Hamiltonian Systems Near Relative Equilibria

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Abstract

We give explicit differential equations for the dynamics near relative equilibria of Hamiltonian systems. These split the dynamics into motion along the group orbit and motion inside a slice transversal to the group orbit. The form of the differential equations that is inherited from the symplectic structure and symmetry properties of the Hamiltonian system is analysed and the effects of time reversing symmetries are included. The results will be applicable to the stability and bifurcation theories of relative equilibria of Hamiltonian systems.

Contents

1 Introduction

Relative equilibria of equivariant dynamical systems are group orbits which are invariant under the flow of the dynamical system. In physical applications they typically correspond to constant 'shape' solutions which evolve by rotating or translating in space. Relative equilibria have been studied in many examples of Hamiltonian systems including rigid bodies [27, 32], underwater vehicle dynamics [20], gravitational N-body problems [34], molecules [13, 15, 38], systems of point vortices [14, 25, 28, 47, 49], liquid drops [18, 23, 24], self-gravitating fluid masses [3, 51] and problems from elasticity theory [26, 35]. Theoretical work has included the development of a variety of techniques for determining the stability of relative equilibria, (see [31] for a review and [21, 36, 43, 45, 46] for more recent work) and a number of results on their persistence and bifurcation [21, 24, 36, 38, 42, 46, 48, 50]. However stability, persistence and bifurcations are still a long way from being properly understood, especially in the presence of nontrivial isotropy subgroups and noncompact group actions.

Our aim in this paper is to develop a Hamiltonian analogue of a tool that has proved extremely useful for determining the stability and bifurcations of relative equilibria in general dynamical systems. This is the description of the dynamics in Palais slice coordinates developed by Krupa [17] for compact group actions and by Fiedler et al. [6] for general proper group actions. These give a decomposition of an equivariant vector field in a neighbourhood of a group orbit into a vector field along the group orbit and a vector field on a slice transverse to the group orbit. The group orbit is a relative equilibrium if and only if the corresponding point in the slice is an equilibrium point of the slice vector field. The study of stability properties and bifurcations of the relative equilibrium can be reduced to that of the slice equilibrium point, while changes in the 'drift' in the group orbit direction are given by the vector field along the group orbit.

The analogous theory for Hamiltonian systems has been developed to a much lesser extent. The structure of equivariant Hamiltonian vectorfields in slice coordinates has been studied by Mielke [35], Roberts and Sousa Dias [50] and Ortega and Ratiu [41, 43]. However their results are either only valid for specific cases or only define the slice vector field implicitly. In this paper we give a detailed explicit decomposition that is valid for Hamiltonian vector fields which are equivariant with respect to any proper Hamiltonian group action on a finite dimensional symplectic manifold.

We also incorporate the effects of time-reversing symmetries by allowing the Hamiltonian functions which generate the vector fields to be invariant under 'semisymplectic' group actions. This is done in a very general way which does not assume, for example, that the time-reversing symmetries are given by an involution that commutes with the time-preserving symmetries. The analogous result for general vector fields has been described by Lamb and Wulff [19].

In the rest of this introduction we briefly outline the general setting for the paper, starting with a short review of the theory for relative equilibria of general (typically dissipative and non-Hamiltonian) systems.

Relative equilibria in general systems

We consider an ordinary differential equation on a manifold $\mathcal{M}$

$$\frac{dx}{dt} = f(x), \quad x \in \mathcal{M} \tag{1.1}$$

that is equivariant with respect to the smooth action of a finite dimensional Lie group Γ,

$$\gamma f(x) = f(\gamma x) \quad \text{for all} \quad \gamma \in \Gamma. \tag{1.2}$$

This implies that $\gamma x(t)$ is a solution whenever $x(t)$ is. We will always assume that the action of Γ is *proper*, which implies that its isotropy subgroups are compact.

We also include the possibility that the differential equation is *reversible*, ie there exists a *reversing symmetry* ρ such that

$$\rho f(x) = -f(\rho x), \tag{1.3}$$

which implies that whenever $x(t)$ is a solution, then so is $\rho x(-t)$. Note that if ρ is a reversing symmetry of (1.1) then so is $\rho\gamma$ for every $\gamma \in \Gamma$.

In the reversible equivariant case, the symmetries (equivariances) and reversing symmetries form a *reversing symmetry group* G such that the group of equivariances Γ is a normal subgroup of G of index two, ie the quotient G/Γ is isomorphic to $\mathbb{Z}_2$. It turns out to be useful to describe this structure by introducing a *reversible sign* (or *temporal character* [52]). This is a character (group homomorphism) $\chi : G \mapsto \{\pm 1\}$, such that

$\chi(\gamma) = 1$ for all $\gamma \in \Gamma$, and $\chi(\rho) = -1$ for all $\rho \in G \setminus \Gamma$. Using this equations (1.2) and (1.3) are equivalent to the single equation

$$gf(x) = \chi(g)f(gx) \quad \text{for all} \quad g \in G. \tag{1.4}$$

We say that f is a (G, χ)-reversible-equivariant vector field or, alternatively, that f is (G, χ)-*semiequivariant*. We usually omit the (G, χ) prefix when it is obvious from the context.

A solution $x(t)$ with initial condition $x(0) = p$ lies on a *relative equilibrium* Gp whenever the group orbit Gp is invariant under the flow of (1.1). This means that $f(p) = \xi p := (\frac{d}{ds}\exp(s\xi).p)_{s=0}$ for some ξ which lies in the Lie algebra $\mathbf{g}$ of G. We call the element ξ the *drift velocity* of the relative equilibrium. We denote the isotropy subgroup of p by G_p, so $G_p = \{g \in G \mid gp = p\}$. Correspondingly we define $\Gamma_p = G_p \cap \Gamma$. The relative equilibrium is said to be *reversible* if G_p contains a reversing symmetry, so that Γ_p is a normal subgroup of G_p of index two, and *nonreversible* if $G_p = \Gamma_p$.

If Γ acts properly on $\mathcal{M}$ then so does G and by the slice theorem of Palais [44] sufficiently small neighbourhoods U of the group orbit Gp have the bundle structure

$$U \;=\; (G \times N)/G_p \;=\; G \times_{G_p} N. \tag{1.5}$$

where N is a local section (also called *slice*) transversal to Gp at p, and the quotient by G_p corresponds to the identifications

$$(g, v) = (gg_p^{-1}, g_p v) \quad \text{for all} \quad g_p \in G_p. \tag{1.6}$$

To analyze the dynamics near, and bifurcations from, relative equilibria, it has proved very useful to model the flow in a neighbourhood of the relative equilibrium by differential equations on the space $G \times N$:

$$\begin{aligned} \dot{g} &= \chi(g) g f_G(v) \\ \dot{v} &= \chi(g) f_N(v) \end{aligned} \tag{1.7}$$

where $f_G : N \to \mathbf{g}$ and $f_N : N \to N$. Here $\mathbf{g}$ denotes the Lie algebra of G. In the local coordinates (g, v) we have $p = (\mathrm{id}, 0)$, so that $f_N(0) = 0$, $f_G(0) = \xi$. The identifications (1.6) imply that f_G and f_N must be (G_p, χ)-semiequivariant:

$$f_G(g_p v) \;=\; \chi(g_p)\; \mathrm{Ad}_{g_p} f_G(v), \qquad f_N(g_p v) \;=\; \chi(g_p)\; g_p f_N(v), \qquad \text{for all} \quad g_p \in G_p, \tag{1.8}$$

where Ad_g is the adjoint action of G on its Lie algebra $\mathbf{g}$: $\mathrm{Ad}_g \xi = g\xi g^{-1}$, $g \in G, \xi \in \mathbf{g}$.

These results are due to Krupa [17] for compact Lie groups, Fiedler et al. [6] for equivariant flows and noncompact Lie groups and Lamb and Wulff [19] for reversible equivariant flows and noncompact groups. The dynamics near a reversible relative equilibrium are in fact determined by the restriction of the equations (1.7) to $\Gamma \times N$, and this is how they appear in [19]. For these restricted equations the sign $\chi(g)$ is always $+1$ and so 'disappears' from the equations.

Note that in any case $\chi(g(t))$ is independent of t and so the slice equation on N does not depend on G. This means that the equations (1.7) have a *skew-product* structure and so, for example, a relative equilibrium is stable in the space of group orbits if it is a stable equilibrium of the slice equation. Moreover the flow on N is given by a general G_p-(semi-)equivariant vector field, where G_p is a compact group even if the original symmetry group G is noncompact. It follows that the study of bifurcations from the relative equilibrium reduces (modulo drift along the G orbit) to the analysis of bifurcations from typically isolated equilibria of flows which are (semi-)equivariant with respect to an action of a compact Lie group.

Hamiltonian setting

The main aim of this paper is to derive a Hamiltonian version of the local equations for relative equilibria (1.7) on the lifted bundle $G \times N$. The results of [17, 6, 19] on the general bundle structure naturally also apply to the Hamiltonian setting. However, it is important to understand how the symplectic structure of a Hamiltonian flow effects the form of the differential equations on $G \times N$ to obtain insights into generic dynamical phenomena near relative equilibria in Hamiltonian systems, including local bifurcations, drift and stability.

We now give a brief introduction to the setting for Hamiltonian differential equations. The starting point is a Hamiltonian ordinary differential equation on a smooth finite-dimensional symplectic manifold $\mathcal{M}$ with (local) symplectic two-form ω_x $(x \in \mathcal{M})$.

We say that a finite dimensional Lie group G acts χ-*semisymplectically* on $\mathcal{M}$ if

$$\omega_{gx}(gu, gv) \;=\; \chi(g)\;\omega_x(u,v) \qquad \text{for all } \; x \in \mathcal{M},\; g \in G,\; u,v \in T_x\mathcal{M}.$$

A Hamiltonian vector field

$$\dot{x} = f_H(x) \tag{1.9}$$

is generated by a smooth function (the *Hamiltonian*), $H : \mathcal{M} \mapsto \mathbb{R}$, via the relationship

$$\omega_x(f_H(x), v) \;=\; \mathrm{D}H(x)v \qquad x \in \mathcal{M},\; v \in T_x\mathcal{M}. \tag{1.10}$$

If H is invariant under the action of G then the vector field f_H is (G, χ)-semiequivariant.

In this paper we derive explicit differential equations describing the flow near relative equilibria of Hamiltonian systems. In other words we describe the additional Hamiltonian structure on the vector fields f_G and f_N in the general slice equations (1.7). By Noether's theorem (see eg [1, 32]), the continuous part of the symmetry group Γ causes the Hamiltonian system to possess conserved quantities, called *momenta*. The symplectic manifold is therefore partitioned into flow-invariant subsets, the *level sets* of the momenta. The way these level sets intersect the slice N can be complicated, and this leads to a nontrivial structure on the slice bundle (1.5) which has been described by Guillemin and Sternberg [10, 11] and Marle [30]. This in turn leads to a nontrivial structure on the lifted bundle equations which is described in §3 of this paper.

Related results on Hamiltonian relative equilibria

There have been several previous descriptions of differential equations near relative equilibria of equivariant Hamiltonian systems. Mielke [35] applied the slice theorem of Palais to derive reduced differential equations under some simplifying assumptions on the isotropy subgroup of the relative equilibrium, but did not investigate explicitly the Poisson structure of the generalized Hamiltonian vectorfield (see subsection 3.3 for more details). Roberts and de Sousa Dias [50] derived equations describing the dynamics on the slice N in the case of compact symmetry groups by embedding N in a larger space. This approach is extended to noncompact groups in this paper and then used to find the equations of motions in the original slice N. Ortega and Ratiu [43] also considered proper actions of noncompact groups and in particular derived 'reconstruction equations' that must be satisfied by the lifted vector field on $G \times N$. Our results in this paper can be viewed as a solution of these reconstruction equations.

In his work Mielke [35] included a discussion of time-reversibility. However our study appears to be the first that systematically treats a general class of systems that are both equivariant and time-reversible. This is particularly important from the viewpoint of applications, as many (perhaps 'most') Hamiltonian systems in applications have time-reversal symmetries.

Our approach combines ideas and techniques from the hitherto rather separate theories for general (typically dissipative) and Hamiltonian dynamical systems. In particular we build on results on slice equations for general and reversible dynamical systems [17, 6, 19] and for Hamiltonian systems with compact symmetry groups [50] and results on local normal forms for symplectic group actions [10, 11, 30, 2].

Organization of the paper

The remainder of this paper is organized as follows. In §2 we recall ideas and results (mainly from Hamiltonian systems theory) that are needed for our main theorems. These are momentum maps, the Witt decomposition, and the definitions and some properties of an operator $\overline{\mathrm{ad}}_\xi^*$ and a map η_μ. Then, in §3, we present our main results, Theorem 3.1 and Theorem 3.2, giving explicit descriptions of the differential equations governing the dynamics near a Hamiltonian relative equilibrium, and compare these results with related results in the literature. In §4 we discuss some important special cases for which the differential equations simplify and apply the results to systems with Euclidean symmetry groups. The final section, §5, is devoted to the proof of our main results.

2 Prerequisites

In this section we recall some notions from Hamiltonian systems theory (momentum maps and Witt decomposition) and introduce an operator $\overline{\mathrm{ad}}^*_\xi$ and a map η_μ. These are necessary for the formulation of our main results in Section 3.

2.1 Momentum maps

Let $(\mathcal{M}, \omega)$ be a symplectic manifold and G a Lie group that acts semisymplectically on $\mathcal{M}$ with respect to a character $\chi : G \to \mathbb{Z}_2$. For simplicity we assume throughout this paper that the symplectic action of the group $\Gamma = \ker \chi$ is *Hamiltonian.* This means that for any $\xi \in \mathbf{g}$ the vector field $p \mapsto \xi p$ on $\mathcal{M}$ is a Hamiltonian vector field with a Hamiltonian which we denote by $\mathbf{J}_\xi : \mathcal{M} \to \mathbb{R}$. The function $\mathbf{J}_\xi$ is linear in ξ and so defines a *momentum map* $\mathbf{J} : \mathcal{M} \to \mathbf{g}^*$ by $\mathbf{J}(p)(\xi) = \mathbf{J}_\xi(p)$. In fact, since we are working locally, we really only need this to hold on some G-invariant neighbourhood of a group orbit Gp.

If H is a Γ-invariant Hamiltonian on $\mathcal{M}$ and $\xi \in \mathbf{g}$ then the Γ-invariance of H implies that $\mathrm{D}H(p).\xi p = 0$ at any point $p \in \mathcal{M}$. It follows that the evolution of $\mathbf{J}_\xi$ along trajectories of the vector field f_H generated by H is given by

$$\dot{\mathbf{J}}_\xi(p) \;=\; \mathrm{D}\mathbf{J}_\xi(p).f_H(p) \;=\; \omega_p(\xi p, f_H(p)) \;=\; -\mathrm{D}H(p).\xi p \;=\; 0$$

and so the components of the momentum map are conserved functions for the Hamiltonian vector field generated by H. This is Noether's theorem [1, 32].

We will also assume throughout this paper that the momentum map $\mathbf{J}$ is equivariant with respect to the symplectic action of Γ on $\mathcal{M}$ and the coadjoint action of Γ on $\mathbf{g}$:

$$\mathbf{J}(gp) \;=\; \mathrm{Ad}^*_{g^{-1}}\mathbf{J}(p) \qquad \forall\, g \in \Gamma$$

where Ad^*_g is the dual operator to Ad_g, ie $\mathrm{Ad}^*_g(\mu(\xi)) = \mu(\mathrm{Ad}_g(\xi))$ for all $\xi \in \mathbf{g}$, $\mu \in \mathbf{g}^*$ and $g \in \Gamma$. Such a momentum map always exists if either Γ is a compact group acting linearly on a symplectic vector space or $\mathcal{M} = T^*\mathcal{Q}$ is the cotangent bundle of a smooth configuration space $\mathcal{Q}$ and the action of Γ on $\mathcal{M}$ is the cotangent lift of a smooth action on $\mathcal{Q}$.

The coadjoint action of Γ extends to the coadjoint action of the full group G on $\mathbf{g}$. However it is *not* true that $\mathbf{J}$ is equivariant with respect to the semisymplectic action of G on $\mathcal{M}$ and the standard coadjoint action of G on $\mathbf{g}$. Instead the coadjoint action has to be 'twisted' by the character χ. More precisely, for any representation V of a group G given by a group homomorphism $\phi : G \to GL(V)$ we define the χ-dual representation or action of G on V to be that given by the homomorphism $\phi_\chi : g \mapsto \chi(g)\phi(g)$. We define the χ-coadjoint representation of G on $\mathbf{g}^*$ as the χ-dual of the usual coadjoint representation of G on $\mathbf{g}^*$, i.e.,

$$(g, \mu) \;\mapsto\; \chi(g)\mathrm{Ad}^*_{g^{-1}}(\mu) \qquad \forall\, g \in G, \mu \in \mathbf{g}^*.$$

For $\mu \in \mathbf{g}^*$ and $g \in G$ we will sometimes use the notation $g\mu$ for $\chi(g)\mathrm{Ad}^*_{g^{-1}}(\mu)$. Note that if G acts by its adjoint action on $\mathbf{g}$ and by the χ-coadjoint action on $\mathbf{g}^*$ then the product action is χ-semisymplectic on $\mathbf{g} \oplus \mathbf{g}^*$ with its canonical symplectic form $\omega((\xi_1, \nu_1), (\xi_2, \nu_2)) = \nu_1(\xi_2) - \nu_2(\xi_1)$.

We will assume throughout the paper that the momentum map $\mathbf{J}$ is equivariant with respect to its semisymplectic action on $\mathcal{M}$ and its χ-coadjoint action on $\mathbf{g}^*$.

$$\mathbf{J}(gp) \;=\; \chi(g)\; \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}(p)) \qquad \forall\, g \in G.$$

If G is compact and there exists a (possibly non-equivariant) momentum map $\mathbf{J}$ then a G-equivariant momentum map $\mathbf{J}_{\mathrm{new}}$ can be obtained by averaging over the Haar measure m on G:

$$\mathbf{J}_{\mathrm{new}}(x) = \int_G \chi(g)\mathrm{Ad}^*_{g^{-1}}\mathbf{J}(g^{-1}x)\mathrm{d}m(g).$$

More generally one may obtain a G-equivariant momentum map whenever G is a compact extension of a group G' for which there exists a G'-equivariant momentum map. In particular if there exists a Γ-equivariant momentum map for the symplectic action of Γ on $\mathcal{M}$ then there exists a G-equivariant momentum map for the semisymplectic action of G.

