sigma \neq \tilde{\Sigma} \text{ then } \Sigma/\tilde{\Sigma} \simeq \mathbb{Z}_2, \quad \Sigma = \tilde{\Sigma} \cup R\tilde{\Sigma}, \tag{2.3}$$

where $R \in \Sigma - \tilde{\Sigma}$ is a reversing symmetry.

Often, the symmetry properties of a map $f$ are described satisfactory in terms of its symmetries and reversing symmetries. However, this is not always the case. Namely, it may happen that the $k$th iterate of a map $f$ has more symmetries and reversing symmetries than the original map $f$, in which case we speak of *$k$-symmetry*.

**Definition 2.1** A map $S$ is called a $k$-symmetry of $f$ if $k$ is the smallest positive integer for which $S$ is a symmetry of $f^k$, i.e.

$$f^k \circ S = S \circ f^k. \tag{2.4}$$

Similarly, a map $R$ is called a reversing $k$-symmetry of $f$ if $k$ is the smallest positive integer for which

$$f^k \circ R = R \circ f^{-k}. \tag{2.5}$$

The occurrence of $k$-symmetries and reversing $k$-symmetries induces the definition of a map $\phi_f$, where

$$\begin{aligned} f \circ S &= \phi_f(S) \circ f, \quad \text{when } S \text{ is a symmetry of } f^k, & (2.6)\\ f \circ R &= \phi_f(R) \circ f^{-1}, \quad \text{when } R \text{ is a reversing symmetry of } f^k. & (2.7) \end{aligned}$$

One should note that $\phi_f$ maps $k$-symmetries of $f$ to $k$-symmetries of $f$ and reversing $k$-symmetries of $f$ to reversing $k$-symmetries of $f$.

Consider a reversing symmetry group $\Sigma$ of $f^q$ for some value of $q$. Then, $U \in \Sigma$ is a (reversing) $k$-symmetry of $f$ with $k$ being the smallest positive integer for which $\phi_f^k(U) = U$, i.e. (reversing) $k$-symmetries come in $\phi_f$-orbits of length $k$. Necessarily, $k$ must divide $q$.

In describing the symmetry properties of a map $f$, we want to omit trivial symmetries and reversing symmetries of $f^k$ (such as the map $f$ itself). Analogously to the fact that one sensibly studies symmetries and reversing symmetries occurring in groups instead of sets that are not closed under composition, in the $k$-symmetric context we preferably deal with sets of symmetries and reversing symmetries of $f^k$ that are not only groups, but are also closed under $\phi_f$.

**Definition 2.2** A (reversing) symmetry group $\Sigma$ of $f^q$ is called a *(reversing) $k$-symmetry group* of $f$ when $k$ is the smallest positive integer for which each element of $\Sigma$ is a (reversing) symmetry of $f^k$ <u>and</u> $\phi_f : \Sigma \mapsto \Sigma$, with $\phi_f$ defined on symmetries and reversing symmetries of $f^k$ as in (2.6) and (2.7).

In shorthand notation we denote the symmetry properties of a map $f$ – (reversing) $k$-symmetry group $\Sigma$ and $\phi_f$-orbit structure – as $(\Sigma, \phi_f)$. The theoretical framework for $k$-symmetry as presented here was first developed in [20], and further explored and refined in [3, 15–18, 21].

In this paper, we will focus on the occurrence of $k$-symmetric maps primarily in the context of return maps of space-time symmetric flows of time-periodic vector fields in $\mathbb{R}^n$.

# 3 Return maps of time-periodic flows with spatiotemporal symmetries

Consider a dynamical system on a manifold $\Omega$ that is the flow of a time-periodic (nonautonomous) vector field $F$, described by the ordinary differential equation

$$\frac{d}{dt}\mathbf{x} = F(\mathbf{x}, t). \tag{3.1}$$

In describing the symmetry properties of such a dynamical system we consider spatiotemporal symmetries of the form

$$U \otimes \tau : \Omega \times \mathbb{R} \mapsto \Omega \times \mathbb{R}, \tag{3.2}$$

where we take $U \otimes \tau$ to be a homeomorphism, acting on $(\mathbf{x}, t) \in \Omega \times \mathbb{R}$ as $U \otimes \tau(\mathbf{x}, t) = (U(\mathbf{x}), \tau(t))$. We say that $U \otimes \tau$ is a spatiotemporal symmetry of (3.1) if

$$\frac{d}{d(\tau(t))}U(\mathbf{x}) = F(U(\mathbf{x}), \tau(t)). \tag{3.3}$$

From (3.1) and (3.3) it directly follows that if $\mathbf{x}(t)$ is a solution of (3.1), then so is

$$U \otimes \tau(\mathbf{x}(t)) := U(\mathbf{x}(\tau^{-1}(t))). \tag{3.4}$$

We will study flows of ordinary differential equations of the form (3.1) with spatiotemporal symmetries that satisfy four hypotheses, of which the first one is:

**H1** The flow is one-periodic in time and nonautonomous, i.e. $\mathrm{Id} \otimes \sigma_1 : (\mathbf{x}, t) \mapsto (\mathbf{x}, t+1)$ is a time-shift symmetry ($F(\mathbf{x}, t) = F(\mathbf{x}, t+1)$ for all $t \in \mathbb{R}$), and there is no time-shift symmetry of the form $\mathrm{Id} \otimes \sigma_s$ with $0 < s < 1$.

By [**H1**], we will study the flow on $\Omega \times \mathbb{R}/\mathbb{Z} \simeq \Omega \times S^1$ (with $S^1$ denoting the unit circle).

The second hypothesis is:

**H2** The flow possesses a compact Lie group $\Gamma$ of spatiotemporal symmetries, acting on $\Omega \times S^1$ as $U \otimes \tau$ (3.2).[2]

[2] We will primarily be concerned with $\Omega = \mathbb{R}^n$, and $\Gamma$ acting linearly as an orthogonal representation. Also, note that in hypothesis [**H2**] we need not really restrict ourselves to compact Lie groups: the only thing we will actually use is that the temporal part of the spatiotemporal symmetry group forms a finite subgroup of $O(2)$. For instance, the spatial part of the spatiotemporal symmetry group $\Gamma$ may be taken to be a planar crystallographic group [2].

The temporal part $\{\tau \mid U \otimes \tau \in \Gamma \text{ for some } U\}$ of the spatiotemporal symmetry transformations forms a closed subgroup of $O(2)$, and we may hence choose a time-coordinate frame such that the action of this subgroup is linear.

In this setting, we distinguish the following three types of spatiotemporal symmetries:

$$\begin{array}{rlcl} \text{Symmetry:} & S \otimes 1 & : & (\mathbf{x}, t) \mapsto (S(\mathbf{x}), t), \\ \text{Time-shift symmetry:} & A \otimes \sigma_s & : & (\mathbf{x}, t) \mapsto (A(\mathbf{x}), t+s), \\ \text{Reversing symmetry:} & R \otimes \rho_s & : & (\mathbf{x}, t) \mapsto (R(\mathbf{x}), -t+s). \end{array} \tag{3.5}$$

Our third hypothesis is:

**H3** The temporal parts of the spatiotemporal symmetry transformations form a finite subgroup of $O(2)$.

By [**H3**] there must be a smallest positive value of $|s|$ for which $A \otimes \sigma_s \in \Gamma$ for some $A$. Because of [**H1**] this value must be rational, $|s| = \frac{1}{q}$ for some $q \in \mathbb{Z}^+$, such that the temporal transformations form a group isomorphic to $\mathbb{Z}_q$ or $\mathbb{D}_q$. [**H3**] is motivated by the fact that in case the temporal parts of $\Gamma$ form a closed subgroup of $O(2)$ that is not finite, then it is either $SO(2) \simeq S^1$ (without time-reversal) or $O(2)$ (with time-reversal). In the context of local bifurcation theory, the latter situations have been addressed in [9] (without time-reversal) and [11, 30] (with time-reversal).

Finally, our fourth and last hypothesis is:

**H4** $(\mathrm{Id} \otimes \rho_s) \notin \Gamma$ for all $s \in \mathbb{R}$.

In case [**H4**] is not satisfied, the return maps we intend to study will be involutory (their square is the identity). By requiring [**H4**] we avoid this degenerate situation.

For the moment we will not further specify the spatial parts of the symmetry transformations or the manifold on which the flow is defined. In examples, however, we will mainly focus on the case that $\Omega = \mathbb{R}^n$ and $\Gamma \leq O(n) \times \mathbb{D}_q$ acting linearly on $\mathbb{R}^n \times S^1$.

Let us consider the evolution operator of the dynamical system (3.1),

$$\mathcal{X}_{[a,b]} : \Omega \mapsto \Omega \tag{3.6}$$

Considering a point on position $\mathbf{x}$ at $t = a$, then $\mathcal{X}_{[a,b]}(\mathbf{x})$ is the position of same point at $t = b$, assuming it moves on $\Omega$ according to the equations of motion (3.1).

It is readily checked that the evolution operator and any spatiotemporal symmetry $U \otimes \tau$ satisfy the relation

$$U \circ \mathcal{X}_{[a,b]} = \mathcal{X}_{[\tau(a),\tau(b)]} \circ U, \tag{3.7}$$

thus in particular (with the notation of (3.5)):

$$S \circ \mathcal{X}_{[a,b]} = \mathcal{X}_{[a,b]} \circ S, \tag{3.8}$$

$$A \circ \mathcal{X}_{[a,b]} = \mathcal{X}_{[a+s,b+s]} \circ A, \tag{3.9}$$

$$R \circ \mathcal{X}_{[a,b]} = \mathcal{X}_{[-a+s,-b+s]} \circ R. \tag{3.10}$$

By [**H1**], $\mathrm{Id} \otimes \sigma_1$ is a symmetry so that

$$\mathcal{X}_{[a,b]} = \mathcal{X}_{[a+1,b+1]}, \tag{3.11}$$

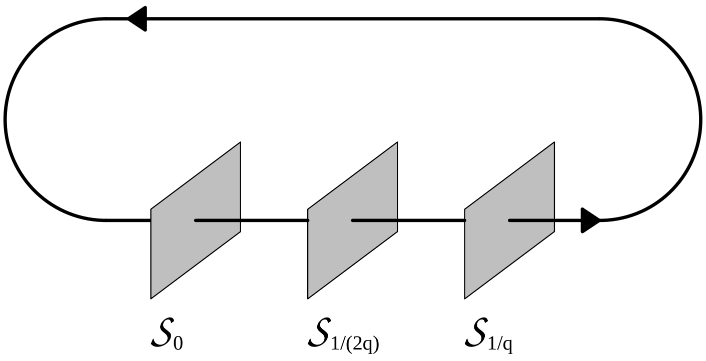



Figure 2: Schematic impression of surfaces of section for a time-one periodic flow: $\mathcal{S}_a := \{(\mathbf{x}, t) \in \Omega^n \times S^1 \mid t = a\}$.

and $\mathcal{X}_{[a,a+n]} = \mathcal{X}^n_{[a,a+1]}$ for all $n \in \mathbb{Z}$. This enables one to study the time-dependent flow of (3.1), by a time-one return map which does no longer explicitly depend on the time variable. The time-one map $\mathcal{X}_{[a,a+1]}$ is a return map of the nonautonomous flow with *surface of section* $\mathcal{S}_a := \{(\mathbf{x}, t) \in \Omega \times S^1 \mid t = a\}$, cf. Figure 2.

In the group of spatiotemporal symmetries $\Gamma$ we distinguish the following subgroups:

$$\hat{\Gamma} := \{U \otimes \tau \in \Gamma \mid \tau = \sigma_s,\ s \in [0,1\rangle\}, \tag{3.12}$$
$$\tilde{\Gamma} := \{U \otimes \tau \in \Gamma \mid \tau = 1\}, \tag{3.13}$$
$$\Gamma_s := \{U \otimes \tau \in \Gamma \mid \tau = \rho_{2s}\} \cup \tilde{\Gamma}. \tag{3.14}$$

$\tilde{\Gamma}$ is a symmetry group of (3.1), $\hat{\Gamma}$ is a time-shift symmetry group of (3.1), and $\Gamma_s$ is the subgroup of $\Gamma$ that fixes the surface of section $\mathcal{S}_s$ setwise. It is readily verified that these subgroups of $\Gamma$ satisfy the following relations:

$$\tilde{\Gamma} \trianglelefteq \hat{\Gamma} \trianglelefteq \Gamma, \tag{3.15}$$
$$\hat{\Gamma}/\tilde{\Gamma} \simeq \mathbb{Z}_q, \tag{3.16}$$
$$\Gamma/\hat{\Gamma} \simeq \mathbb{Z}_2 \text{ when } \Gamma \neq \hat{\Gamma}, \tag{3.17}$$
$$\tilde{\Gamma} \trianglelefteq \Gamma, \tag{3.18}$$
$$\Gamma/\tilde{\Gamma} \simeq \mathbb{D}_q \text{ when } \Gamma \neq \hat{\Gamma}. \tag{3.19}$$

and

$$\tilde{\Gamma} \trianglelefteq \Gamma_s, \tag{3.20}$$
$$\Gamma_s/\tilde{\Gamma} \simeq \mathbb{Z}_2 \text{ when } \Gamma_s \neq \tilde{\Gamma}. \tag{3.21}$$

Let us now consider a return map, assuming we choose the surface to be $\mathcal{S}_0$, gauging our time-coordinate frame such that whenever $\Gamma$ contains reversing symmetries, at least one of them fixes $\mathcal{S}_0$ setwise (i.e. $\Gamma \neq \hat{\Gamma} \Rightarrow \Gamma_0 \neq \tilde{\Gamma}$).

