On symmetric ω-limit sets in reversible flows

Jeroen S.W. Lamb & Matthew Nicol

Nonlinear Systems Laboratory,
Mathematics Institute, University of Warwick,
Coventry CV4 7AL, United Kingdom

September 22, 1995

Abstract

Let $\Gamma \subset O(n)$ be a finite group acting orthogonally on $\mathbb{R}^n$. We say that Γ is a reversing symmetry group of the flow f^t if Γ has an index two subgroup $\tilde{\Gamma}$ whose elements commute with f^t and for all elements $\rho \in \Gamma - \tilde{\Gamma}$ and all t, $f^t \circ \rho(x) = \rho \circ f^{-t}(x)$. In dimensions $n = 1, 2$ we describe all symmetry groups of ω-limit sets for such reversible flows. In case $n \geq 1$ we give group and representation theoretic restrictions on possible symmetry groups and show that for subgroups of $\tilde{\Gamma}$ our conditions are necessary and sufficient. We also describe in detail the possible symmetries of periodic orbits.

Finally, we show that if a Liapunov stable ω-limit set is fixed setwise by a reversing symmetry then it is transitive.

1 Introduction and outline.

In this paper we will be concerned with the symmetry properties of ω-limit sets of reversible flows. For completeness, in this introductory discussion we will mention earlier results pertaining to the symmetry properties of ω-limit sets of mappings as well. In recent years, a number of studies [2, 5, 16, 17] have resulted in necessary and sufficient conditions for stable ω-limit sets to have symmetries in equivariant dynamical systems. In this paper we will discuss extensions of these results in the flow context by taking into account the presence of reversing symmetries.

We will write invertible dynamical systems in the form $f^t : \mathbb{R}^n \mapsto \mathbb{R}^n$, with the time variable $t \in \mathbb{Z}$ (discrete time) or $t \in \mathbb{R}$ (continuous time), such that f^t is continuous on $\mathbb{R}^n$, $f^{t_1} \circ f^{t_2} = f^{t_1+t_2}$ for all t_1, t_2, and $f^0 = \mathrm{id}$. In the case of discrete time we thus have the dynamics generated by a homeomorphism $f = f^1$ (with $f^n = f \circ \ldots \circ f$ (n times)). In case f is moreover continuously differentiable it is a diffeomorphism. In the case of continuous time f^t is the flow of a vector field in $\mathbb{R}^n$.

We will distinguish two types of symmetry properties. The homeomorphism $\sigma : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a *symmetry* of f^t if for all t

$$f^t \circ \sigma = \sigma \circ f^t, \tag{1}$$

where $\circ$ denotes composition. When f^t has a symmetry σ it is also called σ-*equivariant*. The collection of all symmetries of f^t forms a group under composition. This group is usually called the *symmetry group* of f^t.

In addition to symmetries, f^t may also possess *reversing symmetries*: a homeomorphism $\rho : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a reversing symmetry of f^t if for all t

$$f^t \circ \rho = \rho \circ f^{-t}. \tag{2}$$

As the composition of two reversing symmetries yields a symmetry and the composition of a reversing symmetry and a symmetry yields a reversing symmetry, the union of the sets of symmetries and reversing symmetries of f^t form a group, which we will call the *reversing symmetry group* of f^t [9]. It is easily checked that any reversing symmetry group contains a normal subgroup of index two which contains only symmetries. A dynamical system which possesses a reversing symmetry will be called *reversible*. It should be noted that we do not require reversing symmetries to be involutions. Recall that a map ρ is called an involution if $\rho^2 = \mathrm{id}$.

The ω-limit set of a point $\mathbf{x} \in \mathbb{R}^n$ consists of all $\mathbf{y} \in \mathbb{R}^n$ for which there exists a strictly increasing set of positive times $\{t_k\}$ (with either $t_k \in \mathbb{Z}$ or $t_k \in \mathbb{R}$ and $\lim_{k\to\infty} t_k = \infty$) such that $f^{t_k}(\mathbf{x})$ converges to $\mathbf{y}$. The ω-limit set of a flow is always connected. We will be concerned with sets A which are ω-limit sets of a reversible flow $f^t : \mathbb{R}^n \mapsto \mathbb{R}^n$, i.e. $A = \omega(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^n$.

An ω-limit set A is called *Liapunov stable* if for any open neighbourhood V of A there exists an open neighbourhood $U \subset V$ of A such that $f^t(U) \subset V$ for all $t \geq 0$. Liapunov stability should be clearly distinguished from the stronger notion of asymptotic stability. An ω-limit set A is called *asymptotically stable* if there exists an open neighbourhood U of A such that $\omega(x) \subseteq A$ for all $\mathbf{x} \in U$.

Our interest in this paper will be with Liapunov stable ω-limit sets of flows on $\mathbb{R}^n$ having a reversing symmetry group Γ that is a finite subgroup of $O(n)$. Throughout this paper we will assume Γ to be acting as a linear representation on $\mathbb{R}^n$ (and orthogonal with respect to the natural inner product on $\mathbb{R}^n$).

We define the *symmetry group* $\Sigma \leq \Gamma$ of a subset $A \subset \mathbb{R}^n$ as

$$\Sigma = \{\gamma \in \Gamma \mid \gamma(A) = A\}. \tag{3}$$

Let $T \leq \Sigma$ denote the subgroup of instantaneous symmetries, i.e. the symmetries which fix A pointwise. Since in describing the symmetry of the set A it is relevant to know both Σ and T, we indicate the symmetry of A by the pair (Σ, T). Note that for all such pairs, T must be a normal subgroup of Σ.

In this context we have the following notion of *admissibility*:

Definition 1.1 Suppose that Σ is a subgroup of Γ. We say that Σ is *admissible* for flows (homeomorphisms, diffeomorphisms) if there exists a flow (homeomorphism, diffeomorphism) with reversing symmetry group Γ and a Liapunov stable ω-limit set with symmetry $(\Sigma, 1)$.

The central question we would like to address is:[1]

> *What subgroups of reversing symmetry groups are admissible as symmetry groups of Liapunov stable ω-limit sets for reversible dynamical systems ?*

In the present paper we will address this question in the context of flows. The context of reversible mappings will be treated in a forthcoming paper [11].

The above question has been answered in a variety of equivariant contexts. An important observation is that the symmetry group of a periodic orbit must be a cyclic extension of an isotropy subgroup (see [17]). However, if the asymptotic dynamics are more complicated than this then the phenomenon of *symmetry on average* may occur in which an ω-limit set displays greater symmetry than that obtainable by periodic orbits.

In the continuous category necessary and sufficient conditions have been given for admissibility in the work of Melbourne *et al.* [17] and Ashwin and Melbourne [2]. Group elements which act as reflections play the crucial role in determining admissibility. In this context we call $\tau : \mathbb{R}^n \mapsto \mathbb{R}^n$ a reflection if $\tau^2 = \mathrm{id}$ and $\dim \mathrm{Fix}(\tau) = n - 1$. The fixed point subspace of τ is called a *reflection hyperplane* and divides $\mathbb{R}^n$ in two.

If Δ is a subgroup of Γ we let L_Δ denote the union of reflection hyperplanes corresponding to group elements not contained in Δ. A key result of [17] was that if Σ is admissible then necessarily there is a subgroup Δ such that

a) Δ is a normal subgroup of Σ,

b) Σ/Δ is cyclic,

c) Δ fixes a connected component of $\mathbb{R}^n - L_\Delta$.

