Area-preserving dynamics that is not reversible

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Abstract

In this paper we address some global dynamical features in area-preserving dynamical systems on the plane $\mathbb{R}^2$, which obstruct the presence of a time-reversal symmetry that is a reflection. We show that in the case of flows, reversibility and non-reversibility are non-generic properties. In the case of maps we show that reversibility is a non-generic property by constructing an example of a persistently non-reversible area-preserving map.

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1 Introduction

In physics, the conventional notion of time-reversal symmetry applies to dynamical phenomena for which one cannot decide from a motion picture whether it is played in forward or backward direction. For instance, from a motion picture of an *idealized* bouncing ball (bouncing elastically on a flat surface without loss of energy) we cannot determine in what direction it is shown to us. However, if friction is taken into account, the amplitude of the ball's bounces will gradually decrease and eventually the ball will come to rest. The reversed motion picture of the latter phenomenon shows the ball bouncing by itself gradually higher and higher. The latter phenomenon is clearly not in correspondence with our physical experience, and in this way the playing direction of the motion picture can be identified (in which case the system is not reversible in the conventional sense).

In the Hamiltonian formulation of classical mechanics, the above described conventional type of reversibility occurs when the Hamiltonian $H(\mathbf{p},\mathbf{q})$, which is the energy functional as a function of momenta $\mathbf{p}$ and positions $\mathbf{q}$, is invariant under the transformation $(\mathbf{p},\mathbf{q}) \mapsto (-\mathbf{p},\mathbf{q})$, i.e.

$$H(\mathbf{p},\mathbf{q}) = H(-\mathbf{p},\mathbf{q}). \tag{1}$$

This invariance causes the equations of motion

$$\frac{d}{dt}\mathbf{p} = -\frac{\partial}{\partial \mathbf{q}} H(\mathbf{p},\mathbf{q}), \quad \frac{d}{dt}\mathbf{q} = \frac{\partial}{\partial \mathbf{p}} H(\mathbf{p},\mathbf{q}), \tag{2}$$

to be invariant under the transformation $(\mathbf{p},\mathbf{q},t) \mapsto (-\mathbf{p},\mathbf{q},-t)$, corresponding precisely to inverting the playing direction of a motion picture. This symmetry property implies that if $(\mathbf{p}(t),\mathbf{q}(t))$ is a trajectory of the system then $(-\mathbf{p}(-t),\mathbf{q}(-t))$ is also a trajectory. In the $\mathbf{p}$-$\mathbf{q}$ phase space this corresponds to the fact that the transformation $(\mathbf{p},\mathbf{q}) \mapsto (-\mathbf{p},\mathbf{q})$ maps the flow to its inverse. We call such a transformation a *reversing symmetry* of the dynamical system. A dynamical system possessing a reversing symmetry will be called *reversible*.

An important class of reversible systems are derived from Hamiltonians of the form

$$H(\mathbf{p},\mathbf{q}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}), \tag{3}$$

which includes most systems studied in mechanics.

Consequences of conventional time-reversal symmetry are widely discussed in the physics literature. In statistical physics, for instance, there have been extensive discussions about the paradox that arises from Boltzmann's observation that a macroscopic number of particles, individually obeying reversible equations of motion, collectively display irreversible behaviour (entropy always increases) [4]. A quantum mechanical version of the classical conventional time reversal operator was introduced in the 1930s by Wigner [35]. With it, he explained the two-fold degeneracy of energy levels that was reported by Kramers [13] in systems with an odd number of electrons, in the absence of a magnetic field.

In dynamical systems theory, the importance of time-reversal symmetry was recognized already by Birkhoff, who used reversibility in his study of the three body problem [3]. Later, particularly in the work of Devaney [7], it became clear that the consequences of conventional time-reversal symmetry are shared by dynamical systems which have a different but equivalent reversing symmetry. The importance of recognizing non-conventional reversing