We will now examine in what way the spatiotemporal symmetries of the periodic flow emerge in the return map. While studying the symmetry properties of the return map, we are primarily interested in the spatial components of spatiotemporal symmetries of the flow. We let $\Sigma$ denote the group consisting of the spatial parts of the spatiotemporal symmetries in $\Gamma$,

$$\Sigma := \{U \mid U \otimes \tau \in \Gamma \text{ for some } \tau\}. \tag{3.22}$$

Similarly, $\hat{\Sigma}$, $\tilde{\Sigma}$ and $\Sigma_s$ are the subgroups corresponding to the spatial parts of the spatiotemporal transformations of $\hat{\Gamma}$, $\tilde{\Gamma}$ and $\Gamma_s$. Clearly, by **[H1]** and **[H4]**, $\Gamma \mapsto \Sigma$ is a projection and an isomorphism so that $\Gamma \simeq \Sigma$, $\hat{\Gamma} \simeq \hat{\Sigma}$, $\tilde{\Gamma} \simeq \tilde{\Sigma}$, and $\Gamma_s \simeq \Sigma_s$, and $\Sigma$, $\hat{\Sigma}$, $\tilde{\Sigma}$ and $\Sigma_s$ satisfy relations analogous to those of $\Gamma$, $\hat{\Gamma}$, $\tilde{\Gamma}$ and $\Gamma_s$ in (3.15)-(3.21):

$$\begin{array}{rcl} \tilde{\Sigma} & \trianglelefteq & \hat{\Sigma} \trianglelefteq \Sigma, \\ \hat{\Sigma}/\tilde{\Sigma} & \simeq & \mathbb{Z}_q, \\ \Sigma/\hat{\Sigma} & \simeq & \mathbb{Z}_2 \text{ when } \Sigma \neq \hat{\Sigma}, \\ \tilde{\Sigma} & \trianglelefteq & \Sigma, \\ \Sigma/\tilde{\Sigma} & \simeq & \mathbb{D}_q \text{ when } \Sigma \neq \hat{\Sigma}, \\ \tilde{\Sigma} & \trianglelefteq & \Sigma_s, \\ \Sigma_s/\tilde{\Sigma} & \simeq & \mathbb{Z}_2 \text{ when } \Sigma_s \neq \tilde{\Sigma}. \end{array} \tag{3.23}$$

.

**Theorem 3.1** *Let $\mathcal{X}_{[0,1]}$ be the time-one map of the flow of a vector field (3.1), satisfying the hypotheses* **[H1-4]**, *adopting the notation introduced in (3.5)-(3.22). Suppose $A \otimes \sigma_{-\frac{1}{q}}$ is a time-shift symmetry of this flow. Then,*

$$\mathcal{X}_{[0,1]} = A^{-q} \circ (A \circ \mathcal{X}_{[0,\frac{1}{q}]})^q. \tag{3.24}$$

*Let $f := A \circ \mathcal{X}_{[0,\frac{1}{q}]}$, then for all $S \in \tilde{\Sigma}$*

$$\phi_f(S) := f \circ S \circ f^{-1} = A \circ S \circ A^{-1}, \tag{3.25}$$

*and for all $R \in \Sigma_0 - \tilde{\Sigma}$*

$$\phi_f(R) := f \circ R \circ f = A \circ R \circ A. \tag{3.26}$$

**Proof.** In order to obtain (3.24), we write

$$\mathcal{X}_{[0,1]} = \mathcal{X}_{[\frac{q-1}{q},1]} \circ \ldots \circ \mathcal{X}_{[0,\frac{1}{q}]}, \tag{3.27}$$

and use (3.9) to obtain

$$\mathcal{X}_{[\frac{m}{q},\frac{m+1}{q}]} = A^{-m} \circ \mathcal{X}_{[0,\frac{1}{q}]} \circ A^m, \tag{3.28}$$

from which we readily verify (3.24). The relations (3.25)-(3.26) follow directly from (3.9). □

**Remark 3.2** Theorem 3.1 establishes the link between return maps of space-time symmetric flows and $k$-symmetry. Namely, $\Sigma_0$ is a reversing $k$-symmetry group of $f$, with $k$ being the smallest integer for which $A^k$ has $\Sigma_0$ as a reversing symmetry group, and $\phi_f = \phi_A$.

This connection was first established in [16], where an abridged version of this theorem was presented. In that paper we focussed on the case that $A^q = \mathrm{Id}$, in which case the time-one map is precisely the $q$th iterate of a $k$-symmetric map $f$, for some value of $k$ dividing $q$.[3]

[3]Note that when $A^a = \mathrm{Id}$ for some $a \in \mathbb{Z}$, then $k$ and $q$ must divide $a$.

We thus find the spatiotemporal symmetry properties of the flow represented in terms of the reversing $k$-symmetry group $\Sigma_0$ of the map $f$ <u>and</u> its $\phi_f$-orbit structure. By keeping track of the precise $k$-symmetry properties of the map $f$ we retain important information about the spatiotemporal symmetry properties of the underlying flow.

In case $A$ is of infinite order, although $\phi_f : \Sigma_0 \mapsto \Sigma_0$ and $A^q \in \Sigma_0$, there might not exist a positive integer $k$ for which $\phi_f^k(U) = U$ for all $U \in \Sigma$. However, the $(\Sigma_0, \phi_f)$ symmetry property still imposes a lot of structure on the dynamics of $f$. We will refer to this case as $k = \infty$.

**Remark 3.3** Every map $f$ that is reversible with respect to a reversing symmetry $R$, can be written as $f = U \circ R^{-1}$ with $R^2 = U^2$ [13]. Maps possessing reversing $k$-symmetries have analogous decomposition properties. For instance, writing $f = U \circ R^{-1}$ with $R$ a reversing $k$-symmetry of $f$ one obtains

$$U^2 = \phi_f(R) \circ R. \tag{3.29}$$

In general, the full structure of the reversing $k$-symmetry group will impose more constraints complicating the decomposition property further, see e.g. the detailed treatment of the decomposition property in the case of cyclic reversing $k$-symmetry groups in [21].

**Remark 3.4** One may also decompose the map $f = A \circ \mathcal{X}_{[0,\frac{1}{q}]}$ using relations between the evolution operator and reversing symmetries following from (3.9). In particular, if $\Sigma_0 \neq \tilde{\Sigma}$ then for all $R \in \Sigma_0 - \tilde{\Sigma}$

$$\begin{aligned} f &= A \circ C(\mathcal{X}_{[\frac{1}{2q},\frac{1}{q}]}, R \circ A) & (3.30)\\ &= A \circ C(R \circ A, \mathcal{X}_{[\frac{1}{2q},0]}), & (3.31)\\ &= \mathcal{X}_{[-\frac{1}{2q},0]} \circ A \circ R \circ \mathcal{X}^{-1}_{[-\frac{1}{2q},0]} \circ R^{-1}, & (3.32)\end{aligned}$$

with the commutator $C(M,N) := M \circ N \circ M^{-1} \circ N^{-1}$.

# 4 Choices of surface of section and dual reversing $k$-symmetry groups

Instead of considering the evolution operator $\mathcal{X}_{[0,1]}$ one may as well consider any evolution operator over one time-period $\mathcal{X}_{[a,a+1]}$. In principle, such a choice does not change any essential dynamical feature of the map, as $\mathcal{X}_{[0,1]}$ is smoothly conjugate to $\mathcal{X}_{[a,a+1]}$.

However, in case we have reversing symmetries (i.e. $\hat{\Gamma} \neq \Gamma$) certain surfaces of section will be fixed by a reversing symmetry and others not. This has consequences for the way reversing symmetries appear in the return map of the surface of section of a reversible flow.

We consider a time-periodic flow with spatiotemporal symmetry group $\Gamma$. We consider the return map $\mathcal{X}_{[a,a+1]}$ of a section $\mathcal{S}_a := \{(\mathbf{x}, t) \in \Omega \times S^1 \mid t = a\}$ to itself.

We first note that if $U \otimes \tau \in \Gamma$, then $U \circ \mathcal{X}_{[a,a+1]} = \mathcal{X}_{[a,a+1]} \circ U$ if $\tau = \mathrm{Id}$, and $U \circ \mathcal{X}_{[a,a+1]} = \mathcal{X}^{-1}_{[a,a+1]} \circ U$ if $\tau = \rho_{2a}$. In particular, $U \in \Sigma_a$ if and only if $U \otimes \tau$ fixes $\mathcal{S}_a$ setwise. This follows directly from (3.7). In particular, we thus find that $\tilde{\Sigma} \trianglelefteq \Sigma_a$ for all $a$. So, only the reversing symmetries in $\Sigma_a$ depend on the choice of the surface of section. Time-shift symmetries do not fix any surface of section, so the spatial component of a time-shift symmetry never arises as an element of $\Sigma_a$.

In the presence of reversing symmetries, we preferably consider a surface of section that is setwise fixed by at least one reversing symmetry. However, there will always exist other reversing symmetries that do not fix this particular surface of section. In particular, we find that there are (at most) three types of subgroups $\Sigma_a$ of $\Sigma$ acting naturally on a surface of section. This is the content of the following theorem.

**Theorem 4.1** *Consider a dynamical system as in Theorem 3.1, with $\hat{\Gamma}/\tilde{\Gamma} \simeq \mathbb{Z}_q$, $\Sigma_0 \neq \tilde{\Sigma}$ and $\Sigma_0 = \tilde{\Sigma} \cup R\tilde{\Sigma}$. If $\Sigma_0$ is a reversing $k$-symmetry group of the map $f = A \circ \mathcal{X}_{[0,\frac{1}{q}]}$ (with $A \otimes \sigma_{-\frac{1}{q}}$ a time-shift symmetry in $\Gamma$), then for all $n \in \mathbb{Z}$,*

(*i*) $\Sigma_{\frac{n}{q}} \simeq \Sigma_0$.

(*ii*) $\Sigma_{\frac{2n+1}{2q}} \simeq \Sigma_{\frac{1}{2q}} = \tilde{\Sigma} \cup (R \circ A)\tilde{\Sigma}$.

(*iii*) $\Sigma_a = \tilde{\Sigma}$ *when* $a \neq \frac{n}{2q}$.

(*iv*) $\Sigma_0 \simeq \Sigma_{\frac{n}{2q}}$ *when* $\gcd(q,k)$ *is odd.*

**Proof.** Let $R$ be a reversing symmetry in $\Sigma_0$. Then $\Sigma_0 = \tilde{\Sigma} \cup R\tilde{\Sigma}$. We now examine the subgroups of $\Gamma$ fixing $\mathcal{S}_a$ for values of $a$ not equal to 0. The reversing symmetries inside $\Gamma$ are all of the form $U \otimes \rho_{\frac{n}{q}}$ for some $n \in \mathbb{Z}$. Hence $\Sigma_a = \tilde{\Sigma}$ whenever $a \neq \frac{n}{2q}$ for some integer value of $n$, proving (*iii*).

When $a = \frac{n}{q}$ then $\mathcal{X}_{[a,a+\frac{1}{q}]} = A^n \circ \mathcal{X}_{[0,\frac{1}{q}]} \circ A^{-n}$ and consequently $\Sigma_{\frac{n}{q}} \simeq \Sigma_0$, proving (*i*).