Ashwin and Melbourne [2] showed that this representation-theoretic condition is also sufficient by using a construction involving Cayley graphs. In particular they showed that all cyclic subgroups of Γ were admissible, recapturing a result of King and Stewart [8]. They also showed that if a Σ-symmetric attractor is

[1] We would like to note that the context of Liapunov stable ω-limit sets is chosen in connection with the setting in previous work on symmetries of attractors in [2, 5, 16, 17]. However, in particular in the reversible case, other types of ω-limit sets may be equally (or more) relevant. In this respect it is important to note that most of our results apply to ω-limit sets in general, rather than only to Liapunov stable ones. For further remarks we refer the reader to the discussion in Section 7.

connected then Σ must fix a connected component of $\mathbb{R}^n - L_\Sigma$. Furthermore the attractors constructed in the proof of this theorem were topologically transitive, possessed open basins of attraction and the connected attractors were topologically mixing.

When equivariant homeomorphisms or flows are considered then the restrictions on admissibility are more severe. Let L denote the union of the reflection hyperplanes corresponding to reflections in Γ. Melbourne [16] showed that if A is a Σ-symmetric ω-limit set for a Γ-equivariant homeomorphism then either

a) Σ contains no reflections and fixes a connected component of $\mathbb{R}^n - L$, or

b) Σ has an index two subgroup that fixes a connected component of $\mathbb{R}^n - L$

If Σ is to be admissible for flows then Σ must fix a connected component of $\mathbb{R}^n - L$.

Precise necessary and sufficient conditions for admissibility of flows and homeomorphisms are given in [5]. In fact if $n \geq 4$ then the ω-limit sets that realise admissibility can be taken to be Axiom A attractors and in any case they can be taken to be asymptotically stable. The sufficiency conditions for flows is complicated somewhat by a topological obstruction in the case $n = 2$, but otherwise if Σ fixes a connected component of $\mathbb{R}^n - L$ then it is admissible and the admissibility can be realised by asymptotically stable ω-limit sets (which can be taken as Axiom A if $n \geq 5$). In the case of dimension 2 then the only subgroup of $\mathbb{D}_m$ that is admissible for flows is 1 and if $\Gamma = \mathbb{Z}_m$ then the only admissible subgroups are 1 and $\mathbb{Z}_m$.

In order to establish admissibility we use two ingredients: a collection of necessary conditions for admissibility which rule out certain subgroups and for the remaining subgroups a construction of ω-limit sets with the required symmetry.

We will now summarize our main results. We give necessary conditions on finite subgroups of $O(n)$ for admissibility for flows and necessary and sufficient conditions in the case $n \leq 2$. Furthermore, we extend the results of [5] on the admissibility of symmetry groups of ω-limit sets that do not contain reversing symmetries to the reversible flow context. We first introduce some notation and terminology.

We let $\tilde{\Gamma}$ denote the symmetry group *associated* to Γ, i.e. $\tilde{\Gamma}$ is the largest subgroup of Γ containing no reversing symmetries. Then, $\tilde{\Gamma}$ is a normal subgroup of Γ and when $\Gamma \neq \tilde{\Gamma}$ it is a subgroup of index two.

In equivariant dynamical systems restrictions on symmetries for ω-limit sets occur due to the presence of symmetries which are reflections. As in the equivariant case we let L denote the union of all the reflection hyperplanes of Γ, i.e.

$$L := \bigcup_{\text{refl } \tau \in \Gamma} \mathrm{Fix}(\tau). \tag{4}$$

We define $\tilde{L}$ to be the union of all the reflection hyperplanes of $\tilde{\Gamma}$, i.e.

$$\tilde{L} := \bigcup_{\text{refl } \tau \in \tilde{\Gamma}} \text{Fix}(\tau). \tag{5}$$

If Γ does not contain reversing symmetries, the definitions of L and $\tilde{L}$ coincide.

We first note that, as a consequence of the results of [5], if Γ is a finite subgroup of $O(n)$ $(n \geq 1)$, $\Sigma \leq \Gamma$ and A is a Σ-symmetric ω-limit set for a flow $f^t : \mathbb{R}^n \mapsto \mathbb{R}^n$ with reversing symmetry group Γ then Σ must fix a connected component of $\mathbb{R}^n - \tilde{L}$.

In case Γ does not contain reversing symmetries it has been shown in [5] that subgroups Σ are admissible for flows only if they fix a connected component of $\mathbb{R}^n - L$ and that all such subgroups are admissible whenever $n \geq 3$. For subgroups Σ that do not contain reversing symmetries, this result extends to the reversible case:

Theorem 1.2 *Suppose Γ is a finite subgroup of $O(n)$, $n \geq 3$ and $\Sigma \leq \Gamma$. Then, a subgroup Σ that does not contain a reversing symmetry is admissible for flows $f^t : \mathbb{R}^n \mapsto \mathbb{R}^n$ with reversing symmetry group Γ if and only if Σ fixes a connected component of $\mathbb{R}^n - L$. The ω-limit sets that realize admissibility can be taken to be asymptotically stable. Moreover, if $n \geq 5$ the ω-limit sets that realize admissibility can be taken to be Axiom A attractors.*

We now give necessary and sufficient conditions for admissibility of subgroups that do not fix a connected component of $\mathbb{R}^n - L$. For an explanation of our notation of reversing symmetry groups we refer the reader to the Appendix.

Theorem 1.3 *Suppose Γ is a finite subgroup of $O(n)$ $n \geq 2$, and $\Sigma \leq \Gamma$. Then, a subgroup Σ that does not fix a connected component of $\mathbb{R}^n - L$ is admissible for flows $f^t : \mathbb{R}^n \mapsto \mathbb{R}^n$ with reversing symmetry group Γ if and only if*

(i) Σ fixes a connected component of $\mathbb{R}^n - \tilde{L}$

(ii) $\Sigma \simeq \mathbb{Z}_k \wedge \mathbb{D}'_1$ (km') where the subgroup isomorphic to $\mathbb{D}'_1$ (m') is generated by a reversing reflection ρ and the cyclic subgroup isomorphic to $\mathbb{Z}_k$ (k) is a maximal cyclic subgroup of $\tilde{\Gamma}$.[2]

(iii) when $k = 2$ in (ii), the subgroup of $\tilde{\Gamma}$ isomorphic to $\mathbb{Z}_2$ acts freely on the connected components of $\mathbb{R}^n - \text{Fix}(\rho)$.

The ω-limit sets that realise admissibility must be periodic orbits.

Further discussion on those subgroups for which admissibility may be realised by periodic orbits is given in Section 3, Remark 3.1.

[2] A maximal cyclic subgroup of $\tilde{\Gamma}$ is a cyclic subgroup that is not contained strictly in another cyclic subgroup of $\tilde{\Gamma}$.

In $\mathbb{R}$, the situation is very special due to topological restrictions.

Remark 1.4 Suppose Γ is a finite subgroup of $O(1)$, and $\Sigma \leq \Gamma$. Then the only admissible subgroup Σ for flows $f^t : \mathbb{R} \mapsto \mathbb{R}$ with reversing symmetry group Γ is $\mathbf{1}$.

