Pinning and locking of discrete waves

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Abstract

We discuss the behavior of rotating and traveling waves – such as traveling fronts and pulses – in partial differential equations with continuous symmetry groups under perturbations that break the continuous symmetries but preserve some discrete (lattice) symmetries. Our theory provides a rigorous and universal framework for understanding pinning, depinning and locking of discrete rotating and traveling waves.

Key words: pattern formation, mode locking, pinning, bifurcation
PACS: 47.20.Ky, 02.30, 47.54.+r, 82.40.Ck.

1 Introduction

Rotating and traveling waves are observed as solutions of dissipative partial differential equations (pde's) with continuous Lie symmetries that are relevant to many applications in physics, chemistry and biology. Traveling pulses and fronts are important examples of traveling wave solutions. They are known to exist in one-dimensional reaction-diffusion systems [1] that model nerve axons [2] and non-equilibrium dynamics in chemical reactions [3]. Examples of rotating waves include spiral waves in two-dimensional reaction-diffusion systems such as the Belousov-Zhabotinsky reaction [1], rotating convection patterns [4] and wavy vortices in the Taylor-Couette problem [5] (a fluid between two counter rotating cylinders). Dissipative dynamical systems with discrete symmetries may have *discrete* rotating and traveling wave solutions. Examples include reaction-diffusion systems with discrete sources (such as models for

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calcium dynamics [6,7]) or lattice differential equations [8] (such as systems of coupled oscillators with applications to stick-and-slip models for friction and earthquakes [9]). Discrete waves display certain dynamical behaviors that are directly related to their discreteness. Among these behaviors are *pinning*, ie a transition of a discrete wave to a "frozen" steady state with zero velocity, and *locking*, which is alignment of waves with the discrete symmetries of the system. Pinning (or *propagation failure*) has been observed and studied in the context of various applications, see for instance [7,10,4]. In Fig. 1 we have sketched a pinning-depinning transition in a one-dimensional system with a traveling pulse solution.

Symmetry methods have been very useful in understanding the dynamics of rotating and traveling waves in dissipative systems with Euclidean and other continuous symmetry groups [11,12]. In this Letter we will show how symmetry considerations can be used to understand pinning and locking of discrete rotating and traveling waves. Our aim here is to understand the underlying general mechanism, rather than to study specific models. The numerical and analytical study of a depinning transition of a wave front by Mitkov *et al.* [7] initiated this work.

Rotating and traveling waves arise in systems with continuous symmetry groups. They are solutions whose time-evolution is generated by symmetry elements of the equations of motion, i.e. $u(t) = \exp(\xi t)u(0)$ where ξ is an element of the Lie algebra of the symmetry group. If the time-orbit is a circle (S^1) we speak of a rotating wave, if it is a line ($\mathbb{R}$) we speak of a traveling wave. They are also called *relative equilibria* because they are stationary in a comoving frame (moving with velocity ξ).

In systems with only discrete (lattice) symmetries, solutions may arise for which there exists a smallest τ such that $u(t+\tau) = \sigma u(t)$ for some discrete symmetry σ. We call such a solution a *discrete* rotating wave if it is periodic and a *discrete* traveling wave otherwise.

In systems with continuous symmetries, rotating waves and traveling waves typically move with nonzero velocity $\xi \neq 0$. In fact, in a generic parameter-dependent system with rotational symmetry a frozen rotating wave, ie a rotating wave with vanishing rotation frequency (a group orbit of equilibria), is a codimension 1 phenomenon. Similarly, in a generic system with translational symmetry a frozen travelling wave, ie a travelling wave with vanishing propagation velocity, is a codimension 1 phenomenon.

We note that in previous work, pinning has also been studied due to forced symmetry breaking from a system that possesses frozen waves generically (ie with codimension 0). This occurs for instance in a system with $O(2)$ or $E(1)$ symmetry which has a group orbit of equilibria on which each point is invariant