It is readily verified that if $R \in (\Sigma_0 - \tilde{\Sigma})$, then $R \circ A$ is an element of $\Sigma_{\frac{1}{2q}} - \tilde{\Sigma}$. Namely, using (3.9) one verifies that

$$\begin{aligned} R \circ A \circ \mathcal{X}_{[\frac{1}{2q},1+\frac{1}{2q}]} &= \mathcal{X}_{[\frac{1}{2q},-1+\frac{1}{2q}]} \circ R \circ A \\ &= \mathcal{X}^{-1}_{[\frac{1}{2q},1+\frac{1}{2q}]} \circ R \circ A \end{aligned}$$

Hence, $\Sigma_{\frac{1}{2q}} = \tilde{\Sigma} \cup (R \circ A)\tilde{\Sigma}$. Because $\mathcal{X}_{[\frac{2n+1}{2q},-1+\frac{2n+1}{2q}]} = A^n \circ \mathcal{X}_{[\frac{1}{2q},-1+\frac{1}{2q}]} \circ A^{-n}$ for all $n$, we furthermore obtain that $\Sigma_{\frac{1}{2q}} \simeq \Sigma_{\frac{2n+1}{2q}}$. This proves (*ii*).

If $q$ is odd, then we claim that

$$A^{-\frac{q+1}{2}} \circ R \circ A^{\frac{q+1}{2}} = R \circ A \circ S \tag{4.1}$$

for some $S \in \tilde{\Sigma}$. Before we prove this assertion we first show that (4.1) implies that $\Sigma_0 \simeq \Sigma_{\frac{1}{2q}}$. Namely, when (4.1) holds we find that $\Sigma_0 A^{\frac{q+1}{2}} = A^{\frac{q+1}{2}} \Sigma_{\frac{1}{2q}}$ which in turn implies that $\Sigma_0 \simeq \Sigma_{\frac{1}{2q}}$. To prove (4.1), we note that every spatial part of a reversing symmetry in $\Sigma$ can be written as $R \circ A^p \circ S$ for some $p \in \mathbb{Z}$ and some $S \in \tilde{\Sigma}$. Moreover, taking account of the (virtual) temporal components of the left hand side of (4.1) it follows immediately that $p = 1$ in the right hand side (since $\sigma_{\frac{q+1}{2q}} \circ \rho_0 \circ \sigma_{-\frac{q+1}{2q}} = \rho_0 \circ \sigma_{-\frac{1}{q}}$).

Finally, to prove (*iv*) we have to show that $\Sigma_0 \simeq \Sigma_{\frac{1}{2q}}$ when $k$ is odd. One first verifies, using the fact that $\Sigma_0$ is a reversing symmetry group of $A^k$, that

$$\tilde{\Sigma} \cup R\tilde{\Sigma} \simeq \tilde{\Sigma} \cup (R \circ A^k)\tilde{\Sigma}.$$

Moreover, along the lines of the proof of (4.1) we further obtain that for all integer $n$

$$\tilde{\Sigma} \cup R\tilde{\Sigma} \simeq \tilde{\Sigma} \cup (R \circ A^{2n})\tilde{\Sigma}, \quad \tilde{\Sigma} \cup (R \circ A)\tilde{\Sigma} \simeq \tilde{\Sigma} \cup (R \circ A^{2n+1})\tilde{\Sigma}. \tag{4.2}$$

Hence, when $k$ is odd, one readily obtains that $\Sigma_0 \simeq \Sigma_{\frac{1}{2q}}$. This concludes the proof. □

**Remark 4.2 (Dual reversing $k$-symmetry groups)** In Theorem 4.1, when $k$ and $q$ are even, in general one expects to find that $\Sigma_0 \not\simeq \Sigma_{\frac{1}{2q}}$.

Although only (reversing) symmetries inside the subgroup $\Sigma_a$ of $\Sigma$ act naturally as (reversing) symmetries of the return map $\mathcal{X}_{[a,a+1]}$ from $\mathcal{S}_a$ to itself, the reversing symmetries in $\Sigma$ that are not in $\Sigma_a$ also give rise to reversing symmetries of this return map. However, these reversing symmetries do not act naturally (as the spatial parts of the reversing symmetries in $\Gamma$). When $R$ fixes $\mathcal{S}_b$, then $\mathcal{X}_{[a,b]}^{-1} \circ R \circ \mathcal{X}_{[a,b]}$ fixes $\mathcal{S}_a$ and is a reversing symmetry of the return map $\mathcal{X}_{[a,a+1]}$.

Hence, in one return map we find more than one reversing symmetry group acting simultaneously. One of them acts naturally (as the spatial components of a subgroup of the spatiotemporal symmetry group $\Gamma$ that fixes the surface of section setwise), while the others in general do not.

Recall that the map $f_a = A \circ \mathcal{X}_{[a,a+\frac{1}{q}]}$ is $(\Sigma_a, \phi_A)$ $k$-symmetric, cf. Theorem 3.1. We will refer to the simultaneous occurrence of two nonisomorphic types of reversing $k$-symmetry groups in maps possessing a reversing $k$-symmetry group as the *duality* of reversing $k$-symmetry groups. This phenomenon was first identified in some examples of $k$-symmetric maps in [15, 21], in which it was noticed that in the presence of dual reversing $k$-symmetry groups by choosing a coordinate frame one can usually make only one of the dual reversing $k$-symmetry groups acting linearly. Our discussion here provides an explanation of this phenomenon in the context of return maps of space-time symmetric flows.

The occurrence of dual reversing $k$-symmetry groups is related to the fact that the temporal part of the spatiotemporal symmetry group $\Gamma$ is isomorphic to the dihedral group $\mathbb{D}_q$ (for some value of $q$). Considering the natural action of $\mathbb{D}_q$ on the circle, we find that points of nontrivial isotropy come in two equivalence (conjugacy) classes when $q$ is even, but only in one equivalence class if $q$ is odd. The equivalence classes of points with nontrivial isotropy correspond to equivalence classes of subgroups fixing surfaces of sections. So we have two equivalence classes of isotropy subgroups of surfaces of section that contain a reversing symmetry if $q$ is even, but only one if $q$ is odd. All surfaces of section $\mathcal{S}_a$ related to the same equivalence class have isomorphic reversing symmetry groups $\Sigma_a$ with equivalent (conjugate) representations.

**Example 4.3 (Spatiotemporal symmetry groups in $\mathbb{R}^2$.)** As an example, we examine the occurrence of finite subgroups of $O(2)$ arising as spatiotemporal symmetry groups.

In Table 1 we list the possible combinations of groups $\Sigma$, $\tilde{\Sigma}$, and $\hat{\Sigma}$ and corresponding pairs of reversing symmetry groups $\Sigma_0$ and $\Sigma_{\frac{1}{2q}}$ (assuming whenever $\Sigma \neq \hat{\Sigma}$ that $\Sigma_0$ contains at least one reversing symmetry). Note that pairs of nonisomorphic dual reversing $k$-symmetry groups only arise in the last entry of this table, when $p > 1$.

As an example of such a group with $p = 2$, let us consider the time-one map of a time-periodic vector field with spatiotemporal symmetry group $\Gamma$ of which the spatial parts form the group $\Sigma = \mathbb{D}_4$, such that $\hat{\Sigma} = \mathbb{D}_2$ and $\tilde{\Sigma} = \mathbb{Z}_2$.

| $\Sigma$ | $\hat{\Sigma}$ | $\tilde{\Sigma}$ | $q$ | $k$ | $\Sigma_0$ | $\Sigma_{\frac{1}{2q}}$ |
|---|---|---|---|---|---|---|
| $\mathbb{Z}_{p\cdot q}$ | $\mathbb{Z}_{p\cdot q}$ | $\mathbb{Z}_p$ | $q$ | $1$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $\mathbb{D}_p$ | $\mathbb{D}_p$ | $\mathbb{Z}_p$ | $2$ | $\begin{cases} 1 & (p<3) \\ 2 & (p\geq 3) \end{cases}$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ |
| $\mathbb{D}_{2p}$ | $\mathbb{D}_{2p}$ | $\mathbb{D}_p$ | $2$ | $\begin{cases} 1 & (p=1) \\ 2 & (p\geq 2) \end{cases}$ | $\mathbb{D}_p$ | $\mathbb{D}_p$ |
| $\mathbb{D}_{2p}$ | $\mathbb{D}_{2p}$ | $\mathbb{D}_p$ | $2$ | $1$ $(p\ odd)$ | $\mathbb{D}_p$ | $\mathbb{D}_p$ |
| $\mathbb{D}_p$ | $\mathbb{Z}_p$ | $\mathbb{Z}_p$ | $1$ | $1$ | $\mathbb{D}_p$ | $\mathbb{D}_p$ |
| $\mathbb{D}_{2p}$ | $\mathbb{D}_p$ | $\mathbb{D}_p$ | $1$ | $1$ | $\mathbb{D}_{2p}$ | $\mathbb{D}_{2p}$ |
| $\mathbb{D}_{p\cdot q}$ | $\mathbb{Z}_{p\cdot q}$ | $\mathbb{Z}_p$ | $q$ | $1$ | $\mathbb{D}_p$ | $\mathbb{D}_p$ |
| $\mathbb{D}_{2p}$ | $\mathbb{D}_p$ | $\mathbb{Z}_p$ | $2$ | $\begin{cases} 1 & (p=1) \\ 2 & (p\geq 2) \end{cases}$ | $\mathbb{D}_p$ | $\mathbb{Z}_{2p}$ |

Table 1: Possible combinations of finite subgroups of $O(2)$: $\Sigma$, $\tilde{\Sigma}$, and $\hat{\Sigma}$ satisfying the mutual relations (3.23). Also listed are pairs of reversing symmetry groups $\Sigma_0$ and $\Sigma_{\frac{1}{2q}}$ arising in the return maps of surfaces of section $\mathcal{S}_0$ and $\mathcal{S}_{\frac{1}{2q}}$ (choosing the time-coordinate such that whenever $\Sigma \neq \hat{\Sigma}$, $\Sigma_0$ contains a reversing symmetry). Here, $\mathbb{Z}_n$ and $\mathbb{D}_n$ denote the standard representation of the cyclic and dihedral groups in $\mathbb{R}^2$ (i.e. the cyclic group generated by a rotation and the dihedral group generated by reflections), and $p \in \mathbb{Z}^+$.

The example is constructed from the maps:

$$A : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}, \quad R : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x \end{pmatrix},$$

and

$$\mathcal{X}_{[-\frac{1}{4},0]} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \frac{1}{2}g(y) \\ y \end{pmatrix},$$

so that with $\mathcal{X}_{[-\frac{1}{4},\frac{1}{4}]} = (R\circ A)\circ \mathcal{X}_{[0,-\frac{1}{4}]} \circ (R\circ A)^{-1} \circ \mathcal{X}_{[-\frac{1}{4},0]}$ and $g(y) = -g(-y)$, the map

$$f_{-\frac{1}{4}} = A \circ \mathcal{X}_{[-\frac{1}{4},\frac{1}{4}]} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x + g(y) \end{pmatrix} \tag{4.3}$$

is $(\Sigma_{-\frac{1}{4}}, \phi_A)$ 2-symmetric, with $\mathcal{X}_{[-\frac{1}{4},\frac{3}{4}]} = f^2_{-\frac{1}{4}}$ and $\Sigma_{-\frac{1}{4}} = \Sigma_{\frac{1}{4}} = \mathbb{Z}_4$, cf. Theorem 3.1 and Remark 3.4. Alternatively, the map $f_0 = A \circ \mathcal{X}_{[0,\frac{1}{2}]} = \mathcal{X}_{[-\frac{1}{4},0]} \circ f_{-\frac{1}{4}} \circ \mathcal{X}^{-1}_{[-\frac{1}{4},0]}$ is $(\Sigma_0, \phi_A)$ 2-symmetric, with $\mathcal{X}_{[0,1]} = f_0^2$ and $\Sigma_0 = \mathbb{D}_2$. In Fig. 1 of Section 1 phase portraits of (a) $f_0$, and (b) $f_{-\frac{1}{4}}$, with $g(y) = 0.25\sin(2\pi y)$ illustrate the dual symmetry properties.[4]

**Remark 4.4 (Dual representations)** When $\Sigma_0$ is isomorphic to $\Sigma_{\frac{1}{2q}}$, i.e. $\Sigma_0 \simeq \Sigma_{\frac{1}{2q}}$ then it may still be the case that these groups do not have equivalent representations. In particular, this happens when $A$ is a transformation that itself has $\Sigma_0$ as a reversing symmetry

[4]Clearly, these maps possess more symmetry than discussed here. In fact, they possess crystallographic reversing $k$-symmetry groups, cf. [2].

group. Namely, in that case $\phi_A(U) = U$ for all $U \in \Sigma_0$. Whether or not the groups $\Sigma_0$ and $\Sigma_{\frac{1}{2q}}$ have equivalent representations depends on the specific form of $A$ (the spatial part of the time-shift symmetry). Recall in this respect from the proof of Theorem 4.1 that whenever $q$ is odd, $\Sigma_0$ is conjugate to $\Sigma_{\frac{1}{2q}}$ so that whenever $q$ is odd these groups always have equivalent representations.