Also in $\mathbb{R}^2$, extra restrictions arise. In summary, they lead to the following result, with reference to the tables in Section 5:

Theorem 1.5 *Suppose Γ is a finite subgroup of $O(2)$, and $\Sigma \leq \Gamma$. Then all subgroups Σ which are admissible for flows $f^t : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with reversing symmetry group Γ are listed in Table 1 and Table 2. The ω-limit sets that realize admissibility can be taken to be Liapunov stable periodic orbits.*

This paper is organized as follows. In Section 2 we are concerned with symmetry properties of periodic orbits. In Section 3 we prove Theorem 1.2 which concerns the admissibility of subgroups that do not contain reversing symmetries by showing that the constructions of [5] extend to the reversible case. We also prove Theorem 1.3 concerning the admissibility of subgroups for ω-limit sets that intersect more than one connected component of $\mathbb{R}^n - L$, exploiting the fact that these ω-limit sets must be periodic orbits. In the same section, in Remark 3.1 we address the general question of for which symmetry groups admissibility is realisable by periodic orbits. In Section 4 we show that Liapunov stable ω-limit sets that are symmetric with respect to a reversing symmetry must be transitive. We also show that if A is a Liapunov stable ω-limit set and $\gamma(A) \cap A \neq \emptyset$ (where γ is a symmetry or a reversing symmetry) then $\gamma(A) = A$. We then focus on the construction of symmetric ω-limit sets that are periodic orbits in the context of $\mathbb{R}^2$ and establish in Section 5 the admissibility of subgroups of $O(1)$ and $O(2)$. In Section 6 we explain how our results extend to the case of symmetric ω-limit sets possessing nontrivial instantaneous symmetries. The paper is concluded with a discussion.

2 Symmetric periodic orbits

In this section we will focus on symmetric periodic orbits. They are the simplest examples of transitive ω-limit sets. We will include, for completeness, results concerning periodic orbits of reversible mappings as well.

An orbit of f^t is defined as

$$o(\mathbf{x}) := \{f^t(\mathbf{x})\}_{t\in\mathbb{R} \text{ or } t\in\mathbb{Z}}. \tag{6}$$

An orbit $o(\mathbf{x})$ is called periodic with period p if p is the smallest positive real (or integer) for which $f^p(\mathbf{x}) = \mathbf{x}$.

Our main result on symmetric periodic orbits is contained in the following proposition:[3]

Proposition 2.1 *Let $f^t : \mathbb{R}^n \to \mathbb{R}^n$ be a dynamical system with reversing symmetry group Γ (which is finite subgroup of $O(n)$) and associated symmetry group $\tilde{\Gamma}$. Suppose f^t has a periodic orbit and symmetry (Σ, T). Let $\tilde{\Sigma} := \Sigma \cap \tilde{\Gamma}$ and $\tilde{T} := T \cap \tilde{\Gamma}$. Then we have one of the following two situations*

(a) $T = \tilde{T}$ and either $\Sigma/T \simeq \mathbb{Z}_k$ (k), if $\Sigma = \tilde{\Sigma}$, or $\Sigma/T \simeq \mathbb{Z}_k \wedge \mathbb{D}_1'$ (km'), if $\Sigma \neq \tilde{\Sigma}$, and in the discrete time case k divides the period p of the periodic orbit.

(b) $T \neq \tilde{T}$ and in the discrete time case the period p is either 1 or 2 and $\Sigma/T \simeq \mathbb{Z}_k$ (k dividing p), while in the flow case $\Sigma/T \simeq 1$ and the periodic orbit is a fixed point.

A further discussion on for which groups Σ admissibility may be achieved by periodic orbits is presented in Section 3, Remark 3.1.

The proof of Proposition 2.1 uses the following Lemma.

Lemma 2.2 ([12]) *If an orbit $o(\mathbf{x})$ is symmetric with respect to a reversing symmetry ρ, then $o(\mathbf{x}) \subseteq \mathrm{Fix}(\rho^2)$.*

Proof. The proof is elementary. If $o(\mathbf{x})$ is symmetric with respect to ρ then for all $\mathbf{y} \in o(\mathbf{x})$, $\rho(\mathbf{y}) = f^t(\mathbf{y})$ for some t. Hence $\rho^2(\mathbf{y}) = \rho \circ f^t(\mathbf{y}) = f^{-t} \circ \rho(\mathbf{y}) = \mathbf{y}$ for all $\mathbf{y} \in o(\mathbf{x})$. □

Proof of Proposition 2.1 We consider case (b) first. Suppose T contains a reversing symmetry ρ, then $f^{-t} \circ \rho(x) = f^{-t}(x) = \rho \circ f^t(x) = f^t(x)$. Hence in the discrete time case the periodic orbit has period 1 or 2 (and Σ/T is isomorphic to 1 or $\mathbb{Z}_2$) and in the flow case the periodic orbit is a fixed point.

Now we consider case (a) where $\tilde{T} = T$. Suppose that $\Sigma = \tilde{\Sigma}$. Then by [17] we have $\Sigma/T \simeq \mathbb{Z}_k$ (k), where k divides the period p in the discrete time case. Suppose now that $\Sigma \neq \tilde{\Sigma}$ then $\Sigma = \tilde{\Sigma} \cup \rho\tilde{\Sigma}$ for some reversing symmetry ρ. Now $\tilde{\Sigma}/T$ is a normal subgroup of Σ/T and the group generated by $\rho T/T$ is of order 2 (since $\rho^2 \in \tilde{T}$)and together with $\tilde{\Sigma}/T$ generates Σ/T. Thus Σ/T is the semi-direct product of $\tilde{\Sigma}/T$ by $\langle \rho T/T \rangle$. As before $\tilde{\Sigma}/T \simeq \mathbb{Z}_k$ (k) for some integer k (which divides the period p in the discrete time case). The order 2 subgroup generated by $\rho T/T$ is isomorphic to $\mathbb{D}_1'$ (m'). Note that the group $\mathbb{Z}_k \wedge \mathbb{D}_1'$ (km') is isomorphic to the dihedral group $\mathbb{D}_k$ if one ignores the difference between symmetries and reversing symmetries. □

Finally, we recall a well-known characterization of orbits of flows that are symmetric with respect to reversing symmetries, cf. e.g. [4, 10], and a useful Corollary.

[3] For an explanation of our notation of reversing symmetry groups the reader is referred to the Appendix.

Theorem 2.3 *If f^t is a flow with reversing symmetry ρ and orbit $o(\mathbf{x})$, then $o(\mathbf{x})$ is symmetric with respect to ρ if and only if $o(\mathbf{x})$ intersects* $\mathrm{Fix}(\rho)$*. The orbit is nonperiodic if and only if the intersection consists of only one point and $o(\mathbf{x}) \not\subseteq$* $\mathrm{Fix}(\rho)$*. The orbit is periodic (and not stationary) if and only if $o(\mathbf{x})$ intersects* $\mathrm{Fix}(\rho)$ *in precisely two points.*

Corollary 2.4 *Suppose ρ is a reflection and a reversing symmetry of a flow $f^t : \mathbb{R}^n \mapsto \mathbb{R}^n$ possessing an ω-limit set that is symmetric with respect to ρ. Then this ω-limit set is either a fixed point (contained in* $\mathrm{Fix}(\rho)$*) or a periodic orbit (transversally intersecting* $\mathrm{Fix}(\rho)$ *in two points).*

3 Stable symmetric ω-limit sets

In this section we establish Theorem 1.2 and Theorem 1.3. We will begin with the proof of Theorem 1.3, which is based upon the ideas of the preceding section and [5].