If $\mathbf{J}$ is such a G-equivariant momentum map then for any $\mu \in \mathbf{g}^*$ the fibre $\mathbf{J}^{-1}(\mu)$ is invariant under the isotropy subgroup G_μ for the χ-coadjoint representation of G on $\mathbf{g}^*$:

$$G_\mu = \{g \in G \mid \chi(g)\mathrm{Ad}^*_{g^{-1}}(\mu) = \mu\}.$$

Throughout the paper the notation 'G_μ' will refer to this group and $\mathbf{g}_\mu$ to its Lie algebra. Note that $\Gamma_\mu = G_\mu \cap \Gamma$ may be either equal to G_μ or be a normal subgroup of index 2. For a further discussion of momentum maps and phase space reduction for semisymplectic group actions see [39].

2.2 The Witt decomposition

In this section we describe a natural decomposition of the tangent space $T_p\mathcal{M}$ and its symplectic form ω_p for any point $p \in \mathcal{M}$. For notational convenience we drop the subscript p from ω_p in the rest of this section.

Define the ω-orthogonal complement of any subspace W of $T_p\mathcal{M}$ to be

$$W^\omega \;=\; \{u \in T_p\mathcal{M} \;:\; \omega(u,v) = 0 \;\; \forall\, v \in T_p\mathcal{M}\;\}\,.$$

Let G_p denote the isotropy subgroup of the point p and $\mathbf{g}_p$ its Lie algebra. Since G_p is compact we can choose a G_p-invariant inner product on $T_p\mathcal{M}$. We denote the orthogonal complement of any subspace $W \subset T_p\mathcal{M}$ with respect to this inner product by $W^\perp$.

Now let $T = \mathbf{g}p$ be the tangent space to the group orbit Gp in p and define the following subspaces of $T_p\mathcal{M}$:

$$T_0 = T \cap T^\omega, \quad T_1 = T \cap T_0^\perp, \quad N_0 = (T + T^\omega)^\perp, \quad N_1 = T^\omega \cap T_0^\perp.$$

It is easily checked that these are all G_p-invariant subspaces of $T_p\mathcal{M}$ and that

$$T_p\mathcal{M} = T_0 \oplus T_1 \oplus N_0 \oplus N_1.$$

Further properties of this decomposition are summarized in the following proposition.

Proposition 2.1

a) The symplectic form ω on $T_p\mathcal{M}$ restricts to symplectic forms ω_0 on $T_0 \oplus N_0$, ω_{T_1} on T_1 and ω_{N_1} on N_1. The actions of G_p on these spaces are χ-semisymplectic with respect to the restricted forms.

b) The symplectic form ω defines a G_p-equivariant isomorphism between the induced G_p-action on N_0 and the χ-dual of the induced action on T_0^*. Under this isomorphism the symplectic form ω_0 is the natural symplectic form on $T_0 \oplus T_0^*$:

$$\omega_0\left((u_1,\nu_1),(u_2,\nu_2)\right) \;=\; \nu_2(u_1) \;-\; \nu_1(u_2)$$

c) Let $\mathbf{J} : \mathcal{M} \to \mathbf{g}^*$ be a G-equivariant momentum map (with respect to the χ-coadjoint action on $\mathbf{g}^*$), $\mathbf{J}(p) = \mu$ and G_μ be the isotropy subgroup at μ. Then:

(i) $T^\omega = \ker \mathrm{D}\mathbf{J}(p)$.

(ii) $\mathrm{D}\mathbf{J}(p)$ maps T_1 isomorphically to $T_\mu(G\mu) \cong \mathbf{g}/\mathbf{g}_\mu$ and ω_{T_1} to the Kostant-Kirillov-Souriau (KKS) form ω_μ:

$$\omega_{T_1}(\xi_1.p,\xi_2.p) \;=\; \omega_\mu(\xi_1,\xi_2) \;:=\; \mu\left([\xi_1,\xi_2]\right) \tag{2.1}$$

where $\xi_i \in \mathbf{g}$, $i = 1,2$, and $[\cdot,\cdot]$ is the Lie bracket on $\mathbf{g}$.

(iii) The map $\mathbf{g} \to T_p\mathcal{M}$ given by $\xi \mapsto \xi p$ induces an isomorphism $\mathbf{g}_\mu/\mathbf{g}_p \cong T_0$.

Proofs of this proposition can be found in [40] and [2] for the case that G acts symplectically. The extension to semisymplectic group actions is straightforward.

Note that $T = T_0 \oplus T_1$ and $N = N_0 \oplus N_1$ are the tangent and normal spaces at p to the G orbit through p. The spaces T_1 and N_1 are the (uniquely defined) maximal symplectic subspaces of these spaces and we refer to them as the *symplectic tangent space* and *symplectic normal space*, respectively.

Remark 2.2 The symplectic form ω_{N_1} on N_1 depends on the symplectic manifold $\mathcal{M}$ and every symplectic form for which the G_p action is χ-semisymplectic occurs for some $\mathcal{M}$. If a G_p-invariant inner product $< \cdot,\cdot >_{N_1}$ is fixed on N_1, then (G_p,χ)-semisymplectic forms $\omega(\cdot,\cdot)_{N_1}$ are given by

$$\omega(w_1,w_2)_{N_1} \;=\; \langle w_1, Jw_2\rangle_{N_1} \qquad w_1,w_2 \in N_1 \tag{2.2}$$

where the linear maps J are orthogonal and skew-symmetric with respect to the inner product and are (G_p, χ)-semiequivariant:

$$J g_p w \;=\; \chi(g_p) g_p J w \qquad\qquad g_p \in G_p, w \in N_1.$$

Such maps are called *complex structures* on N_1. Any two out of the three objects J, $\omega(\cdot,\cdot)_{N_1}$ and $< \cdot,\cdot >_{N_1}$ satisfying (2.2) determine the third uniquely. Normal forms for (G_p, χ)-semisymplectic complex structures are classified in [12]. The results can be regarded either as a classification of semisymplectic representations or, equivalently, as a semisymplectic Darboux lemma.

Since the linear action of the compact group G_p on N_1 is semisymplectic, by the remarks in section 2.1 there exists a momentum map $\mathbf{J}_{N_1} : N_1 \to \mathbf{g}_p^*$ which is G_p-equivariant with respect to the χ-coadjoint action of G_p on $\mathbf{g}_p^*$. We now discuss the embedding of this momentum map as a mapping into $\mathbf{g}_\mu^*$. Since G_p is compact we can choose a G_p-invariant complement $\mathbf{m}_\mu$ to $\mathbf{g}_p$ in $\mathbf{g}_\mu$. Then $\mathbf{g}_\mu = \mathbf{g}_p \oplus \mathbf{m}_\mu$. By the natural projection from $\mathbf{g}_\mu$ to $\mathbf{g}_\mu/\mathbf{g}_p$ we obtain a G_p-equivariant isomorphism $\mathbf{m}_\mu \cong \mathbf{g}_\mu/\mathbf{g}_p$. Accordingly, we also have $\mathbf{g}_\mu^* = \mathrm{ann}_{\mathbf{g}_\mu^*}(\mathbf{g}_p) \oplus \mathrm{ann}_{\mathbf{g}_\mu^*}(\mathbf{m}_\mu)$, where the annihilator of $\mathbf{g}_p$ in $\mathbf{g}_\mu^*$ is defined as

$$\mathrm{ann}_{\mathbf{g}_\mu^*}(\mathbf{g}_p) = \{\nu \in \mathbf{g}_\mu^* \mid \nu(\xi) = 0, \ \forall \xi \in \mathbf{g}_p\},$$

and other annihilators in a similar way. Now, $\mathrm{ann}_{\mathbf{g}_\mu^*}(\mathbf{g}_p)$ is the subspace $(\mathbf{g}_\mu/\mathbf{g}_p)^* \subset \mathbf{g}_\mu^*$, while $\mathrm{ann}_{\mathbf{g}_\mu^*}(\mathbf{m}_\mu)$ is the image of an $\mathbf{m}_\mu$-dependent linear embedding of $\mathbf{g}_p^*$ in $\mathbf{g}_\mu^*$. Using this embedding we may regard the momentum map $\mathbf{J}_{N_1} : N_1 \to \mathbf{g}_p^*$ for the action of G_p on N_1 as a mapping from N_1 into $\mathbf{g}_\mu^*$.

2.3 Definition of $\overline{\mathrm{ad}}_\xi^*$

To describe the form of the vector field f_N in the Hamiltonian case it is useful to introduce a generalization of the Lie bracket for the quotient space $\mathbf{g}_\mu/\mathbf{g}_p$. In fact the construction will apply to the quotient space $\mathbf{g}/\mathbf{k}$ for any Lie algebra $\mathbf{g}$ of a (finite dimensional) Lie group G and subalgebra $\mathbf{k}$ of a *compact* Lie subgroup K. A more detailed discussion, including a generalization to noncompact subgroups K and applications to mechanical systems on homogeneous spaces, is given in [22].

Since K is compact there exists a K-invariant complement $\mathbf{m}$ to $\mathbf{k}$ in $\mathbf{g}$, so that $\mathbf{g} = \mathbf{k} \oplus \mathbf{m}$. The natural projection $\pi : \mathbf{g} \to \mathbf{g}/\mathbf{k}$ has kernel $\mathbf{k}$ and maps $\mathbf{m}$ isomorphically to $\mathbf{g}/\mathbf{k}$.

We construct a bracket on $\mathbf{g}/\mathbf{k}$ by lifting elements of $\mathbf{g}/\mathbf{k}$ to $\mathbf{m}$, applying the Lie bracket $[\cdot,\cdot]$ of $\mathbf{g}$, and then projecting the result back to $\mathbf{g}/\mathbf{k}$. In other words, let $\tilde{\xi}, \tilde{\eta}$ be the representatives in $\mathbf{m}$ of $\xi, \eta \in \mathbf{g}/\mathbf{k}$. Define the bracket $[\cdot,\cdot]_{\mathbf{m}}$ on $\mathbf{g}/\mathbf{k}$ by

$$[\xi, \eta]_{\mathbf{m}} \;=\; \pi\left([\tilde{\xi}, \tilde{\eta}]\right). \tag{2.3}$$

This bracket is equivariant with respect to the adjoint action of K on $\mathbf{g}/\mathbf{k}$:

$$[g\xi, g\eta]_{\mathbf{m}} \;=\; g[\xi,\eta]_{\mathbf{m}} \qquad \text{for all} \quad g \in K.$$

The subscript $\mathbf{m}$ is used in the notation $[\cdot,\cdot]_{\mathbf{m}}$ to emphasize that the bracket depends on the choice of $\mathbf{m}$. Moreover, it is not a Lie bracket since it satisfies all the axioms of a Lie bracket *except* the Jacobi identity. Only the restriction of the bracket to the fixed point set $\mathrm{Fix}_{\mathbf{g}/\mathbf{k}}(K) = \{\xi \in \mathbf{g}/\mathbf{k} : g\xi = \xi \ \forall g \in K\}$ is again a Lie bracket.

Lemma 2.3 *If* $\mathbf{n}$ *is the Lie algebra of the normalizer* $N(K)$ *of* K *in* G *then:*

a) $[\mathbf{m}\cap\mathbf{n}, \mathbf{k}] = 0$*;*

b) $\mathrm{Fix}_{\mathbf{g}/\mathbf{k}}(K) = \mathbf{n}/\mathbf{k}$*;*

c) The restriction of $[\cdot,\cdot]_{\mathbf{m}}$ *to* $\mathrm{Fix}_{\mathbf{g}/\mathbf{k}}(K)$ *is the natural Lie bracket on the Lie algebra* $\mathbf{n}/\mathbf{k}$ *of the group* $N(K)/K$*. In particular, if* K *is a normal subgroup of* G *then* $[\cdot,\cdot]_{\mathbf{m}}$ *satisfies the Jacobi identity, is independent of the choice of* $\mathbf{m}$ *and is the natural Lie bracket on* $\mathbf{g}/\mathbf{k}$*.*

Proof.

a) The K-invariance of $\mathbf{m}$ implies that $[\mathbf{m},\mathbf{k}] \subset \mathbf{m}$. If $g \in N(K)$ then $gKg^{-1} = K$ and so $[\mathbf{n},\mathbf{k}] \subset \mathbf{k}$. It follows that $[\mathbf{m}\cap\mathbf{n},\mathbf{k}] \subset \mathbf{m}\cap\mathbf{k} = \{0\}$.

b) We show first that $\mathbf{n}/\mathbf{k} \subset \mathrm{Fix}_{\mathbf{g}/\mathbf{k}}(K)$. Let $\eta \in \mathbf{n}/\mathbf{k}$ and let $\tilde\eta$ be the representative of η in $\mathbf{m}\cap\mathbf{n}$. The fact that $\tilde\eta \in \mathbf{n}$ implies that $ge^{t\tilde\eta}g^{-1}e^{-t\tilde\eta} \in K$ and so $\mathrm{Ad}_g\tilde\eta - \tilde\eta \in \mathbf{k}$ for all $g \in K$. Since $\tilde\eta \in \mathbf{m}$ we also have $\mathrm{Ad}_g\tilde\eta - \tilde\eta \in \mathbf{m}$. It therefore follows from part a) that $\mathrm{Ad}_g\tilde\eta = \tilde\eta$ and so $\eta \in \mathrm{Fix}_{\mathbf{g}/\mathbf{k}}(K)$. This statement was proved by Field [7].

Conversely, let $\eta \in \mathrm{Fix}_{\mathbf{g}/\mathbf{k}}(K)$. Since $\mathbf{g}/\mathbf{k} \cong \mathbf{m}$ and $\mathbf{m}$ is K-invariant we have $\mathrm{Fix}_{\mathbf{g}/\mathbf{k}}(K) \cong \mathrm{Fix}_{\mathbf{m}}(K)$ and can choose a representative $\tilde\eta$ of η such that $\tilde\eta \in \mathrm{Fix}_{\mathbf{m}}(K)$. Then

$$\mathrm{e}^{-t\tilde\eta} g \mathrm{e}^{t\tilde\eta} \;=\; \mathrm{e}^{-t\tilde\eta}\mathrm{e}^{t\mathrm{Ad}_g\tilde\eta} g \;=\; \mathrm{e}^{-t\tilde\eta}\mathrm{e}^{t\tilde\eta} g \;=\; g$$

for all $g \in K$, and so $\tilde\eta \in \mathbf{n}\cap\mathbf{m}$ and $\eta \in \mathbf{n}/\mathbf{k}$. This proves part b).

c) Let $\eta_1, \eta_2 \in \mathrm{Fix}_{\mathbf{g}/\mathbf{k}}(K)$. If $\eta_i = \tilde\eta_i + \mathbf{k}$, where $\tilde\eta_i \in \mathbf{n}\cap\mathbf{m}$, then the Lie bracket on the Lie algebra $\mathbf{n}/\mathbf{k}$ is defined by

$$[\eta_1,\eta_2]_{\mathbf{n}/\mathbf{k}} \;=\; [\tilde\eta_1,\tilde\eta_2] \;+\; \mathbf{k} \;=\; \pi\left([\tilde\eta_1,\tilde\eta_2]\right) \;=\; [\eta_1,\eta_2]_{\mathbf{m}}$$

and being the Lie bracket on the Lie algebra of the group $N(K)/K$ it satisfies the Jacobi identity. ∎

This lemma is used in subsection 4.3 below.

For $\xi \in \mathbf{g}/\mathbf{k}$ we also define an operator $\overline{\mathrm{ad}}_\xi : \mathbf{g}/\mathbf{k} \to \mathbf{g}/\mathbf{k}$ by $\overline{\mathrm{ad}}_\xi(\eta) = [\xi,\eta]_{\mathbf{m}}$ and denote the dual of this by $\overline{\mathrm{ad}}_\xi^* : (\mathbf{g}/\mathbf{k})^* \to (\mathbf{g}/\mathbf{k})^*$ where $(\mathbf{g}/\mathbf{k})^* \cong \mathrm{ann}(\mathbf{k})$, the annihilator of $\mathbf{k}$ in $\mathbf{g}^*$. If $\mathbf{k}$ is normal in $\mathbf{g}$ then these give the usual adjoint action of

$\mathbf{g}/\mathbf{k}$ on itself and coadjoint action of $\mathbf{g}/\mathbf{k}$ on its dual. Unravelling the definition of $\overline{\mathrm{ad}}^*_\xi$ we find that

$$\overline{\mathrm{ad}}^*_\xi(\nu)(\eta) \;=\; \nu([\xi,\eta]_{\mathbf{m}}) \tag{2.4}$$

where $\nu \in (\mathbf{g}/\mathbf{k})^*$, $\xi, \eta \in \mathbf{g}/\mathbf{k}$ and $[\cdot,\cdot]_{\mathbf{m}}$ is the bracket on $\mathbf{g}/\mathbf{k}$, as defined in (2.3). The K-equivariance of $[\cdot,\cdot]_{\mathbf{m}}$ implies that

$$\overline{\mathrm{ad}}^*_{\mathrm{Ad}_g(\xi)} \;=\; \mathrm{Ad}^*_g \circ \overline{\mathrm{ad}}^*_\xi \circ \mathrm{Ad}^*_{g^{-1}} \qquad\qquad g \in K \tag{2.5}$$

where $\mathrm{Ad}^*_{g^{-1}}$ is the action of $g \in K$ on $(\mathbf{g}/\mathbf{k})^* \subset \mathbf{g}^*$ obtained by restricting the coadjoint action on $\mathbf{g}^*$.

Using a K-invariant inner product we can identify $\mathbf{g}^*$ with $\mathbf{g}$ and the coadjoint action of K on $\mathbf{g}^*$ with the adjoint action of K on $\mathbf{g}$. Since $\mathbf{m}$ is K-invariant we have the K-equivariant identifications $(\mathbf{g}/\mathbf{k})^* \cong \mathbf{g}/\mathbf{k} \cong \mathbf{m}$. Moreover, $\mathrm{ann}(\mathbf{m}) \cong \mathbf{k}$. Note that in general the identification of $\mathbf{g}$ with $\mathbf{g}^*$ will *not* be G-equivariant. However if the inner product is G-invariant then this identification is G-equivariant, the adjoint and coadjoint action of G are the same, and we have the following result.

Lemma 2.4 *Assume that there exists a G-invariant inner product on* $\mathbf{g}$. *If $\xi \in \mathbf{g}/\mathbf{k}$ and $\zeta \in (\mathbf{g}/\mathbf{k})^*$ then under the identification $\mathbf{g}/\mathbf{k} \cong (\mathbf{g}/\mathbf{k})^*$, $\overline{\mathrm{ad}}^*_\xi(\zeta)$ is given by*

$$\overline{\mathrm{ad}}^*_\xi(\zeta) \;=\; [\zeta,\xi]_{\mathbf{m}}.$$

Proof. Let $< \cdot,\cdot >$ denote the G-invariant inner product on $\mathbf{g}$. Recall that G-invariance implies that for all $\xi_1, \xi_2, \xi_3 \in \mathbf{g}$ we have $\langle \xi_1, [\xi_2,\xi_3]\rangle = \langle \xi_3, [\xi_1,\xi_2]\rangle$ and so, under the identification $\mathbf{g}^* \cong \mathbf{g}$ induced by the inner product, $\mathrm{ad}^*_{\xi_2}(\xi_1) = [\xi_1,\xi_2]$.