From Table 1 one obtains that in the case of $O(2)$ in $\mathbb{R}^2$ there is only one situation in which one finds nonequivalent representations: when $\Sigma = \mathbb{D}_2$, $\hat{\Sigma} = \mathbb{D}_1$ and $\tilde{\Sigma} = \mathrm{Id}$, we find dual nonequivalent representations of the cyclic group of order two: $\mathbb{Z}_2$ and $\mathbb{D}_1$.

**Example 4.5 (Dual representations of $\mathbb{Z}_2$ in $\mathbb{R}^2$)** Reversible maps in $\mathbb{R}^2$ may be such that $f$ is the product of two involutions $f := I_0 \circ I_1$, where $\dim \mathrm{Fix} I_0 = 0$ and $\dim \mathrm{Fix} I_1 = 1$. Hence both involutions are not conjugate and therefore the map cannot be the return map of a time-periodic or autonomous reversible flow in $\mathbb{R}^2$. However, when $I_0$ and $I_1$ are both smoothly conjugate to a linear involution, then (in a well-chosen coordinate frame) $f$ is the square root of the return map of a time-one periodic flow with reversing symmetry $M_y \otimes \rho_0$, time-shift symmetry $M_x \otimes \sigma_{\frac{1}{2}}$ and reversing symmetry $R_\pi \otimes \rho_{\frac{1}{2}}$ (where $M_x$ and $M_y$ denote the linear reflections in $x$-axis and $y$-axis and $R_\pi$ is the rotation over $\pi$ around the origin).

An example of such a map is the area-preserving orientation-reversing Hénon map [10]

$$f_0 \; : \; \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 + y - ax^2 \\ x \end{pmatrix}. \tag{4.4}$$

This map can be decomposed as

$$f_0 = \mathcal{X}_{[-\frac{1}{2q},0]} \circ A \circ R \circ \mathcal{X}^{-1}_{[-\frac{1}{2q},0]} \circ R^{-1}, \tag{4.5}$$

with

$$A \; : \; \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}, \quad R \; : \; \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ -x \end{pmatrix},$$

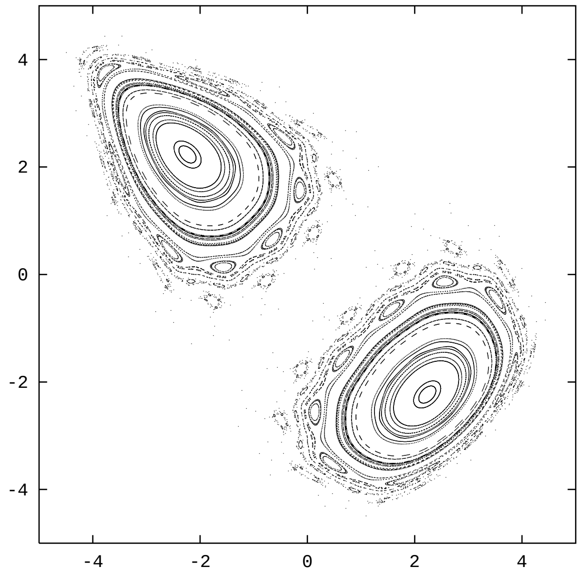
and

$$\mathcal{X}_{[-\frac{1}{2q},0]} \; : \; \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \frac{1}{2}(1 - ay^2) \\ y \end{pmatrix}.$$

Indeed, $\mathcal{X}_{[-\frac{1}{2q},0]}$ is isotopic to the identity and hence the evolution operator between $t = -\frac{1}{2q}$ and $t = 0$ of the flow of a nonautonomous vector field in $\mathbb{R}^2$. Hence, the area-preserving orientation-reversing Hénon map $f_0$ is the square root of the return map of a nonautonomous flow with spatiotemporal symmetry group $\Gamma = \langle A \otimes \sigma_{\frac{1}{2}}, R \otimes \rho_0 \rangle \simeq \mathbb{D}_2$. The different type of involutions constituting the map $f$ refer to the nonequivalent dual representations of the reversing symmetry groups (cyclic group of order 2) arising in the return maps of the surfaces of section $\mathcal{S}_0$ and $\mathcal{S}_{\frac{1}{4}}$.

In Fig. 3 phase portraits of $f_0 := A \circ \mathcal{X}_{[0,\frac{1}{2}]}$ and $f_{-\frac{1}{4}} := A \circ \mathcal{X}_{[-\frac{1}{4},\frac{1}{4}]} = \mathcal{X}_{[0,-\frac{1}{2q}]} \circ f_0 \circ \mathcal{X}_{[-\frac{1}{2q},0]}$ illustrate the dual representations of $\mathbb{Z}_2$.

(a)




(b)

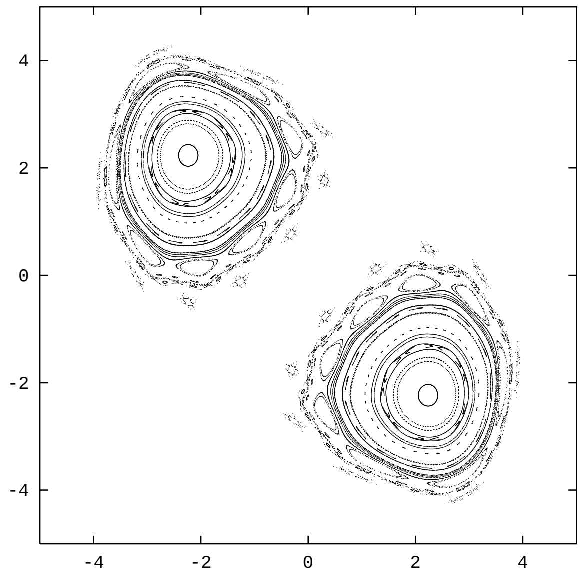



Figure 3: Phase portraits of the area-preserving orientation-reversing Hénon map (a) $f_0$ (4.4) and (b) $f_{-\frac{1}{4}} := \mathcal{X}_{[0,-\frac{1}{4}]} \circ f_0 \circ \mathcal{X}_{[-\frac{1}{4},0]}$ with $a = 0.2$. In the phase portrait (a) of $f_0$ the reflection in the line $x = -y$ ($(x, y) \mapsto (-y, -x)$) is a linear time-reversal symmetry, whereas in the phase portrait (b) of $f_{-\frac{1}{4}}$ the twofold rotation around $(0, 0)$ ($(x, y) \mapsto (-x, -y)$) is a linear time-reversal symmetry.

## 5 Symmetric versus $k$-symmetric periodic orbits

In this section we will discuss the correspondence between symmetric periodic orbits of flows and periodic orbits of $k$-symmetric maps that are related to their return maps in the spirit of Theorem 3.1. In the flow $\mathcal{X}$ of an ODE (3.1), the symmetry properties of an orbit $\mathbf{x}(t)$ are specified by a subgroup of $\Gamma$ containing spatiotemporal symmetries $U \otimes \tau$:

**Definition 5.1** Let $\mathbf{x}(t)$ be an orbit of a (nonautonomous or autonomous) flow. Then this orbit is symmetric with respect to a spatiotemporal symmetry $U\otimes\tau$ when $U\otimes\tau(\mathbf{x}(t)) = \mathbf{x}(t)$ for all $t \in \mathbb{R}$. [5]

An orbit $\mathbf{x}(t)$ of the differential equation (3.1) is said to be periodic with period $p$ ($p > 0$) when $p$ is the smallest real number for which $\mathbf{x}(t) = \mathbf{x}(t+p)$ for all $t \in \mathbb{R}$. From the observation that $\mathbf{x}(t) = \mathbf{x}(t+p)$ for some value of $t$, the implication that $\mathbf{x}(t) = \mathbf{x}(t+p)$ for all $t \in \mathbb{R}$ follows if and only $\mathrm{Id}\otimes\sigma_p$ is a time-shift symmetry of the equations of motion. It is useful to focus on periodic orbits of this type, so that the periodicity of an orbit corresponds to a (time-shift) symmetry property of the equations of motion. Considering time-periodic nonautonomous equations with minimal pure time-shift symmetry $\mathrm{Id}\otimes\sigma_1$ (cf. [**H1**]), we will hence focus on measuring periods of orbits in integer numbers. These periodic flow-orbits correspond to periodic orbits of the time-one return map $\mathcal{X}_{[0,1]}$.

We will now examine how symmetric periodic orbits of a flow $\mathcal{X}$ with spatiotemporal symmetry group $\Gamma$ can be recognized and analyzed in terms of periodic orbits of the $(\Sigma_0, \phi_A)$ $k$-symmetric map $f := A \circ \mathcal{X}_{[0,\frac{1}{q}]}$, and vice versa.

The following Proposition summarizes a few connections between orbits of a $k$-symmetric map $f$ and the flow orbits they represent. A proof of this Proposition follows straightforwardly by application of the definitions, and is therefore omitted.

**Proposition 5.2** *Let $A \otimes \sigma_{-\frac{1}{q}}$ be a time-shift symmetry, $S \otimes 1$ a symmetry, and $R \otimes \rho_0$ a time-reversal symmetry of the flow $\mathcal{X}$ of an ordinary differential equation with minimal pure time-shift symmetry $\mathrm{Id} \otimes \sigma_1$. Let $f := A \circ \mathcal{X}_{[0,\frac{1}{q}]}$, and $\mathbf{x}(t)$ be an orbit of the flow with $\mathbf{x}(0) =: \mathbf{x}_0$. Then,*

(i) $f \circ \mathcal{X}_{[0,1]} = \mathcal{X}_{[0,1]} \circ f$.

(ii) $f^p(\mathbf{x}_0) = \mathbf{x}_0 \quad \Leftrightarrow \quad (A^p \otimes \sigma_{-\frac{p}{q}})\mathbf{x}(t) = \mathbf{x}(t) \quad \forall t \in \mathbb{R}$.

(iii) $(f^n \circ S)\mathbf{x}_0 = \mathbf{x}_0 \quad \Leftrightarrow \quad ((A^n \circ S) \otimes \sigma_{-\frac{n}{q}})\mathbf{x}(t) = \mathbf{x}(t)\ \forall\ t \in \mathbb{R}$,
$(f^n \circ R)\mathbf{x}_0 = \mathbf{x}_0 \quad \Leftrightarrow \quad ((A^n \circ R) \otimes \rho_{-\frac{n}{q}})\mathbf{x}(t) = \mathbf{x}(t)\ \forall\ t \in \mathbb{R}$.

**Remark 5.3** Let us discuss the interpretation and some of the consequences of the properties listed in Proposition 5.2. Property (i) states that $f$ is a symmetry of the time-one map $\mathcal{X}_{[0,1]}$, and is a manifestation of the time-shift symmetry $A \otimes \sigma_{-\frac{1}{q}}$. In fact, the $f$-orbit $o(\mathbf{x}_0) := \{f^n(\mathbf{x}) \mid n \in \mathbb{Z}\}$ is related to the $\langle A \otimes \sigma_{-\frac{1}{q}} \rangle$ group orbit of the periodic orbit $\mathbf{x}(t)$ of the flow.[6] Namely, let $\mathbf{y}_n(0) := f^n(\mathbf{x}_0)$ (for some $n \in \mathbb{Z}$) then

$$\mathbf{y}_n(t) := \mathcal{X}_{[0,t]}\mathbf{y}_n(0) = (A^n \otimes \sigma_{-\frac{n}{q}})\mathbf{x}(t),\ \forall\ t \in \mathbb{R}. \tag{5.1}$$

[5] Recall that $(U \otimes \tau)\mathbf{x}(t) := U(\mathbf{x}(\tau^{-1}(t)))$.

[6] $\langle a_1, \dots, a_n \rangle$ denotes the group generated by the generators $a_1, \dots, a_n$.

Property (ii) states that the period $p$ of the $f$-orbit $o(\mathbf{x}_0)$ (i.e. the smallest positive integer for which $f^p(\mathbf{x}_0) = \mathbf{x}_0$) is the smallest positive integer for which $A^p \otimes \sigma_{-\frac{p}{q}}$ is a time-shift symmetry of $\mathbf{x}(t)$. The number $p$ also equals the number of different periodic orbits in the $\langle A \otimes \sigma_{-\frac{1}{q}} \rangle$ group orbit of $\mathbf{x}(t)$.