Proof of Theorem 1.3 Suppose Σ is admissible for flows. Then Σ must fix a connected component of $\mathbb{R}^n - \tilde{L}$ by the results of [5]. Suppose Σ does not fix a connected component of $\mathbb{R}^n - L$. Then any flow which realises admissibility of Σ must intersect $\mathrm{Fix}(\rho)$ for some reversing reflection ρ. Thus by Theorem 2.3 the ω-limit of the flow is ρ-symmetric and so by Corollary 2.4 the ω-limit set must be a periodic orbit (as it is assumed to have trivial instantaneous symmetry). Proposition 2.1 shows, as Σ contains a reversing symmetry and $T = \tilde{T} = 1$, that $\Sigma \simeq \mathbb{Z}_k \wedge \mathbb{D}_1'\ (km')$ for some $k \geq 1$, where the subgroup isomorphic to $\mathbb{D}_1'\ (m')$ is generated by the reversing reflection ρ.

Moreover, if the ω-limit set intersects the fixed sets of two reversing reflections then the symmetry group of the ω-limit set is precisely the group generated by these two reflections. Therefore, if $k \geq 2$, the ω-limit set (which is a periodic orbit) must connect two different faces of a connected component of $\mathbb{R}^n - L$ originating from two different reversing reflection hyperplanes. Two of such reversing reflections generate a dihedral group whose nonreversing cyclic subgroup is maximal.

Conversely, if Γ contains a subgroup $\Sigma \simeq \mathbb{Z}_k \wedge D_1'\ (km')$ $k \geq 3$ that fixes a connected component of $\mathbb{R}^n - \tilde{L}$, but not of $\mathbb{R}^n - L$, with the group isomorphic to $\mathbb{D}_1'\ (m')$ being generated by a reversing reflection ρ and the subgroup of $\tilde{\Gamma}$ isomorphic to $\mathbb{Z}_k\ (k)$ being a maximal cyclic subgroup, one connected component of $\mathbb{R}^n - L$ will possess two faces made up of the fixed point subspaces of reversing reflections ρ_1, ρ_2 generating Σ. A flow with a Σ-symmetric (Liapunov stable) periodic orbit is constructed as follows: inside the connected component of $\mathbb{R}^n - L$ with faces consisting of parts of $\mathrm{Fix}(\rho_1)$ and $\mathrm{Fix}(\rho_2)$, these two faces are connected by a flow-line (satisfying the boundary condition of orthogonal impact on $\mathrm{Fix}(\rho_1)$ and $\mathrm{Fix}(\rho_2)$) avoiding intersections with fixed point subspaces of other (reversing) symmetries. This flow-line can be smoothly embedded in a flow in the connected component of $\mathbb{R}^n - L$ satisfying appropriate boundary conditions at the fixed point subspaces. From here, the flow in all other connected components of $\mathbb{R}^n - L$ follows by symmetry, such that the flow-line intersecting ρ_1 and ρ_2 extends to a Σ-symmetric periodic orbit in $\mathbb{R}^n$. It is obvious that the embedding can be constructed in such a way that the orbit will be Liapunov stable. Note that the constructed periodic orbit cannot have more symmetry than Σ.

In case $k = 2$, one must add the requirement that the subgroup of $\tilde{\Gamma}$ isomorphic to $\mathbb{Z}_2$ (2) acts freely on the connected components of $\mathbb{R}^n - \mathrm{Fix}(\rho)$. Namely, if it does not, then Σ cannot be the symmetry of a periodic orbit. This follows from the fact that $\mathrm{Fix}(\rho)$ divides any ρ-symmetric orbit in two halves. Let σ denote generator of the cyclic subgroup of $\tilde{\Gamma}$ isomorphic to $\mathbb{Z}_2$ (2), and suppose the period of the symmetric periodic orbit is p, then for any point $\mathbf{x}(t)$ on this periodic orbit $\sigma(\mathbf{x}(t)) = \mathbf{x}(t + p/2)$. However, $\rho(\mathbf{x}(t)) = \mathbf{x}(a - t)$ for some $a \in \mathbb{R}$. Hence, if $\mathbf{x}(t)$ is in one connected component of $\mathbb{R}^n - \mathrm{Fix}(\rho)$ then $\mathbf{x}(t + p/2)$ must be in the other connected component. Moreover, σ fixes $\mathrm{Fix}(\rho)$ and thus should permute the connected components of $\mathbb{R}^n - \mathrm{Fix}(\rho)$.

In case $k \geq 3$, however, this requirement is redundant, as it follows directly from the (dihedral) group structure of Σ that the $\mathbb{Z}_k$ action does not fix $\mathrm{Fix}(\rho)$.

Finally, in case $k = 1$ the construction of a symmetric periodic orbit consists of constructing and embedding a flow line connecting two points on $\mathrm{Fix}(\rho)$ inside one connected component of $\mathbb{R}^n - L$. □

Remark 3.1 In the case of flows, it is not difficult to obtain from Proposition 2.1 and Theorem 2.3 necessary and sufficient conditions for subgroups Σ of Γ containing a reversing symmetry for which admissibility can be realized by periodic orbits. As discussed in the proof of Theorem 1.3, for the construction of symmetric periodic orbits it suffices to construct a flow containing one flow-line connecting two points of the fixed sets of reversing symmetries generating Σ. As the periodic orbit must intersect each fixed set twice it is necessary that the fixed point subspaces of the reversing symmetries have dimensions greater or equal to 1. The case that Σ does not fix a connected component of $\mathbb{R}^n - L$ has been treated in Theorem 1.3. In the remaining case, in which Σ fixes a connected component of $\mathbb{R}^n - L$ and $\Sigma \simeq \mathbb{Z}_k \wedge \mathbb{D}'_1$ (km') one needs to consider the cases $k = 1$, $k = 2$ and $k \geq 3$ separately.

If $k = 1$ (and $\Sigma \simeq \mathbb{D}'_1$) a necessary and sufficient condition for the possibiity of constructing a symmetric periodic orbit is that the fixed set of the reversing symmetry generating Σ has dimension greater or equal to 1. Within one connected component of $\mathbb{R}^n - L$ it is possible to locally construct a flow-line connecting two points of $\mathrm{Fix}(\rho)$ without intersecting any other fixed point subspace and this can be embedded in a flow which has a reversing symmetry group Γ.

If $k \geq 3$ it is easily verified from the (dihedral) group structure of Σ that indeed all reversing symmetries in Σ must have fixed point subspaces of dimension greater or equal to 1. Namely, the only linear involution in $O(n)$ that has a zerodimensional fixed-point subspace is $-\mathrm{id}$, and $-\mathrm{id}$ does commute with all k-fold rotations in $O(n)$. Admissibility for these subgroups can always be realized by symmetric periodic orbits, in a similar way as described above.

In case $k = 2$ one has to take more care and check explicitly that the two reversing symmetries generating Σ indeed both have fixed point subspaces of dimension greater or equal to 1.

In a similar way, it can also be shown that for every cyclic subgroup of $\tilde{\Gamma}$ that fixes a connected component of $\mathbb{R}^n - L$, admissibility can be realized by a periodic orbit. Let us say that this cyclic group is isomorphic to $\mathbb{Z}_k$ and generated by σ. Then, inside one component one constructs a flow with a flow-line connecting two $\mathbb{Z}_k$-equivalent points (i.e. a point $\mathbf{x}$ and a point $\sigma(\mathbf{x})$), that does not intersect any fixed point subspace. This flow-line can be embedded in a Γ-symmetric flow by first embedding the flow line in a Γ-symmetric flow inside the connected component of $\mathbb{R}^n - L$ and then extending the flow by symmetry extends to a flow in $\mathbb{R}^n$. Taking appropriate boundary conditions for the initial flow-line (to avoid the occurrence of a singularity at $\mathbf{x}$) will then lead to a flow with a $\mathbb{Z}_k$-symmetric periodic orbit. Liapunov stability can also be realized in the embedding.

