Resonant driving and k-symmetry

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Abstract

A particular type of 'resonant' periodic driving of a symmetric dynamical system may cause the time-one map of the driven system to have a non-trivial root that possesses less symmetry than the time-one map itself. The latter phenomenon is called k-symmetry. Some examples and consequences to the dynamical behaviour are pointed out briefly.

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1 Introduction

Consider a dynamical system on a phase space Ω that is the flow of a time-dependent (nonautonomous) vector field F, described by the ordinary differential equation

$$\frac{d}{dt}\mathbf{x} = F(\mathbf{x}, t). \tag{1}$$

In describing the symmetry properties it is important to consider reversing symmetries in addition to the more usual notion of ordinary symmetries. A transformation $M : \Omega \mapsto \Omega$ is a symmetry of the dynamical system (1) if

$$\frac{d}{dt}(M(\mathbf{x})) = F(M(\mathbf{x}), t), \tag{2}$$

and $S : \Omega \mapsto \Omega$ is a reversing symmetry if

$$\frac{d}{dt}(S(\mathbf{x})) = -F(S(\mathbf{x}), -t). \tag{3}$$

In case of a discrete time dynamical system described by a map $L : \Omega \mapsto \Omega$, a symmetry M commutes with L,

$$M \circ L = L \circ M, \tag{4}$$

and a reversing symmetry S conjugates L to its inverse

$$S \circ L^{-1} = L \circ S. \tag{5}$$

In the present context we will restrict ourselves to invertible dynamical systems and invertible symmetries and reversing symmetries. Then, all symmetries and reversing symmetries together form a group, which is called the *reversing symmetry group* of the dynamical system [5]. The reversing symmetry group combines in a natural way the established notions of equivariance [2] and (weak) reversibility [12, 13].

In various recent studies [3, 4, 6, 9, 11, 14] it was found that a map L may possess less symmetries and reversing symmetries than its kth iterate L^k. If k is the smallest positive integer for which a map U is a (reversing) symmetry of the kth iterate of L, then U is called a (reversing) k-symmetry of L [9]. Along the same lines, a group Σ is called a reversing k-symmetry group of the map L if k is the smallest positive integer for which L^k possesses Σ as a reversing symmetry group and Σ satisfies some invariance condition.[1] In this letter we will show how a specific type of periodic driving may lead to k-symmetric features of the time-one map. The main result of this letter

[1] Σ should also be invariant under the action of an operator ϕ_L, that will be defined later on, describing a permutation on Σ that is induced by L.

is summarized in the following theorem:

Theorem *If the equations of motion of a nonautonomous dynamical system are invariant under a transformation* $(\mathbf{x},t) \mapsto (U(\mathbf{x}), t-1/k)$*, with* $U^k = \mathrm{Id}$*, then the time-one map can be written as the* k*-th iterate of a map* L*, such that the reversing symmetry group of the nonautonomous dynamical system is a reversing* $\tilde{k}$*-symmetry group of* L*, for some value of* $\tilde{k}$ *dividing* k*.*

Thus, the occurrence of a combined spatial-timeshift symmetry causes the time-one map of the vector field to be the kth iterate of a map L such that symmetries and reversing symmetries of the vector field appear as (reversing) $\tilde{k}$-symmetries of the map L, with $\tilde{k}$ dividing k. This may have nontrivial consequences for the dynamics of the time-one map.

The mechanism described above has an interpretation of k-symmetry occurring due to a kind of resonant driving, as an intermediate phenomenon between symmetry preserving and symmetry breaking driving. Namely, starting from a symmetric autonomous dynamical system one may apply an external time-periodic driving. Usually, such a driving either preserves or breaks (reversing) symmetries of the equations of motion. Yet, it may happen that the driving 'resonates' with a broken symmetry in the sense that an effectively broken symmetry transformation appears as the spatial part of a combined spatial-timeshift symmetry.

In [3, 4, 6] phenomena related to k-symmetry have been observed to occur in time-one maps of vector fields suffering a periodic sequence of instantaneous 'kicks'. In such kicked systems the occurrence of k-symmetry is found often to coincide with resonances between the number of kicks per unit time and the natural frequency of the free motion [8]. In this respect, one should note that the type of resonant driving discussed in this letter has a different physical interpretation as it is not tuned by the frequency of the driving, but rather by other parameters describing the type of driving.