Property (iii) identifies spatiotemporal symmetry properties of flow orbits as $k$-symmetry properties of the $f$-orbits. We will discuss this relationship in more detail below.

The (integer) period $P$ of a flow orbit $\mathbf{x}(t)$ can be derived by recognizing the periodicity as a time-shift symmetry property. In describing time-shift symmetries of a periodic flow-orbit $\mathbf{x}(t)$ with $\mathbf{x}(0) =: \mathbf{x}_0$ it is of interest to find the smallest positive integer $n$ for which $(f^n \circ S)\mathbf{x}_0 = \mathbf{x}_0$ for some $k$-symmetry $S$ of $f$. Namely, by Proposition 5.2 (iii) this implies that $(A^n \circ S) \otimes \sigma_{-\frac{n}{q}}$ is a "minimal" time-shift symmetry of the periodic orbit (i.e. a time-shift symmetry with minimal time-shift component).

Now, let $l$ be the smallest positive integer multiple of $\frac{q}{\gcd(n,q)}$ such that $((A^n \circ S)^l \otimes 1)\mathbf{x}(t) = \mathbf{x}(t)$ for all $t \in \mathbb{R}$. Then, the period $P$ of $\mathbf{x}(t)$ is given by

$$P = l \cdot \frac{n}{q}. \tag{5.2}$$

Note that $P$ is always an integer multiple of $\frac{n}{\gcd(n,q)}$. In comparison, recall that the period $p$ of the $f$-orbit $o(\mathbf{x}_0)$ is the smallest positive integer for which $A^p \otimes \sigma_{-\frac{p}{q}}$ is a time-shift symmetry of $\mathbf{x}(t)$.

Periodic orbits of $k$-symmetric maps were previously studied by Brands *et al.* [3]. In that paper the attention was focussed on *$k$-symmetric* periodic orbits:

**Definition 5.4** [20] Let $U$ be a (reversing) $k$-symmetry of a map $f$. An orbit $o(\mathbf{x})$ of $f$ is *$k$-symmetric* with respect to $U$ if $(f^n \circ U)\mathbf{y} = \mathbf{y}$ for some $\mathbf{y} \in o(\mathbf{x})$ and some $n \in \mathbb{Z}$.

From Proposition 5.2 (iii) it is clear that there is an intimate relationship between $k$-symmetric orbits of $k$-symmetric maps and space-time symmetric orbits of the flows they represent. In the remainder of this section we will discuss this relationship in more detail in relation to the characterization of periodic orbits possessing reversing $k$-symmetries.

Our first result characterizes periodic orbits that are symmetric with respect to a reversing symmetry.

**Theorem 5.5** *Let $\mathbf{x}(t)$ be an orbit of a time-one periodic flow with spatiotemporal symmetry group $\Gamma$ containing at least one reversing symmetry fixing the surface of section $\mathcal{S}_0$. Let $A \otimes \sigma_{-\frac{1}{q}}$ be a time-shift symmetry of the flow and let $p$ be the smallest positive integer such that $A^p \otimes \sigma_{-\frac{p}{q}}$ is a time-shift symmetry of the orbit $\mathbf{x}(t)$.*

*Suppose $\mathbf{x}(t)$ is symmetric with respect to a reversing symmetry $(A^m \circ R) \otimes \rho_{-\frac{m}{q}}$ (for some $m \in \mathbb{Z}$). Then in the $\langle A \otimes \sigma_{-\frac{1}{q}} \rangle$ group orbit of $\mathbf{x}(t)$ there exists an orbit $\mathbf{y}(t)$ with a reversing symmetry inside the set:*

$$\begin{array}{ll} \{\phi_A^n(R) \otimes \rho_0 \mid n \in \{0, \ldots, \gcd(p,k) - 1\}\} & \textit{if } p \textit{ is odd,} \\ \{\phi_A^n(R) \otimes \rho_0,\ (A \circ \phi_A^n(R)) \otimes \rho_{-\frac{1}{q}} \mid n \in \{0, \ldots, \gcd(\frac{p}{2}, k) - 1\}\} & \textit{if } p \textit{ is even,} \end{array}$$

*where $k$ is the smallest positive integer for which*

$$\phi_A^k(R) := A^k \circ R \circ A^k = R. \tag{5.3}$$

**Proof.** Suppose that $\mathbf{x}(t)$ is symmetric with respect to the reversing symmetry $(A^m \circ R) \otimes \rho_{-\frac{m}{q}}$, i.e.

$$((A^m \circ R) \otimes \rho_{-\frac{m}{q}})\mathbf{x}(t) = \mathbf{x}(t), \ \forall\, t \in \mathbb{R}. \tag{5.4}$$

Then, the orbit $\mathbf{y}_l(t) := A^l(\mathbf{x}(t+\frac{l}{q}))$ is symmetric with respect to $(A^{l+m} \circ R \circ A^{-l}) \otimes \rho_{-\frac{2l+m}{q}}$:

$$((A^{2l+m} \circ \phi_A^{-l}(R)) \otimes \rho_{-\frac{2l+m}{q}})\mathbf{y}_l(t) = \mathbf{y}_l(t), \ \forall\, t \in \mathbb{R}. \tag{5.5}$$

Now we use the fact that $A^p \otimes \sigma_{-\frac{p}{q}}$ is a time-shift symmetry of $\mathbf{x}(t)$ and also of $\mathbf{y}_l(t)$ for all $l \in \mathbb{Z}$, and that $\phi_A^{nk-l}(R) = \phi_A^{-l}(R)$ for all $n \in \mathbb{Z}$. We address the question whether there is an appropriate choice of $l$ such that $\mathbf{y}_l(t)$ has a reversing symmetry $\phi_A^{-l}(R) \otimes \rho_0$ fixing the surface of section $\mathcal{S}_0$ setwise. From (5.5) we obtain that such an $l$ must satisfy the equation

$$2l + m = 0 \mod \gcd(p, 2k). \tag{5.6}$$

The equation $2l + m = 0$ is to be evaluated modulo $p$ because $A^p \otimes \sigma_{-\frac{p}{q}}$ is a time-shift symmetry so that $(A^p \otimes \sigma_{-\frac{p}{q}})\mathbf{y}_l(t) = \mathbf{y}_l(t)$, and modulo $2k$ because $l$ may be considered modulo $k$ since $\phi_A^k(R) = R$.

In case $p$ is odd, then (5.6) can always be solved: if $m$ is even then $l = -\frac{m}{2} \mod \gcd(p, k)$, and if $m$ is odd we obtain $l = -\frac{m+\gcd(p,k)}{2} \mod \gcd(p, k)$.

In case $p$ is even and $m$ is also even, (5.6) has solutions $l = -\frac{m}{2} \mod \gcd(\frac{p}{2}, k)$. However, if $m$ is odd and $p$ is even there is no integer $l$ that solves (5.6). Alternatively, we can identify an orbit with a reversing symmetry of the form $(A \circ \phi_A^{-l}(R)) \otimes \rho_{-\frac{1}{q}}$, as there are solutions to

$$2l + m = 1 \mod \gcd(p, 2k). \tag{5.7}$$

Namely, $l = -\frac{m-1}{2} \mod \gcd(\frac{p}{2}, k)$. (It should be noted that orbits $\mathbf{y}_l(t)$ with a reversing symmetry of the form $(A \circ \phi_A^{-l}(R)) \otimes \rho_{-\frac{1}{q}}$ can be identified as well in case $p$ is odd. Only when $p$ and $m$ are both even, there is no orbit which such a reversing symmetry.) $\square$

We compare the content of Theorem 5.5 to the results obtained in [3], concerning a characterization of periodic orbits that are $k$-symmetric with respect to a reversing $k$-symmetry. The results in [3] were formulated from the point of view of $k$-branches $o^{(k)}(\mathbf{y}) := \{f^{kn}(\mathbf{y}) \mid n \in \mathbb{Z}\}$, with $\mathbf{y} \in o(\mathbf{x}_0)$, but Theorem 5.5 motivates a simpler formulation:

**Theorem 5.6** *Let $f$ be a map with reversing $k$-symmetry $R$ and a periodic orbit $o(\mathbf{x}_0)$ with period $p$. Then, $o(\mathbf{x}_0)$ is $k$-symmetric with respect to $R$ if and only if there exists a point $\mathbf{x}_i \in o(\mathbf{x}_0)$ such that*

$$\mathbf{x}_i \in \mathrm{Fix}(R') \cap \mathrm{Fix}(f^p \circ R')$$

*for some reversing $k$-symmetry*

$$\begin{array}{ll} R' \in \{\phi_f^n(R) \mid n \in \{0, \dots, \gcd(p, k) - 1\}\} & \textit{if } p \textit{ is odd,} \\ R' \in \{\phi_f^n(R),\ f \circ \phi_f^n(R) \mid n \in \{0, \dots, \gcd(\frac{p}{2}, k) - 1\}\} & \textit{if } p \textit{ is even.} \end{array}$$

**Proof.** We will make a comparison with the result obtained in [3] and show that Theorem 5.6 is indeed in correspondence with Theorem 3.2 of [3]. We briefly recall the content of the latter theorem:

**Theorem 5.7 ([3, Theorem 3.2])** *Let $f$, $p$, $R$, and $o(\mathbf{x}_0)$ be as in Theorem 5.6. Then, $o(\mathbf{x}_0)$ is $k$-symmetric with respect to $R$ if and only if there exists a point $\mathbf{x}_i \in o(\mathbf{x}_0)$ such that*

$$\begin{array}{ll} \mathbf{x}_i \in \mathrm{Fix}(R') \cap \mathrm{Fix}(f^p \circ R'), & \textit{if } p/\gcd(p,k) \textit{ is odd,} \\ \mathbf{x}_i \in \mathrm{Fix}(R') \cap \mathrm{Fix}(f^p \circ R') \text{ or } \mathbf{x}_i \in \mathrm{Fix}(f^k \circ R') \cap \mathrm{Fix}(f^{k+p} \circ R'), & \textit{if } p/\gcd(p,k) \textit{ is even,} \end{array}$$

*for some reversing $k$-symmetry*

$$\begin{array}{ll} R' \in \{\phi_f^n(R) \mid n \in \{0, \dots, \gcd(p,k)-1\}\} & \textit{if } \gcd(p,k) \textit{ is odd,} \\ R' \in \{\phi_f^n(R),\ f \circ \phi_f^n(R') \mid n \in \{0, \dots, \frac{\gcd(p,k)}{2}-1\}\} & \textit{if } \gcd(p,k) \textit{ is even.} \end{array}$$

We first discuss the situation when $p$ is odd. Then, $p/\gcd(p,k)$ and $\gcd(p,k)$ are odd as well, so that one recaptures directly the result of Theorem 5.6.

In case $p$ is even, we first note that

$$\gcd(\frac{p}{2}, k) = \begin{cases} \gcd(p,k)/2 & \text{if } p/\gcd(p,k) \text{ is odd,} \\ \gcd(p,k) & \text{if } p/\gcd(p,k) \text{ is even.} \end{cases} \tag{5.8}$$

Then, in case $p/\gcd(p,k)$ is odd we find that $\gcd(p,k)$ must be even since $p$ is even, and one obtains from Theorem 5.7 directly the result of Theorem 5.6.

In case $p$ and $p/\gcd(p,k)$ are both even there are two subcases to be considered corresponding to $k$ being either odd or even. (Namely, when $p$ is even $\gcd(p,k)$ is odd if and only if $k$ is odd.) In case $k$ is odd, we find that $\mathrm{Fix}(f^k \circ R') = f^{(k-1)/2}\mathrm{Fix}(f \circ \phi_f^{(k-1)/2}(R'))$. Now we note that if $o(\mathbf{x}_0)$ is $k$-symmetric with respect to $R'$, and has a point in $\mathrm{Fix}(R')$ or $\mathrm{Fix}(f \circ R')$, then it also $k$-symmetric with respect to $\phi_f^n(R')$ for all $n = 0 \mod \gcd(p,k)$ and hence has a point in $\mathrm{Fix}(\phi_f^n(R'))$ or $\mathrm{Fix}(f \circ \phi_f^n(R'))$, cf. [3, Theorem 2.3]. As $(k-1) \mod \gcd(p,k) \in \{0, \dots, \gcd(p,k)-1\}$ we thus find that the orbit has a point in $\mathrm{Fix}(R')$ or $\mathrm{Fix}(f \circ R')$ with $R' \in \{\phi_f^n(R),\ f \circ \phi_f^n(R) \mid n \in \{0, \dots, \gcd(p,k)-1\}\}$ as stated in the Theorem.