Proof of Theorem 1.2 We will first establish the necessity of the condition. In the case that $L = \tilde{L}$ then the results of [5] show that Σ must fix a connected component of $\mathbb{R}^n - L$. So we suppose that $L \neq \tilde{L}$. Note that, in this case, if Σ does not fix a connected component of $\mathbb{R}^n - L$ then the flow which realises the admissibilty of Σ must intersect $\mathrm{Fix}(\rho)$ for some reversing reflection ρ and hence Σ contains ρ (by Theorem 2.3). This contradicts the assumption that Σ contains no reversing symmetries. So the condition is necessary.

The proof of sufficiency involves an obvious modification to the corresponding proofs of [5] and we present in detail only the construction of Axiom A attractors for flows in dimension $n \geq 5$. The case $n = 3, 4$ (in which we may construct asymptotically stable attractors) is handled with the same (mutatis mutandi) modification to the constructions given in [5].

Let $\tilde{\Gamma}$ be the symmetry group associated to Γ and $\tilde{\Sigma}$ be a subgroup of $\tilde{\Gamma}$ which fixes a connected component of $\mathbb{R}^n - L$. Let ρ be a reversing symmetry (recall that $\Gamma = \rho\tilde{\Gamma} \cup \tilde{\Gamma}$). Note that , by the results of [5], there exists a Γ- equivariant flow which has an attractor with symmetry group $\tilde{\Sigma}$. Hence there exists a $\tilde{\Sigma}$ invariant connected open set O such that $\gamma(O) \cap O = \emptyset$ for all $\gamma \in \Gamma - \tilde{\Sigma}$.

The existence of such an open set O shows that we may as in [5] construct a $\tilde{\Gamma}$-equivariant flow f^t with a $\tilde{\Sigma}$-symmetric attractor $A \subset O$ (note that $\tilde{\Sigma}$ is Class I in the terminology of [5]). This attractor has $|\tilde{\Gamma} : \tilde{\Sigma}|$ conjugate attractors. In the construction of f^t and A we may require that A has a $\tilde{\Sigma}$ invariant neighbourhood $U \subset O$ such that $f^t(U) = U$ and $f^t|_{\tilde{\Gamma}U^c} = \mathrm{id}|_{\tilde{\Gamma}U^c}$ (this follows from the construction of [5]).

We may smoothly isotope $f^t|_U$ to a flow g^t which is the identity map on O^c (since by construction in [5] we may take f^t Γ-equivariantly isotopic to the identity and $f^t(U) = U$ for all t) and which has the properties

a) $g^t_{|U} = f^t_{|U}$ for all t

b) $g^t(O) = O$ for all t.

Now we extend g^t Γ-equivariantly to a flow h^t. Finally define a flow F^t by $F^t(x) = h^t(x)$ if $x \in \tilde{\Gamma}(O)$ and $F^t(x) = h^{-t}(x)$ if $x \in \rho\tilde{\Gamma}(O)$. Note that F^t has reversing symmetry group Γ and F^t has a $\tilde{\Sigma}$ symmetric Axiom A attractor contained in O. □

4 Marginal stability and transitivity

In this section we will show that Liapunov stable ω-limit sets that are symmetric with respect to a reversing symmetry must be transitive. Recall that a set A is called transitive if $A = \omega(\mathbf{x})$ for some $\mathbf{x} \in A$.

Theorem 4.1 *Suppose that A is a Liapunov stable ω-limit set of a dynamical system f^t with reversing symmetry ρ. Then,*

$$\rho(A) = A \quad \Rightarrow \quad A \text{ is transitive.}$$

The proof of Theorem 4.1 will follow directly from Lemma 4.3 and Lemma 4.4 below.

A common notion of attractor in the literature is that of an asymptotically stable ω-limit set (for a definition see Section 1). In the context of symmetric (equivariant) dynamical systems, in order to establish admissibility of symmetry groups for attractors in [5] asymptotically stable ω-limit sets were constructed, although the main results were obtained under the assumption of only Liapunov stability.

However, in reversible systems an ω-limit set that is symmetric with respect to a reversing symmetry cannot be asymptotically stable, it can at most be *marginally stable*.

Definition 4.2 An ω-limit set A of a dynamical system f^t is *marginally stable* if for all open neighbourhoods V of A there exists an open neighbourhood U of A such that $f^t(U) \subset V$ for all $t \in \mathbb{R}$ (in the case of continuous time) or $t \in \mathbb{Z}$ (in the case of discrete time).

Lemma 4.3 *Suppose we have the setting of Theorem 4.1. Then,*

$$\rho(A) = A \quad \Rightarrow \quad A \text{ is marginally stable.}$$

Proof. Because A is Liapunov stable, by definition we have that for all neighbourhoods V there exists a neighbourhood $U \subset V$ such that $f^t(U) \subset V$ for all $t \geq 0$. We are thus left to prove the assertion for $t < 0$.

Suppose the assertion does not hold, then $\rho(A) = A$ and there exists a neighbourhood V such that for all $U \subset V$ there exists a $t < 0$ such that $f^t(U) \not\subset V$ (note that without loss of generality we may assume that all neighbourhoods V, U in the following discussion are ρ-symmetric). Thus, given $U \subset V$, there exists a point $\mathbf{x} \notin V$ for which $f^t(\mathbf{x}) = \tilde{\mathbf{x}} \in U \subset V$ for some $t > 0$. But then also $\rho(\tilde{\mathbf{x}}) = \rho(f^t(\mathbf{x})) = f^{-t}(\rho(\mathbf{x}))$, which implies that $f^t(\rho(\tilde{\mathbf{x}}))$ is not in V and hence A is not Liapunov stable since $\rho(\tilde{\mathbf{x}}) \in U$ and $U \subset V$ was arbitrary. □

Lemma 4.4 *If an ω-limit set is marginally stable then it is also transitive.*

Proof. Suppose $A = \omega(\mathbf{x})$ and A is marginally stable. We show that $\mathbf{x} \notin A$ leads to a contradiction. Every neighbourhood $V \supset A$ contains a point of the f^t-orbit of $\mathbf{x}$. In turn this implies that there exists a neighbourhood V' for which there does not exist a smaller neighbourhood U such that $f^t(U) \subset V'$ for all $t < 0$. This implies that A cannot be marginally stable. □

We now prove a theorem which is the reversible analogue to a result of [3, 17].

Theorem 4.5 *Suppose $f^t : I\!R^n \to I\!R^n$ has a Liapunov stable ω-limit set A. Suppose γ is either a symmetry or a reversing symmetry of f^t. Then if $\gamma(A) \cap A \neq \emptyset$ then $\gamma(A) = A$.*

Proof. The case where γ is a reversing symmetry of f^t has been covered in [17]. Suppose that γ is a reversing symmetry of f^t. For convenience we let $\omega_g(\mathbf{x})$ denote the ω-limit set of $\mathbf{x}$ under g^t (either a map or flow). Let $\mathbf{x}$ be such that $\omega_f(\mathbf{x}) = A$. Then, as γ is a reversing symmetry of f^t, $\omega_{f^{-1}}(\gamma(\mathbf{x})) = \gamma(A)$. Suppose that $\gamma(A)$ and A are not disjoint and $\gamma(A) \neq A$. We will derive a contradiction.