In the next section we discuss how k-symmetry may arise from a spatial-timeshift symmetry, in the spirit of the above theorem. In section 3 the occurrence of k-symmetry due to resonant driving will be illustrated with some examples in $\mathbb{R}^2$. These examples will be constructed from autonomous symmetric vector fields by adding a time-dependent driving giving rise to a combined spatial-timeshift symmetry of which the spatial part already appeared as a symmetry of the autonomous system. Furthermore, to motivate the relevance of recognizing k-symmetry, in section 4 we will point out some nontrivial dynamical consequences of k-symmetry.

2 Symmetry properties of the time-one map

Let us consider symmetry properties of nonautonomous dynamical systems of the form (1) in terms of transformations acting in space-time $\Omega \times \mathbb{R}$ as a tensor product of two transformations of which one acts on the phase space Ω, and the other on the time space $\mathbb{R}$:

$$U \otimes \tau : \Omega \times \mathbb{R} \mapsto \Omega \times \mathbb{R}, \tag{6}$$

with

$$(U \otimes \tau)(\mathbf{x}, t) = (U(\mathbf{x}), \tau(t)). \tag{7}$$

Such a space-time transformation is a space-time symmetry of (1) if

$$\frac{d}{d(\tau(t))}(U(\mathbf{x})) = F(U(\mathbf{x}), \tau(t)). \tag{8}$$

From this point of view, symmetries M correspond to space-time transformations $M \otimes 1$ and reversing symmetries S correspond to $S \otimes -1$, with '1' and '-1' acting as multiplication on $\mathbb{R}$. The composition properties of symmetries and reversing symmetries follow as special cases from the general composition properties of space-time symmetries of the above form:

$$(U_1 \otimes \tau_1) \circ (U_2 \otimes \tau_2) = (U_1 \circ U_2) \otimes (\tau_1 \circ \tau_2). \tag{9}$$

It is obvious that much more complicated space-time symmetries than ordinary symmetries and reversing symmetries may occur in a dynamical system. In this letter we will focus on systems possessing a combined spatial-timeshift symmetry of the form

$$(U \otimes \sigma_{-\frac{1}{k}})(\mathbf{x}, t) := (U(\mathbf{x}), t - 1/k), \tag{10}$$

with $\sigma_\kappa(t) := t + \kappa$ and $U^k = \mathrm{Id}$, such that U itself is not a symmetry of the dynamical system. Note that the presence of the space-time symmetry (10) requires that the system also possesses the timeshift symmetry $\sigma_1 : t \mapsto t+1$.

Let us now consider the time-evolution, i.e. the flow of the vector field. We write $\mathcal{X}_{[a,b]} : \Omega \mapsto \Omega$ for the time-evolution on Ω between $t = a$ and $t = b$. In case a dynamical system possesses the timeshift symmetry σ_1, we find that

$$\mathcal{X}_{[0,1]} = \mathcal{X}_{[n,n+1]}, \quad \forall n \in \mathbb{Z}. \tag{11}$$

This provides the possibility to reduce the nonautonomous continuous time dynamical system to an autonomous discrete time dynamical system by considering only the action of the time-one map.

In the presence of a spatial-timeshift symmetry of the form (10), the time-one map $\mathcal{X}_{[0,1]}$ of (1) is the kth iterate of a map L that is defined as

$$L := U \circ \mathcal{X}_{[0,\frac{1}{k}]}. \tag{12}$$

Namely,

$$\mathcal{X}_{[0,1]} = \mathcal{X}_{[\frac{k-1}{k},1]} \circ \ldots \circ \mathcal{X}_{[0,\frac{1}{k}]} \tag{13}$$

and because of the spatial-timeshift symmetry (10) the different time-$1/k$ maps are related by U:

$$\mathcal{X}_{[\frac{i+1}{k},\frac{i+2}{k}]} = U^{-1} \circ \mathcal{X}_{[\frac{i}{k},\frac{i+1}{k}]} \circ U. \tag{14}$$

With $U^k = \mathrm{Id}$, this immediately leads to

$$\mathcal{X}_{[0,1]} = (U \circ \mathcal{X}_{[0,\frac{1}{k}]})^k. \tag{15}$$

Now, we will study the consequences in case the vector field F possesses next to the spatial-timeshift symmetry (10) a reversing symmetry group Σ. Let $M \in \Sigma$ denote a symmetry and $S \in \Sigma$ denote a reversing symmetry, then for all $\tau \in \mathbb{R}$

$$\begin{aligned} M \circ \mathcal{X}_{[0,\tau]} &= \mathcal{X}_{[0,\tau]} \circ M, & (16)\\ S \circ \mathcal{X}_{[0,\tau]} &= \mathcal{X}_{[0,-\tau]} \circ S = \mathcal{X}^{-1}_{[-\tau,0]} \circ S. & (17) \end{aligned}$$