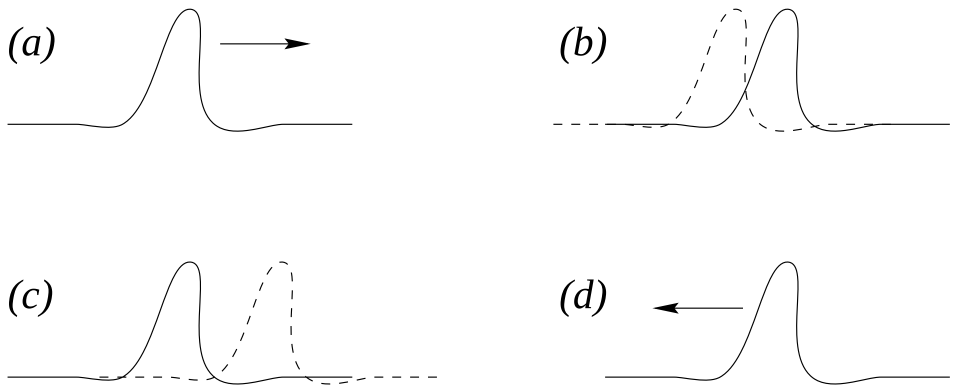

Fig. 1. Schematic sketch of a pinning-depinning transition for a discrete travelling pulse in one dimension. Initially, we consider a stable solution with one pulse that travels to the right (a). Then after changing a parameter the pulse freezes in space. At the same time that this happens, a different unstable frozen pulse solution comes to existence (by the mechanism of a saddle-node bifurcation). With changing parameter the unstable frozen pulse solution moves away from the stable frozen pulse (b). But at the same time another unstable frozen pulse solution (symmetric copy) approaches from the opposite direction (c). At some point the stable and unstable frozen pulse solutions merge in a saddle node bifurcation, which leads to depinning: a stable pulse solution travelling now to the left (d). The underlying dynamics on the circle for these transitions is sketched in Fig.5.

under some reflection symmetry, see eg the examples in [4,13]. (Recall that $O(2)$ and $E(1)$ are the symmetry groups of the circle and the line, respectively.)

In systems with discrete symmetries, discrete rotating and discrete traveling waves may get pinned.

In this Letter, we discuss the transitions from rotating and traveling waves to their discrete counterparts, when a system with continuous symmetry is perturbed to lose the continuous symmetry while preserving some discrete (lattice) symmetry. This is relevant for numerical studies of rotating and traveling waves in systems with continuous symmetry where discretizations inevitably occur, and for understanding the behavior of discrete systems near a known continuum limit, cf [7]. In particular, we will study pinning phenomena near frozen rotating and traveling waves and relate them to saddle-node bifurcations on invariant circles.

Effects of forced rotational symmetry breaking on flow-invariant solutions have been analyzed before in the literature [14]. Moreover the idea that a pair of locked states can coalesce in a saddle-node bifurcation leading to a nonuniformly propagating wave has been around for some time, see [13,4] for small-amplitude rotating waves and [7] for travelling waves. However whereas the references [4,13,7,10] concentrate on special cases our approach provides a universal framework for understanding pinning and locking in dissipative systems with discrete symmetries.

2 Pinning

We will now derive a simple model for pinning-depinning transitions. We start from a parameter-dependent semilinear parabolic pde

$$\partial_t u = -Au + F(u, \mu) + \varepsilon G(u, \varepsilon, \mu) \tag{1}$$

that has a rotating or traveling wave solution $u(t)$ with continuous circular (S^1) respectively translational ($\mathbb{R}$) symmetry when $\varepsilon = 0$. We take G to be such that when $\varepsilon \neq 0$ the equation has only discrete circular ($\mathbb{Z}_n$) or translational ($\mathbb{Z}$) symmetry.

So ε controls the symmetry breaking and μ parametrizes a family of rotating respectively travelling waves of the unperturbed system $\varepsilon = 0$.

Examples include traveling waves of one-dimensional reaction-diffusion equations of the form

$$\partial_t u = \partial_x^2 u + F(u, \mu) + \varepsilon G(u, x, \varepsilon, \mu)$$

where G is a source periodic in the space variable x. Examples include furthermore a spiral wave in a two-dimensional reaction-diffusion system on a disk with circular symmetry or rotating wave solutions of a Taylor-Couette problem, under the influence of small symmetry breaking forces and rotating convection patterns subject to non-homogeneous boundary conditions [4].

We assume that the relative equilibria of the unperturbed system ($\varepsilon = 0$) are exponentially stable for every μ. This means that every solution nearby a relative equilibrium converges exponentially fast to the relative equilibrium. The relative equilibrium is thus isolated in phase space. This assumption is satisfied for stable rotating wave solutions in bounded domains and stable rotating and traveling wave solutions in unbounded domains that converge exponentially to homogeneous stable equilibria at $x = \pm\infty$, such as wave front or pulse solutions, cf [15]. When a relative equilibrium is exponentially stable, under small perturbations it persists as an invariant manifold [16], here a circle or a line. However, the dynamics on this invariant manifold may change.