In this case there exists a neighbourhood V of A such that $\gamma(A) \cap V^c \neq \emptyset$, where V^c denotes the complement of V in $\mathbb{R}^n$. Then there exists a neighbourhood $U \subset V$ such that $f^t(U) \subset V$ for all $t \geq 0$. There exists an increasing sequence $\{t_k\}$ such that $f^{-t_k}(\gamma(\mathbf{x})) \in V^c$ (since $\omega_{f^{-1}}(\gamma(\mathbf{x})) = \gamma(A)$). But there exists a time $t' > t_1$ such that $f^{-t'}(\gamma(\mathbf{x})) \in U$ and hence $f^t(f^{-t'}(\gamma(\mathbf{x}))) \in V$ for all positive t. However if $t = t' - t_1 > 0$ then $f^t(f^{-t'}(\gamma(\mathbf{x}))) \in V^c$, which is a contradiction. Thus $\gamma(A) = A$ if $\gamma(A) \cap A \neq \emptyset$. □

5 Constructions in $\mathbb{R}$ and $\mathbb{R}^2$

In this section we outline the construction of symmetric periodic orbits in $\mathbb{R}$ and $\mathbb{R}^2$.

Before treating the constructions in $\mathbb{R}^2$, we first prove the results on admissibility in the one-dimensional case.

Proof of Remark 1.4, In the case that $\Gamma = \mathbb{Z}_2$ and does not contain a reversing symmetry, it is easily checked that its only admissible subgroup is 1, cf. [5]. We are left to consider only one group Γ containing a reversing symmetry, namely $\mathbb{Z}'_2 := \langle \rho \rangle$, with $\rho = -\mathrm{id}$. In the case of a flow it follows immediately that the only possible $\mathbb{Z}'_2$-symmetric ω-limit set is a stationary point at $\mathrm{Fix}(\rho)$. □

The rest of this section will be devoted to constructions of symmetric periodic orbits in $\mathbb{R}^2$.

Theorem 5.1 *Suppose A is a Liapunov stable ω-limit set of a flow f^t in $\mathbb{R}^2$ with reversing symmetry ρ. Then, if $\rho(A) = A$, A is a periodic orbit.*

Proof. By Theorem 4.1, A is transitive. Hence A is an ω-limit set for f^t as well as for f^{-t}. For flows in $\mathbb{R}^2$ it is well known, cf. [13], that this implies that A must be a periodic orbit. □

We now discuss the question of the admissibility of subgroups of $O(2)$.

From Theorem 5.1 and Theorem 2.3 it follows that $\mathbb{Z}'_n$ is not admissible for flows for any $n \geq 2$. In more detail, Theorem 5.1 shows that a Liapunov stable ω-limit set must be a periodic orbit and Theorem 2.3 shows that this periodic orbit must intersect $\mathrm{Fix}(\rho)$ (the origin) in precisely two points unless it is a fixed point at the origin (and hence possesses nontrivial instantaneous symmetry).

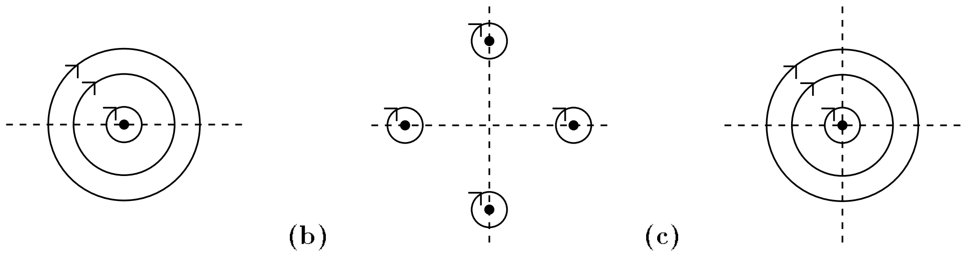
Now we look at the subgroups which fix a connected component of $\mathbb{R}^n - \tilde{L}$. The subgroup $\mathbb{D}'_1$ plays a distinguished role. The local construction of a $\mathbb{D}'_1$ symmetric periodic orbit is given in Fig. 1(a). The periodic orbit may be taken to be marginally stable. The constructions in Fig. 1 show that $\mathbb{D}'_1$ is admissible in case Γ is $\mathbb{D}'_1$ (m') or $\mathbb{Z}_n \wedge \mathbb{D}'_1$ (nm'). In a similar way, one finds that $\mathbb{D}'_1$ is admissible in the case that $\Gamma = \mathbb{D}_n \wedge \mathbb{D}'_1$ $((2n)'m)$.

In the case of $\mathbb{Z}_n \wedge \mathbb{D}'_1$ it follows from Theorem 1.3 that the subgroups $\mathbb{Z}_k$ $(k|n,\ k \neq 1)$ and $\mathbb{Z}_k \wedge \mathbb{D}'_1$ $(k|n, k \neq 1, n)$ are not admissible, while $\mathbb{Z}_n \wedge \mathbb{D}'_1$ is admissible. The construction of a $\mathbb{Z}_n \wedge \mathbb{D}'_1$ symmetric Liapunov stable attractor consists of a $\mathbb{D}'_1$ symmetric periodic orbit with $\mathbb{Z}_n$ symmetry, cf. Fig. 1(c).

The results are summarized in Table 1. Together with the results of [5] – which are summarized in Table 2 – they cover all admissible finite subgroups of $O(2)$ for flows in $\mathbb{R}^2$.

6 Symmetric ω-limit sets in fixed point subspaces

In this paper we have concentrated mainly on symmetric ω-limit sets with trivial instantaneous symmetry, i.e. in the notation of Section 1 we have studied the admissibility of subgroup (Σ, T) with $T = 1$. In this section we discuss how the results obtained in the previous sections extend to the case of symmetry groups (Σ, T) with $T \neq 1$, giving a reversible extension to the exposition in [5].



(a) (b) (c)

Figure 1: Schematic illustration of the construction of flows f^t with a periodic orbit that is symmetric with respect to a reversing symmetry ρ. The strategy is to construct flows constituting a local rotation in a small domain isotoped to the identity outside this domain. Some examples: **(a)** Local rotation constituting $\mathbb{D}_1'$ (m') symmetric marginally stable ω-limit sets. **(b)** Construction with $\mathbb{D}_1'$ (m') symmetry when $\Gamma \simeq \mathbb{Z}_2 \times \mathbb{D}_1'$ $(2m')$. **(c)** Construction with $\mathbb{Z}_2 \times \mathbb{D}_1'$ $(2m')$ symmetry when $\Gamma \simeq \mathbb{Z}_2 \times \mathbb{D}_1'$ $(2m')$.