If $\xi, \eta \in \mathbf{g}/\mathbf{k}$ and $\zeta \in (\mathbf{g}/\mathbf{k})^*$ then by definition $\overline{\mathrm{ad}}^*_\xi(\zeta)(\eta) = \zeta([\xi,\eta]_{\mathbf{m}})$. Let $\tilde{\xi}$ and $\tilde{\eta}$ be representatives of ξ and η in $\mathbf{m}$. The isomorphism $\mathbf{g} \cong \mathbf{g}^*$ identifies $\pi^*\zeta \in \mathbf{g}^*$ with an element $\tilde{\zeta} \in \mathbf{m}$. Then

$$\overline{\mathrm{ad}}^*_\xi(\zeta)(\eta) = \zeta(\pi([\tilde{\xi},\tilde{\eta}])) = \pi^*\zeta([\tilde{\xi},\tilde{\eta}]) = \langle \tilde{\zeta}, [\tilde{\xi},\tilde{\eta}]\rangle = \langle \tilde{\eta}, [\tilde{\zeta},\tilde{\xi}]\rangle = \langle \eta, [\zeta,\xi]_{\mathbf{m}}\rangle_{\mathbf{g}/\mathbf{k}}$$

where $\langle\cdot,\cdot\rangle_{\mathbf{g}/\mathbf{k}}$ is the induced inner product on $\mathbf{g}/\mathbf{k}$. This gives the required result. ∎

2.4 Definition of $\eta_\mu(\xi,\zeta)$

For any $\mu \in \mathbf{g}^*$ let $\mathbf{n}_\mu$ denote a complement to $\mathbf{g}_\mu$ in $\mathbf{g}$, so that $\mathbf{g} \cong \mathbf{g}_\mu \oplus \mathbf{n}_\mu$. If the annihilators of $\mathbf{g}_\mu$ and $\mathbf{n}_\mu$ in $\mathbf{g}^*$ are denoted by $\mathrm{ann}(\mathbf{g}_\mu)$ and $\mathrm{ann}(\mathbf{n}_\mu)$, respectively, then $\mathbf{g}^* \cong \mathrm{ann}(\mathbf{g}_\mu) \oplus \mathrm{ann}(\mathbf{n}_\mu)$. The dual of the natural projection from $\mathbf{g}$ to $\mathbf{g}/\mathbf{g}_\mu$ is the natural inclusion of $(\mathbf{g}/\mathbf{g}_\mu)^*$ into $\mathbf{g}^*$, the image of which is $\mathrm{ann}(\mathbf{g}_\mu)$. We will therefore usually identify $\mathrm{ann}(\mathbf{g}_\mu)$ with $(\mathbf{g}/\mathbf{g}_\mu)^*$. The natural inclusion $\mathbf{g}_\mu \subset \mathbf{g}$ induces a natural projection $\mathbf{g}^* \to \mathbf{g}^*_\mu$ which restricts to a linear isomorphism between $\mathrm{ann}(\mathbf{n}_\mu)$ and $\mathbf{g}^*_\mu$. Thus choosing a complement $\mathbf{n}_\mu$ determines an (unnatural) embedding of $\mathbf{g}^*_\mu$ into $\mathbf{g}^*$.

In this section, to emphasize the fact that this embedding depends on the choice of $\mathbf{n}_\mu$, we *won't* identify its image $\operatorname{ann}(\mathbf{n}_\mu)$ with $\mathbf{g}_\mu^*$.

The tangent space at μ to the coadjoint orbit $G\mu$ is the image of the map $\mathbf{g} \to \mathbf{g}^*$ given by $\xi \mapsto \operatorname{ad}_\xi^*(\mu)$ and so equals $\operatorname{ann}(\mathbf{g}_\mu) \equiv (\mathbf{g}/\mathbf{g}_\mu)^*$. The subspace $\operatorname{ann}(\mathbf{n}_\mu)$ is therefore a complement in $\mathbf{g}^*$ to the tangent space $T_\mu G\mu$. If $\mathbf{n}_\mu$ can be chosen so that $\mu + \operatorname{ann}(\mathbf{n}_\mu)$ is a G_μ-invariant slice transverse to $G\mu$ at μ then for all ζ sufficiently close to the origin in $\operatorname{ann}(\mathbf{n}_\mu)$ we have $\mathbf{g}_{\mu+\zeta} \subset \mathbf{g}_\mu$. However if G_μ is not compact such slices need not exist.

The following proposition introduces a map which measures the extent to which $\mu + \operatorname{ann}(\mathbf{n}_\mu)$ fails to be a slice at μ. This map appears in the formulation of Theorems 3.1 and 3.2 and its properties have important consequences for applications of the theorem.

Proposition 2.5 *Let G be a Lie group, $\mathbf{g}$ its Lie algebra and μ any point in $\mathbf{g}^*$. Let $\mathbf{n}_\mu$ be a complement to $\mathbf{g}_\mu$ in $\mathbf{g}$ and $\mathrm{P}_{\operatorname{ann}(\mathbf{g}_\mu)}$ be the projection from $\mathbf{g}^*$ to $\operatorname{ann}(\mathbf{g}_\mu)$ with kernel $\operatorname{ann}(\mathbf{n}_\mu)$. Then for each ζ sufficiently close to 0 in $\operatorname{ann}(\mathbf{n}_\mu)$ and each $\xi \in \mathbf{g}_\mu$ the equation*

$$\mathrm{P}_{\operatorname{ann}(\mathbf{g}_\mu)}\left(\operatorname{ad}_{\xi+\eta}^*(\mu+\zeta)\right) \;=\; 0 \tag{2.6}$$

has a unique solution $\eta = \eta_\mu(\xi,\zeta) \in \mathbf{n}_\mu$. The map $\eta_\mu : \mathbf{g}_\mu \oplus \operatorname{ann}(\mathbf{n}_\mu) \to \mathbf{n}_\mu$, defined on the whole of $\mathbf{g}_\mu$ and a neighbourhood of $0 \in \operatorname{ann}(\mathbf{n}_\mu)$, is smooth, linear in ξ, and satisfies $\eta_\mu(\xi,0) = 0$ for all $\xi \in \mathbf{g}_\mu$ and $\eta_{\lambda\mu}(\xi,\lambda\zeta) = \eta_\mu(\xi,\zeta)$ for all $\lambda \in \mathbb{R}$.

Proof. For fixed $\xi \in \mathbf{g}_\mu$ and $\zeta \in \operatorname{ann}(\mathbf{n}_\mu)$ equation (2.6) can be written as an inhomogeneous linear system of equations for $\eta \in \mathbf{n}_\mu$:

$$\mathrm{P}_{\operatorname{ann}(\mathbf{g}_\mu)}\left(\operatorname{ad}_\eta^*(\mu+\zeta)\right) \;=\; -\mathrm{P}_{\operatorname{ann}(\mathbf{g}_\mu)}\left(\operatorname{ad}_\xi^*(\mu+\zeta)\right). \tag{2.7}$$

It follows that there will be a unique solution for $\eta = \eta_\mu(\xi,\zeta)$ if and only if the linear map $\eta \mapsto \mathrm{P}_{\operatorname{ann}(\mathbf{g}_\mu)}\left(\operatorname{ad}_\eta^*(\mu+\zeta)\right)$ is injective (and hence also surjective). To prove injectivity note that

$$\begin{aligned}\mathrm{P}_{\operatorname{ann}(\mathbf{g}_\mu)}\left(\operatorname{ad}_\eta^*(\mu+\zeta)\right) = 0 \quad &\Leftrightarrow \quad \operatorname{ad}_\eta^*(\mu+\zeta)(\psi) = 0 \quad \text{for all } \psi \in \mathbf{n}_\mu \\ &\Leftrightarrow \quad \operatorname{ad}_{\mathbf{n}_\mu}^*(\mu+\zeta)(\eta) = 0.\end{aligned}$$

Now $\operatorname{ad}_{\mathbf{n}_\mu}^*(\mu) = T_\mu G\mu = \operatorname{ann}(\mathbf{g}_\mu)$ and so $\mathbf{g}^* = \operatorname{ad}_{\mathbf{n}_\mu}^*(\mu) \oplus \operatorname{ann}(\mathbf{n}_\mu)$. If ζ is sufficiently close to 0 then $\operatorname{ad}_{\mathbf{n}_\mu}^*(\mu+\zeta)$ is a small perturbation of $\operatorname{ad}_{\mathbf{n}_\mu}^*(\mu)$ and we still have $\mathbf{g}^* = \operatorname{ad}_{\mathbf{n}_\mu}^*(\mu+\zeta) \oplus \operatorname{ann}(\mathbf{n}_\mu)$. It follows that if $\eta \in \mathbf{n}_\mu$ satisfies $\operatorname{ad}_{\mathbf{n}_\mu}^*(\mu+\zeta)(\eta) = 0$ then $\eta = 0$. Thus $\eta \mapsto \mathrm{P}_{\operatorname{ann}(\mathbf{g}_\mu)}\left(\operatorname{ad}_\eta^*(\mu+\zeta)\right)$ is injective and there exists a unique solution $\eta = \eta_\mu(\xi,\zeta)$ to (2.6).

The coefficients of the linear system (2.7) depend smoothly on ξ and ζ and hence so also will the solution $\eta_\mu(\xi,\zeta)$. If $\zeta = 0$ then the righthand side of (2.7) is 0 and so $\eta = \eta_\mu(\xi,0) = 0$. If $\eta_\mu(\xi_i,\zeta)$ is a solution of (2.7) with $\xi = \xi_i$, $i = 1,2$ then for all

$\lambda_1, \lambda_2 \in \mathbb{R}$ the sum $\lambda_1 \eta_\mu(\xi_1, \zeta) + \lambda_2 \eta_\mu(\xi_2, \zeta)$ is a solution for $\xi = \lambda_1 \xi_1 + \lambda_2 \xi_2$. Hence, by uniqueness,

$$\eta_\mu(\lambda_1 \xi_1 + \lambda_2 \xi_2, \zeta) \;=\; \lambda_1 \eta_\mu(\xi_1, \zeta) + \lambda_2 \eta_\mu(\xi_2, \zeta).$$

Similarly, if $\lambda \in \mathbb{R}$ and $\eta_\mu(\xi, \zeta)$ is a solution for particular values for μ and ν, then it also is for $\lambda\mu$ and $\lambda\zeta$, and so $\eta_{\lambda\mu}(\xi, \lambda\zeta) = \eta_\mu(\xi, \zeta)$. ∎

The following proposition describes some further properties of $\eta_\mu(\xi, \zeta)$ that are useful in applications.

Proposition 2.6

a) *Let $\chi : G \to \{\pm 1\}$ be any group homomorphism. If K is a subgroup of*

$$\{g \in G \;:\; \chi(g)\mathrm{Ad}^*_{g^{-1}}(\mu) = \mu\}$$

and $\mathbf{n}_\mu$ is invariant under the restriction to K of the adjoint action of G on $\mathbf{g}$ then, for all ζ near $0 \in \mathrm{ann}(\mathbf{n}_\mu)$, η_μ is K-equivariant in the sense that

$$\eta_\mu(\mathrm{Ad}_g(\xi), \chi(g)\mathrm{Ad}^*_{g^{-1}}(\zeta)) = \mathrm{Ad}_g(\eta_\mu(\xi, \zeta)) \quad \textit{for all } g \in K. \tag{2.8}$$

Moreover we have

$$\eta_\mu(\xi, \zeta) = 0 \qquad \textit{for all } \xi \in \mathbf{k}. \tag{2.9}$$

b) *Let G^0_μ denote the identity component of G_μ. If $\mathbf{n}_\mu$ is a G^0_μ-invariant complement to $\mathbf{g}_\mu$ in $\mathbf{g}$ then $\eta_\mu \equiv 0$.*

Proof.

a) If $\mathbf{n}_\mu$ is K-invariant then for all $g \in K$

$$\mathrm{P}_{\mathrm{ann}(\mathbf{g}_\mu)}\left(\mathrm{ad}^*_{\mathrm{Ad}_g(\xi+\eta)}(\mu + \chi(g)\mathrm{Ad}^*_{g^{-1}}(\zeta))\right) = \chi(g)\mathrm{Ad}^*_{g^{-1}}\mathrm{P}_{\mathrm{ann}(\mathbf{g}_\mu)}\left(\mathrm{ad}^*_{\xi+\eta}(\mu+\zeta)\right).$$

It follows that if $\eta_\mu(\xi, \zeta)$ is a solution of (2.7) for some ξ, ζ, then for all $g \in K$ so is $(\mathrm{Ad}_g)^{-1}(\eta_\mu(\mathrm{Ad}_g(\xi), \chi(g)\mathrm{Ad}^*_{g^{-1}}(\zeta)))$. Hence, by uniqueness,

$$(\mathrm{Ad}_g)^{-1}\left(\eta_\mu(\mathrm{Ad}_g(\xi), \chi(g)\mathrm{Ad}^*_{g^{-1}}(\zeta))\right) \;=\; \eta_\mu(\xi, \zeta),$$

giving (2.8). The K-invariance of $\mathbf{n}_\mu$ also implies that $[\mathbf{k}, \mathbf{n}_\mu] \subset \mathbf{n}_\mu$ and so, if $\xi \in \mathbf{k}$, $\zeta \in \mathrm{ann}(\mathbf{n}_\mu)$ and $\psi \in \mathbf{n}_\mu$, we have

$$\mathrm{ad}^*_\xi(\mu+\zeta)(\psi) \;=\; \mathrm{ad}^*_\xi(\zeta)(\psi) \;=\; \zeta([\xi, \psi]) \;=\; 0.$$

Hence $\mathrm{P}_{\mathrm{ann}(\mathbf{g}_\mu)}\left(\mathrm{ad}^*_\xi(\mu+\zeta)\right) = 0$ and $\eta_\mu(\xi, \zeta) = 0$.

b) This follows immediately from (2.9) in part a) with $K = G^0_\mu$.

∎

We will follow §2.3 of Guillemin et al. [9] and say that μ is *split* if there exists a G_μ^0 invariant complement $\mathbf{n}_\mu$ to $\mathbf{g}_\mu$ in $\mathbf{g}$. This implies that η_μ is identically zero. If G_μ^0 is compact then of course μ is split. An example of a split μ with noncompact G_μ^0 is given in §4.6. We also give an example of a μ that is not split.

More generally, if K is any compact subgroup of G_μ then $\mathbf{n}_\mu$ can be chosen to be at least K-invariant. In particular, in the rest of this paper we will always choose it to be invariant under the action of the compact phase space isotropy subgroup G_p.

We note that it is almost, but not quite, true that if $\mathbf{n}_\mu$ is G_μ-invariant then it provides a slice for the coadjoint action at μ. For precise statements and a discussion see §2.3 of [9]. There it is also shown that if $\mu+\mathrm{ann}(\mathbf{n}_\mu)$ is a slice at μ and ζ is sufficiently close to the origin in $\mathrm{ann}(\mathbf{n}_\mu)$ then the coadjoint orbit $G(\mu+\zeta)$ fibres symplectically over $G\mu$ with fibre a coadjoint orbit of G_μ.

For each ζ sufficiently close to 0 in $\mathrm{ann}(\mathbf{n}_\mu)$ we now define a linear map $j = j_\mu(\zeta)$: $\mathbf{g}_\mu \to \mathbf{g}$ by $j_\mu(\zeta)\xi = \xi + \eta_\mu(\zeta, \xi)$. If $\mathbf{n}_\mu$ is K invariant for some subgroup K of G_μ then (2.9) implies that $\eta_\mu(\zeta, \cdot)$ and $j_\mu(\zeta)$ descend to give well defined linear maps from $\mathbf{g}_\mu/\mathbf{k}$ to $\mathbf{n}_\mu$ and $\mathbf{g}/\mathbf{k}$, respectively. In particular this can always be assumed to hold for $K = G_p$. If μ is split then $j_\mu(\zeta)$ is just the inclusion of $\mathbf{g}_\mu$ into $\mathbf{g}$ and descends to the inclusion of $\mathbf{g}_\mu/\mathbf{k}$ into $\mathbf{g}/\mathbf{k}$.

3 Hamiltonian systems near group orbits

In this section we present our main results on the structure of a reversible equivariant Hamiltonian system (1.9) in a neighbourhood of a group orbit. The local parametrization (1.7) given by the slice theorem near a group orbit still holds. However the symplectic form imposes additional structure and there are further decompositions of the vector field f_G along the group orbit and the normal vector field f_N.

3.1 The main results

Let $(\mathcal{M}, \omega)$ be a symplectic manifold with a smooth proper χ-semisymplectic action of the Lie group G. We consider a point p in $\mathcal{M}$ with isotropy G_p. It follows from the Witt decomposition of the symplectic form at p, as described in Proposition 2.1, that the normal space N can be decomposed as:

$$N \;=\; N_0 \oplus N_1 \;\cong\; (\mathbf{g}_\mu/\mathbf{g}_p)^* \oplus N_1$$

where N_1 is the symplectic normal space at p with its induced χ-semisymplectic linear action of the isotropy subgroup G_p. Note that by Proposition 2.1 the action of G_p on N_0 is the χ-coadjoint action of G_p on $(\mathbf{g}_\mu/\mathbf{g}_p)^*$. We fix a G_p-invariant inner product $< \cdot, \cdot >_{N_1}$ on N_1 and a (G_p, χ)-semiequivariant complex structure J satisfying equation (2.2).

Choose G_p-invariant complements $\mathbf{m}_\mu$ to $\mathbf{g}_p$ in $\mathbf{g}_\mu$ and $\mathbf{n}_\mu$ to $\mathbf{g}_\mu$ in $\mathbf{g}$. Then $\mathbf{g} = \mathbf{g}_p \oplus \mathbf{m}_\mu \oplus \mathbf{n}_\mu$ and $\mathbf{g}^* = \mathrm{ann}(\mathbf{m}_\mu \oplus \mathbf{n}_\mu) \oplus \mathrm{ann}(\mathbf{g}_p \oplus \mathbf{m}_\mu) \oplus \mathrm{ann}(\mathbf{g}_p \oplus \mathbf{m}_\mu)$. These choices of complements also define G_p-equivariant linear isomorphisms (as discussed in Section 2.2):

$$\begin{array}{lclcl}
\mathrm{ann}(\mathbf{n}_\mu) & \cong & \mathbf{g}_\mu^* & & \\
\mathrm{ann}(\mathbf{m}_\mu \oplus \mathbf{n}_\mu) & \cong & \mathrm{ann}_{\mathbf{g}_\mu^*}(\mathbf{m}_\mu) & \cong & \mathbf{g}_p^* \\
\mathrm{ann}(\mathbf{g}_p \oplus \mathbf{n}_\mu) & \cong & \mathrm{ann}_{\mathbf{g}_\mu^*}(\mathbf{g}_p) & \cong & (\mathbf{g}_\mu/\mathbf{g}_p)^* \\
\mathrm{ann}(\mathbf{g}_p \oplus \mathbf{m}_\mu) & \cong & \mathbf{n}_\mu^* & &
\end{array}$$

where $\mathrm{ann}(\cdot)$ denotes an annihilator in $\mathbf{g}^*$ and $\mathrm{ann}_{\mathbf{g}_\mu^*}(\cdot)$ an annihilator in $\mathbf{g}_\mu^*$. Let $H : \mathcal{M} \to \mathbb{R}$ denote the smooth G-invariant Hamiltonian on $\mathcal{M}$ generating the Hamiltonian vectorfield (1.9).