For the map L (12), this implies that for a symmetry M

$$L \circ M = \phi_U(M) \circ L, \quad \text{with} \quad \phi_U(M) := U \circ M \circ U^{-1}, \tag{18}$$

and for a reversing symmetry S

$$L \circ S = \phi_U(S) \circ L^{-1}, \quad \text{with} \quad \phi_U(S) := U \circ S \circ U. \tag{19}$$

Eq. (18) follows directly from (16), and (19) is derived as follows:

$$\begin{aligned} L \circ S &= U \circ \mathcal{X}_{[0,\frac{1}{k}]} \circ S \circ U \circ U^{-1} \\ &= U \circ S \circ \mathcal{X}_{[0,-\frac{1}{k}]} \circ U \circ U^{-1} \\ &= U \circ S \circ U \circ \mathcal{X}^{-1}_{[0,\frac{1}{k}]} \circ U^{-1} \\ &= \phi_U(S) \circ L^{-1}. \end{aligned} \tag{20}$$

The operator ϕ_U has been defined on symmetries and reversing symmetries in (18) and (19) in analogy to the definition of ϕ_L in [9]. The operator ϕ_U acts as a permutation on the symmetries and the reversing symmetries of the nonautonomous dynamical system (1). Namely,

$$\begin{aligned} \phi_U(M) \otimes 1 &= (U \otimes \sigma_{-\frac{1}{k}}) \circ (M \otimes 1) \circ (U \otimes \sigma_{-\frac{1}{k}})^{-1}, & (21)\\ \phi_U(S) \otimes -1 &= (U \otimes \sigma_{-\frac{1}{k}}) \circ (S \otimes -1) \circ (U \otimes \sigma_{-\frac{1}{k}}). & (22) \end{aligned}$$

Hence, in case the reversing symmetry group Σ under consideration is not invariant under ϕ_U, then Σ is a subgroup of a reversing symmetry group $\tilde{\Sigma}$ that satisfies

$$\phi_U(\tilde{\Sigma}) = \tilde{\Sigma}. \tag{23}$$

Without loss of generality, we will hence assume that the reversing symmetry group Σ satisfies (23).

In connection to the theoretical framework for k-symmetry as developed in [9] it is important to note that that the relations (18) and (19) imply that

$$\phi_U = \phi_L. \tag{24}$$

Reversing k-symmetries and k-symmetries correspond to periodic orbits of ϕ_L of length k in Σ:

$$M \circ L^k = L^k \circ M \quad \Leftrightarrow \quad \phi_L^k(M) = M, \tag{25}$$

$$S \circ L^{-k} = L^k \circ S \quad \Leftrightarrow \quad \phi_L^k(S) = S. \tag{26}$$

ϕ_L orbits provide a natural way to describe the occurrence of (reversing) k-symmetries for a map L.

Thus, in case Σ is a reversing symmetry group of (1) satisfying (23), then Σ is a reversing $\tilde{k}$-symmetry group of L (12), where $\tilde{k}$ divides k. Firstly, from (16) and (17), we find that Σ is a reversing symmetry group of the time-one map $\mathcal{X}_{[0,1]}$, thus also of L^k. Secondly (23) ensures that ϕ_L induces a permutation within Σ. This is an additional requirement for a group to be called a reversing $\tilde{k}$-symmetry group for some value of $\tilde{k}$, cf. [9]. This proves the theorem we presented in the Introduction.

In case ϕ_U does not act as the identity transformation on Σ we have $\tilde{k} > 1$, i.e. a genuine reversing $\tilde{k}$-symmetry group. In case $\tilde{k} = 1$, Σ is a reversing symmetry group of L. We are particularly interested in the case that $\tilde{k} > 1$.[2]

3 Some examples in $\mathbb{R}^2$

Let us now discuss some examples of periodically driven vector fields in $\mathbb{R}^2$ possessing a reversing symmetry group Σ of which the time-one map is written as L^k such that Σ is a reversing $\tilde{k}$-symmetry group of L with $\tilde{k} > 1$. The examples will be constructed as follows:

[2]Note that if U is a symmetry of (1), U commutes with $\mathcal{X}_{[0,\frac{1}{k}]}$ and nothing new is learned from writing the time-one map as the kth iterate of L, even in case we obtain a $\tilde{k} > 1$. In that case we should take $U \in \Sigma$ and study the dynamics of $\mathcal{X}_{[0,\frac{1}{k}]}$ in the presence of Σ. Only if the spatial part of (10) is not a symmetry, genuine k-symmetric phenomena may occur.