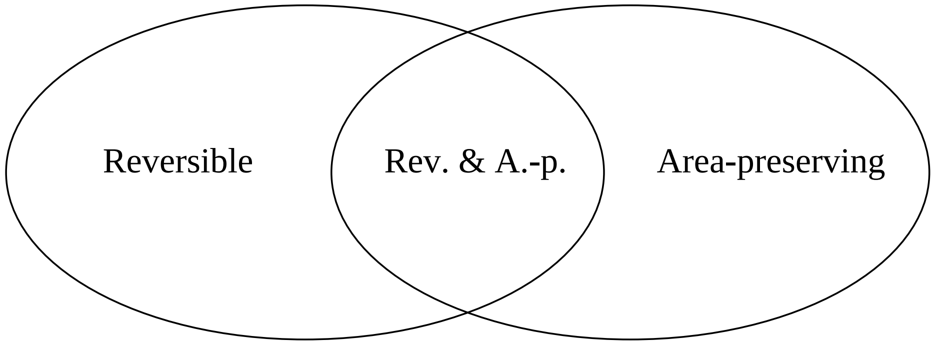

Figure 1: Venn diagram showing the intersection of the classes of reversible systems and area-preserving systems in the plane $\mathbb{R}^2$, illustrating that reversible systems need not be area-preserving and area-preserving systems need not be reversible.

symmetries has become apparent in many applications, from determining the proper universality class of level spacing distributions in 'chaotic' quantum systems [12], to acknowledging the full symmetry properties of phase portraits of dynamical systems [15].

Because of the importance of the class of Hamiltonians (3), reversibility has become a property in mechanics which is sometimes almost taken for granted. Furthermore, it turns out that Hamiltonian and reversible dynamical systems have many features in common, cf. e.g. [7, 32, 33]. This has lead people to address the question whether all Hamiltonian systems are reversible, in the sense that they possess some reversing symmetry which is not necessarily the conventional one, and whether all reversible systems are Hamiltonian. The answer to both questions is negative, e.g. all regions in the Venn-diagram of Fig. 1 are nonempty. Note in this respect that for maps and flows on the plane $\mathbb{R}^2$ being area-preserving is equivalent to being Hamiltonian (symplectic).

Reversible systems that are not conservative were encountered by Politi *et al.* [26] and studied in more detail by Roberts & Quispel [32]. Typically, such systems show a mixture of conservative (families of invariant tori) and dissipative dynamical behaviour (attractor-repeller pairs).

In response to the question whether all Hamiltonian systems in $\mathbb{R}^2$ are reversible, Mather constructed an example of a non-reversible flow of a divergence free (Hamiltonian) vector field, cf. [21]. Other examples were given by Arnol'd & Sevryuk [2] (cf. also [32]) and by Roberts & Capel [31]. The latter constructed examples of non-reversible area-preserving (symplectic) maps exploiting the fact that certain types of degenerate (Birkhoff) normal forms are incompatible with reversibility, cf. also [20]. A drawback of the examples by Arnol'd & Sevryuk and Roberts & Capel is that their constructions require degeneracies, leaving open the question of persistence of non-reversibility. Mather's example does not rely on degeneracies, and will be discussed in more detail in Section 3.2.

Although it is well known that all the regions in the Venn-diagram in Fig. 1 are nonempty, less is known about the nature of the intersection. For instance, it is not known whether the reversible and area-preserving systems intersect in an open subset of the class of area-preserving dynamical systems, or whether reversibility or non-reversibility are *generic* properties in the classes of area-preserving flows and diffeomorphisms. Recall that a property $\mathcal{P}$ is called a generic property in a class of systems when the property $\mathcal{P}$ is satisfied in a subset of this class that contains a countable intersection of open en dense sets (a so-called *residual* *set*). Hence, if a property does not hold in an open subset of a certain class of systems, then this property is not *generic* within this class. For a readable introduction into this notion of genericity, see e.g. [5].

Clearly, as the class of reversible dynamical systems contains open sets of systems that are not area-preserving, area-preservation is a non-generic property in this class.

In this paper we will address the question whether being reversible or being not reversible are generic properties of area-preserving dynamical systems on the plane $\mathbb{R}^2$. We will be concerned with continuously differentiable (C^1) area-preserving flows and diffeomorphisms and will deal with both contexts separately. Moreover, we focus on reversibility with respect to reversing symmetries that are continuous reflections in curves: $\rho : \mathbb{R}^2 \mapsto \mathbb{R}^2$, with $\rho^2 = \mathrm{id}$ and $\dim \mathrm{Fix}(\rho) = 1$. For such reflections it is readily verified from the continuity of ρ that $\mathrm{Fix}(\rho)$ has no self-intersections and no endpoints, and consequently that we can write $\mathbb{R}^2 = U_1 \cup U_2 \cup \mathrm{Fix}(\rho)$ with $U_1 \cap U_2 = \emptyset$ and U_1 and U_2 are simply connected, such that $\rho(U_1) = U_2$. In particular, it follows that such a reflection ρ is topologically conjugate to a linear reflection $r : (x, y) \mapsto (x, -y)$, i.e. that there exists an invertible continuous transformation (homeomorphism) $h : \mathbb{R}^2 \mapsto \mathbb{R}^2$ such that $r = h \circ \rho \circ h^{-1}$.