Γ	Σ
$\mathbb{Z}_n'$ (n'), $n \geq 1$	1
$\mathbb{D}_1'$ (m')	$\mathbb{D}_1'$ (m')
$\mathbb{Z}_n \wedge \mathbb{D}_1'$ (nm'), $n \geq 1$	$\mathbb{Z}_n \wedge \mathbb{D}_1'$ (nm'), $\mathbb{D}_1'$ (m'), 1
$\mathbb{D}_n \wedge \mathbb{D}_1'$ $((2n)'m)$, $n \geq 1$	$\mathbb{D}_1'$ (m'), 1

Table 1: Subgroups Σ of reversing symmetry groups Γ that are admissible for flows in the case that Γ is a finite subgroup of $O(2)$. Groups labeled with a prime indicate groups generated by a reversing symmetry. The notation between brackets follows conventions in crystallography (see the Appendix or [14] for more details).

Γ	Σ
$\mathbb{Z}_n$ (n), $n \geq 1$	$\mathbb{Z}_n$ (n)
$\mathbb{D}_n$ $n \geq 1$	1

Table 2: Subgroups Σ of symmetry group Γ which are admissible for flows in the case that Γ is a finite subgroup of $O(2)$ (taken from [5]).

We first note T must be a normal subgroup of Σ and that T must be an isotropy subgroup (cf. [16]).[4] Moreover, if T contains a reversing symmetry then the ω-limit set must be a fixed point (and hence Σ must be an isotropy subgroup of Γ). The situation may become more interesting when T does not contain a reversing symmetry, and from now on we will assume that this is the case. Note that then $\text{Fix}(T)$ is an invariant subspace for f^t.

One obtains necessary conditions for admissibility of a pair of subgroups (Σ, T) by restricting to $\text{Fix}(T)$, which is isomorphic to $\mathbb{R}^{n_T}$ for some $n_T \leq n$.

Let $N(T)$ denote the normalizer of T in Γ, i.e.

$$N(T) := \{\gamma \in \Gamma \mid \gamma \circ \tau \circ \gamma^{-1} \in T \text{ for all } \tau \in T\}. \tag{7}$$

Then a flow f^t on $\mathbb{R}^n$ with reversing symmetry group Γ restricts to a flow with reversing symmetry group $\Gamma_T := N(T)/T$ on $\text{Fix}(T)$. Then it follows that if (Σ, T) is admissible for flows in $\mathbb{R}^n$ with reversing symmetry group Γ, then also $(\Sigma_T, 1)$ is admissible for flows in $\mathbb{R}^{n_T}$ with reversing symmetry group Γ_T, where $\Sigma_T := \Sigma/T$. However, this observation does not lead to an optimal result due to the presence of so-called *hidden symmetries* [7] which are elements γ of $\Gamma - N(T)$ with the property that $\gamma(\text{Fix}(T)) \cap \text{Fix}(T) \neq \emptyset$.

In order to determine the admissibility of a pair of subgroups (Σ, T) it suffices to take account of *hidden reflections*, which are elements $\gamma \in \Gamma - N(T)$ such that $\text{Fix}(\gamma)$ intersects $\text{Fix}(T)$ in a codimension one subspace. Following [5] we denote the set of $\gamma \in \Gamma$ for which $\text{Fix}(\gamma)$ intersects $\text{Fix}(T)$ in a codimension one subspace as K_T.

Hidden reflections in Γ give rise to invariant subspaces within $\text{Fix}(T)$ separating $\text{Fix}(T)$ in two connected components. In particular, hidden reflections due to reversing symmetries give rise to invariant subspaces that are fixed pointwise by the flow f^t.

Lemma 6.1 *Let $f^t : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a flow with reversing symmetry group Γ that is a finite subgroup of $O(n)$. Furthermore, let $T \leq \tilde{\Gamma}$ be an isotropy subgroup and $\rho \in \Gamma - \tilde{\Gamma}$ be a reversing symmetry that is not in $N(T)$. Suppose that $\text{Fix}(\rho)$ intersects $\text{Fix}(T)$ in a codimension one subspace, then for all $\mathbf{x} \in \text{Fix}(\rho) \cap \text{Fix}(T)$ we have $f^t(\mathbf{x}) = \mathbf{x}$ for all t.*

Proof. For all $\mathbf{x} \in \text{Fix}(T)$ it follows from the fact that $T \leq \tilde{\Gamma}$, that $f^t(\mathbf{x}) \in \text{Fix}(T)$ for all t. Using the fact that Γ is a finite subgroup of $O(n)$ we furthermore obtain that if $\text{Fix}(\rho)$ intersects $\text{Fix}(T)$ in a codimension one subspace and $\rho \notin N(T)$, then for all $\mathbf{x} \in \text{Fix}(T) - \text{Fix}(T) \cap \text{Fix}(\rho)$, $\rho(\mathbf{x}) \notin \text{Fix}(T)$.

Now consider some point $\mathbf{x} \in \text{Fix}(\rho) \cap \text{Fix}(T)$. For this point it follows that $f^t(\mathbf{x}) \in \text{Fix}(T)$ and $f^t(\mathbf{x}) = \rho(f^{-t}(\mathbf{x}))$ for all t. As for all $\mathbf{x} \in \text{Fix}(T) - \text{Fix}(T) \cap \text{Fix}(\rho)$, $\rho(\mathbf{x}) \notin \text{Fix}(T)$, this implies that $f^t(\mathbf{x}) \in \text{Fix}(\rho)$ for all t. In turn, this implies that $f^t(\mathbf{x}) = f^{-t}(\mathbf{x})$ for all t, and hence that $f^t(\mathbf{x}) = \mathbf{x}$ for all t. □

[4] A group $T \leq \Gamma$ is called an isotropy subgroup of Γ if there exists a point $\mathbf{x} \in \mathbb{R}^n$ such that $T = \{\gamma \in \Gamma \mid \gamma(\mathbf{x}) = \mathbf{x}\}$.

Accordingly we define $\tilde{K}_T := K_T \cap (\Gamma - N(T) + N(T) \cap \tilde{\Gamma})$ which is precisely the subgroup of K_T giving rise to invariant codimension one subspaces within $\mathrm{Fix}(T)$.

In analogy to (4)-(5) one now defines

$$L_T := \bigcup_{\tau \in K_T} \mathrm{Fix}(\tau), \tag{8}$$

$$\tilde{L}_T := \bigcup_{\tau \in \tilde{K}_T} \mathrm{Fix}(\tau). \tag{9}$$

From here it is straighforward to show that the results of Theorem 1.2 and Theorem 1.3, as well as the results on symmetric periodic orbits in Section 2 and Section 3, Remark 3.1, translate directly to the context of ω-limit sets in fixed point subspaces by simply replacing $\mathbb{R}^n$ by $\mathrm{Fix}(T)$, L by L_T, and $\tilde{L}$ by $\tilde{L}_T$.

7 Discussion

We have given necessary and sufficient conditions for a subgroup to be admissible for flows in dimensions $n = 1, 2$.

We have also given restrictions on admissibility in all dimensions and shown ($n > 1$) that a subgroup containing no reversing symmetries is admissible for flows if and only if it fixes a connected component of $\mathbb{R}^n - L$. Moreover, we have given a full description of the symmetry groups for which admissibility can be realized by periodic orbits. It should be noted that in the case of subgroups Σ that do fix a connected componentof $\mathbb{R}^n - L$ and contain a reversing symmetry, our conditions are only shown to be necessary and not yet to be sufficient if Σ is such that admissibility cannot be realized by periodic orbits. Further insight in the structure of marginally stable ω-limit sets other than periodic orbits will be needed to fully treat these cases.