- Firstly, we consider an autonomous vector field with a reversing symmetry group $\tilde{\Sigma}$.
- Secondly, we apply a symmetry-breaking periodic driving to the system in such a way that at least one of the generators of the reversing symmetry group disappears from the reversing symmetry group, but remains as the spatial part of a combined spatial-timeshift symmetry. In that case, the driving *resonates* in a particular way with the reversing symmetry group $\tilde{\Sigma}$ that exists in the autonomous case.

We will illustrate this mechanism of resonant driving on two examples in $\mathbb{R}^2$. In both cases the autonomous system possesses a reversing symmetry group isomorphic to the dihedral group D_4, that is generated by a fourfold rotation $R_{\pi/2} : (x, y) \mapsto (-y, x)$ and a mirror $M_v : (x, y) \mapsto (y, x)$. We will treat two different cases. First, I discuss an example in which the fourfold rotation is a reversing symmetry and the mirror is a symmetry. In that case $\tilde{\Sigma}$ is denoted $4'm$. In the second example both generators are symmetries. In that case $\tilde{\Sigma}$ is denoted $4m$.

We use crystallographic notations $4'm$ and $4m$ for the dihedral groups. In this notation one specifies the generators of the group. Integers denote rotations and m denotes a mirror. To distinguish symmetries from reversing symmetries, reversing generators are labeled with a prime. This notation originates from the context of magnetic groups and is also used in dichromatic colour groups [10].

As a first example we start from a time-independent (autonomous) dynamical system in $\mathbb{R}^2$ with $4'm$ symmetry

$$\left\{ \begin{array}{rcl} \dot{x} & = & yf(x^2, y^2), \\ \dot{y} & = & xf(y^2, x^2). \end{array} \right. \tag{27}$$

One easily verifies that this system has a reversing symmetry group generated by the reversing symmetry $R_{\pi/2} : (x, y) \mapsto (-y, x)$ and the symmetry M_v : $(x, y) \mapsto (y, x)$.

We consider the following periodic driving with amplitude ε and period one on the dynamical system (27):

$$\left\{ \begin{array}{rcl} \dot{x} & = & yf(x^2, y^2)(1 + \varepsilon \cos(2\pi t)), \\ \dot{y} & = & xf(y^2, x^2)(1 - \varepsilon \cos(2\pi t)). \end{array} \right. \tag{28}$$

The autonomous system (27) corresponds to the case $\varepsilon = 0$. If $\varepsilon \neq 0$, the driving destroys the symmetry M_v, but M_v is retained as the spatial part of a combined spatial-timeshift symmetry $M_v \otimes \sigma_{-\frac{1}{2}}$. Hence, the time-one map $\mathcal{X}_{[0,1]}$ of (28) can be written as

$$\mathcal{X}_{[0,1]} = L^2, \tag{29}$$

where

$$L := M_v \circ \mathcal{X}_{[0,\frac{1}{2}]}. \tag{30}$$

Due to the breaking of the symmetry M_v, on the level of the nonautonomous vector field and its time-one map we are left with the reversing symmetry group $2m'$ which is a subgroup of the reversing symmetry group $4'm$ that was present in the autonomous system. The group $2m'$, that is generated by the reversing symmetry $M_x : (x, y) \mapsto (x, -y)$ and the symmetry $-\mathrm{Id} : (x, y) \mapsto (-x, -y)$, is a reversing 2-symmetry group of L. Namely, from the fact that $\phi_L = \phi_{M_v}$ one easily checks that

$$\phi_L^2(M_x) = \phi_L(M_y) = M_x, \quad \text{and} \quad \phi_L(-\mathrm{Id}) = -\mathrm{Id}, \tag{31}$$

where $M_y : (x, y) \mapsto (-x, y)$ denotes the mirror in the y-axis.

As a second example, we may consider the dynamical system

$$\left\{ \begin{array}{rcl} \dot{x} & = & xf(x^2, y^2)(1 + \varepsilon \cos(2\pi t)), \\ \dot{y} & = & yf(y^2, x^2)(1 - \varepsilon \cos(2\pi t)). \end{array} \right. \tag{32}$$

If $\varepsilon = 0$, this dynamical system possesses $4m$ symmetry, generated by the symmetries M_v and $R_{\pi/2}$.