It is worthwhile noting that such reflections are *orientation-reversing.* To denote this specific type of reversibility we will follow the terminology of Devaney [7]:

Definition 1.1 A dynamical system on the plane $\mathbb{R}^2$ is called R-reversible if it possesses a reversing reflection symmetry ρ (with $\dim \mathrm{Fix}(\rho) = 1$ and $\rho^2 = \mathrm{id}$).

Throughout this paper when we refer to "reversible" systems, we will mean "R-reversible" systems, unless explicitly stated otherwise.

In this paper we will present examples of persistently non-reversible area-preserving flows and diffeomorphisms. We show persistence with respect to continuously differentiable (C^1) perturbations that preserve the area-preserving property. We will also show that there exist many persistently reversible flows. In summary, our examples provide the proofs for the following two theorems:

Theorem 1.2 *Being R-reversible and being not R-reversible are non-generic properties in the class of (C^1) area-preserving flows of (divergence free, Hamiltonian) vector fields in* $\mathbb{R}^2$.

Theorem 1.3 *Being R-reversible is a non-generic property in the class of (C^1) area-preserving (symplectic) diffeomorphisms in* $\mathbb{R}^2$.

Naturally, in both results we refer to genericity with respect to the C^1-topology. Throughout this paper we will work within the classes of continuously differentiable flows and diffeomorphisms, and hence from now on we will quietly assume that our dynamical systems (and perturbations) are C^1.

The paper is organized as follows. In the next section we describe the setting of the problem, defining our notion of reversibility in a more precise way, and recall some dynamical consequences of the presence of reversing symmetries, in particular in relation to periodic, homoclinic, and heteroclinic orbits.

Thereafter, in Section 3 we will discuss the case of flows and show that being reversible and being not reversible are both non-generic properties of area-preserving flows (Theorem 1.2). This result is obtained by pointing out the existence of persistently non-reversible and persistently reversible area-preserving flows (the non-reversible example being originally proposed by Mather).

In Section 4 we will be primarily concerned with the construction of an example of an area-preserving diffeomorphisms that is not reversible in a persistent way. In Section 3 it is argued that local obstructions against reversibility are usually not persistent in the case of area-preserving flows. However, as we will show, in the context of diffeomorphisms this is no longer the case. The starting point for the example is the flow of a vector field as sketched in Fig. 6. This flow is not reversible. By perturbing the time-one map of this flow in such a way that the stable and unstable manifolds of adjacent saddle points intersect transversally, a persistently non-reversible area-preserving diffeomorphism is constructed, providing the proof of Theorem 1.3.

The last section of this paper contains some concluding remarks.

2 Reversing symmetries and their dynamical consequences

We consider a continuously differentiable invertible dynamical system on the plane $\mathbb{R}^2$, i.e. a one-parameter family of diffeomorphisms $f^t : \mathbb{R}^2 \mapsto \mathbb{R}^2$ (t being the parameter representing *time*) such that $f^{t_1} \circ f^{t_2} = f^{t_1+t_2}$, and $f^0 = \mathrm{id}$. In the continuous time case ($t \in \mathbb{R}$), f^t is the flow of a vector field. In the discrete time case ($t \in \mathbb{Z}$) the dynamics is generated by a single diffeomorphism $f := f^1$. In the following discussion we will denote a dynamical system as f^t, both in the discrete and continuous time case.