Theorem 3.1 *Let (g, ν, w) be the bundle coordinates from the slice theorem parametrizing a G-invariant neighbourhood of Gp and let $h(\nu, w)$ be the restriction of the Hamiltonian H to the slice $N = (\mathbf{g}_\mu/\mathbf{g}_p)^* \oplus N_1$. Define $\tilde{h} : \mathbf{g}_\mu^* \oplus N_1 \to \mathbb{R}$ by $\tilde{h}(\zeta, w) = h(\nu, w)$ where $\zeta = \nu + \lambda \in \mathbf{g}_\mu^*$, $\nu \in (\mathbf{g}_\mu/\mathbf{g}_p)^*$ and $\lambda \in \mathbf{g}_p^*$. Then*

$$g(t) \in G, \quad \zeta(t) = \nu(t) + \mathbf{J}_{N_1}(w(t)) \in \mathbf{g}_\mu^*, \quad w(t) \in N_1$$

satisfy the differential equations

$$\begin{array}{rcl}
\dot{g} & = & \chi(g)\ g\ j_\mu(\zeta)\mathrm{D}_\zeta \tilde{h} \\
\dot{\zeta} & = & \chi(g)\ \mathrm{ad}^*_{j_\mu(\zeta)\mathrm{D}_\zeta \tilde{h}}(\mu + \zeta) \\
\dot{w} & = & \chi(g)\ J D_w \tilde{h}(\zeta, w)
\end{array}$$

where

$$j_\mu(\zeta)\mathrm{D}_\zeta \tilde{h} \ = \ \mathrm{D}_\zeta \tilde{h}(\zeta, w) \ + \ \eta_\mu(\mathrm{D}_\zeta \tilde{h}(\zeta, w), \zeta)$$

and $\eta_\mu : \mathbf{g}_\mu \oplus \mathbf{g}_\mu^* \to \mathbf{n}_\mu$ is the map defined in §2.4.

A proof of Theorem 3.1 is given in Section 5. Note that if μ is split, for example if G_μ is compact, then by Proposition 2.5 we have $\eta_\mu \equiv 0$ and $j_\mu \mathrm{D}_\zeta \tilde{h} = \mathrm{D}_\zeta \tilde{h}$.

Projecting the ζ-equation in Theorem 3.1 back to N_0 we obtain the following explicit differential equations in the bundle coordinates (g, ν, w):

Theorem 3.2 *Coordinates (g, ν, w) can be chosen on $G \times N = G \times ((\mathbf{g}_\mu/\mathbf{g}_p)^* \oplus N_1)$ so that the restriction of the (G, χ)-semiequivariant Hamiltonian system (1.9) to a neighbourhood of Gp can be lifted to the following system on $G \times N$:*

$$\dot{g} \ = \ \chi(g) g \left(D_\nu h(\nu, w) + \hat{\eta}(\nu, w)\right) \tag{3.1}$$

$$\begin{aligned}\dot{\nu} \ = \ \chi(g) \Big(&\overline{\mathrm{ad}}^*_{D_\nu h(\nu,w)}(\nu) + \mathrm{ad}^*_{D_\nu h(\nu,w)}(\mathbf{J}_{N_1}(w)) \\ &+ \mathrm{P}(\mathrm{ad}^*_{\hat{\eta}(\nu,w)}(\nu + \mathbf{J}_{N_1}(w)))\Big)\end{aligned} \tag{3.2}$$

$$\dot{w} \ = \ \chi(g) J D_w h(\nu, w) \tag{3.3}$$

where $h(\nu,w)$ is the function obtained by restricting the Hamiltonian H to the slice N, the map $\hat{\eta} : N \to \mathbf{n}_\mu$ is given by $\hat{\eta}(\nu,w) = \eta_\mu(D_\nu h(\nu,w), \nu + \mathbf{J}_{N_1}(w))$ and P is the projection from $\mathbf{g}^*$ to $\mathrm{ann}(\mathbf{g}_p + \mathbf{n}_\mu) \cong (\mathbf{g}_\mu/\mathbf{g}_p)^*$ with kernel $\mathrm{ann}(\mathbf{m}_\mu)$.

Details of the proof of Theorem 3.2 are given in Section 5.5 below.

The derivative $\mathrm{D}_\nu h(\nu,w)$ is an element of $\mathbf{g}_\mu/\mathbf{g}_p$ and so can also be identified with an element of $\mathbf{m}_\mu$. In equation (3.2) $\overline{\mathrm{ad}}^*_{D_\nu h(\nu,w)}$ is the linear operator from $(\mathbf{g}_\mu/\mathbf{g}_p)^*$ to itself defined in §2.3 with $\mathbf{g}$ replaced by $\mathbf{g}_\mu$ and $\mathbf{k}$ by $\mathbf{g}_p$. The operator $\mathrm{ad}^*_{D_\nu h(\nu,w)}$ is the usual coadjoint operator from $\mathbf{g}^*_\mu$ to itself, with $D_\nu h(\nu,w)$ being regarded as an element of $\mathbf{m}_\mu$. The map $\mathbf{J}_{N_1}$ takes values in $\mathbf{g}^*_\mu$ via the embedding $\mathbf{g}^*_p \cong \mathrm{ann}_{\mathbf{g}^*_\mu}(\mathbf{m}_\mu)$ described in §2.2. We will prove in Lemma 5.5e) below that $\mathrm{ad}^*_{D_\nu h(\nu,w)}(\mathbf{J}_{N_1}(w)) \in (\mathbf{g}_\mu/\mathbf{g}_p)^*$.

The map $\hat{\eta} : (\mathbf{g}_\mu/\mathbf{g}_p)^* \oplus N_1 \to \mathbf{n}_\mu$ is defined using the map η_μ defined and discussed in §2.4. The operator $\mathrm{ad}^*_{\hat{\eta}(\nu,w)}$ in equation (3.2) is the coadjoint operator from $\mathbf{g}^*$ to itself. The projection P maps the image of this operator into $(\mathbf{g}_\mu/\mathbf{g}_p)^*$.

Remark 3.3

a) (Relative equilibria) *As in the general case mentioned in the introduction the orbit Gp is a relative equilibrium if and only if $(\nu,w) = (0,0)$ is an equilibrium point of the slice equations and so if and only if $\mathrm{D}_w h(0,0) = 0$. The drift velocity of the relative equilibrium is $\xi = \mathrm{D}_\nu h(0,0)$.*

b) (Interpretation) *In a typical application, for example a gravitational or molecular N-body problem, the component N_1 of the slice describes the shape dynamics or internal vibrations of the system, while N_0 describes the motion of the angular momentum in body coordinates, that is in a frame moving with the velocity $g^{-1}(t)\dot{g}(t)$ of the motion $g(t)$ of the center of mass. Because of this we call the momentum $\mathbf{J}_{N_1}(w)$ the vibrational angular momentum. For a thorough discussion of the separation of shape and rotational dynamics in the context of N-body problems see [29].*

3.2 Symmetry properties

As in (1.7), along a trajectory of the vector field the function $\chi(g(t))$ is independent of t and so the equations (3.1, 3.2, 3.3) are essentially independent of $g(t)$. Taking $\chi(g) = 1$ the vector field f_G along the group orbit decomposes into a component in the $\mathbf{g}_\mu/\mathbf{g}_p$ direction and a component in the $\mathbf{n}_\mu$ direction:

$$f_G(v) \;=\; f_G(\nu,w) \;=\; j_\mu(\nu + \mathbf{J}_{N_1}(w))\mathrm{D}_\nu h(\nu,w) \;=\; \mathrm{D}_\nu h(\nu,w) + \hat{\eta}(\nu,w). \qquad (3.4)$$

Equations (3.2, 3.3) form an autonomous system on the normal space N called the slice equations. The vector field f_N is given by:

$$f_N(\nu,w) = \begin{pmatrix} \overline{\mathrm{ad}}^*_{D_\nu h(\nu,w)}(\nu) + \mathrm{ad}^*_{D_\nu h(\nu,w)}(\mathbf{J}_{N_1}(w)) + \mathrm{P}(\mathrm{ad}^*_{\hat\eta(\nu,w)}(\nu + \mathbf{J}_{N_1}(w))) \\ JD_w h(\nu,w) \end{pmatrix}. \tag{3.5}$$

The action of G_p on the slice $N = (\mathbf{g}_\mu/\mathbf{g}_p)^* \oplus N_1$ is given by $g_p(\nu,w) = (\chi(g_p)\mathrm{Ad}^*_{g_p^{-1}}\nu, g_p w)$. As in the general case (1.8) we have

Proposition 3.4 *The vectorfields f_G from (3.4) and f_N from (3.5) are (G_p,χ)-semiequivariant.*

Proof. The function $h(\nu,w)$ is G_p-invariant with respect to this action, and so $\mathrm{D}_\nu h : (\mathbf{g}_\mu/\mathbf{g}_p)^* \oplus N_1 \to \mathbf{g}_\mu/\mathbf{g}_p$ is (G_p,χ)-semiequivariant if G_p acts on $\mathbf{g}_\mu/\mathbf{g}_p$ by the standard adjoint action.

The map $\hat\eta$ is also (G_p,χ)-semiequivariant. This follows from the calculation:

$$\begin{aligned} \hat\eta\left(\chi(g_p)\mathrm{Ad}^*_{g_p^{-1}}(\nu),\ g_p.w\right) &= \eta_\mu\left(\chi(g_p)\mathrm{Ad}_{g_p}(\mathrm{D}_\nu h(\nu,w)),\ \chi(g_p)\mathrm{Ad}^*_{g_p^{-1}}(\nu + \mathbf{J}_{N_1}(w))\right) \\ &= \chi(g_p)\eta_\mu\left(\mathrm{Ad}_{g_p}(\mathrm{D}_\nu h(\nu,w)),\ \chi(g_p)\mathrm{Ad}^*_{g_p^{-1}}(\nu + \mathbf{J}_{N_1}(w))\right) \\ &= \chi(g_p)\mathrm{Ad}_{g_p}\left(\eta_\mu\left(\mathrm{D}_\nu h(\nu,w),\ \nu + \mathbf{J}_{N_1}(w)\right)\right) \\ &= \chi(g_p)\mathrm{Ad}_{g_p}\left(\hat\eta(\nu,w)\right). \end{aligned}$$

The first equality follows from the definition of $\hat\eta$, the (G_p,χ)-semiequivariance of $\mathrm{D}_\nu h(\nu,w)$ and the (G_p,χ)-semiequivariance of $\mathbf{J}_{N_1}$. The second equality is a consequence of the linearity of $\eta_\mu(\xi,\zeta)$ in ξ and the third equality follows from statement 2 of Proposition 2.6 and the fact that $\chi(g_p)\mathrm{Ad}^*_{g_p^{-1}}\mu = \mu$, $g_p \in G_p$.

It follows from this that f_G is (G_p,χ)-semiequivariant, as for general reversible equivariant systems (1.7). Similarly the (G_p,χ)-semiequivariance of $f_N(\nu,w)$ follows from the (G_p,χ)-semiequivariance of $\mathrm{D}_\nu h(\nu,w)$ and $\hat\eta(\nu,w)$, the (G_p,χ)-semiequivariance of $\mathbf{J}_{N_1}$ and J, and the G_p-equivariance of $\mathrm{D}_w h(\nu,w)$, ad^*, $\overline{\mathrm{ad}}^*$ (see 2.5) and the projection P. ∎

There are a number of important special cases for which one or more of the terms in the expressions for f_G and f_N given by equations (3.4) and (3.5) vanish. If μ is *split* then $\hat\eta(\nu,w) = 0$ and $\mathrm{P}(\mathrm{ad}^*_{\hat\eta(\nu,w)}(\nu + \mathbf{J}_{N_1}(w))) = 0$, as will be described in §4.1. The term $\mathrm{ad}^*_{D_\nu h(\nu,w)}(\mathbf{J}_{N_1}(w))$ is identically zero if $\mathbf{g}_p$ is a *conormal* subalgebra of $\mathbf{g}_\mu$, while $\overline{\mathrm{ad}}^*_{D_\nu h(\nu,w)}(\nu)$ is identically zero if $\mathbf{g}_p$ is a *symmetric* subalgebra of $\mathbf{g}_\mu$. These cases are discussed in §4.3 and §4.4, respectively. If μ is a *minimal* element of $\mathbf{g}^*$, ie the momentum isotropy subgroup G_μ has minimal dimension, then all three of these terms vanish, see §4.2.

In the cases that all three terms in the $\dot\nu$ equation vanish the 'angular momentum in body coordinates' ν is conserved and so can be regarded as a parameter in the

$\dot{w}$ equation. Investigations of bifurcations of relative equilibria (and more complex trajectories) from Gp can be reduced to studies of the dynamics of the ν-dependent $\dot{w}$-equation with Hamiltonian $h_\nu(w) = h(\nu, w)$ on N_1. In particular, it follows immediately that if the relative equilibrium is nondegenerate, in the sense that the second derivative $\mathrm{D}^2_{ww}h(0,0)$ of h with respect to w has maximal rank, then there exists a smooth family of nearby relative equilibria parametrised by $\nu \in (\mathbf{g}_\mu/\mathbf{g}_p)^*$.

When $\dot{\nu} \equiv 0$ results on bifurcations from equilibrium points of Hamiltonian systems can also be applied directly to the $\dot{w}$ equation to obtain analogous results on bifurcations from relative equilibria. Since the family of Hamiltonians $h(\nu, w)$ is invariant under the action of G_p on the slice, results on bifurcations from equilibria of symmetric Hamiltonian systems are particularly relevant [8, 4, 40, 53]. Note however that the individual Hamiltonian $h_\nu(w)$ is only invariant under the isotropy subgroup $(G_p)_\nu$ and so the effects of symmetry breaking in families of Hamiltonians needs to be taken into account (see for example [37]).

3.3 Related results in the literature

The $(\dot{\zeta}, \dot{w})$ equations in Theorem 3.1 are an extension to noncompact semisymplectic group actions of a result given in [50]. In the absence of time-reversal symmetries the equations of Theorem 3.2 can be regarded as a particular solution of the 'reconstruction equations' of Ortega and Ratiu [41, 43].

The case where the (time-reversal) symmetry group G decomposes into a product of the isotropy group G_p of the relative equilibrium and another group $\widetilde{\Gamma}$ was treated by Mielke [35]. This is an example of the conormal isotropy subalgebra case treated in section 4.3 below. Mielke derived the corresponding differential equations, but did not make explicit the Poisson structure of the generalized Hamiltonian vectorfield on the slice $N = N_0 \oplus N_1$. His equations [35, (5.6), (5.7)] appear as follows:

$$\dot{g} = gJ_{gy}(y)\mathrm{D}_y H, \quad \dot{y} = J_y(y)\mathrm{D}_y H \tag{3.6}$$

where

$$J(g, y) = \begin{pmatrix} J_g(y) & J_{gy}(y) \\ -J_{gy}^T(y) & J_y(y) \end{pmatrix}$$

generates the symplectic form ω on $\mathcal{M}$. Here $y \in N$ and the $\dot{y}$-equation is a Poisson-system. Mielke also treats the case of time-reversal symmetries which are involutions and respect the decomposition of G into $G_p \times \widetilde{\Gamma}$.

Note that from Theorem 3.2 we see that the Hamiltonian equations near group orbits indeed always have the form (3.6) and that the theorem provides explicit expressions for the matrix $J(g, y)$:

$$J_{gy}(y) = j_\mu(\nu + \mathbf{J}_{N_1}(w)), \quad J_y(y) = \begin{pmatrix} \overline{\mathrm{ad}}^*_\cdot \nu + \mathrm{ad}^*_\cdot \mathbf{J}_{N_1}(w) & 0 \\ 0 & J \end{pmatrix} \quad \text{where} \quad y = (\nu, w).$$

Finally $J_g(y)$ is determined by the KKS form on T_1 evaluated at $\mu + \nu + \mathbf{J}_{N_1}(w)$, as described in §2.2.

Mielke [35] studied Hamiltonian systems on cotangent bundles in more detail, and in the case of a relative equilibrium with $\mu = 0$ he obtained our explicit differential equations (3.1, 3.2, 3.3).

3.4 Conservation of momentum

In the remainder of this section we discuss issues related to the conservation of momenta. A (G, χ)-reversible equivariant flow generated by an invariant Hamiltonian on $\mathcal{M}$ preserves the level sets of the momentum map $\mathbf{J} : \mathcal{M} \to \mathbf{g}^*$. The following proposition of Guillemin, Sternberg and Marle describes the form this takes in the coordinates used in Theorem 3.2. Recall that we have fixed G_p-invariant complements to $\mathbf{g}_p$ in $\mathbf{g}_\mu$ and $\mathbf{g}_\mu$ in $\mathbf{g}$ and hence also linear embeddings $\mathbf{g}_p^* \subset \mathbf{g}_\mu^* \subset \mathbf{g}^*$. We will use these implicitly throughout the remainder of this section.

Proposition 3.5 *([10, 11, 30]) In the coordinates used in Theorem 3.2 the momentum map $\mathbf{J}$ near an orbit Gp with $\mathbf{J}(p) = \mu$ is given by:*

$$\mathbf{J}(g, \nu, w) \;=\; \chi(g)\mathrm{Ad}^*_{g^{-1}}\left(\mu + \nu + \mathbf{J}_{N_1}(w)\right).$$

This proposition is reproved in §3.5 for the sake of completeness. It follows that $\mathrm{Ad}^*_{g^{-1}}\left(\mu + \nu + \mathbf{J}_{N_1}(w)\right)$ must be constant along the trajectories of the equations in Theorem 3.2. The following result describes the momentum conservation properties of the slice equations (3.2, 3.3).