We have concentrated on the symmetry properties of reversible flows. We address the issue of the symmetry properties of ω-limit sets of reversible maps in a forthcoming paper [11]. Similar questions could be asked about the symmetry properties of the ω-limit sets of symplectic maps and Hamiltonian flows with reversing symmetry groups. In this context it is worthwhile to note that the results on admissibility of Theorem 1.5 also apply in the case of Hamiltonian flows. Indeed, the constructions of periodic orbits in Section 5 can be done using solely Hamiltonian flows. For related work on the symmetries of periodic orbits for Hamiltonian flows see e.g. [6, 18].

In the Hamiltonian context (and indeed also in the reversible context) more consideration should be given to the notion of stability that is used. One could ask simply for the possible symmetries of ω-limit sets with no additional structure but to reflect the behavior of physical systems some form of stability is desirable.

For Hamiltonian systems with no more than two degrees of freedom the concept of Liapunov stability is reasonable, e.g. in connection with the stability results implied by KAM-theory, cf. Moser [19]. However, especially in higher dimensional systems the phenomenon of Arnol'd diffusion makes topological notions less adequate. Thom [20] has proposed the measure theoretic notion of a *vague attractor* as a Hamiltonian analog of the attractors of differentiable dynamics (cf. also [1, p.583]). Similar considerations exist in the work of Melbourne [15] concerning the notion of *essential asymptotic* stability.

Another interesting line of investigation is the relation between symmetry and mixing properties in reversible (symplectic, Hamiltonian) systems. An elementary result in this direction is Theorem 4.1. Earlier work on equivariant systems has shown that topological mixing is incompatible with attractors having certain symmetries [17, 5]. As Hamiltonian systems have a natural invariant measure, relations between ergodic properties of the system and symmetry properties would be expected.

Acknowledgements

We thank Ian Melbourne for a helpful discussion and pointing out an error in an earlier version of this paper. We acknowledge the support of the University of Warwick and the University of Amsterdam for mutual visits in the early stages of this work. The research of JSWL is supported by the European Union through its Human Capital & Mobility research fellowship scheme (ERBCHBICT941533).

Appendix: notation

In this appendix we briefly explain our notation of reversing symmetry groups.

In the notation of reversing symmetry groups one needs to take account of the difference between symmetries and reversing symmetries of Γ. One way to do this is to adopt the notation used in crystallography in the description of magnetic groups, cf. [14]. In this notation, the finite subgroups of $O(2)$ are denoted in the following way. A single integer 'n' denotes a n-fold rotation, and the letter 'm' denotes a reflection (mirror). As all finite subgroups of $O(2)$ are generated by rotations and reflections they can just be denoted by their generators. Adopting the convention of denoting the cyclic subgroups of $O(2)$ that are generated by an n-fold rotation by $\mathbb{Z}_n$ and denoting the diheral subgroups of $O(2)$ generated by a reflection and an n-fold rotation by $\mathbb{D}_n$ we then have $n = \mathbb{Z}_n$ and $nm = \mathbb{D}_n$. In the case of reversing symmetry groups, in the crystallographic notation one labels reversing generators with a prime, e.g. m' or $2'm$.

Alternatively, we may write Γ as a group product $\Gamma = \mathcal{G} \cdot \mathcal{H}'$, with $\mathcal{G} \leq \tilde{\Gamma}$, $\mathcal{H}'$ is a cyclic group generated by a reversing symmetry (as indicated by the prime), and the product '$\cdot$' being either semi-direct '$\wedge$' or direct '$\times$'. Note that in case Γ contains a reversing symmetry that is an involution, we may choose $\mathcal{G} = \tilde{\Gamma}$ and

$\mathcal{H} \simeq \mathbb{Z}_2$. This notation has the advantage that it shows how reversing symmetry groups can be viewed as cyclic extensions of symmetry groups. In the present discussion we will use primarily the latter notation, followed by the crystallographic notation between brackets, e.g. $\mathbb{D}_1 \times \mathbb{D}_1'$ $(2'm)$.

In denoting the symmetry of an ω-limit set it is in principle not relevant to distinguish between symmetries and reversing symmetries. It is however instructive to use a notation which shows the origin of the symmetry. One easily verifies the symmetry of the set by ignoring the primes in the notation.

References

[1] R. Abraham and J.E. Marsden. *Foundations of mechanics.* Addison-Wesley, Redwood City CA, 2nd edition, 1978. (updated 1985 printing).

[2] P. Ashwin and I. Melbourne. Symmetry groups of attractors. *Arch.Rat.Mech.Anal.*, 126:59–78, 1994.

[3] P. Chossat and M. Golubitsky. Symmetry-increasing bifurcation of chaotic attrcators. *Physica D*, 32:423–436, 1988.

[4] R.L. Devaney. Reversible diffeomorphisms and flows. *Trans.Am.Math.Soc.*, 218:89–113, 1976.

[5] M. Field, I. Melbourne, and M. Nicol. Symmetric attractors for diffeomorphisms and flows. Proceedings of the LMS, to appear, 1994.

[6] M. Golubitsky, J.-M. Mao, and M. Nicol. Symmetries of periodic solutions for planar potential systems. Proceedings of the AMS, to appear, 1994.

[7] M. Golubitsky, J.E. Marsden, and D.G. Schaeffer. Bifurcation problems with hidden symmetries. In W.E. Fitzgibbon III, editor, *Partial differential equations and dynamical systems*, volume 101 of *Research notes in mathematics*, pages 181–210. Pitman, San Fransisco, 1984.

[8] G.P. King and I.N. Stewart. Symmetric chaos. In W.F. Ames and C.F. Rogers, editors, *Nonlinear equations in the applied sciences*, pages 257–315. Academic Press, 1991.

[9] J.S.W. Lamb. Reversing symmetries in dynamical systems. *J.Phys.A:math.gen.*, 25:925–937, 1992.

[10] J.S.W. Lamb. *Reversing symmetries in dynamical systems.* PhD thesis, University of Amsterdam, 1994.

[11] J.S.W. Lamb and M. Nicol. On symmetric ω-limit sets in reversible dynamical systems. in preparation, 1995.

[12] J.S.W. Lamb and G.R.W. Quispel. Reversing k-symmetries in dynamical systems. *Physica D*, 73:277–304, 1994.

[13] S. Lefschetz. *Differential equations: geometric theory*, volume VI of *Pure and Applied Mathematics.* Wiley Interscience, New York, 2nd edition, 1963.

[14] A.L. Loeb. *Color and symmetry.* Wiley & Sons, New York, 1971.

[15] I. Melbourne. An example of a non-asymptotically stable attractor. *Nonlinearity*, 4:835–844, 1991.

[16] I. Melbourne. Generalizations of a result on symmetry groups of attractors. In J. Chadam and W. Langford, editors, *Pattern formation: symmetry methods and applications*, Fields Institute Communications. Am.Math.Soc., Providence RI, 1994. to appear.

[17] I. Melbourne, M. Dellnitz, and M. Golubitsky. The structure of symmetric attractors. *Arch.Rat.Mech.Anal.*, 123:75–98, 1993.

[18] J.A. Montaldi, R.M. Roberts, and I.N. Stewart. Existence of nonlinear normal modes of symmetric hamiltonian systems. *Nonlinearity*, 3:695–730, 1990.

[19] J. Moser. *Stable and random motions in dynamical systems*, volume 77 of *Annals of Mathematics studies*. Princeton University Press, Princeton NJ, 1973.

[20] R. Thom. *Structural stability and morphogenesis: an outline of a general theory of models*. Benjamin-Cummings, Reading MA, 1975.