In case $p$, $p/\gcd(p,k)$ and $\gcd(p,k)$ are all even, it remains to be shown that the fixed sets $\mathrm{Fix}(R')$ with $R' \in \{\phi_f^n(R),\ y \circ \phi_f^n(R) \mid n \in \{0, \dots, \gcd(p,k)-1\}\}$ and $Fix(R') \cup \mathrm{Fix}(f^k \circ R')$ with $R' \in \{\phi_f^n(R),\ y \circ \phi_f^n(R) \mid n \in \{0, \dots, \gcd(p,k)/2-1\}\}$ are identical. To show this, we note that when $\gcd(p,k)$ is even $k$ must be even, and $\mathrm{Fix}(f^k \circ R') = f^{k/2}\mathrm{Fix}(\phi_{k/2}(R'))$. We assert that $k$ is an odd multiple of $\gcd(p,k)$ and hence that $k/2 \neq 0 \mod \gcd(p,k)$. If the assertion is true, it follows indeed that the above sets are identical as in that case $\{\phi_f^{k/2+n}(R) \mid n \in \{0, \dots, \gcd(p,k)/2-1\}\} = \{\phi_f^n(R) \mid n \in \{\gcd(p,k)/2, \dots, \gcd(p,k)-1\}\}$. To prove the assertion, we note that $p/\gcd(p,k)$ is even, from which it immediately follows that $k/\gcd(p,k)$ must be odd.

The proof is concluded by observing that a periodic orbit of a map $f$ with a point inside the fixed set $\mathrm{Fix}(U)$, is of period $p$ if and only if this point is also inside $\mathrm{Fix}(U) \cap \mathrm{Fix}(f^p \circ U)$. □

**Remark 5.8** The formulation of Theorem 5.6 improves the formulation in Theorem 5.7 ([3, Theorem 3.2]). This is of practical relevance, as these results underly the Fixed Set Iteration (FSI) method for finding $k$-symmetric periodic orbits: $k$-symmetric periodic orbits can be found as intersections of iterates of fixed sets, exploiting the fact that $\mathrm{Fix}(f^p \circ U) = f^{p/2}\mathrm{Fix}(\phi_f^{p/2}(U))$ if $p$ is even, and $\mathrm{Fix}(f^p \circ U) = f^{(p-1)/2}\mathrm{Fix}(f \circ \phi_f^{(p-1)/2}(U))$ if $p$ is odd, cf. [3] (and references therein) for illustrations.

# 6 From $k$-symmetric maps to space-time symmetric flows

In the previous sections we have mainly focussed on the construction of $k$-symmetric maps from return maps of space-time symmetric flows. It is however also of interest to raise the question whether we can relate every $k$-symmetric map to a space-time symmetric flow. Interestingly, we will state in Theorem 6.2 that locally around a $k$-symmetric fixed point $\mathbf{x}_0$ of a $k$-symmetric diffeomorphism $f$ (including the case $k=1$) the diffeomorphism is always related to the return map of a space-time symmetric flow with the fixed point $\mathbf{x}_0$ representing a space-time symmetric periodic orbit. In Section 7 we will discuss how the latter result underlies the occurrence of *formal* symmetry properties of (Birkhoff) normal forms and can be applied in the analysis of local bifurcations.

Examining the construction of $k$-symmetric maps from space-time symmetric flows, one finds that the reverse procedure can be applied only when $f$ satisfies some additional conditions.

**Theorem 6.1 (Global constructions)** *Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a $(\Sigma, \phi_A)$ $k$-symmetric diffeomorphism with $A \in O(n)$ and $\Sigma \leq O(n)$, such that $\langle A, \tilde{\Sigma}\rangle/\tilde{\Sigma} \simeq \mathbb{Z}_q$ and $\langle A, \Sigma\rangle/\tilde{\Sigma} \simeq \mathbb{D}_q$ in case $\Sigma \neq \tilde{\Sigma}$ (for some value or $q \in \mathbb{Z}^+$). Then, there exists a flow $\mathcal{X}$ of a time-one periodic ordinary differential equation in $\mathbb{R}^n$ with spatiotemporal symmetry group $\Gamma$ such that $f := A \circ \mathcal{X}_{[0,\frac{1}{q}]}$, $A \otimes \sigma_{-\frac{1}{q}} \in \Gamma$ and $S \otimes 1 \in \tilde{\Gamma}$ for all $S \in \tilde{\Sigma}$, if and only if $g := A^{-1} \circ f$ is isotopic to the identity transformation.*

*The flow $\mathcal{X}$ can be constructed to have reversing symmetries $R \otimes \rho_0$ for all $R \in \Sigma - \tilde{\Sigma}$ if and only if $g$ can be further decomposed as $g := (R \circ A) \circ h \circ (R \circ A)^{-1} \circ h^{-1}$, where $h : \mathbb{R}^n \mapsto \mathbb{R}^n$ is isotopic to the identity transformation.*

**Proof.** Let $g : \mathbb{R}^n \mapsto \mathbb{R}^n$. If $g$ is isotopic to the identity, then $g$ is the time-one return map of the flow $\hat{\mathcal{X}}$ of a (time-one periodic) vector field in $\mathbb{R}^n$. As $g$ is $\tilde{\Sigma}$-equivariant, the flow $\hat{\mathcal{X}}$ can be constructed in such a way that whenever $S \circ g = g \circ S$ for some $S \in \tilde{\Sigma}$, then $S \circ \hat{\mathcal{X}}_{[t_1,t_2]} = \hat{\mathcal{X}}_{[t_1,t_2]} \circ S$ for all $t_1, t_2 \in [0, 1\rangle$.

From the fact that $f$ is $(\Sigma, \phi_A)$ $k$-symmetric it is easily verified that whenever $R \in \Sigma_0 - \tilde{\Sigma}$, then $R \circ A$ is a reversing symmetry of $g$. This reversing symmetry of $g$ can be respected in the construction of the flow $\hat{\mathcal{X}}$ if and only if $g := (R \circ A) \circ h \circ (R \circ A)^{-1} \circ h^{-1}$ for some $h$ that is isotopic to the identity transformation (with $h$ representing $\hat{\mathcal{X}}_{[\frac{1}{2},0]}$), cf. (3.31) in Remark 3.4.

We define the map

$$F := A^{-q} \circ f^q = (A^{1-q} \circ g \circ A^{q-1}) \circ (A^{2-q} \circ g \circ A^{q-2}) \circ \ldots \circ (A^{-1} \circ g \circ A) \circ g, \qquad (6.1)$$

noting that $A^q \in \tilde{\Sigma}$, and construct a time-one periodic flow $\mathcal{X}$ for $F$ using the flow $\hat{\mathcal{X}}$ we previously constructed for $g$ ($g$ now being the first hit map between the surfaces of section at $t = 0$ and $t = \frac{1}{q}$). Let $g := \mathcal{X}_{[0,\frac{1}{q}]}$, then by construction $F = \mathcal{X}_{[0,1]}$ is the time-one map of a flow $\mathcal{X}$ that automatically has a time-shift symmetry $A \otimes \sigma_{-\frac{1}{q}}$. The symmetries and reversing symmetries of $g$, together with the time-shift symmetry obtained in the construction above, form the spatiotemporal symmetry group of the flow $\mathcal{X}$. $\square$

It should be noted that the conditions listed in Theorem 6.1 explicitly require that $g$ and $h$ are isotopic to the identity transformation. As a result, not every $k$-symmetric map is directly related to the return map of a space-time symmetric flow in $\mathbb{R}^n \times S^1$. In this respect it should be noted that $f$ can always be constructed to be the return map of a flow, also if $g$ is not isotopic to the identity. However, in that case $g$ is not the return map of a flow in $\mathbb{R}^n \times S^1$ but only of a flow in an $(n+1)$-dimensional space with a more complicated topology (cf. the discussion in [25, §5.8.1]). It would be interesting to further explore the existence of these topologically nontrivial suspension flows, but this is beyond the scope of the present paper.

In a local context, it turns out that with a suitable choice of $A$ the conditions set out in Theorem 6.1 can always be satisfied.

**Theorem 6.2 (Local constructions)** *Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a diffeomorphism with reversing $k$-symmetry group $\Sigma \leq O(n)$ and fixed point $\mathbf{x}_0$ that is invariant under $\Sigma$. Let $g := L_s^{-1} \circ f$, with $L_s$ being the semisimple part of $df|_{\mathbf{x}_0}$.*[7] *Then, locally in a neighbourhood of the fixed point $\mathbf{x}_0$, $g = \mathcal{X}_{[0,\frac{1}{q}]}$ is the evolution operator of a flow $\mathcal{X}$ of a time-one periodic vector field in $\mathbb{R}^n$, with spatiotemporal symmetry group $\Gamma$, where $S \otimes 1 \in \Gamma$ for all $S \in \tilde{\Sigma}$, $R \otimes \rho_0 \in \Gamma$ for all $R \in \Sigma - \tilde{\Sigma}$, and $L_s \otimes \sigma_{-\frac{1}{q}} \in \Gamma$. The fixed point $\mathbf{x}_0$ represents a $\Gamma$-symmetric periodic orbit $\mathbf{x}(t)$ of the flow $\mathcal{X}$, with $\mathbf{x}(0) = \mathbf{x}_0$.*

**Proof.** The proof consists of checking that the conditions of Proposition 6.1 are satisfied locally. The proof partly relies on results that were obtained previously in [17].

First of all we note that when $f$ is $(\Sigma, \phi_f)$ $k$-symmetric, it is also $(\Sigma, \phi_{L_s})$ $k$-symmetric [17, Lemma 2.2], and hence that $\Sigma' := \tilde{\Sigma} \cup (R \circ L_s)\tilde{\Sigma}$ is a reversing symmetry group of $g := L_s^{-1} \circ f$. We then verify that $g$ is isotopic to the identity transformation. Namely, $dg|_{\mathbf{x}_0} = \exp(J)$ where $J$ is a nilpotent linear transformation [29]. We are left to verify that $(R \circ L_s)^{-1} \circ g$ is locally smoothly conjugate to $(R \circ L_s)^{-1}$. Since $dg|_{\mathbf{x}_0} = \exp(J)$ and $g$ has $R \circ L_s$ as a reversing symmetry, it follows that [17]

$$(R \circ L_s) \circ \exp(J) \circ (R \circ L_s)^{-1} = \exp(-J) \quad \Leftrightarrow \quad (R \circ L_s) \circ J \circ (R \circ L_s)^{-1} = -J, \tag{6.2}$$

which implies that

$$(R \circ L_s)^{-1} \circ (dg|_{\mathbf{x}_0}) = \exp(-J/2) \circ (R \circ L_s)^{-1} \circ \exp(J/2). \tag{6.3}$$

Then, noting that the reversibility property implies that [13]

$$((R \circ L_s)^{-1} \circ g)^2 = (R \circ L_s)^{-2}, \tag{6.4}$$

so that $\langle (R \circ L_s)^{-1} \rangle$ is isomorphic to $\langle (R \circ L_s)^{-1} \circ g \rangle$, it follows that indeed locally (in a neighbourhood of $\mathbf{x}_0$), $(R \circ L_s) \circ g$ is smoothly conjugate to $(R \circ L_s)$, i.e. locally there exists a diffeomorphism $h$ such that

$$(R \circ L_s)^{-1} \circ g = h \circ (R \circ L_s)^{-1} \circ h^{-1}. \tag{6.5}$$

[7] Let $L := df|_{\mathbf{x}_0}$ denote the (Frechet) derivative of $f$ in $\mathbf{x}_0$, then $L$ can be splitted uniquely in its semisimple and nilpotent part, $L := L_s + L_n$, such that $L_n$ and $L_s$ commute. Recall that a linear operator $L$ is semisimple if it is $\mathbb{C}$-diagonizable, and nilpotent if $L^n = 0$ for some $n \in \mathbb{Z}^+$.

Moreover, as $dh|_{\mathbf{x}_0} = \exp(-J/2)$, $h$ is indeed isotopic to the identity transformation. $\square$

## 7 Formal normal form symmetry and local bifurcations

In this section we will discuss the relationship between the local constructions described in Theorem 6.2 and the phenomenon of *formal* normal form symmetry. We will also discuss the consequences of this relationship for local bifurcation theory.

The (Birkhoff) normal form of a map $f$ at a fixed point $\mathbf{x}_0$ is obtained by bringing the formal Taylor expansion of $f$ at $\mathbf{x}_0$ into a *simple* form by choosing a convenient set of local coordinates. The main idea behind normal forms is to study aspects of the local dynamics around $\mathbf{x}_0$ by means of properties of the (truncated) Taylor series expansion of the map around $\mathbf{x}_0$. However, one should be careful in using this approach as formal Taylor expansions of $f$ at $\mathbf{x}_0$ may display dynamical properties that are different from the local dynamics of $f$ around $\mathbf{x}_0$.