Proposition 3.6 *Assume the hypotheses and notation of Theorem 3.2.*

a) If μ is split and $g(0) \in G_\mu$ then $g(t) \in G_\mu$ for all t.

b) If μ is split and $(\nu(t), w(t))$ is a solution of the equations (3.2, 3.3) then

$$(\nu(t), \mathbf{J}_{N_1}(w(t))) \;\in\; G_\mu(\nu(0), \mathbf{J}_{N_1}(w(0))) \;\subset\; \mathbf{g}_\mu^*$$

for all t.

c) In general the evolution of the vibrational angular momentum $J_{N_1}(w)$ along solutions of equations (3.2, 3.3) is given by:

$$\dot{\mathbf{J}}_{N_1}(w) \;=\; \widehat{\mathrm{P}}\left(\mathrm{ad}^*_{D_\nu h(\nu,w)}(\nu)\right).$$

where $\widehat{\mathrm{P}}$ is the projection from $\mathbf{g}_\mu^*$ to $\mathbf{g}_p^*$.

If G_p^0 is a normal subgroup of G_μ^0 (for example if G_μ^0 is Abelian) then Prop. 3.6 c) and the fact that $[\mathbf{g}_p, \mathbf{m}_\mu] = 0$, by Lemma 2.3 a), imply that $\dot{\mathbf{J}}_{N_1}(w) = 0$ and so the vibrational angular momentum is preserved. Note that this is always the case if G_μ has minimal dimension, see §4.2. There we will see that such μ are generic in $\mathbf{g}^*$ and that in this case $\dot{\nu} = 0$, so the momentum ν in body coordinates is also conserved.

Proof. If μ is split then $\hat{\eta}(\nu, w) \equiv 0$ and so equation (3.1) implies that $\dot{g} \in g\mathbf{g}_\mu$ at all points (g, ν, w). It follows that $G_\mu \times N$ is invariant under the flow of the equations (3.1, 3.2, 3.3) and so $g(t) \in G_\mu$ for all t if $g(0) \in G_\mu$. This proves part a) of the proposition.

Using part a) of the proposition, together conservation of $\mathbf{J}$ and Proposition 3.5, gives (with $(g(0) = \mathrm{id})$

$$\mu + (\nu(0), \mathbf{J}_{N_1}(w(0))) = \mathrm{Ad}^*_{g(t)}(\mu + (\nu(t), \mathbf{J}_{N_1}(w(t))) = \mu + \mathrm{Ad}^*_{g(t)}(\nu(t), \mathbf{J}_{N_1}(w(t))$$

for all t.

Statement c) can be deduced from Theorem 3.1. Details are postponed to §5.6. ∎

4 Special cases and examples

In this section we describe a number of special cases for which the form of the equations of motion in Theorem 3.2 simplifies. The terms in the $\dot{\nu}$ equations (3.2) which vanish for each of these special cases are summarised in Table 1. The final two subsections shows how these special cases arise naturally in examples with $\mathbf{g} = \mathbf{se}(2)$ and $\mathbf{se}(3)$, the Lie algebras of the special Euclidean groups SE(2) and SE(3).

4.1 Split momenta

If μ is split then by Proposition 2.6 b) the map $\hat{\eta}$ in Theorem 3.2 is identically zero and so $f_G(\nu, w) = \mathrm{D}_\nu h(\nu, w) \in \mathbf{g}_\mu/\mathbf{g}_p$ and the term $\mathrm{P}(\mathrm{ad}^*_{\hat{\eta}(\nu,w)}(\nu + \mathbf{J}_{N_1}(w)))$ in the $\dot{\nu}$ equation is identically zero. In particular this simplification will occur if there exists a G^0_μ-invariant inner product on $\mathbf{g}$, which in turn occurs if G^0_μ is compact. In §4.5 we show that it also occurs if G^0 is the special Euclidean group SE(2), while in §4.6 we show that for $G^0 = \mathrm{SE}(3)$ there are momenta $\mu \in \mathbf{g}^*$ which are not split.

If in addition to μ being split the isotropy subalgebra $\mathbf{g}_\mu$ is Abelian, then (3.2) reduces to $\dot{\nu} = 0$. For compact groups these two conditions always hold for minimal momentum values μ. The following section shows that a similar simplification holds for minimal μ for arbitrary Lie groups, even though in general such μ need not be split.

4.2 Minimal momenta

Let $r = \min\{\dim \mathbf{g}_\mu \mid \mu \in \mathbf{g}^*\}$. We say that $\mu \in \mathbf{g}^*$ is *minimal* if $\dim \mathbf{g}_\mu = r$. The following result is due to Duflo and Vergne [5].

Proposition 4.1

a) The set of μ which are minimal is open and dense in $\mathbf{g}^*$.

b) If μ is minimal then $\mathbf{g}_\mu$ is Abelian.

For a proof see also [32](Theorem 9.3.10). In the case of minimal momentum values the momentum ν in body coordinates is preserved:

Proposition 4.2 *If μ is minimal, then $\dot{\nu} \equiv 0$.*

Proof. We show that $\mathrm{ad}^*_{\xi+\eta_\mu(\xi,\zeta)}(\mu+\zeta) = 0$ for all $\xi \in \mathbf{g}_\mu$ and ζ sufficiently close to 0 in $\mathrm{ann}(\mathbf{n}_\mu)$. If μ is minimal then by Proposition 4.1 a) the dimension of any coadjoint orbit close to $G\mu$ is equal to that of $G\mu$ and so, for $\zeta \in \mathrm{ann}(\mathbf{g}_\mu)$ close to 0, the tangent space $\mathrm{ad}^*_{\mathbf{g}}(\mu+\zeta)$ is a subspace of $\mathbf{g}^*$ which has the same dimension as $\mathrm{ad}^*_{\mathbf{g}}(\mu) = \mathrm{ann}(\mathbf{g}_\mu)$ and is a small perturbation of it. It follows that for $\xi \in \mathbf{g}_\mu$ and $\eta \in \mathbf{n}_\mu$ we have $\mathrm{P}_{\mathrm{ann}(\mathbf{g}_\mu)}(\mathrm{ad}^*_{\xi+\eta}(\mu+\zeta)) = 0$ if and only if $\mathrm{ad}^*_{\xi+\eta}(\mu+\zeta) = 0$. The result therefore follows from the definition of η_μ given in Proposition 2.5. ∎

The fact that the isotropy subalgebra $\mathbf{g}_\mu$ is Abelian implies that a rather stronger statement is true, namely that each of the three terms on the righthand side of the $\dot{\nu}$-equation (3.2) vanishes individually. This is true even though in general $\hat{\eta}(\nu, w) \neq 0$. The following example shows that non-split minimal μ do exist.

Example 4.3 Let $G = \mathrm{SL}(2,\mathbb{R})$. A basis for the Lie algebra $\mathbf{sl}(2,\mathbb{R})$ is given by

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the Lie bracket by

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = 2e_1, \quad [e_2, e_3] = 2e_2.$$

Denote the dual basis to $\{e_1, e_2, e_3\}$ by $\{e_1^*, e_2^*, e_3^*\}$. It is easy to verify that $\mu = \mu_1 e_1^*$ is not split (see [9]). Indeed $\mathbf{g}_\mu = \mathrm{span}(e_2)$, but since $[e_2, \cdot] = \mathrm{ad}_{e_2}$ is not semisimple there is no $\mathbf{g}_\mu$-invariant complement $\mathbf{n}_\mu$ to $\mathbf{g}_\mu$ in $\mathbf{sl}(2,\mathbb{R})$. However $\mathbf{g}_\mu$ does have minimal dimension.

4.3 Conormal isotropy subalgebras

We will say that $\mathbf{g}_p$ is a *conormal* subalgebra of $\mathbf{g}_\mu$ if there exists a G_p-invariant complement $\mathbf{m}_\mu$ to $\mathbf{g}_p$ in $\mathbf{g}_\mu$ such that $[\mathbf{m}_\mu, \mathbf{m}_\mu] \subset \mathbf{m}_\mu$. Since the G_p-invariance of $\mathbf{m}_\mu$ implies that $[\mathbf{g}_p, \mathbf{m}_\mu] \subset \mathbf{m}_\mu$, this is equivalent to requiring that $\mathbf{m}_\mu$ be a normal subalgebra of $\mathbf{g}_\mu$. It follows immediately that for any $\xi \in \mathbf{g}_\mu$ and $\lambda \in \mathrm{ann}_{\mathbf{g}_\mu^*}(\mathbf{m}_\mu) \cong \mathbf{g}_p^*$ we have $\mathrm{ad}^*_\xi(\lambda) = 0$ and so the term $\mathrm{ad}^*_{\mathrm{D}_\nu h(\nu,w)}(\mathbf{J}_{N_1}(w))$ in equation (3.2) is identically zero.

Two important examples of conormal subalgebras are given in the following lemma.

Lemma 4.4

a) *If G_μ^0 is a semidirect product $G_p^0 \ltimes H$ for some normal subgroup H of G_μ^0 then $\mathbf{g}_p$ is a conormal subalgebra of $\mathbf{g}_\mu$ with normal complement $\mathbf{m}_\mu$ equal to the Lie algebra of H.*

b) *If there exists a G_μ^0-invariant inner product on $\mathbf{g}_\mu$ and $\mathbf{g}_p$ is a normal subalgebra of $\mathbf{g}_\mu$ then $\mathbf{g}_p$ is also a conormal subalgebra of $\mathbf{g}_\mu$.*

Proof.

a) This is clear.

b) Since $\mathbf{m}_\mu$ is G_p-invariant we always have $[\mathbf{g}_p, \mathbf{m}_\mu] \subset \mathbf{m}_\mu$ and so, if $\mathbf{g}_p$ is a normal subalgebra of $\mathbf{g}_\mu$, then $[\mathbf{g}_p, \mathbf{m}_\mu] = \{0\}$. Let $\langle \cdot, \cdot \rangle$ denote the G^0_μ-invariant inner product on $\mathbf{g}_\mu$. Then for all ξ, ζ and η in $\mathbf{g}_\mu$ we have: $\langle \xi, [\zeta, \eta] \rangle = -\langle [\zeta, \xi], \eta \rangle = \langle \eta, [\xi, \zeta] \rangle$. Let $\mathbf{m}_\mu$ be the orthogonal complement to $\mathbf{g}_p$ in $\mathbf{g}_\mu$ with respect to the inner product. Then putting $\xi \in \mathbf{g}_p$, $\zeta \in \mathbf{m}_\mu$ and $\eta \in \mathbf{g}_\mu$ in the above calculation, and using the fact that $[\mathbf{g}_p, \mathbf{m}_\mu] = \{0\}$, shows that $[\mathbf{m}_\mu, \mathbf{g}_\mu] \subset \mathbf{m}_\mu$ and hence that $\mathbf{m}_\mu$ is a normal subalgebra of $\mathbf{g}_\mu$.

∎

Now suppose that $\mathbf{g}_p$ satisfies the stronger condition of having complements $\mathbf{m}_\mu$ and $\mathbf{n}_\mu$ such that

$$[\mathbf{m}_\mu, \mathbf{m}_\mu + \mathbf{n}_\mu] \ \subset \ \mathbf{m}_\mu + \mathbf{n}_\mu. \tag{4.1}$$

In particular this is true if $\mathbf{g}_p$ is a conormal subalgebra of the whole of $\mathbf{g}$, so that the complements $\mathbf{m}_\mu$ and $\mathbf{n}_\mu$ can be chosen so that $\mathbf{m}_\mu + \mathbf{n}_\mu$ is a normal subalgebra of $\mathbf{g}$. Then a similar argument shows that the term $\mathrm{P}(\mathrm{ad}^*_{\hat\eta(\nu,w)}(\mathbf{J}_{N_1}(w)))$ in equation 3.2 also vanishes identically. In this case the equations (3.1), (3.2) and (3.3) reduce to

$$\begin{aligned}
\dot g &= \chi(g) g \left(D_\nu h(\nu, w) + \hat\eta(\nu, w)\right) \\
\dot\nu &= \chi(g) \left(\overline{\mathrm{ad}}^*_{D_\nu h(\nu,w)}(\nu) \ + \ \mathrm{P}(\mathrm{ad}^*_{\hat\eta(\nu,w)}(\nu))\right) \\
\dot w &= \chi(g) J D_w h(\nu, w).
\end{aligned}$$

An example for which $\mathbf{g}_p$ has complements satisfying (4.1), but which is not conormal in $\mathbf{g}$, is given in §4.6. If $\mathbf{g}_p$ is conormal in $\mathbf{g}$ then we see from the definition of $\eta_\mu(\xi, \zeta)$ that $\eta_\mu(\xi, \nu + \mathbf{J}_{N_1}(w)) = \eta_\mu(\xi, \nu)$ and $j_\mu(\nu + \mathbf{J}_{N_1}(w)) = j_\mu(\nu)$ so that the $\dot g$-equation and the $\dot\nu$-equation do not depend explicitly on $\mathbf{J}_{N_1}(w)$.

If, as in statement b) of the Lemma, $\mathbf{g}_p$ is a normal subalgebra of $\mathbf{g}_\mu$, then by Lemma 2.3 the bracket $[\cdot, \cdot]_{\mathbf{m}_\mu}$ induced on $\mathbf{g}_\mu / \mathbf{g}_p$ by a G_p-invariant complement $\mathbf{m}_\mu$ to $\mathbf{g}_p$ in $\mathbf{g}_\mu$ is just the natural Lie bracket on the quotient algebra. As a consequence the operator $\overline{\mathrm{ad}}^*_{D_\nu h(\nu,w)}$ becomes the natural $\mathrm{ad}^*_{D_\nu h(\nu,w)}$ operator on $(\mathbf{g}_\mu / \mathbf{g}_p)^*$. If there is a G^0_μ-invariant inner product on $\mathbf{g}_\mu$ as in statement b) of the lemma then by Lemma 2.4 we can identify the coadjoint and adjoint G^0_μ-actions and, if in addition μ is split, we get the simple equation $\dot\nu = [\mathrm{D}_\nu h(\nu, w), \nu]$ for the momenta in body coordinates.

If $\mathbf{g}_p$ is trivial then of course it is both normal and conormal in $\mathbf{g}$. More generally this is true if G^0 is the direct product of G^0_p and another group H, say. This is the case considered by Mielke [35].

4.4 Symmetric isotropy subalgebras

We say that $\mathbf{g}_p$ is a *symmetric* subalgebra of $\mathbf{g}_\mu$ if there exists a G_p-invariant complement $\mathbf{m}_\mu$ to $\mathbf{g}_p$ in $\mathbf{g}$ such that $[\mathbf{m}_\mu, \mathbf{m}_\mu] \subset \mathbf{g}_p$. If G_μ and G_p are connected then this is equivalent to the quotient manifold G_μ/G_p being a symmetric space in the usual sense. The simplest example, and an important one for applications, is given by $\mathbf{g}_\mu = \mathbf{so(3)}$ and $\mathbf{g}_p = \mathbf{so(2)}$.

For symmetric subalgebras it follows from the definition that the bracket $[\cdot,\cdot]_{\mathbf{m}_\mu}$ on $\mathbf{g}_\mu/\mathbf{g}_p$ is identically zero and hence so are the $\overline{\mathrm{ad}}^*$ operators on $(\mathbf{g}_\mu/\mathbf{g}_p)^*$. Thus the $\dot\nu$ equation (3.2) (with $\chi(g) = 1$) becomes:

$$\dot\nu = \mathrm{ad}^*_{D_\nu h(\nu,w)}(\mathbf{J}_{N_1}(w)) + \mathrm{P}(\mathrm{ad}^*_{\hat\eta(\nu,w)}(\nu + \mathbf{J}_{N_1}(w))).$$

If $\mathbf{g}_p$ is a symmetric subalgebra of the whole of $\mathbf{g}$ or, more generally, if

$$[\mathbf{m}_\mu, \mathbf{m}_\mu + \mathbf{n}_\mu] \ \subset \ \mathbf{g}_p + \mathbf{n}_\mu. \tag{4.2}$$

then the term $\mathrm{P}(\mathrm{ad}^*_{\hat\eta(\nu,w)}(\nu))$ also vanishes. If μ is split then of course the whole of $\mathrm{P}(\mathrm{ad}^*_{\hat\eta(\nu,w)}(\nu + \mathbf{J}_{N_1}(w)))$ is zero, and if in addition there is a G^0_μ-invariant inner product on $\mathbf{g}_\mu$ then by Lemma 2.4 we can identify the coadjoint and adjoint G^0_μ-actions, and the $\dot\nu$-equation simply becomes $\dot\nu = [\mathrm{D}_\nu h, \mathbf{J}_{N_1}(w)]$.

For the particular case of $\mathbf{g}_\mu = \mathbf{so(3)}$ and $\mathbf{g}_p = \mathbf{so(2)}$ the Lie bracket is just the vector product. We can take $\nu = (\nu_1, \nu_2)$ to lie in the two dimensional subspace orthogonal to $\mathbf{so(2)}$ in $\mathbf{so(3)} \cong \mathbb{R}^3$ and so, for example if μ is split (eg $\mathbf{g} = \mathbf{g}_\mu$), the $\dot\nu$ equation becomes:

$$\begin{pmatrix} \dot\nu_1 \\ \dot\nu_2 \end{pmatrix} = \mathbf{J}_{N_1}(w) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial \nu_1}(\nu, w) \\ \frac{\partial h}{\partial \nu_2}(\nu, w) \end{pmatrix}. \tag{4.3}$$

Equations of this form are well known in, for example, the study of dynamics near linear configurations of molecules [54, 16].