We will call a dynamical system $f^t : \mathbb{R}^2 \mapsto \mathbb{R}^2$ *area-preserving* if $\det(Df^t|_{\mathbf{x}}) = 1$ for all $\mathbf{x} \in \mathbb{R}^2$, where $Df^t|_{\mathbf{x}}$ denotes the (Frechet) derivative of f^t evaluated at the point $\mathbf{x}$. It is easily verified that the area-preserving property in $\mathbb{R}^2$ is equivalent to the property that f^t is *symplectic*, i.e. that $Df^t|_{\mathbf{x}} \circ J \circ Df^t|_{\mathbf{x}} = J$ where $J(x, y) = (y, -x)$. If f^t is the flow of a vector field, then, accordingly, this vector field can be generated by a Hamiltonian function $H(x, y)$, cf. (2).

Let $\rho : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a homeomorphism. Then, we say that ρ is a *reversing symmetry* of the dynamical system f^t if for all t

$$f^t \circ \rho = \rho \circ f^{-t}. \tag{4}$$

From (4) it follows that reversing symmetries map trajectories (time-parameterized evolutions in the state space) of f^t to trajectories of f^{-t}, or equivalently that reversing symmetries map trajectories of f^t to other trajectories of f^t apart from a change in the direction in which they are traversed in the course of time. This is illustrated in Fig. 2(a). Note that in the continuous time case, when the dynamical system is the flow of a vector field F

$$\frac{d}{dt}\mathbf{x} = F(\mathbf{x}), \tag{5}$$

ρ is a reversing symmetry if and only if

$$\frac{d}{dt}(\rho(\mathbf{x})) = -F \circ \rho(\mathbf{x}). \tag{6}$$

Up to now, in the literature most attention has been given to reversing symmetries that are their own inverse (so-called *involutions*), mainly motivated by the involutory nature of the conventional time-reversal symmetry $(\mathbf{p}, \mathbf{q}) \mapsto (-\mathbf{p}, \mathbf{q})$. It should be noted, however, that reversing symmetries need not be involutions [14]. To illustrate this point, in Fig. 2(b) the flow of a vector field in $\mathbb{R}^2$ is sketched that possesses a non-involutory reversing symmetry,

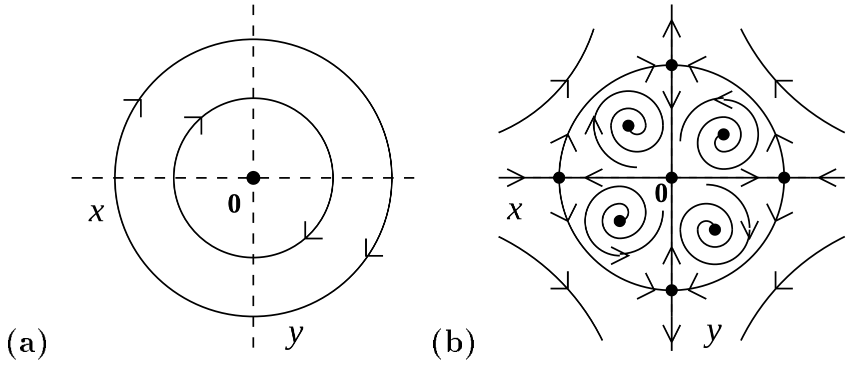

Figure 2: Schematic phase portraits of the flow of a vector field in $\mathbb{R}^2$. **(a)** possesses a reversing symmetry that is the reflection in the x-axis and **(b)** possesses a non-involutory reversing symmetry, being the rotation over 90 degrees around the origin $\mathbf{0}$, but no involutory reversing symmetry.

namely the rotation over 90 degrees around the origin $\mathbf{0}$, but no involutory one. It should also be noted that we do not require reversing symmetries to be linear transformations.

In contrast to reversing symmetries, dynamical systems may also possess (ordinary) symmetries: a homeomorphism σ is called a symmetry of f^t if for all t

$$f^t \circ \sigma = \sigma \circ f^t. \tag{7}$$

Alternatively, a dynamical system satisfying (7) is also called *σ-equivariant*. Equivariant dynamical systems have been studied extensively in recent years, cf. e.g. [6, 10, 11, 24]. The difference between symmetries and reversing symmetries is that symmetries map a trajectory of f^t to a trajectory of f^t inclusive the directions of time-parameterization, whereas in the case of reversing symmetries the latter directions are opposite.