By an appropriate choice of coordinates, normal forms can always be chosen to be equivariant with respect to the semisimple part of the linear part of the map at the fixed point:

**Theorem 7.1 (Normal form [17])** *Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a diffeomorphism with reversing k-symmetry group $\Sigma \leq O(n)$. Let $f$ be such that it has a fixed point $\mathbf{x}_0$ that is invariant under $\Sigma$. Then, in case $f$ is at least $C^m$, there exists a $\Sigma$-equivariant polynomial coordinate transformation $T$ of degree $m$ such that*

$$T^{-1} \circ f \circ T \qquad \text{and} \qquad L_s \circ \mathcal{N}_1$$

*have the same Taylor expansion up to order $m$ at $\mathbf{x}_0$, where $\mathcal{N}_1$ denotes the time-one map of an autonomous flow $\mathcal{N}$ in $\mathbb{R}^n$ that is $L_s$-equivariant (with $L_s$ being the semisimple part of $df|_{\mathbf{x}_0}$) and moreover has $\Sigma$ as a reversing symmetry group.*

The derivation of Theorem 7.1 follows an approach originally due to Takens [29]. The $L_s$-equivariance of $\mathcal{N}_1$ is referred to as the *formal* symmetry of the normal form. Namely, it is a symmetry property of the formal Taylor expansion (up to any desired order in the expansion), but does in general not correspond to a true symmetry property of the local dynamics around the fixed point, cf. [4].

Theorem 6.2 provides a dynamical interpretation of the origin of the formal normal form symmetry:

**Corollary 7.2 (Local normal form flow constructions)** *Let $f$ be a diffeomorphism as described in Theorem 7.1, such that $\langle \tilde{\Sigma}, L_s \rangle / \tilde{\Sigma} \simeq \mathbb{Z}_q$ for some $q \in \mathbb{Z}^+$. Then, there exists a neighbourhood $U$ of the fixed point $\mathbf{x}_0$ for which $f_{loc} := f|_U$ can be written as*

$$f_{loc} := L_s \circ \mathcal{X}_{[0,\frac{1}{q}]}, \tag{7.1}$$

*where $\mathcal{X}$ is a flow in $\mathbb{R}^n \times S^1$ with spatiotemporal symmetry group $\Gamma$ containing the symmetries $S \otimes 1$ for all $S \in \tilde{\Sigma}$, reversing symmetries $R \otimes \rho_0$ for all $R \in \Sigma - \tilde{\Sigma}$, and time-shift symmetry $L_s \otimes \sigma_{-\frac{1}{q}}$.*

Importantly, in the linear unfolding of $df|_{\mathbf{x}_0}$ without loss of generality we may consider small perturbations entirely in the nilpotent part while respecting its $L_s$-equivariance (cf. e.g. [29]). Hence, the local flow construction remains valid throughout the study of local bifurcations (arising from linear unfoldings of $df|_{\mathbf{x}_0}$). Hence, in this context we find that the bifurcation theory for a fixed point $\mathbf{x}_0$ that is invariant under the action of a reversing symmetry group $\Sigma$ of a map $f$ with linear part $df|_{\mathbf{x}_0}$, is equivalent to the local bifurcation theory of a $\Gamma$-symmetric periodic orbit of a flow with spatiotemporal symmetry group $\Gamma$ that is generated by symmetries and reversing symmetries in $\Sigma$ as well as a time-shift symmetry $L_s \otimes \sigma_{-\frac{1}{q}}$.

From the flow $\mathcal{X}$ constructed in Corollary 7.2 the normal form flow $\mathcal{N}$ described in Theorem 7.1 can be obtained by performing coordinate transformations that remove all time-dependent terms in the Taylor expansion of $\mathcal{X}_{[0,\frac{1}{q}]}$ to arbitrary high order, or by the procedure of averaging. In that way, the time-shift symmetry $L_s \otimes \sigma_{-\frac{1}{q}}$ of $\mathcal{X}$ automatically leads to a symmetry $L_s$ of the normal form flow $\mathcal{N}$. In this respect it is interesting to note $R$ becomes a reversing symmetry of the normal form flow because of this $L_s$-equivariance as $\mathcal{X}_{[0,\frac{1}{q}]}$ has reversing symmetry $R \circ L_s$ but is in general not $L_s$-equivariant).

Importantly, the equivalence between the bifurcation problem of the fixed point $\mathbf{x}_0$ of the map $f$ and the space-time symmetric periodic orbit of a flow, establishes a geometrical argument for the occurrence of the *formal* symmetry $L_s$ as a symmetry property of the bifurcation equations.

**Remark 7.3** We want to emphasize that the result described in Corollary 7.2 has been obtained in the context of a tradition of approaches towards a geometric interpretation of formal normal form symmetry, obtained in various case-studies of normal form and local bifurcation theory. In particular we mention the discussions in Arnol'd *et al.* [1] and Broer & Vegter [5], in which the derivations of normal forms for periodic orbits with *resonant* Floquet multipliers have been discussed using intermediate flows admitting time-shift symmetries. In case the return map of a period orbit has resonant Floquet multipliers on the unit circle this time-shift symmetric vector field can be obtained by an explicit transformation to a rotating coordinate frame (*Seifert* foliation). A single multiplier equal to $-1$ naturally leads to a flow on a Möbius band (Möbius foliation). Normal forms (flows of autonomous vector fields) are from there obtained by averaging or by coordinate transformations that move the time-dependent terms to arbitrarily high order in the Taylor expansion. Corollary 7.2 presented above contains a systematic extension of these results into the context of equivariant, reversible, and $k$-symmetric diffeomorphisms, and provides an alternative route towards proving Theorem 7.1.

Finally, we would like to discuss how local steady-state bifurcations in the normal form flow $\mathcal{N}$ of the map $f$ relate to local bifurcations of the space-time symmetric (nonautonomous) flow $\mathcal{X}$ from which the map $f := A \circ \mathcal{X}_{[0,\frac{1}{q}]}$ is derived.

**Proposition 7.4 (Symmetry interpretation)** *Let $\mathbf{y}_0$ be a steady-state solution of the $L_s$-equivariant normal form flow $\mathcal{N}$ with reversing symmetry group $\Sigma_0$, as obtained in Theorem 7.1 from a $k$-symmetric diffeomorphism $f := A \circ \mathcal{X}_{[0,\frac{1}{q}]}$ that is derived from a time-one periodic flow $\mathcal{X}$ with spatiotemporal symmetry group $\Gamma$. If $\mathbf{y}_0$ is invariant under the symmetry $L_s^n \circ S$ or reversing symmetry $L_s^n \circ R$ (for some $n \in \mathbb{Z}$ and $S \in \tilde{\Sigma}$ or $R \in \Sigma_0 - \tilde{\Sigma}$),* *then the steady state $\mathbf{y}_0$ represents a periodic solution $\mathbf{y}(t)$ of the flow $\mathcal{X}$ with a (time-shift) symmetry $(A^n \circ S) \otimes \sigma_{-\frac{n}{q}}$ or reversing symmetry $(A^n \circ R) \otimes \rho_{-\frac{n}{q}}$, and $\mathbf{y}(0) = \mathbf{y}_0$.*

**Proof.** Let $\mathbf{y}_0$ be a steady state of the normal form flow, i.e. $\mathcal{N}_1(\mathbf{y}_0) = \mathbf{y}_0$, and $U \in \Sigma_0$, then

$$(L_s^n \circ U)\mathbf{y}_0 = \mathbf{y}_0 \quad \Leftrightarrow \quad (L_s^n \circ \mathcal{N}_1^n \circ U)\mathbf{y}_0 = \mathbf{y}_0, \tag{7.2}$$

$$\Leftrightarrow \quad ((L_s \circ \mathcal{N}_1)^n \circ U)\mathbf{y}_0 = \mathbf{y}_0. \tag{7.3}$$

The normal form symmetry of the steady state in (7.3) represents, in analogy to Proposition 5.2, a periodic orbit of the original flow $\mathcal{X}$ with spatiotemporal symmetry $(A^n \circ U) \otimes \tau$ with $\tau = \sigma_{-\frac{n}{q}}$ if $U \in \tilde{\Sigma}$ and $\tau = \rho_{-\frac{n}{q}}$ if $U \in \Sigma_0 - \tilde{\Sigma}$.

It should be noted that $A$ need not be identical to $L_s$. □

In this way, the isotropy subgroups of bifurcating steady states in the normal form flow provide full information on the spatiotemporal symmetry properties of periodic solutions in the original flow that they represent.

General results on generic local bifurcations can be obtained, e.g. using the equivariant branching lemma (following the ideas set out in [9]). However, a full treatment along these lines is beyond the scope of the present paper. Instead, we will illustrate the procedure by a few examples.

**Example 7.5** We consider the return map $\mathcal{X}_{[0,1]} : \mathcal{S}_0 \mapsto \mathcal{S}_0$ of a time-one periodic flow on the plane $\mathbb{R}^2$ with spatiotemporal symmetry group $\Gamma = \langle M_x \otimes \rho_0, -\mathrm{Id} \otimes \sigma_{\frac{1}{2}} \rangle \simeq \mathbb{D}_2$ (with $M_x$ denoting the reflection in the $x$-axis and $-\mathrm{Id}$ denoting a rotation of $\pi$ around the origin $\mathbf{0}$) and a $\Gamma$-symmetric periodic orbit $\mathbf{x}(t)$ with period one and $\mathbf{x}(0) := \mathbf{x}_0 = \mathbf{0}$.

We write $\mathcal{X}_{[0,1]} = f^2$ with $f = A \circ \mathcal{X}_{[0,\frac{1}{2}]}$ with $A = -\mathrm{Id}$. The map $f$ has reversing symmetry group $\Sigma = \langle M_x \rangle$. This allows $L_s$ to be any rotation. We focus on the resonant rotations $L_s = R_{\frac{2\pi p}{q}}$ with $p, q \in \mathbb{Z}^+$ and $\gcd(p,q) = 1$. At these resonances, one finds normal form symmetry $\langle L_s, M_x \rangle = \mathbb{D}_q$, with generically bifurcating steady state solutions that are invariant under the action of a reversing reflection symmetry $R_{\frac{2\pi np}{q}} \circ M_x$ corresponding to periodic solutions of the flow with reversing symmetry $M_x \otimes \rho_{\frac{n}{2}}$ if $n$ is even and $(-\mathrm{Id} \circ M_x) \otimes \rho_{\frac{n}{2}}$ if $n$ is odd. The periods of these solutions are $\frac{q}{2}$ when $q$ is even: the steady states are invariant under $L_s^q$, implying a $\mathrm{Id} \otimes \sigma_{\frac{q}{2}}$ time-shift symmetry for the corresponding periodic orbits of the flow $\mathcal{X}$. In case $q$ is odd the solutions of $\mathcal{X}$ have period $q$: invariance of the steady states under $L_s^q$ implies that the corresponding periodic solutions of $\mathcal{X}$ have a time-shift symmetry $-\mathrm{Id} \otimes \sigma_{\frac{q}{2}}$ and hence have period $q$.

Hence we observe a clear distinction between resonances with $q$ even and $q$ odd: when $q$ is even the bifurcating periodic orbits have a spatiotemporal symmetry group isomorphic to $\mathbb{Z}_2$ (generated by a reversing symmetry), while in case $q$ is odd the bifurcating periodic orbits have a spatiotemporal symmetry group isomorphic to $\mathbb{D}_2$ (generated by a reversing symmetry and a time-shift symmetry).

From the Birkhoff normal form analysis one can derive the stability properties of the bifurcating solutions. We will only discuss the supercritical bifurcations starting from an elliptic solution (Floquet multipliers on the unit circle). Local phase portraits of the normal form flow before and after the local bifurcations are sketched for some representative values of $q$ in Fig. 4.