4.5 Example: $\mathbf{G} = \mathbf{SE(2)}$

The special Euclidean group $\mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ is the group of all orientation preserving isometries of $\mathbb{R}^2$. We write $g = (\phi, a)$ where $\phi \in \mathrm{SO}(2)$ and $a \in \mathbb{R}^2$. The Lie bracket on $\mathbf{se}(2) = \mathbf{so}(2) \oplus \mathbb{R}^2 \cong \mathbb{R}^3$ is given by

$$[(\xi^r, \xi^a_1, \xi^a_2), (\eta^r, \eta^a_1, \eta^a_2)] \ = \ (0\ ,\ -\xi^r \eta^a_2 + \eta^r \xi^a_2\ ,\ \xi^r \eta^a_1 - \eta^r \xi^a_1).$$

Using the standard inner product on $\mathbb{R}^3$ to identify it with $(\mathbb{R}^3)^*$, the coadjoint action of $\mathbf{se}(2) = \mathbf{so}(2) \oplus \mathbb{R}^2 \cong \mathbb{R}^3$ on $\mathbf{se}(2)^* = \mathbf{so}(2)^* \oplus (\mathbb{R}^2)^* \cong (\mathbb{R}^3)^*$ is given by

$$-\mathrm{ad}^*_\xi(\mu) \ = \ (\ \xi^a_1 \mu^a_2 - \xi^a_2 \mu^a_1\ ,\ -\xi^r \mu^a_2\ ,\ \xi^r \mu^a_1\ )$$

	$\overline{\mathrm{ad}}^*_{\mathrm{D}_\nu h}(\nu)$	$\mathrm{ad}^*_{\mathrm{D}_\nu h}(\mathbf{J}_{N_1})$	$\mathrm{P}(\mathrm{ad}^*_{\hat\eta}(\nu))$	$\mathrm{P}(\mathrm{ad}^*_{\hat\eta}(\mathbf{J}_{N_1}))$
μ split			0	0
$\mathbf{g}_\mu$ minimal	0	0	0	0
$\mathbf{g}_p$ conormal in $\mathbf{g}_\mu$		0		
$[\mathbf{m}_\mu, \mathbf{m}_\mu + \mathbf{n}_\mu] \subset \mathbf{m}_\mu + \mathbf{n}_\mu$		0		0
$\mathbf{g}_p$ symmetric in $\mathbf{g}_\mu$	0			
$[\mathbf{m}_\mu, \mathbf{m}_\mu + \mathbf{n}_\mu] \subset \mathbf{g}_p + \mathbf{n}_\mu$	0		0	

Table 1: Cases for which the various individual terms in the $\dot\nu$ equation (3.2) are identically zero. The vanishing of a term is indicated by a 0.

where $\mu = (\mu^r, \mu^a)$ with $\mu^r \in \mathbb{R}$ and $\mu^a = (\mu^a_1, \mu^a_2) \in \mathbb{R}^2$. If $\mu^a = 0$ then $G_\mu = G$. If $\mu^a \neq 0$ then without loss of generality we may take $\mu^a = (0,1)$, in which case $G_\mu = \{(0,s) \in \mathbb{R}^2 \subset \mathrm{SE}(2) : s \in \mathbb{R}\} \cong \mathbb{R}$. Let $\mathbf{n}_\mu$ denote the orthogonal complement in $\mathbf{se}(2) \cong \mathbb{R}^3$ to $\mathbf{g}_\mu = \mathbb{R}_3 = \{(0,0,s) \in \mathbb{R}^3 \cong \mathbf{se}(2) : s \in \mathbb{R}\}$ with respect to the standard inner product. Then $\mathbf{n}_\mu$ is G_μ invariant. It follows that every $\mu \in \mathbf{g}^*$ is split and so $\hat\eta(\nu, w) \equiv 0$. The $\dot g$-equation (3.1) takes the form

$$\dot\phi = \mathrm{D}_{\nu^r} h(\nu, w), \quad \dot a = R_\phi \mathrm{D}_{\nu^a} h(\nu, w)$$

where R_ϕ denotes a rotation by ϕ in $\mathbb{R}^2$. The assumption that G_p is compact implies that if $\mathbf{g}_\mu = \mathbf{se}(2)$ then $\mathbf{g}_p$ can be either $\{0\}$ or $\mathbf{so}(2)$, while if $\mathbf{g}_\mu = \mathbb{R}_3$ then $\mathbf{g}_p$ must be $\{0\}$. We treat each case in turn.

4.5.1 $\mathbf{g}_\mu = \mathbf{se(2)}$

a) $\mathbf{g_p} = \{\mathbf{0}\}$: In this case $\mathbf{g}_p$ is trivially both normal and conormal in $\mathbf{g}_\mu = \mathbf{g}$ and the $\dot\nu$ equation on $(\mathbf{g}_\mu/\mathbf{g}_p)^* = \mathbf{se}(2)^*$ becomes

$$\begin{aligned} \dot\nu^r &= \frac{\partial h}{\partial \nu^a_2}\nu^a_1 - \frac{\partial h}{\partial \nu^a_1}\nu^a_2 \\ \begin{pmatrix} \dot\nu^a_1 \\ \dot\nu^a_2 \end{pmatrix} &= \frac{\partial h}{\partial \nu^r}\begin{pmatrix} \nu^a_2 \\ -\nu^a_1 \end{pmatrix}. \end{aligned}$$

b) $\mathbf{g_p} = \mathbf{so(2)}$: Here we can take $\mathbf{m}_\mu$ to be $\{0\} \oplus \mathbb{R}^2 \subset \mathbb{R}^3$. This is an Abelian normal subalgebra of $\mathbf{se}(2)$ and so $\mathbf{g}_p$ is both conormal and symmetric in $\mathbf{g}_\mu = \mathbf{g}$. It follows that $\dot\nu = 0$ on $(\mathbf{g}_\mu/\mathbf{g}_p)^* \cong \mathbb{R}^2$. Note that for $\nu = 0$ the $\dot w$-equation is SO(2)-equivariant, while for fixed $\nu \neq 0$ the SO(2) symmetry is broken.

4.5.2 $\mathbf{g}_\mu = \mathbb{R}$

This is the generic case of minimal momentum isotropy algebras and, as we discussed in general in section 4.2, we have $\dot{\nu} = 0$ on $\mathbf{g}_\mu^* \cong \mathbb{R}$. In this case we must have $\mathbf{g}_p = \{0\}$ and the equations on the group simplify to $\dot{\phi} = 0, \quad \dot{a} = R_{\phi(0)} \mathrm{D}_{\nu_2^a} h$.

4.6 Example: $\mathbf{G} = \mathbf{SE(3)}$

Finally we take G to be the special Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$. This symmetry group arises in the study of the Saint Venant's problem of rod theory and in this context was considered in [35] in the case of trivial isotropy $\mathbf{g}_p = \{0\}$. The symmetry group SE(3) also plays an important role in the analysis of underwater vehicle dynamics given in [20] where isotropy subalgebras $\mathbf{g}_p = \{0\}$ and $\mathbf{g}_p = \mathbf{so}(2)$ were considered.

We write $g = (R, a)$ where $R \in \mathrm{SO}(3)$ and $a \in \mathbb{R}^3$. The Lie algebra $\mathbf{se(3)}$ can be decomposed as $\mathbf{so(3)} \oplus \mathbb{R}^3 \cong \mathbb{R}^3 \oplus \mathbb{R}^3$. With respect to this decomposition the Lie bracket is given by

$$\mathrm{ad}_{(\xi^r, \xi^a)}(\eta^r, \eta^a) \;=\; [\xi, \eta] \;=\; (\ \xi^r \times \eta^r\ ,\ \xi^r \times \eta^a - \eta^r \times \xi^a\ )$$

for $\xi^r,\ \eta^r \in \mathbf{so(3)}$ and $\xi^a,\ \eta^a \in \mathbb{R}^3$. Using the standard inner product on $\mathbb{R}^3 \oplus \mathbb{R}^3$ to identify $\mathbf{se(3)}$ with its dual and then taking the transpose of ad_ξ gives

$$\mathrm{ad}_\xi^*(\mu) \;=\; -(\ \xi^r \times \mu^r + \xi^a \times \mu^a,\ \xi^r \times \mu^a\ )$$

for $\mu = (\mu^r, \mu^a) \in \mathbf{se(3)}^* = \mathbb{R}^3 \oplus \mathbb{R}^3$. The adjoint and coadjoint actions of SE(3) on $\mathbf{se}(3)$ and $\mathbf{se}(3)^*$ are respectively

$$\begin{aligned} \mathrm{Ad}_{(R,a)}\xi &= (\ R\xi^r\ ,\ R\xi^a - R\xi^r \times a\ ) \\ \mathrm{Ad}^*_{(R,a)^{-1}}\mu &= (\ R\mu^r + a \times R\mu^a\ ,\ R\mu^a\ ) \end{aligned}$$

where $(R, a) \in \mathrm{SO}(3) \ltimes \mathbb{R}^3$.

Let SO(2) denote the subgroup of SO(3) consisting of rotations which fix the vector $(0,0,1) \in \mathbb{R}^3$ and $\mathbf{so}(2)$ its Lie algebra. Let $\mathbb{R}$ denote the subspace of $\mathbb{R}^3$ spanned by $(0,0,1)$. Then the isotropy subgroups of the coadjoint action of SE(3) on $\mathbf{se}(3)^*$ are conjugate to SE(3) if $\mu = 0$, $\mathrm{SO}(2) \ltimes \mathbb{R}^3$ if $\mu^a = 0$ and $\mathrm{SO}(2) \times \mathbb{R}$ if $\mu^a \neq 0$.

In the first case $\mathbf{g}_\mu = \mathbf{g}$ and so $\mathbf{n}_\mu = \{0\}$ and μ is trivially split. In the third case we can take $\mathbf{n}_\mu = \mathbf{so}(2)^\perp \oplus \mathbb{R}^2$ where $\mathbf{so}(2)^\perp$ is the orthogonal complement of $\mathbf{so}(2)$ in $\mathbf{so}(3) \cong \mathbb{R}^3$ and $\mathbb{R}^2$ is the orthogonal complement of $\mathbb{R}$ in $\mathbb{R}^3$, both with respect to the standard inner product on $\mathbb{R}^3$. This subspace is $\mathrm{SO}(2) \times \mathbb{R}$ invariant and so again μ is split. In the second case, however, μ is not split. We can take μ to be $(\mu^r, 0)$ with $\mu^r = (0, 0, \mu_3^r)$ and $\mathbf{n}_\mu$ to be $\mathbf{so}(2)^\perp \oplus \{0\}$. The identification of $\mathbf{se}(3)$ with its dual identifies $\mathrm{ann}(\mathbf{g}_\mu)$ with $\mathbf{n}_\mu = \mathbf{so}(2)^\perp \oplus \{0\}$ and $\mathrm{ann}(\mathbf{n}_\mu)$ with $\mathbf{g}_\mu = \mathbf{so}(2) \oplus \mathbb{R}^3$.

We consider the forms of the equations (3.1), (3.2) and (3.3) for each of the possible momentum isotropy subalgebras in turn.

4.6.1 $\mathbf{g}_\mu = \mathbf{se(3)}$

Since μ is (trivially) split in this case, we have $\hat{\eta}(\nu, w) \equiv 0$. The isotropy subalgebra $\mathbf{g}_p$ can be $\{0\}$, $\mathbf{so}(2)$ or $\mathbf{so}(3)$.

a) $\mathbf{g_p} = \{\mathbf{0}\}$: If $\mathbf{g}_p = \{0\}$ then $\mathrm{ad}^*(\mathbf{J}_{N_1}(w)) \equiv 0$ and the $\dot{\nu}$ equation (3.2) reduces to

$$\dot{\nu} \;=\; \mathrm{ad}^*_{D_\nu h(\nu,w)}(\nu).$$

Putting $\nu = (\nu^r, \nu^a) \in \mathbf{so}(3) \oplus \mathbb{R}^3$ we obtain the more explicit form

$$\begin{aligned}\dot{\nu^r} &= \nu^r \times \mathrm{D}_{\nu^r} h \;+\; \nu^a \times \mathrm{D}_{\nu^a} h\\ \dot{\nu^a} &= \nu^a \times D_{\nu^r} h.\end{aligned}$$

The equations on the group are given by

$$\dot{R} = R\mathrm{D}_{\nu^r} h, \quad \dot{a} = R\mathrm{D}_{\nu^a} h.$$

These equations have already been obtained by Mielke [35].

b) $\mathbf{g_p} = \mathbf{so(2)}$: The subalgebra $\mathbf{g}_p = \mathbf{so}(2)$ is neither conormal nor symmetric in $\mathbf{g}_\mu$ and the general form of the $\dot{\nu}$ equation is

$$\dot{\nu} \;=\; \overline{\mathrm{ad}}^*_{D_\nu h(\nu,w)}(\nu) + \mathrm{ad}^*_{D_\nu h(\nu,w)}(\mathbf{J}_{N_1}(w))$$

where $h(\nu, w) = h(\nu^r_2, \nu^r_3, \nu^a)$ is SO(2)-invariant. Putting $\nu = (\nu^r, \nu^a) \in \mathbf{so}(2)^\perp \oplus \mathbb{R}^3$ with $\nu^r = (\nu^r_1, \nu^r_2, 0)$ leads to

$$\begin{aligned}\begin{pmatrix}\dot{\nu}^r_1\\ \dot{\nu}^r_2\end{pmatrix} &= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\left\{\nu^a_3\begin{pmatrix}\frac{\partial h}{\partial \nu^a_1}\\ \frac{\partial h}{\partial \nu^a_2}\end{pmatrix} - \frac{\partial h}{\partial \nu^a_3}\begin{pmatrix}\nu^a_1\\ \nu^a_2\end{pmatrix} + \mathbf{J}_{N_1}(w)\begin{pmatrix}\frac{\partial h}{\partial \nu^r_1}\\ \frac{\partial h}{\partial \nu^r_2}\end{pmatrix}\right\}\\ \dot{\nu^a} &= \nu^a \times \left(\frac{\partial h}{\partial \nu^r_1}, \frac{\partial h}{\partial \nu^r_2}, 0\right).\end{aligned}$$

c) $\mathbf{g_p} = \mathbf{so(3)}$: Finally, the subalgebra $\mathbf{g}_p = \mathbf{so}(3)$ has a normal, Abelian complement and so is itself both conormal and symmetric. The $\dot{\nu}$ equation therefore reduces to $\dot{\nu} = 0$, where $\nu = (0, \nu^a) \in \{0\} \oplus \mathbb{R}^3$. Note that $h(\nu^a, w)$ is SO(3)-invariant, but that the $\dot{w}$-equation is only SO(2)-equivariant for fixed $\nu^a \neq 0$.

4.6.2 $\mathbf{g}_\mu = \mathbf{so(2)} \oplus \mathbb{R}^3$

In this case μ is not split and we compute $\hat{\eta}(\nu, w)$ in the following lemma.

Lemma 4.5 *The mapping* $\eta : \mathbf{g}_\mu \oplus \mathrm{ann}(\mathbf{n}_\mu) \cong (\mathbf{so}(2)\oplus\mathbb{R}^3) \oplus (\mathbf{so}(2)\oplus\mathbb{R}^3) \to \mathbf{n}_\mu = \mathbf{so}(2)^\perp \oplus \{0\}$ *from Proposition 2.5 is given by*

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{\zeta_3^r + \mu_3^r} \begin{pmatrix} -\zeta_3^a & 0 & \zeta_1^a \\ 0 & -\zeta_3^a & \zeta_2^a \end{pmatrix} \xi^a$$

where $\xi \in \mathbf{g}_\mu = \mathbf{so}(2)\oplus\mathbb{R}^3$ *and* $\zeta \in \mathrm{ann}(\mathbf{n}_\mu) \cong \mathbf{so}(2)\oplus\mathbb{R}^3$.

Note that η is independent of ζ^r.

Proof. We need to solve (2.6), i.e. $P_{\mathrm{ann}(\mathbf{g}_\mu)}\mathrm{ad}^*_{\xi+\eta}(\mu+\zeta) = 0$. Substituting the expression for the coadjoint action we get

$$\mathrm{P}_{\mathbf{so}(2)^\perp}((\xi^r+\eta^r)\times(\mu^r+\zeta^r) + \xi^a \times \zeta^a) = 0.$$

Remembering that ξ^r, ζ^r and μ^r all lie in $\mathbf{so}(2)$, and that $\eta_=0$, $\eta^r = (\eta_1,\eta_2,0)$, gives

$$-\mathrm{P}_{\mathbf{so}(2)^\perp}((\mu^r+\zeta^r)\times(\eta_1,\eta_2,0)) = \mathrm{P}_{\mathbf{so}(2)^\perp}(\xi^a\times\zeta^a)$$

and hence

$$(\mu_3^r+\zeta_3^r)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \xi_2^a\zeta_3^a - \xi_3^a\zeta_2^a \\ \xi_3^a\zeta_1^a - \xi_1^a\zeta_3^a \end{pmatrix}.$$

Solving for (η_1,η_2) gives the result. ∎

As in the previous case the phase space isotropy subalgebra $\mathbf{g}_p$ can be either $\{0\}$ or $\mathbf{so}(2)$. If $\mathbf{g}_p = \{0\}$ then $\nu = (\nu^r,\nu^a) \in \mathbf{so}(2)\oplus\mathbb{R}^3$ and $\mathrm{D}_\nu h(\nu,w) = (\mathrm{D}_{\nu^r}h(\nu^r,\nu^a,w), \mathrm{D}_{\nu^a}h(\nu^r,\nu^a,w)) \in \mathbf{so}(2)\oplus\mathbb{R}^3$. It follows from the definition of $\hat\eta(\nu,w) \in \mathbf{n}_\mu = \mathbf{so}(2)^\perp$ in Theorem 3.2 that

$$\hat\eta(\nu,w) = \frac{1}{\mu_3^r+\nu_3^r}\begin{pmatrix} -\nu_3^a & 0 & \nu_1^a \\ 0 & -\nu_3^a & \nu_2^a \end{pmatrix}\mathrm{D}_\nu h(\nu_3^r,\nu^a,w). \qquad (4.4)$$

If $\mathbf{g}_p = \mathbf{so}(2)$ then $\nu = (0,\nu^a) \in \{0\}\oplus\mathbb{R}^3$ and $\mathrm{D}_\nu h(\nu,w) = (0,\mathrm{D}_{\nu^a}h(\nu,w)) \in \{0\}\oplus\mathbb{R}^3$, and $\hat\eta$ continues to have the form shown in equation (4.4) with ν_3^r replaced by $\mathbf{J}_{N_1}(w)$.

a) $\mathbf{g_p} = \{\mathbf{0}\}$: In this case the two terms $\mathrm{ad}^*_{\mathrm{D}_\nu h}(\mathbf{J}_{N_1})$ and $\mathrm{P}(\mathrm{ad}^*_{\hat\eta}(\mathbf{J}_{N_1}))$ in equation (3.2) disappear and $\overline{\mathrm{ad}}^*_{\mathrm{D}_\nu h}(\nu)$ becomes $\mathrm{ad}^*_{\mathrm{D}_\nu h}(\nu)$, where the ad^* operator is that for the action of $\mathbf{g}_\mu$ on $\mathbf{g}^*_\mu$. Combining these observations with the calculation of $\hat\eta$ above we obtain the following form for the $\dot\nu$ equation (3.2):

$$\begin{aligned}
\dot{\nu^r} &= \nu_1^a\frac{\partial h}{\partial\nu_2^a} - \nu_2^a\frac{\partial h}{\partial\nu_1^a} \\
\dot{\nu^a} &= \nu^a \times \left\{ \begin{pmatrix} 0 \\ 0 \\ \frac{\partial h}{\partial\nu^r} \end{pmatrix} + \frac{1}{\mu_3^r+\nu^r}\begin{pmatrix} -\nu_3^a & 0 & \nu_1^a \\ 0 & -\nu_3^a & \nu_2^a \end{pmatrix}\mathrm{D}_{\nu^a}h \right\} \\
&= \frac{\partial h}{\partial\nu^r}\begin{pmatrix} \nu_2^a \\ -\nu_1^a \\ 0 \end{pmatrix} - \frac{\nu_3^a}{\mu_3^r+\nu^r}(\,\nu^a \times \mathrm{D}_{\nu^a}h\,).
\end{aligned}$$

b) $\mathbf{g_p} = \mathbf{so(2)}$: The subalgebra $\mathbf{g}_p = \mathbf{so}(2)$ is conormal and symmetric in $\mathbf{g}_\mu$ and so the terms $\overline{\mathrm{ad}}^*_{\mathrm{D}_\nu h}(\nu)$ and $\mathrm{ad}^*_{\mathrm{D}_\nu h}(\mathbf{J}_{N_1})$ both disappear. It also satisfies the condition (4.1) and so $\mathrm{P}(\mathrm{ad}^*_{\hat{\eta}}(\mathbf{J}_{N_1}))$ vanishes identically. A calculation similar to that above shows that the remaining term $\mathrm{P}(\mathrm{ad}^*_{\hat{\eta}}(\nu))$ is nonzero and that the $\dot{\nu}$ equation (3.2) simplifies to

$$\dot{\nu} \;=\; -\;\frac{\nu_3}{\mu_3^r + \mathbf{J}_{N_1}(w)}\;\left(\;\nu \;\times\; \mathrm{D}_\nu h(\nu, w)\;\right).$$

4.6.3 $\mathbf{g}_\mu = \mathbf{so(2)} \oplus \mathbb{R}$

Since μ is split $\hat{\eta}(\nu, w)$ is always identically zero. The phase space isotropy subalgebra can be either $\{0\}$ or $\mathbf{so}(2)$. Since $\mathbf{g}_\mu$ is a minimal momentum isotropy algebra and therefore Abelian it follows that in both cases the equations take the form

$$\dot{R} = R\;\mathrm{D}_{\nu_3^r} h(\nu, w) e_3, \quad \dot{a} = R\;\mathrm{D}_{\nu_3^a} h(\nu, w) e_3, \quad \dot{\nu} = 0, \quad \dot{w} = J\mathrm{D}_w h(\nu, w)$$

where $\nu \in \mathbf{so}(2) \oplus \mathbb{R}$ if $\mathbf{g}_p = \{0\}$ and $\nu \in \mathbb{R}$ if $\mathbf{g}_p = \mathbf{so}(2)$. Note that for $(R(0), a(0)) \in G_\mu$ where $R(0)$ is a rotation by the angle $\phi_3(0)$ around e_3 and $a_1(0) = a_2(0) = 0$ the equations on the group simplify to $\dot{\phi}_3 = \mathrm{D}_{\nu_3^r} h(\nu, w)$, $\dot{a}_3 = \mathrm{D}_{\nu_3^a} h(\nu, w)$.