We now study the consequences of *forced symmetry breaking* from S^1 to the cyclic subgroup $\mathbb{Z}_n$ or from $\mathbb{R}$ to the lattice $\mathbb{Z}$. Under this forced symmetry breaking, rotating waves with S^1-symmetry will become discrete rotating

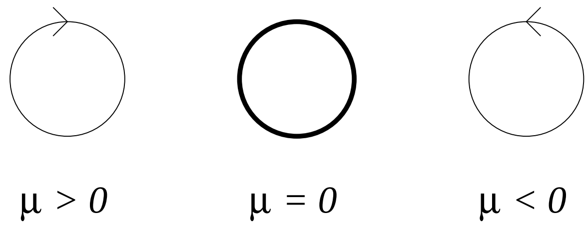

Fig. 2. Orientation reversing transition of (2) on S^1 for $\varepsilon = 0$.

waves with $\mathbb{Z}_n$-symmetry and traveling waves with $\mathbb{R}$-symmetry become discrete traveling waves with $\mathbb{Z}$-symmetry.

Let ϕ denote the position on the group orbit of the rotating rsp. traveling wave. As shown in [11] for S^1-group orbits of equilibria and extended in [12] to general relative equilibria the differential equation on the invariant manifold has the form

$$\dot{\phi} = h(\phi, \varepsilon, \mu) = f(\mu) + \varepsilon g(\phi, \varepsilon, \mu). \tag{2}$$

Here the function g in the symmetry breaking term is 2π-periodic in ϕ/n in the case of the symmetry group $\mathbb{Z}_n$ and 2π-periodic in ϕ in the case of a lattice symmetry $\mathbb{Z}$ (with lattice spacing scaled to 2π). The rotation frequency of the rotating wave or the velocity of the traveling wave of the unperturbed system $\varepsilon = 0$ with parameter μ is given by $f(\mu)$.

In order to understand the pinning-depinning phenomena, we consider an *orientation reversing transition* in the system (1) with continuous symmetry ($\varepsilon = 0$). We start with a group orbit of equilibria, ie a frozen rotating rsp. traveling wave of (1) at $(\varepsilon, \mu) = 0$ and assume that the propagation velocity of the relative equilibria of the unperturbed system changes sign at $\mu = 0$.

For the μ-dependent family of ordinary differential equations (2) at $\varepsilon = 0$ the orientation reversing transition means the following: For $(\varepsilon, \mu) = (0, 0)$ we have a group orbit of equilibria, i.e. $f(0) = 0$. Since generically $f'(0) \neq 0$, by use of the implicit function theorem we can define a modified parameter μ

$$f(\mu) = \mu. \tag{3}$$

Hence at $\mu = 0$ the vector field changes orientation: for $\mu > 0$ the flow is rotating clockwise, and for $\mu < 0$ the flow is rotating counterclockwise, see Fig. 2. A stationary solution at $\mu = 0$ (which is never isolated!) will typically start drifting when the system is exposed to small perturbations that preserve the continuous S^1-symmetry. However, when a perturbation only preserves the symmetries of a (discrete) cyclic subgroup, isolated equilibrium solutions may arise.

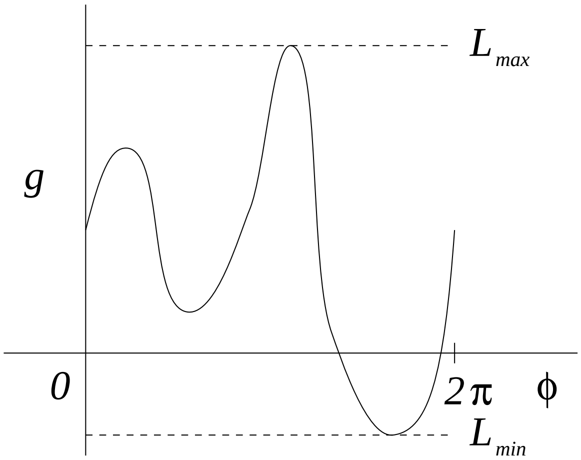

Fig. 3. Sketch of a typical periodic function g.

We now study the unfolding of the orientation-reversing transition of (2) (ε moving through zero) when we move at the same time also μ through zero. We first discuss the case of rotating waves and assume the perturbations break all rotational symmetry ($n = 1$).