Symmetries and reversing symmetries give rise to symmetry properties of phase portraits, though in different ways. For instance, fixed point subspaces of symmetries are invariants of the motion whereas fixed point subspaces of reversing symmetries typically are not. Also, symmetries preserve the stability properties, e.g. symmetries map attractors to attractors and repellers to repellers, whereas reversing symmetries reverse the stability properties, e.g. they map attractors to repellers and vice-versa. From a visual inspection of a phase portrait one can usually see whether symmetry properties of the phase portrait could have been caused by the presence of either symmetries or reversing symmetries of the equations of motion, cf. also [17].

Since phase portraits of maps are usually obtained from following a set of initial points forward in time, repellers are usually less visible than attractors. Because of this reason, in systems possessing dissipative features reversing symmetries may at first sight not be observed. However, in phase portraits of area-preserving systems (that cannot have attractors or repellers) reversing symmetries are often easily recognizable, in particular when they act linearly.

In the following section we will discuss in more detail the interplay between reversing symmetries and some dynamical features.

2.1 Symmetric periodic orbits

One well-appreciated feature in reversible systems is the occurrence of symmetric periodic orbits. We call the set $\{f^t(\mathbf{x}_0) \mid \text{for all } t\}$ the orbit of a point $\mathbf{x}_0$, where $t \in \mathbb{R}$ in the case of flows and $t \in \mathbb{Z}$ in the case of diffeomorphisms. It is called a periodic orbit with period p if $f^p(\mathbf{x}_0) = \mathbf{x}_0$. An orbit is called symmetric with respect to a reversing symmetry ρ whenever ρ maps the orbit to itself, i.e. when $f^t(\mathbf{x}_0) = \rho(\mathbf{x}_0)$ for some t. From now one we will call such orbits *symmetric* orbits.

There are a lot of similarities between symmetric orbits in reversible systems and orbits in Hamiltonian systems. For instance, in both cases orbits cannot be asymptotically stable or asymptotically unstable (i.e. attracting or repelling). In terms of the eigenvalues of $Df^t|_{\mathbf{x}_0}$ in symmetric equilibrium points one finds that eigenvalues come in singlets $\{1\}$ or $\{-1\}$, doublets $\{\lambda, \lambda^{-1}\}$ with $\lambda \in \mathbb{R}$ or λ on the unit circle in the complex plane, and quadruplets $\{\lambda, \lambda^{-1}, \lambda^*, \lambda^{*-1}\}$. In $\mathbb{R}^2$ quadruples cannot occur, and we call an equilibrium point a saddle point if the two eigenvalues are both real and not equal to ± 1, and a centre point if the eigenvalues are on the unit circle in the complex plane and not equal to ± 1. Importantly, the same conditions on the multipliers apply in the case of Hamiltonian dynamical systems. Therefore, in Hamiltonian dynamical systems it is often difficult to rule out reversibility from equilibrium points or periodic orbits alone as they typically have eigenvalues that are compatible with being symmetric periodic orbits. In fact, typically (in case $|\lambda| \neq 1$) (Birkhoff) normal forms for symmetric equilibrium points and equilibrium points in area-preserving dynamical systems are identical to all orders. Only in degenerate cases (when $|\lambda| = 1$) differences may occur. For further reading, see [20, 31, 32].

Other similarities between Hamiltonian and reversible systems are found in the persistence of invariant tori (KAM-theory) [23, 33] and on the types of local bifurcations [18, 34].

For later use, we now recall a well-known characterization of symmetric orbits in flows of vector fields.

Proposition 2.1 *Suppose f^t is the flow of a vector field with reversing symmetry ρ. Then an orbit of this flow is symmetric with respect to ρ if and only if the orbit intersects* $\mathrm{Fix}(\rho)$. *The orbit is non-periodic or stationary if and only if the intersection consists of precisely one point. The orbit is periodic (and not stationary) if and only if the orbit intersects* $\mathrm{Fix}(\rho)$ *in precisely two points.*

For this and related results see e.g. [7, 16].

2.2 Symmetric homoclinic and heteroclinic orbits

Other important types of orbits are homoclinic and heteroclinic orbits.