5 Proofs of Main Results

We begin by constructing 'local models' for $\mathcal{M}/\Gamma$ and $\mathcal{M}$ and then use these to derive the equations given in the statements of Theorems 3.1 and 3.2. We suppose that a G_p-invariant inner product has been fixed on $T_p\mathcal{M}$, giving the Witt decomposition described in Proposition 2.1. We also assume that G_p-invariant complements $\mathbf{n}_\mu$ and $\mathbf{m}_\mu$ have been chosen such that

$$\mathbf{g} = \mathbf{g}_\mu \oplus \mathbf{n}_\mu \quad \text{and} \quad \mathbf{g}_\mu = \mathbf{g}_p \oplus \mathbf{m}_\mu.$$

Throughout this section we will identify $\mathbf{g}_\mu^*$ with the annihilator of $\mathbf{n}_\mu$ in $\mathbf{g}^*$ and $\mathbf{g}_p^*$ with the annihilator of $\mathbf{m}_\mu$ in $\mathbf{g}_\mu^*$ and with the annihilator of $\mathbf{m}_\mu \oplus \mathbf{n}_\mu$ in $\mathbf{g}^*$.

5.1 A local model for $\mathcal{M}/\Gamma$

Recall that Γ is the subgroup of G which acts symplectically on $\mathcal{M}$. Using the complement $\mathbf{m}_\mu$ we can embed $\mathbf{g}_p^*$ into $\mathbf{g}_\mu^*$ and the normal space $N = N_0 \oplus N_1 \cong (\mathbf{g}_\mu/\mathbf{g}_p)^* \oplus N_1$ into $\mathbf{g}_\mu^* \oplus N_1$ by the mapping

$$(\nu, w) \;\mapsto\; (\nu + \mathbf{J}_{N_1}(w), w), \qquad \nu \in (\mathbf{g}_\mu/\mathbf{g}_p)^*,\; w \in N_1.$$

This embedding is G_p-equivariant with respect to the χ-coadjoint action of G_p on $N_0 \cong (\mathbf{g}_\mu/\mathbf{g}_p)^*$, the semisymplectic action of G_p on N_1 and the χ-coadjoint action of G_p on $\mathbf{g}_\mu^*$.

As the direct sum of the dual of a Lie algebra and a symplectic vector space, $\mathbf{g}_\mu^* \oplus N_1$ is naturally a Poisson space with Poisson bracket given by

$$\{f_1, f_2\}(\nu, w) \;=\; -\nu([\mathrm{D}_\nu f_1(\nu, w), \mathrm{D}_\nu f_2(\nu, w)]) \;+\; \omega_{N_1}(J\mathrm{D}_w f_1(\nu, w), J\mathrm{D}_w f_2(\nu, w))$$

for any pair of smooth functions $f_i = f_i(\nu, w)$ on $\mathbf{g}_\mu^* \oplus N_1$. The Poisson bracket is (G_p, χ)-semi-invariant:

$$\{f_1 \circ g, f_2 \circ g\} \;=\; \chi(g)\{f_1, f_2\} \circ g \qquad g \in G_p.$$

Let ι denote the inclusion from $\mathbf{g}_p$ into $\mathbf{g}_\mu$ and define a map

$$\psi : \mathbf{g}_\mu^* \oplus N_1 \to \mathbf{g}_p^*, \qquad \psi(\nu, w) = -\widehat{\mathrm{P}}\nu + \mathbf{J}_{N_1}(w).$$

where $\widehat{\mathrm{P}}$ is the projection from $\mathbf{g}_\mu^*$ to $\mathbf{g}_p^*$ dual to ι. This map is G_p-equivariant with respect to the induced action on N_1 and the χ-coadjoint actions on $\mathbf{g}_\mu^*$ and $\mathbf{g}_p^*$. Moreover it is a momentum map for the G_p-action on the Poisson space $\mathbf{g}_\mu^* \oplus N_1$, i.e. it satisfies [32]:

$$\{f, \psi_\xi\}(\nu, w) = \mathrm{D}f(\nu, w)\xi(\nu, w), \quad \xi \in \mathbf{g}_p, \; (\nu, w) \in \mathbf{g}_\mu^* \oplus N_1, \; f \in C^\infty(\mathbf{g}_\mu^* \oplus N_1).$$

Indeed one computes that

$$\begin{aligned}
\{f, \psi_\xi\}(\nu, w) &= \nu([\mathrm{D}_\nu \psi_\xi, \mathrm{D}_\nu f]) + \omega_{N_1}(J\mathrm{D}_w f, J\mathrm{D}_w \psi_\xi) \\
&= \nu([-\xi, \mathrm{D}_\nu f]) + \langle \mathrm{D}_w f, \xi w\rangle \\
&= \xi\nu(\mathrm{D}_\nu f) + \langle \mathrm{D}_w f, \xi w\rangle.
\end{aligned}$$

The embedded image of N in $\mathbf{g}_\mu^* \oplus N_1$ is the zero level set of this momentum map, $N \cong \psi^{-1}(0)$.

It follows that the quotient variety $N/\Gamma_p \cong \psi^{-1}(0)/\Gamma_p$ has a natural Poisson structure. The group G_p/Γ_p is isomorphic to $\mathbb{Z}_2$ if G_p contains elements that act anti-symplectically on $\mathcal{M}$, and is trivial if it doesn't. In the first case the action of the generator ρ of G_p/Γ_p on $N/\Gamma_p \cong \psi^{-1}(0)/\Gamma_p$ is 'anti-Poisson'.

By the slice theorem N/Γ_p is isomorphic as a set to a neighbourhood of Γp in the orbit space $\mathcal{M}/\Gamma$ and this construction therefore defines a Poisson structure on this neighbourhood. It will follow from the proof below that this structure is isomorphic to that induced directly from $\mathcal{M}$. We note that it is often more convenient to work on the Poisson space $\mathbf{g}_\mu^* \oplus N_1$ rather than directly on the slice N.

5.2 A local model for $\mathcal{M}$

Let $\widetilde{\mathcal{M}}$ denote the manifold $G \times (\mathbf{g}_\mu^* \oplus N_1)$. Define a smooth action of $G \times G_p$ on $\widetilde{\mathcal{M}}$ by

$$(\tilde{g}, g_p).(g, \nu, w) \;=\; (\tilde{g} g g_p^{-1}, \; \chi(g_p)\,\mathrm{Ad}^*_{g_p^{-1}}\nu, \; g_p w), \quad g, \tilde{g} \in G, \; g_p \in G_p \tag{5.1}$$

and a 2-form $\tilde{\omega}$ on $\widetilde{\mathcal{M}}$ by

$$\tilde{\omega}(g,\nu,w) \;=\; \chi(g)\left(\tilde{\omega}_G(g,\nu)+\tilde{\omega}_\mu(g)+\tilde{\omega}_{N_1}\right) \tag{5.2}$$

where:

1. $\tilde{\omega}_G$ is the pullback of the natural symplectic form ω_G on $T^*G \cong G\times \mathbf{g}^*$:

$$\omega_G(g,\nu)\left((g\xi_1,\nu_1),(g\xi_2,\nu_2)\right) \;=\; \nu_2(\xi_1)-\nu_1(\xi_2)+\nu\left([\xi_1,\xi_2]\right)$$

   where $g\in G$, $\nu,\nu_1,\nu_2\in\mathbf{g}^*$, $\xi_1,\xi_2\in\mathbf{g}$ (see [1, Prop. 4.4.1]), by the map $(g,\nu,w)\mapsto(g,\iota_\mu\nu)$, where the inclusion $\iota_\mu:\mathbf{g}_\mu^*\to\mathbf{g}^*$ is induced by the G_p-invariant complement $\mathbf{n}_\mu$ to $\mathbf{g}_\mu$ in $\mathbf{g}$;
2. $\tilde{\omega}_\mu$ is the pullback of the KKS symplectic form (2.1) on the coadjoint orbit $G\mu$ by $(g,\nu,w)\mapsto \mathrm{Ad}^*_{g^{-1}}\mu$;
3. $\tilde{\omega}_{N_1}$ is the pullback of the symplectic form ω_{N_1} on N_1 by $(g,\nu,w)\mapsto w$.

We summarize some properties of the 2-form $\tilde{\omega}$ in the following proposition.

Proposition 5.1

a) The form $\tilde{\omega}$ is a symplectic form on a $G\times G_p$ invariant neighbourhood of $G\times\{(0,0)\}$ in $\widetilde{\mathcal{M}}$. The action of G on this neighbourhood is χ-semisymplectic while the action (5.1) of G_p is symplectic.

b) The restriction of $\tilde{\omega}$ to $T_{(\mathrm{id},0,0)}\widetilde{\mathcal{M}} \;\cong\; \mathbf{g}/\mathbf{g}_\mu\oplus(\mathbf{g}_\mu\oplus\mathbf{g}_\mu^*)\oplus N_1$ is isomorphic to the direct sum of the KKS form on $\mathbf{g}/\mathbf{g}_\mu$, the canonical form on $\mathbf{g}_\mu\oplus\mathbf{g}_\mu^*$ and the form ω_{N_1} on N_1.

Note especially that the G_p-action (5.1) on $\widetilde{\mathcal{M}}$ is symplectic with respect to $\tilde{\omega}$ although the G_p-action on the symplectic slice N_1 is semisymplectic with respect to the symplectic form ω_{N_1}.

A momentum map $\Psi:\widetilde{\mathcal{M}}\to\mathbf{g}_p^*$ for the symplectic action of G_p on $\widetilde{\mathcal{M}}$ is given by

$$\Psi(g,\nu,w) \;=\psi(\nu,w).$$

The map Ψ is G_p-equivariant with respect to the action (5.1) on $\widetilde{\mathcal{M}}$ and the usual coadjoint action of G_p on $\mathbf{g}_p^*$. Because the action of G_p on $\widetilde{\mathcal{M}}$ is free, proper, and symplectic we can reduce by G_p to obtain a natural symplectic structure $\tilde{\omega}_0$ on a G-invariant neighbourhood $\widetilde{U}_0$ of $G\times_{G_p}\{0\}$ in the manifold

$$\widetilde{\mathcal{M}}_0 \;=\; \Psi^{-1}(0)/G_p \;=\; G\times_{G_p}\psi^{-1}(0) \;\cong\; G\times_{G_p}N.$$

There is an induced χ-semisymplectic action of G on $\widetilde{U}_0$.

By the slice theorem $\widetilde{U}_0$ is G-equivariantly diffeomorphic to a G-invariant neighbourhood U of the orbit Gp in $\mathcal{M}$. The following theorem says that this diffeomorphism can be chosen to be a G-equivariant symplectomorphism with respect to the symplectic form ω of $\mathcal{M}$ and the symplectic form $\tilde{\omega}_0$ on $\widetilde{\mathcal{M}}_0$. It is a generalization to reversible actions of the local normal form for symplectic G-manifolds obtained by Guillemin and Sternberg [10, 11], Marle [30] and Bates and Lerman [2].

Theorem 5.2 *There exists a G-equivariant symplectomorphism between a G-invariant open neighbourhood of $G \times_{G_p} \{0\}$ in $\widetilde{\mathcal{M}}_0 = G \times_{G_p} \psi^{-1}(0) \cong G \times_{G_p} N$ and a G-invariant open neighbourhood of Gp in $\mathcal{M}$.*

Proof. A calculation shows that the induced symplectic form on $\widetilde{M}_0$ at the point $G_p \times_{G_p} \{0\}$ is the same as that on $\mathcal{M}$ at the point p. The result then follows from the relative Darboux theorem below with $Y = U \subset \mathcal{M}$, $X = Gp$, $\omega_1 = \omega$ and ω_0 equal to the pullback of $\tilde{\omega}_0$ from $\widetilde{U}_0$ to U.

∎

Theorem 5.3 (Semisymplectic Relative Darboux) *Let the Lie group G act properly and χ-semisymplectically on a symplectic manifold Y with respect to two symplectic forms ω_0 and ω_1. Let X be a G-invariant submanifold such that $\omega_0 = \omega_1$ on X. Then there is a G-equivariant diffeomorphism Φ defined on a tubular neighborhood U of X such that $\Phi^*\omega_1 = \omega_0$ on U.*

Proof. The proof is taken from [2] with minor modifications to deal with the semisymplecticity of the group action. Suppose that there is a (G,χ)-semiinvariant 1-form ζ on U such that $\omega_1 - \omega_0 = \mathrm{d}\zeta$ and $\zeta|_X = 0$. Let $\omega_t = t\omega_0 + (1-t)\omega_1$, $t \in [0,1]$. Then the equation $\omega_t(\xi_t, \cdot) = \zeta$ defines a t-dependent vectorfield ξ_t with $\xi_t|_X = 0$ which is G-equivariant because

$$\chi(g)\omega_t(g^{-1}\xi_t(gx), v) = \omega_t(\xi_t(gx), gv) = \zeta(gx) = \chi(g)\zeta(x) = \chi(g)\omega_t(\xi_t(x), v).$$

As in [2] the corresponding flow Φ_t satisfies $\Phi_t^*(\omega_t) = \omega_0$ and is G-equivariant so that we can define $\Phi = \Phi_1$.

We define ζ as in [2] by the formula

$$\zeta(y) = -\int_0^t \phi_t^*(\omega_1 - \omega_0)(y)(\eta_t(y), \cdot)\mathrm{d}t$$

where ϕ_t is a G-equivariant contraction from the tubular neighbourhood U of X to X and $\eta_t(y)$ is the tangent vector to $\phi_t(y)$. The contraction can be obtained by replacing Y by the normal bundle to X and defining $\phi_t(y) = (1-t)y$, as in [2]. Since η_t and ϕ_t are G-equivariant, and ω_1, ω_0 are G-semi-invariant, the form ζ is also G-semi-invariant.

∎

5.3 Proof of Proposition 3.5

The proof of this result for semisymplectic group actions is essentially the same as for symplectic group actions [10, 11, 30]. We sketch the main ideas.

First we show that the map $\widetilde{\mathbf{J}}(g,\zeta,w) = \chi(g)\mathrm{Ad}^*_{g^{-1}}(\mu+\zeta)$ is a momentum map for the semisymplectic action of G on $\widetilde{\mathcal{M}}$ given by (5.1). In other words we show that

$$\tilde{\omega}(g,\zeta,w)\left(\left(\frac{d}{dt}\left(e^{t\xi}g\right)_{t=0},0,0\right),\left(g\hat{\xi},\hat{\zeta},\hat{w}\right)\right) \;=\; \mathrm{D}\widetilde{\mathbf{J}}_\xi(g,\zeta,w)(g\hat{\xi},\hat{\zeta},\hat{w})$$

for all $g \in G$, $\xi,\hat{\xi} \in \mathbf{g}$, $\zeta,\hat{\zeta} \in \mathbf{g}^*_\mu$ and $w,\hat{w} \in N_1$. Using the definition (5.2) of $\tilde{\omega}$ the lefthand side is equal to

$$\tilde{\omega}((\xi g,0,0),(g\hat{\xi},\hat{\zeta},\hat{w})) = \chi(g)\,(\tilde{\omega}_G + \tilde{\omega}_\mu)\,((\xi g,0),(g\hat{\xi},\hat{\zeta}))$$

where

$$\tilde{\omega}_G(g,\zeta)((\xi g,0),(g\hat{\xi},\hat{\zeta})) = \hat{\zeta}(\mathrm{Ad}_{g^{-1}}(\xi)) + \zeta([\mathrm{Ad}_{g^{-1}}(\xi),\hat{\xi}])$$

and

$$\tilde{\omega}_\mu(g)(\xi g, g\hat{\xi}) = \mu([\mathrm{Ad}_{g^{-1}}(\xi),\hat{\xi}])$$

while the righthand side is

$$\begin{aligned}\mathrm{D}\widetilde{\mathbf{J}}_\xi(g\hat{\xi},\hat{\zeta},\hat{w}) &= \mathrm{D}_g\widetilde{\mathbf{J}}_\xi g\hat{\xi} + \mathrm{D}_\zeta\widetilde{\mathbf{J}}_\xi\hat{\zeta}\\ &= \chi(g)\left(-\mathrm{Ad}^*_{g^{-1}}(\mathrm{ad}^*_{\mathrm{Ad}_g(\hat{\xi})}(\mu+\zeta)) + \mathrm{Ad}^*_{g^{-1}}(\hat{\zeta})\right)(\xi).\end{aligned}$$

Comparing coefficients of $\hat{\zeta}$ and $\mathrm{Ad}_g(\hat{\xi})$ shows that the two expressions are the same.