To analyze the local dynamics, one needs to study the equilibria of (2) with $f(\mu) = \mu$ (this lowest order term suffices), that is, the zeros of h in a small neighbourhood of the origin in the ε-μ plane. They are given by the equation $g(\phi, \varepsilon, \mu) = -\mu/\varepsilon$. So the intersection of the horizontal line at height $-\mu/\varepsilon$ and the graph of g gives the equilibria for the parameters (ε, μ), see Fig. 3.

We consider the situation when ε and μ are close to zero. Near $(0, 0)$ in the ε-μ plane, there exists a region U_0 where h has no zeros and consequently (2) has no equilibria. This region is defined by $\mu > -L_{\min}(\varepsilon, \mu)\varepsilon$ and $\mu < -L_{\max}(\varepsilon, \mu)\varepsilon$, where $L_{\min}(\varepsilon, \mu) := \min_\phi g(\phi, \varepsilon, \mu)$, $L_{\max}(\varepsilon, \mu) := \max_\phi g(\phi, \varepsilon, \mu)$. As typically $L_{\min}(\varepsilon, \mu) < L_{\max}(\varepsilon, \mu)$, near $(\varepsilon, \mu) = (0, 0)$, in linear approximation U_0 consists of two cones, see Fig. 4.

Fixing $\varepsilon < 0$ small, ie considering a small perturbation that breaks the symmetry of the rotating wave, we will sketch a typical scenario of what happens when μ decreases through 0. We assume our starting point in the region U_0 where $\mu > -\varepsilon L_{\min}(\varepsilon, \mu)$ and we diminish μ. In the graph of g in Fig. 3 this roughly means that we consider a varying horizontal line at height $-\mu/\varepsilon$. When $-\mu/\varepsilon = L_{\min}(\varepsilon, \mu)$ two equilibria are born through a saddle-node bifurcation. More precisely, each time our horizontal line passes through a minimum or maximum of g two equilibria are created or annihilated in a saddle-node bifurcation. Finally when $-\mu = \varepsilon L_{\max}(\varepsilon, \mu)$ there is a last saddle-node bifurcation after which no more equilibria exist. Note that typically the function g has only a finite number of maxima and minima and there exists a neighborhood of $(0, 0)$ in the ε-μ plane where the number of maxima and minima of g is constant.

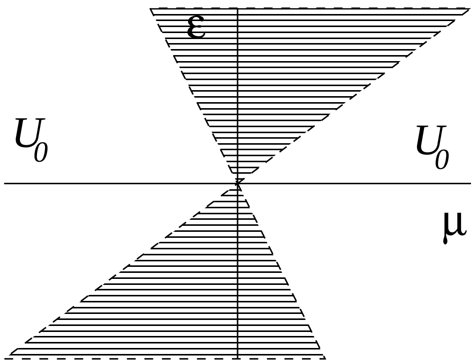

Fig. 4. Sketch of a phase-diagram in the ε-μ plane around $(0, 0)$ for a typical function g. In the unshaded region U_0 no pinning occurs.

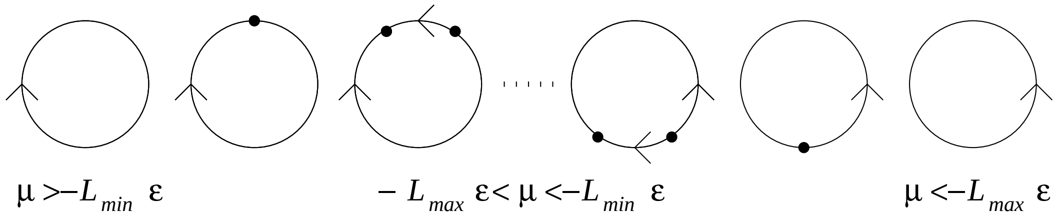

Fig. 5. Sketch of dynamics of (2) on S^1 in the pinning-depinning transition (first and last saddle-node) for $\varepsilon > 0$.

Equilibria on the circle prevent the existence of solutions that drift unboundedly and hence cause *pinning*. A pinned wave thus corresponds to a stable equilibrium point. Each time an equilibrium point associated with a pinned wave annihilates in a saddle-node bifurcation, the wave will temporarily drift until it meets an attracting equilibrium point on the circle and gets pinned again. In fact, we just showed that under symmetry breaking perturbations, the orientation reversing transition typically unfolds into an arbitrary (but finite) series of saddle node bifurcations. The last one constitutes the *depinning* transition, ie after the last saddle-node bifurcation the wave will be able to drift unboundedly, now in the opposite direction as before. The pinning and depinning saddle-node bifurcations are illustrated in Fig. 5. Note that at the pinning or depinning saddle-node bifurcations, the shape of the wave will change continuously with ε.