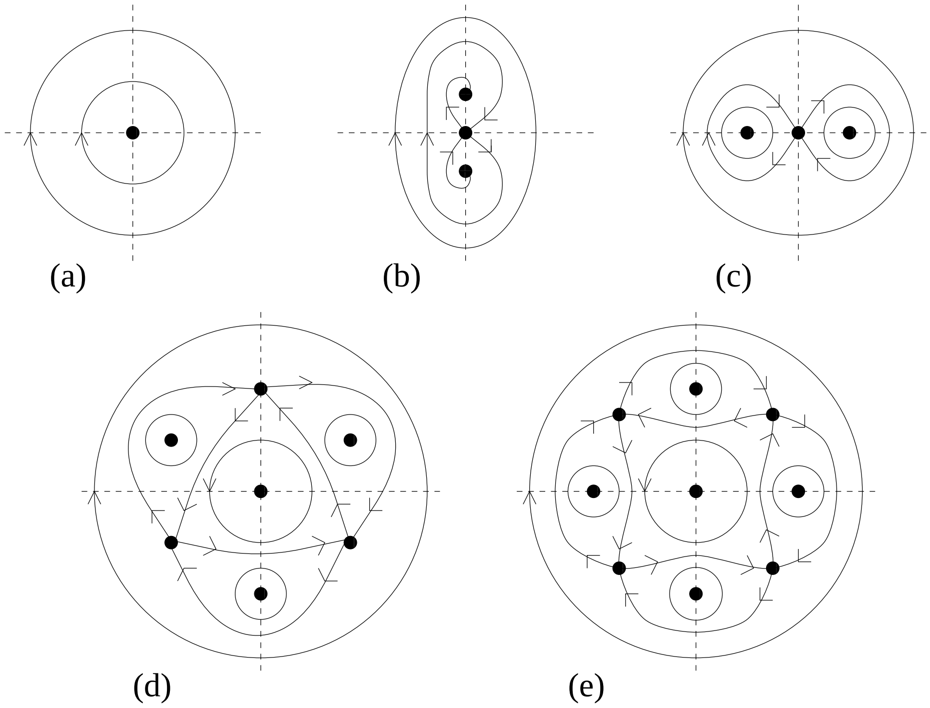



Figure 4: Local phase portraits of the normal form flow around local codimension-one supercritical bifurcations starting from an elliptic steady state (a). The sketches illustrate the local phase portraits after the bifurcation in the presence of reversing symmetry group $\Sigma = \mathbb{D}_q$ (with $\tilde{\Sigma} = \mathbb{Z}_q$) and $q = 1$ (b), $q = 2$ (c), $q = 3$ (d), and $q = 4$ (e).

When $q \geq 3$ the $\Gamma$-symmetric solution remains elliptic while one set of bifurcating periodic solutions is elliptic and the other set is of saddle type (positive real Floquet multipliers, one of modulus larger than one and one of modulus smaller than one). When $q = 2$, the elliptic point loses stability and becomes of saddle type, while the bifurcating periodic solutions are elliptic. When $q = 1$, one generically finds a symmetry breaking bifurcation with the elliptic solution turning into a solution of saddle type while an attractor (two real positive Floquet multipliers with modulus smaller than one) and a repeller (two real positive Floquet multipliers with modulus greater than one) are born.

We will now briefly review the examples presented in Section 1.

**Example 7.6 (Examples 1.1 and 1.2)** In Section 1 we already discussed two examples of how $k$-symmetric maps may be used to study local bifurcations of space-time symmetric flows. Example 1.1 has spatiotemporal symmetry group $\Gamma = \langle M_x \otimes 1, R_{\frac{\pi}{2}} \otimes \sigma_{\frac{1}{2}} \rangle \simeq \mathbb{D}_4$ and Example 1.2 has spatiotemporal symmetry group $\Gamma = \langle M_x \otimes \rho_0, R_{\frac{\pi}{2}} \otimes \rho_{\frac{1}{2}} \rangle \simeq \mathbb{D}_4$. Without loss of generality, in both examples the semi-simple parts $L_s$ of the linear part of the return map may be chosen to be equal to $A$.

# 8 Local return maps for symmetric periodic orbits of autonomous flows

Return maps of time-periodic flows have many properties in common with local return maps for periodic orbits of autonomous flows. Although we have stated our main results in the context of return maps of time-periodic flows, we would like to emphasize their validity and relevance also in the context of local return maps for symmetric periodic orbits of autonomous flows.

The dynamics in the neighbourhood of a periodic orbit of an autonomous flow is conveniently studied by means of a local (first) return map. Such a map $g$ is defined from a section $\mathcal{S}$ to itself, $g : \mathcal{S} \mapsto \mathcal{S}$, where $\mathcal{S}$ is a $(n-1)$-dimensional subspace of $\mathbb{R}^n$ intersecting the periodic orbit in $\mathbb{R}^n$ transversally at some point $\mathbf{x}_0$. If $\mathbf{x}_0$ is a point in which an orbit $\mathbf{x}(t)$ of the flow transversally intersects the section $\mathcal{S}$, then $g(\mathbf{x}_0)$ is defined to be the point in which this orbit intersects $\mathcal{S}$ the next time (i.e. if $\mathbf{x}_0 = \mathbf{x}(0)$, then $g(\mathbf{x}_0) = \mathbf{x}(t_1) \in \mathcal{S}$ such that $\mathbf{x}(t) \notin \mathcal{S}$ for all $t \in \langle 0, t_1 \rangle$). Obviously, such a return map is usually only locally defined. For a more extended introduction on return maps, see e.g. Wiggins [32].

We consider a return map for a single periodic orbit so that the periodic orbit arises as a fixed point of the return map. Without loss of generality we may assume the period of the orbit to be equal to 1, and we conveniently choose the time coordinate so that the point $\mathbf{x}_0$ on the periodic orbit in the surface of section corresponds to the state of the system at time $t = 0$.

We are particularly interested in return maps of *symmetric* periodic orbits, i.e. orbits $\mathbf{x}(t)$ that satisfy $U(\mathbf{x}(t)) = \mathbf{x}(\tau(t))$ for all $t \in \mathbb{R}$ for some $U \otimes \tau \in \Gamma$ (cf. Definition 5.1).

Consequently, every $U \otimes \tau$-symmetric periodic orbit is also setwise invariant under the action of the spatial part $U : \mathbb{R}^n \mapsto \mathbb{R}^n$. It should be noted that in the context of autonomous flows in $\mathbb{R}^n$ one commonly calls an orbit $\mathbf{x}(t)$ symmetric with respect to a (reversing) symmetry $U : \mathbb{R}^n \mapsto \mathbb{R}^n$ whenever $U$ fixes the orbit setwise. This is in implicit agreement with Definition 5.1, as in autonomous flows $\sigma_a : t \mapsto t + a$ is a time-shift symmetry for all $a \in \mathbb{R}$, so that whenever $U\mathbf{x}(0) = \mathbf{x}(a)$ for some $a \in \mathbb{R}$, indeed the periodic orbit is symmetric with respect to the spatiotemporal symmetry $U \otimes \sigma_a$ or $U \otimes \rho_a$. It should be noted, however, that in order to analyse the local dynamics around a symmetric periodic orbit accurately it is important to take notice of the temporal components of the spatiotemporal symmetries of periodic orbits, cf. e.g. also the treatises on bifurcation theory of symmetric periodic orbits of Golubitsky *et al.* [9] and Fiedler [7].

Let $\Gamma$ be the group of spatiotemporal symmetries of the periodic orbit, and let $\tilde{\Gamma}$ denote the subgroup of $\Gamma$ containing symmetries which fix the orbit pointwise. Then $\tilde{\Gamma}$ is a normal subgroup of $\Gamma$, and it is well known that $\Gamma/\tilde{\Gamma} \simeq \mathbb{D}_q$ (for some $q$) when $\Gamma$ contains a reversing symmetry and that $\Gamma/\tilde{\Gamma} \simeq \mathbb{Z}_q$ when it does not contain a reversing symmetry, cf. e.g. [19]. The symmetry elements that do not fix the orbit pointwise are the spatiotemporal symmetries with a nontrivial temporal component. One distinguishes time-shift symmetries $A \otimes \sigma_s$ and reversing symmetries $R \otimes \rho_s$. The group structure of $\Gamma$, with subgroups $\tilde{\Gamma}$, $\hat{\Gamma}$, and $\Gamma_s$ is identical to the description in Section 3, (3.15)-(3.21).

Considering the spatiotemporal symmetries of a periodic orbit with period 1, if the periodic orbit possesses a genuine time-shift symmetry, assuming that the temporal part of $\Gamma$ is finite, there always exists a time-shift symmetry with smallest time-shift $\frac{1}{q}$ for some $q \in \mathbb{Z}^+$.

We thus focus on the local return map of a periodic orbit with a time-shift symmetry $A \otimes \sigma_{-\frac{1}{q}}$. Making use of this time-shift symmetry one may, instead of studying the flow from the full return map of a surface of section $\mathcal{S}$ to itself, study the flow by means of a first-hit map $h$ from the surface of section $\mathcal{S}$ to the surface of section $A^{-1}(\mathcal{S})$,

$$h : \mathcal{S} \mapsto A^{-1}(\mathcal{S}). \tag{8.1}$$

Analogously, we may define $h^{(m)} := A^{-m} \circ h \circ A^m : A^{-m}(\mathcal{S}) \mapsto A^{-m-1}(\mathcal{S})$, such that

$$g = h^{(q-1)} \circ \ldots \circ h = A^{-q} \circ (A \circ h)^q, \tag{8.2}$$

is a return map for the surface of section $\mathcal{S}$. We now choose the surface of section $\mathcal{S}$ such that $\tilde{\Sigma}(\mathcal{S}) = \mathcal{S}$ (this can always be done). Then, in analogy to Theorem 3.1, the map $f : \mathcal{S} \to \mathcal{S}$ given by $f := A \circ h$ is $k$-symmetric with respect to $\bar{\Sigma}_0 := \Sigma_0|_{\mathcal{S}}$ with $\Sigma_0 := \{U \mid U \otimes \tau \in \Gamma,\ U(\mathcal{S}) = \mathcal{S}\}$, and $\phi_f = \phi_A$. Indeed, with $\mathcal{X}_{[0,1]}$ replaced by $g$, and $\mathcal{X}_{[0,\frac{1}{q}]}$ replaced by $h$, the results of Section 3 (in particular Theorem 3.1) equally apply. Importantly, again, the spatiotemporal symmetry properties of the flow induce no further structure on the first hit map than the $k$-symmetry properties described above. Hence, generic bifurcations of the periodic orbit are faithfully represented by generic local bifurcations of $k$-symmetric maps.

In analogy to the discussions in Section 4, one can identify an alternative choice for a surface of section $\mathcal{S}'$ that is not fixed setwise by the reversing symmetry $R$ (that fixes $\mathcal{S}$), but by the reversing symmetry $R \circ A$. If $\mathcal{S}$ contains the point of the periodic orbit at time $t = 0$, then $\mathcal{S}'$ must contain the point of the periodic orbit at time $t = \frac{1}{2q}$ and hence $\mathcal{S}'$ must lie between $\mathcal{S}$ and $A^{-1}(\mathcal{S})$ (that contains the point of the periodic orbit at time $t = \frac{1}{q}$). Note in this respect that $R \circ A(A^n(\mathcal{S})) = A^{-n-1}(\mathcal{S})$ for all $n \in \{1, \ldots, q\}$. Thus, in the presence of a reversing symmetry we can further decompose the first hit map $h$ into $h := h_2 \circ h_1$, with $h_1 : \mathcal{S} \mapsto \mathcal{S}'$ and $h_2 : \mathcal{S}' \mapsto A^{-1}(\mathcal{S})$. We then obtain decomposition properties of the return map analogous to those in Remark 3.4. Similarly, the results of Section 4 (e.g. Theorem 4.1) have their validity readily extended to the context of local return maps.

**Remark 8.1** It should be noted that in previous works by Swift and Wiesenfeld [28], Fiedler [7, Section 5.4] and Nikolaev [23], return maps for symmetric periodic orbits were considered as well (although without taking consideration of reversing symmetries). Swift and Wiesenfeld and Fiedler consider only cyclic groups $\Gamma$ with $\tilde{\Gamma} = 1$ (the latter motivated by the fact that $\Gamma/\tilde{\Gamma} \simeq \mathbb{Z}_q$ in the absence of reversing symmetries), thus avoiding the need to consider $k$-symmetric maps. Nikolaev takes nontrivial instantaneous symmetries of the periodic orbit into account and observes that the first-hit map $h$ induces an automorphism of $\tilde{\Gamma}$, corresponding to the action of $\phi_A$ on $\tilde{\Sigma}$ as discussed above.

Return maps for periodic orbits that are symmetric with respect to a single reversing symmetry have been previously studied in the context of subharmonic bifurcations by Vanderbauwhede [31]. In particular, in that work the decomposition property (3.30)-(3.32) was employed in the special case when $A = \mathrm{Id}$.

The present paper unifies and extends the above approaches into the framework of $k$-symmetric maps.

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