In case $\varepsilon \neq 0$, the symmetry M_v is broken, but again M_v remains as the spatial part of a combined spatial-timeshift symmetry $M_v \otimes \sigma_{-\frac{1}{2}}$. Consequently, we find (29) and (30). Moreover, because $M_v = M_v^{-1}$ and $\phi_L = \phi_{M_v}$, we also have (31), and we find that the group $2m$ generated by M_x and $-\mathrm{Id}$ is a 2-symmetry group of the map L.

Analogously, we may also apply resonant driving to $4'm$ and $4m$ symmetric vector fields in such a way that the subgroups $4'$ and 4 appear as (reversing) 2-symmetry groups of a map L satisfying (29) and (30).

4 Discussion

The dynamics of a time-one map may be influenced by its (reversing) symmetries. If this map has a kth root possessing less (reversing) symmetries, additional consequences may arise. We summarize two aspects with reference to [1, 7]:

(i) For a map L that possesses a reversing k-symmetry group Σ (with $k > 1$) specific relations between the periods of orbits of L^k and their symmetry properties may be different from corresponding relations for periodic orbits of a map $\tilde{L}$ possessing the same reversing symmetry group Σ. This has been observed to occur in a recent study [1] of periodic orbits of a map in $\mathbb{R}^2$ possessing a crystallographic reversing 2-symmetry group.

(ii) Generic local bifurcations of a map $\tilde{L}$ in the presence of a reversing symmetry group Σ differ from the local bifurcations one finds for a map L^k possessing the same reversing symmetry group, where Σ is a reversing k-symmetry group of L (with $k > 1$). In particular, in the examples of section 3 the bifurcation equations (i.e. the normal form) at the origin $\mathbf{0}$ are symmetric with respect to the spatial part U of the combined spatial-timeshift symmetry (10).[3] From the derivation of k-symmetric normal forms in [7] it turns out that local bifurcations of the time-one map of the driven vector field at the origin $\mathbf{0}$ with $\varepsilon > 0$ are closely related to the local bifurcations that one finds in the autonomous dynamical system with $\varepsilon = 0$: in the presence of the reversing 2-symmetry group $2m'$ and the 2-symmetry group $2m$ the generic local bifurcations are well approximated by generic local bifurcations in autonomous vector fields with respectively $4'm$ and $4m$ symmetry.

Usually, if one applies a symmetry-breaking driving this will changes the type of generic local bifurcations severely. However, it turns out that if the driving 'resonates' in the sense of (10) with the reversing symmetry group of the autonomous system, although the symmetry is broken, some relation to the autonomous generic bifurcations may persist. It is worthwhile to note that this persistence does not depend on the amplitude of the driving.

As a simple example of the occurrence of k-symmetry due to a combined spatial-timeshift symmetry in a physically relevant model, consider the time-dependent Hamiltonian

$$H(p, x; t) = \frac{p^2}{2} + A\cos(2\pi x) + \varepsilon x \cos(2\pi t) \qquad (33)$$

describing a particle in a periodic potential experiencing time-periodic external driving with amplitude ε. This Hamiltonian may for instance model the motion of a charged particle through a one-dimensional crystal in an oscillating electric field. The Hamiltonian (33) is invariant under the transformation $(p, x, t) \mapsto (-p, -x, t - \frac{1}{2})$. As a consequence, this transformation arises as a combined spatial-timeshift symmetry in the equations of motion. Accordingly, the translation symmetry $(p, x) \mapsto (p, x + 1)$ of the autonomous system (with $\varepsilon = 0$) appears as a 2-symmetry of the nontrivial square root $-\mathrm{Id} \circ \mathcal{X}_{[0,1/2]}$ of the time-one map of the driven system (with $\varepsilon \neq 0$).[4]

[3]This is obvious in case U is a symmetry of (1), but certainly not if U is only the spatial part of the combined spatial-timeshift symmetry (10).

[4]One easily checks that $(p, x) \mapsto (p, x + 1)$ is a symmetry of the dynamical system governed by the Hamiltonian (33), since $H(p, x + 1; t) = H(p, x; t) + c(t)$.

Along the same lines, other physically relevant examples with more complicated resonances may be found in higher-dimensional systems.

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