For traveling waves the same considerations can be applied because due to $\mathbb{Z}$-symmetry the function h is periodic in ϕ. For the interpretation we only need to unwrap the circle S^1 onto the line $\mathbb{R}$, cf Fig. 1. Likewise, if we have a system with rotational symmetry which is perturbed to a $\mathbb{Z}_n$-symmetric system (with $n > 1$) the same result applies.

The model (2), (3) thus provides a universal framework for understanding pinning-depinning transitions of discrete rotating and traveling waves, close to rotating and traveling waves whose continuous symmetry group is one-dimensional (ie S^1 or $\mathbb{R}$). The pinning-depinning transition is understood in the context of a forced symmetry breaking unfolding of the orientation reversing transition with continuous S^1-symmetry.

It should be noted that for the dynamics on the circle describing the discrete waves, saddle-node bifurcations are predicted by local bifurcation theory [17]. In order to gain control over the global bifurcation that underlies the pinning-depinning transition, we started from the continuous symmetry picture and identified the orientation reversing transition as an organizing centre for pinning-depinning transitions.

3 Locking

If the continuous symmetry group has higher dimension, forced symmetry breaking may lead to more complicated phenomena such as *locking*.

To illustrate this, we will discuss the effects of forced symmetry breaking to traveling waves in a planar reaction-diffusion system with state dependent diffusion. Such a system is $\mathbb{R}^2$-equivariant. We consider symmetry breaking perturbations which only have the lattice symmetry $\mathbb{Z}^2$, that is we consider the differential equation

$$\partial_t u = \mathrm{div}(D(u)\mathrm{grad}(u)) + F(u,\mu) + \varepsilon G(u,x,y,\varepsilon,\mu), \tag{4}$$

where ε is a small parameter, G is periodic in x and y (coordinates in $\mathbb{R}^2$), and $D(u) = \mathrm{diag}(d_1(u),\dots,d_n(u))$ with $u \in \mathbb{R}^n$, $d_1(u),\dots d_n(u) > c > 0$.

Assuming again exponential stability of the traveling wave at $\varepsilon = 0$, the equations of motion effectively reduce to equations on the invariant manifold $\mathbb{R}^2$ of the form

$$\begin{cases} \dot{x}_1 = f_1(\mu) + \varepsilon g_1(x_1,x_2,\varepsilon,\mu), \\ \dot{x}_2 = f_2(\mu) + \varepsilon g_2(x_1,x_2,\varepsilon,\mu), \end{cases} \tag{5}$$

where the functions g_1 and g_2 are periodic in x_1 and x_2, $\mu = (\mu_1,\mu_2)$ and $\varepsilon \in \mathbb{R}$, cf [12]. Because of the periodicity, (5) naturally defines a flow on the two-torus.

Pinning (ie absence of drift) requires the torus flow to possess an attracting equilibrium point. For appropriate choices of g_1 and g_2 such a situation can arise due to an arbitrarily small symmetry breaking perturbation of the codimension-two point μ with $f(\mu) = (f_1(\mu), f_2(\mu)) = (0,0)$ (of the unperturbed system with $\varepsilon = 0$).

In general, however, we will have $f(\mu) \neq (0,0)$. In that case, the unperturbed flow is periodic or quasiperiodic on the two-torus, depending on the rationality or irrationality of the *rotation number* $f_1(\mu)/f_2(\mu)$. The analysis of perturbations of such a flow is classical in dynamical systems theory. One may study the problem using a Poincaré return map that is a *circle map* [18]. The conclusion of the theory is that under the influence of small perturbations, the linear flow on the two-torus generically develops finitely many attracting periodic orbits with the same rotation number. This phenomenon is called *mode-locking.* The associated phase-diagram features the famous *Arnol'd tongues* [18].

In terms of the traveling waves in the planar system, mode-locking of the torus flow means that the traveling wave of the unperturbed system becomes a discrete traveling wave $u(t,x)$ that moves on average in the direction of a lattice vector, i.e. $u(t,x)$ is a solution of (4) which satisfies

$$u(t+\tau, x) = u(t, x+a)$$

where $a \in \mathbb{R}^2$ is a lattice vector. The average velocity of the discrete traveling wave $u(t,x)$ is then $v = \frac{a}{\tau}$. We say that such a propagating wave is *locked* with the lattice. Notice that a locked wave is a discrete wave, ie a solution with discrete spatiotemporal symmetry. Locking here thus coincides with a tendency of solutions to symmetrize.