This action of G on $\widetilde{\mathcal{M}}$ commutes with the free, symplectic action of G_p given by (5.1). Moreover $\widetilde{\mathbf{J}}$ is invariant under this action of G_p. It follows that $\widetilde{\mathbf{J}}$ restricts and descends to a well-defined map $\mathbf{J}$ on the G_p-reduced space $\widetilde{M}_0$ and that this map is a momentum map for the action of G on $\widetilde{M}_0$ [33]. Putting $\zeta = \nu + \mathbf{J}_{N_1}(w)$ gives $\mathbf{J}(g,\nu,w) = \chi(g)\mathrm{Ad}^*_{g^{-1}}(\mu+\nu+\mathbf{J}_{N_1}(w))$ as required.

5.4 Proof of Theorem 3.1

The slice theorem provides a G-equivariant diffeomorphism from a neighbourhood of the zero section of $G\times_{G_p} N$ to a neighbourhood of Gp in $\mathcal{M}$. The pull back of a G-invariant Hamiltonian H on $\mathcal{M}$ under this diffeomorphism is a $G\times G_p$-invariant function h on $G\times N$. Since H is G-invariant we know that h is independent of $g\in G$ and depends only on the slice variables $(\nu,w) \in N_0\oplus N_1 = N$. Via the inverse of the map $(\nu,w)\to(\nu+\mathbf{J}_{N_1}(w),w)$ the function h pulls back to a G_p-invariant function on $\psi^{-1}(0)\subset \mathbf{g}^*_\mu\oplus N_1$. Let $\tilde{h}$ be a G_p-invariant smooth extension of this function to $\mathbf{g}^*_\mu\oplus N_1$, so that $\tilde{h}(\nu+\mathbf{J}_{N_1}(w),w) = h(\nu,w)$. For simplicity we choose $\tilde{h}$ to be the trivial extension: for $\zeta\in\mathbf{g}^*_\mu$ with $\zeta=\nu+\lambda$, where $\nu\in\mathrm{ann}_{\mathbf{g}^*_\mu}(\mathbf{g}_p)\cong\mathbf{g}^*_\mu/\mathbf{g}^*_p$ and $\lambda\in\mathrm{ann}_{\mathbf{g}^*_\mu}(\mathbf{m}_\mu)\cong\mathbf{g}^*_p$ we define $\tilde{h}(\zeta,w)=h(\nu,w)$. We can also regard the function $\tilde{h}$ as a smooth $G\times G_p$-invariant extension of h to $G\times(\mathbf{g}^*_\mu\oplus N_1)$.

By the reduction procedure outlined in §5.2 the flow on a neighbourhood U of Gp in $\mathcal{M}$ is G-equivariantly symplectomorphic to a flow on the neighbourhood $\widetilde{U}_0$ of $G\times_{G_p}\{0\}$ in $\widetilde{\mathcal{M}}_0 = \Psi^{-1}(0)/G_p \cong G \times_{G_p} N$. This flow is obtained by restriction to $\Psi^{-1}(0)$ and reduction by the G_p-action (5.1) of the flow on $\widetilde{\mathcal{M}} = G \times (\mathbf{g}_\mu^* \oplus N_1)$ generated by $\tilde{h}$ and the symplectic form $\tilde{\omega}$.

We are now ready to prove Theorem 3.1 which gives explicit differential equations for the flow on $\widetilde{\mathcal{M}}$. The Hamiltonian system of differential equations on $\widetilde{\mathcal{M}} = G\times(\mathbf{g}_\mu^*\oplus N_1)$ is determined by the equation

$$\widetilde{\omega}(g,\zeta,w)((\dot{g},\dot{\zeta},\dot{w}),(\hat{g},\hat{\zeta},\hat{w})) \;=\; \mathrm{D}\tilde{h}(g,\zeta,w)(\hat{g},\hat{\zeta},\hat{w})$$

where $\dot{g},\hat{g}\in g\mathbf{g}$. From the definition of $\tilde{\omega}$ in §5.2 the lefthand side of this equation is given by

$$\chi(g)\left(\hat{\zeta}(g^{-1}\dot{g}) - \dot{\zeta}(g^{-1}\hat{g}) + (\zeta+\mu)[g^{-1}\dot{g},g^{-1}\hat{g}] + \omega_{N_1}(\dot{w},\hat{w})\right)$$

while the righthand side is

$$\mathrm{D}_\zeta\tilde{h}(\zeta,w)\hat{\zeta} + \mathrm{D}_w\tilde{h}(\zeta,w)\hat{w}.$$

Comparing coefficients of $g^{-1}\hat{g},\hat{\zeta}$ and $\hat{w}$ we obtain, respectively,

$$\begin{aligned}\dot{\zeta} &= \mathrm{ad}^*_{g^{-1}\dot{g}}(\mu+\zeta) &(5.3)\\ \mathrm{P}_{\mathbf{g}_\mu}(g^{-1}\dot{g}) &= \chi(g)\mathrm{D}_\zeta\tilde{h}(\zeta,w) &(5.4)\\ \dot{w} &= \chi(g)J\mathrm{D}_w\tilde{h}(\zeta,w), &(5.5)\end{aligned}$$

where $\mathrm{P}_{\mathbf{g}_\mu}$ is the projection from $\mathbf{g}$ to $\mathbf{g}_\mu$ with kernel $\mathbf{n}_\mu$.

Equation (5.5) for $\dot{w}$ is already in the required form while equation (5.4) gives us the component of $g^{-1}\dot{g}$ along $\mathbf{g}_\mu$. Specifically, if $g^{-1}\dot{g}=\xi+\eta$ where $\xi\in\mathbf{g}_\mu$ and $\eta\in\mathbf{n}_\mu$ then $\xi=\chi(g)\mathrm{D}_\zeta\tilde{h}(\zeta,w)$. To obtain η and $\dot{\zeta}$ we split equation (5.3) into its projections into the two subspaces $\mathrm{ann}(\mathbf{g}_\mu)$ and $\mathrm{ann}(\mathbf{n}_\mu)$ of $\mathbf{g}^*$. Since $\dot{\zeta}$ lies in $\mathrm{ann}(\mathbf{n}_\mu)\cong\mathbf{g}_\mu^*$ these equations are

$$\begin{aligned}0 &= \mathrm{P}_{\mathrm{ann}(\mathbf{g}_\mu)}\left(\mathrm{ad}^*_{\xi+\eta}(\mu+\zeta)\right) &(5.6)\\ \dot{\zeta} &= \mathrm{P}_{\mathrm{ann}(\mathbf{n}_\mu)}\left(\mathrm{ad}^*_{\xi+\eta}(\mu+\zeta)\right). &(5.7)\end{aligned}$$

By Proposition 2.5 equation (5.6) can be solved uniquely for

$$\eta \;=\; \eta_\mu(\xi,\zeta) \;=\; \eta_\mu(\chi(g)\mathrm{D}_\zeta\tilde{h}(\zeta,w),\zeta) \;=\; \chi(g)\eta_\mu(\mathrm{D}_\zeta\tilde{h}(\zeta,w),\zeta),$$

the last equality following from the linearity of $\eta_\mu(\xi,\zeta)$ in ξ. This gives

$$g^{-1}\dot{g} \;=\; \xi+\eta_\mu(\xi,\zeta) \;=\; \chi(g)\left(\mathrm{D}_\zeta\tilde{h}(\zeta,w)+\eta_\mu(\mathrm{D}_\zeta\tilde{h}(\zeta,w),\zeta)\right) \;=\; \chi(g)j_\mu(\zeta)\mathrm{D}_\zeta\tilde{h}$$

as required for the $\dot{g}$ equation. Substituting for $g^{-1}\dot{g}$ in equation (5.7) gives the $\dot{\zeta}$ equation. This completes the proof of Theorem 3.1.

Remark 5.4 In general different choices of the extension $\tilde{h}$ generate different flows on $G \times \left(\mathbf{g}_\mu^* \oplus N_1\right)$ and even on $G \times \psi^{-1}(0)$. However these flows all induce the same quotient flow on the reduced space $G \times_{G_p} \psi^{-1}(0)$.

5.5 Proof of Theorem 3.2

To prove Theorem 3.2 we pull the equations of Theorem 3.1 back to $G \times ((\mathbf{g}_\mu/\mathbf{g}_p)^* \oplus N_1)$ via the map to $G \times \psi^{-1}(0) \subset G \times (\mathbf{g}_\mu^* \oplus N_1)$ given by $(g, \zeta, w) = (g, \nu + \mathbf{J}_{N_1}(w), w)$. Remember that the function $\tilde{h}$ from Theorem 3.1 satisfies $\tilde{h}(\zeta, w) = \tilde{h}(\nu + \mathbf{J}_{N_1}(w), w) = h(\nu, w)$. It follows that

$$\begin{aligned} \mathrm{D}_\zeta \tilde{h}(\zeta, w) &= \mathrm{D}_\nu h(\nu, w) \quad \in \quad \mathbf{m}_\mu \\ \eta_\mu(\mathrm{D}_\zeta \tilde{h}(\zeta, w), \zeta) &= \eta_\mu(\mathrm{D}_\nu h(\nu, w), \nu + \mathbf{J}_{N_1}(w)) \quad \in \quad \mathbf{n}_\mu. \end{aligned}$$

Defining $j = j(\nu, w) = j_\mu(\nu + \mathbf{J}_{N_1}(w))$ the equations in Theorem 3.1 become:

$$\dot{g} = \chi(g) g j \mathrm{D}_\nu h \tag{5.8}$$

$$\dot{\nu} = \chi(g) \mathrm{P}_1 \left(\mathrm{ad}^*_{j\mathrm{D}_\nu h}(\mu + \nu + \mathbf{J}_{N_1}(w))\right) \tag{5.9}$$

$$\dot{\mathbf{J}}_{N_1}(w) = \chi(g) \mathrm{P}_2 \left(\mathrm{ad}^*_{j\mathrm{D}_\nu h}(\mu + \nu + \mathbf{J}_{N_1}(w))\right) \tag{5.10}$$

$$\dot{w} = \chi(g) J D_w h(\nu, w) \tag{5.11}$$

where P_1 is the projection from $\mathbf{g}^*$ to $\mathrm{ann}(\mathbf{g}_p + \mathbf{n}_\mu) \cong (\mathbf{g}_\mu/\mathbf{g}_p)^*$ with kernel $\mathrm{ann}(\mathbf{m}_\mu)$ and P_2 is the projection from $\mathbf{g}^*$ to $\mathrm{ann}(\mathbf{m}_\mu + \mathbf{n}_\mu) \cong \mathbf{g}_p^*$ with kernel $\mathrm{ann}(\mathbf{g}_p)$. We need the following lemma:

Lemma 5.5 *With P_1 and P_2 defined as above the following statements hold:*

a) For $\xi \in \mathbf{g}$ we have $\mathrm{ad}^*_\xi \mu \in \mathrm{ann}(\mathbf{g}_\mu)$ and so $\mathrm{P}_1 \mathrm{ad}^*_\xi \mu = \mathrm{P}_2 \mathrm{ad}^*_\xi \mu = 0$.

b) For $\xi \in \mathbf{m}_\mu \cong \mathbf{g}_\mu/\mathbf{g}_p$ and $\nu \in \mathrm{ann}(\mathbf{g}_p + \mathbf{n}_\mu) \cong (\mathbf{g}_\mu/\mathbf{g}_p)^*$ we have

$$\mathrm{P}_1 \mathrm{ad}^*_\xi(\nu) = \overline{\mathrm{ad}}^*_\xi(\nu).$$

Here the ad^* operator on the left hand side is acting on $\mathbf{g}^*$ while the $\overline{\mathrm{ad}}^*$ operator on the righthand side is the operator on $(\mathbf{g}_\mu/\mathbf{g}_p)^*$ defined in §2.3.

c) For $\eta \in \mathbf{n}_\mu$ the map ad^*_η maps $\mathrm{ann}(\mathbf{n}_\mu) \cong \mathbf{g}_\mu^*$ into $\mathrm{ann}(\mathbf{g}_p) = (\mathbf{g}/\mathbf{g}_p)^*$ and so $\mathrm{P}_2 \mathrm{ad}^*_\eta \nu = 0$ for $\nu \in \mathbf{g}_\mu^*$.

d) For $\xi \in \mathbf{m}_\mu \oplus \mathbf{n}_\mu \cong \mathbf{g}/\mathbf{g}_p$ the map ad^*_ξ maps $\mathrm{ann}(\mathbf{m}_\mu \oplus \mathbf{n}_\mu) \cong \mathbf{g}_p^*$ into $\mathrm{ann}(\mathbf{g}_p) = (\mathbf{g}/\mathbf{g}_p)^*$. Therefore $\mathrm{P}_2 \mathrm{ad}^*_\xi \zeta = 0$ for $\zeta \in \mathbf{g}_p^*$.

e) For $\xi \in \mathbf{m}_\mu \cong \mathbf{g}_\mu/\mathbf{g}_p$ and $\zeta \in \mathrm{ann}(\mathbf{m}_\mu \oplus \mathbf{n}_\mu) \cong \mathbf{g}_p^*$ we have

$$\mathrm{P}_1 \mathrm{ad}^*_\xi \zeta = \mathrm{ad}^*_\xi \zeta$$

where the ad^* on the left hand side is acting on $\mathbf{g}^*$, while that on the righthand side is acting on $\mathbf{g}_\mu^*$.

Proof.

a) For $\xi \in \mathbf{g}$ and $\hat{\xi} \in \mathbf{g}_\mu$ we have $\mathrm{ad}^*_\xi(\mu)(\hat{\xi}) = \mu([\xi, \hat{\xi}]) = -\mathrm{ad}^*_{\hat{\xi}}(\mu)(\xi) = 0$.

b) If $\psi \in \mathbf{m}_\mu \cong \mathbf{g}_\mu/\mathbf{g}_p$ then $\mathrm{ad}^*_\xi(\nu)(\psi) = \nu([\xi, \psi])$. Since ξ also lies in $\mathbf{m}_\mu \cong \mathbf{g}_\mu/\mathbf{g}_p$ and $\nu \in \mathbf{g}^*_\mu/\mathbf{g}^*_p$ this implies that $\mathrm{P}_1(\mathrm{ad}^*_\xi(\nu)) = \overline{\mathrm{ad}}^*_\xi(\nu)$ where ad^* acts on $\mathbf{g}^*$ while $\overline{\mathrm{ad}}^*$ operator acts on $(\mathbf{g}_\mu/\mathbf{g}_p)^*$.

c) Let $\eta \in \mathbf{n}_\mu$, $\zeta \in \mathrm{ann}(\mathbf{n}_\mu)$ and $\xi \in \mathbf{g}_p$. The $\mathbf{g}_p$-invariance of $\mathbf{n}_\mu$ implies that $[\eta, \xi] \in \mathbf{n}_\mu$ and so $\mathrm{ad}^*_\eta(\zeta)(\xi) = \zeta([\eta, \xi]) = 0$.

d) The $\mathbf{g}_p$-invariance of $\mathbf{m}_\mu$ and $\mathbf{n}_\mu$ implies that $[\xi, \mathbf{g}_p] \subset \mathbf{m}_\mu \oplus \mathbf{n}_\mu$ for $\xi \in \mathbf{m}_\mu \oplus \mathbf{n}_\mu$. For $\zeta \in \mathrm{ann}(\mathbf{m}_\mu \oplus \mathbf{n}_\mu)$ we have $\zeta([\xi, \mathbf{g}_p]) = 0$ and so $\mathrm{ad}^*_\xi(\zeta) \in \mathrm{ann}(\mathbf{g}_p)$.

e) We have $\mathrm{P}_1\mathrm{ad}^*_\xi(\zeta) + \mathrm{P}_2\mathrm{ad}^*_\xi(\zeta) = \mathrm{ad}^*_\xi(\zeta)$ where the ad^*_ξ on the left hand side is acting on $\mathbf{g}^*$ and that on the right hand side is acting on $\mathbf{g}^*_\mu$. It is therefore sufficient to show that $\mathrm{P}_2\mathrm{ad}^*_\xi(\zeta) = 0$, which follows from part d).

■

The equations (5.8, 5.11) for $\dot{g}$ and $\dot{w}$ are those required for the statement of Theorem 3.2. By Lemma 5.5a) the first term in the $\dot{\nu}$ equation (5.9) vanishes. We now rewrite (5.9) as

$$\dot{\nu} = \chi(g)\mathrm{P}_1\left(\mathrm{ad}^*_{\mathrm{D}_\nu h}(\nu + \mathbf{J}_{N_1}(w)) + \mathrm{ad}^*_\eta(\nu + \mathbf{J}_{N_1}(w))\right) \tag{5.12}$$

where $\eta = \eta_\mu(\mathrm{D}_\nu h(\nu, w), \nu + \mathbf{J}_{N_1}(w))$. By Lemma 5.5b) the first term in (5.12) is $\mathrm{P}_1(\mathrm{ad}^*_{\mathrm{D}_\nu h}(\nu)) = \overline{\mathrm{ad}}^*_{\mathrm{D}_\nu h}(\nu)$. Similarly, by Lemma 5.5e), we get $\mathrm{P}_1(\mathrm{ad}^*_{\mathrm{D}_\nu h}(\mathbf{J}_{N_1}(w))) = \mathrm{ad}^*_{\mathrm{D}_\nu h}(\mathbf{J}_{N_1}(w))$ where the ad^* on the left hand side is acting on $\mathbf{g}^*$, while that on the righthand side is acting on $\mathbf{g}^*_\mu$. Dropping the subscript 1 from P_1 completes the proof that equations (5.8, 5.9, 5.11) are equivalent to the equations in the statement of Theorem 3.2.

5.6 Proof of Proposition 3.6 c)

Equation (5.10) simplifies to $\dot{\mathbf{J}}_{N_1}(w) = \chi(g)\mathrm{P}_2\left(\mathrm{ad}^*_{\mathrm{D}_\nu h}(\nu)\right)$ as a result of the following observations:

1. The term $\mathrm{P}_2(\mathrm{ad}^*_{j\mathrm{D}_\nu h}\mu)$ in (5.10) vanishes by Lemma 5.5a).
2. The term $\mathrm{P}_2(\mathrm{ad}^*_{j\mathrm{D}_\nu h}\mathbf{J}_{N_1}(w))$ in (5.10) vanishes by Lemma 5.5d)
3. By Lemma 5.5c) we have $\mathrm{P}_2(\mathrm{ad}^*_{j\mathrm{D}_\nu h(\nu,w)}(\nu)) = \mathrm{P}_2(\mathrm{ad}^*_{\mathrm{D}_\nu h(\nu,w)}(\nu))$.

Restricting to $\chi(g) = 1$ and noting that the restriction of P_2 to $\mathrm{ann}(\mathbf{n}_\mu) \cong \mathbf{g}^*_\mu$ is equal to $\widehat{\mathrm{P}}$ proves part c) of Proposition 3.6.

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