4 Pinning and locking of modulated waves

In the above analysis, we focused on relative equilibria. However, along the same lines also pinning and locking behavior of modulated rotating waves or modulated traveling waves can be analyzed. These waves satisfy $u(\tau) = \exp(\xi\tau)u(0)$ for some $\tau > 0$ (called relative period) and some ξ in the Lie algebra of the symmetry group. They are periodic in a comoving frame (with velocity ξ) and accordingly called *relative periodic solutions.*

The equations of motion on the invariant manifold now require apart from the group-orbit variables an additional variable θ to describe the motion on the relative periodic orbit [19]. For instance, if the continuous symmetry group of the unperturbed system is one-dimensional, the equations of motion will take the form

$$\begin{cases} \dot{\phi} = f_\phi(\theta, \mu) + \varepsilon g_\phi(\phi, \theta, \varepsilon, \mu), \\ \dot{\theta} = f_\theta(\theta, \mu) + \varepsilon g_\theta(\phi, \theta, \varepsilon, \mu), \end{cases} \tag{6}$$

with g_ϕ and g_θ 2π-periodic in ϕ and θ, where ϕ is the continuous group variable and θ is the phase of the relative periodic orbit. Note that again (6) defines a flow on a two-torus.

In the present context, pinning means that ϕ is constant *on average*, ie

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \phi(t)\mathrm{d}t = 0.$$

This corresponds to mode-locking of the torus flow with rotation number zero. In the unperturbed system with continuous symmetry ($\varepsilon = 0$), the equations for ϕ and θ are of skew product form (ie the $\dot{\theta}$ equation does not depend on ϕ). Hence for this special type of torus flow mode-locking will not occur and the average velocity of ϕ is expected to change smoothly with ε. Now assume that the parameter μ is chosen in such a way that there is an orientation reversing transition in the average velocity $\dot{\phi}$ of the relative periodic orbit at $\mu = 0$. Unfolding this bifurcation with the symmetry breaking perturbations (small ε in (6)) will in general lead to mode-locking of the flow on the torus at rotation number 0 which in turn implies pinning for the discrete wave. The pinning-depinning transition again involves a finite number of saddle-node bifurcations, now not of a vector field on the circle but of a circle map.

Actually, pinning is only one example of mode-locking. Typically there is frequency-locking on the torus. Locked periodic solutions represent discrete waves with maximal symmetry possible, given their rotation number p/q (with $p, q \in \mathbb{N}$ minimal). To illustrate this, let us focus on the case that the isotropy of the relative periodic orbit in the unperturbed system is trivial. As a result of forced translational symmetry breaking, we obtain a discrete travelling wave $u(t)$, satisfying $u(\tau) = \sigma u(0)$ where $\tau = 2\pi q$ is the relative period, and σ is a translation over $2\pi p$. Forced rotation symmetry breaking to $\mathbb{Z}_n$-symmetry with $n > 1$, leads to locked periodic solutions which correspond to discrete rotating waves with spatiotemporal symmetry $\mathbb{Z}_{\gcd(n,q)}$, ie $u(t)$ satisfies $u(\tau) = \sigma u(0)$ where σ is a rotation by $2\pi p/\gcd(n, q)$. This is a general property of orientation preserving $\mathbb{Z}_n$-equivariant circle maps. So the spatiotemporal symmetries of a discrete wave originating from a relative periodic orbit depend on the rotation number of the mode-locked solution. Again we thus observe a tendency to symmetrize by means of mode-locking.

Finally, we note that generic bifurcations of (discrete, modulated) rotating and traveling waves can be studied within a *symmetry preserving* setting using equivariant bifurcation theory. For recent developments in the part of this theory dealing with spatiotemporally symmetric solutions, see [19].

Acknowledgements

This work was carried out while enjoying the hospitality of the Mathematics Department of the University of Houston. It is a great pleasure to thank Marty Golubitsky and John Pearson for sharing a helpful discussion on a depinning bifurcation of burst waves [7], which was the starting point of our work. The research of JSWL is supported by a Talent Stipendium of the Netherlands Organization for Scientific Research (NWO) and by the UK Engineering and Physical Sciences Research Council (EPSRC GR/L60029). The research of CW is supported by the German Research Foundation (DFG).

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