We consider an equilibrium $\mathbf{x}_0$ of saddle type of a dynamical system f^t, with either discrete or continuous time. Such an equilibrium point possesses stable and unstable manifold $W^s_{\mathbf{x}_0}$ and $W^u_{\mathbf{x}_0}$, that are defined as

$$W^s_{\mathbf{x}_0} := \{\mathbf{x} \in \Omega \mid \lim_{t\to\infty} f^t(\mathbf{x}) = \mathbf{x}_0\}, \tag{8}$$

$$W^u_{\mathbf{x}_0} := \{\mathbf{x} \in \Omega \mid \lim_{t\to-\infty} f^t(\mathbf{x}) = \mathbf{x}_0\}. \tag{9}$$

where $t \in \mathbb{R}$ in the continuous time case and $t \in \mathbb{Z}$ in the discrete time case. The existence and uniqueness of these manifolds can be deduced from the derivatives at the equilibrium point.

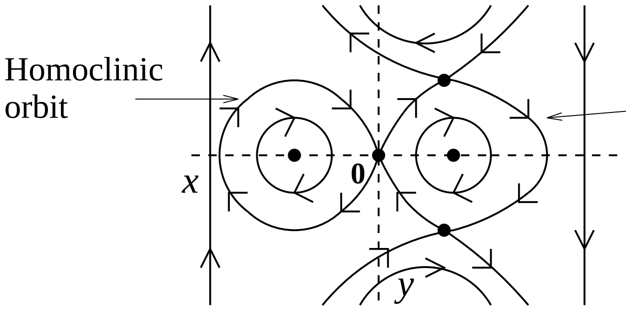

Figure 3: Illustration of symmetric homoclinic and heteroclinic orbits in a schematic phase portrait of the flow of a vector field in $\mathbb{R}^2$ possessing the reflection in the x-axis as a reversing symmetry.

For instance, if $f^t : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is area-preserving and $Df^t|_{\mathbf{x}_0}$ in an equilibrium point $\mathbf{x}_0$ has eigenvalues $\{\lambda, \lambda^{-1}\}$ with $\lambda \in \mathbb{R}$ and $|\lambda| > 1$ then if the dynamical system is C^k $(k \geq 1)$ it has one-dimensional stable and unstable manifolds that are also C^k, cf. e.g. [25]. As we are concerned with C^1 flows and diffeomorphisms, we can rely on the existence of stable and unstable manifolds (and their persistence under small perturbations).

A point $\mathbf{x} \in W^s_{\mathbf{x}_0} \cap W^u_{\mathbf{x}_1}$ is called a homoclinic point if $\mathbf{x}_0 = \mathbf{x}_1 \neq \mathbf{x}$ and a heteroclinic point if $\mathbf{x}_0 \neq \mathbf{x}_1$. The orbit of such a point will be referred to as a homoclinic resp. heteroclinic orbit. Homoclinic and heteroclinic orbits are important in dynamical systems as, for instance, chaotic dynamics is often related to transversal intersections of stable and unstable manifolds of saddle points, see e.g. [8, 25].

The following properties of stable and unstable manifolds in the presence of reversing symmetries are easily verified:

Proposition 2.2 *Let ρ be a reversing symmetry of a dynamical system f^t with equilibrium point $\mathbf{x}_0$, then*

$$W^s_{\mathbf{x}_0} \cap \mathrm{Fix}(\rho) = W^u_{\rho(\mathbf{x}_0)} \cap \mathrm{Fix}(\rho), \tag{10}$$

$$W^u_{\mathbf{x}_0} \cap \mathrm{Fix}(\rho) = W^s_{\rho(\mathbf{x}_0)} \cap \mathrm{Fix}(\rho), \tag{11}$$

and

$$\rho(W^u_{\mathbf{x}_0}) = W^s_{\mathbf{x}_0} \quad \Leftrightarrow \quad \mathbf{x}_0 \in \mathrm{Fix}(\rho) \quad \Leftrightarrow \quad \rho(W^s_{\mathbf{x}_0}) = W^u_{\mathbf{x}_0}. \tag{12}$$

As an illustration, in Fig. 3 a phase portrait of the flow of a vector field is sketched possessing homoclinic and heteroclinic orbits that are symmetric with respect to the reflection in the x-axis that is a reversing symmetry.

We conclude with two results that are special to the continuous time case and will be of use later on.

Proposition 2.3 *Suppose f^t is the flow of a vector field with reversing symmetry ρ. Then, a homoclinic (heteroclinic) orbit of such a flow is symmetric with respect to ρ if and only if the homoclinic (heteroclinic) orbit intersects $\mathrm{Fix}(\rho)$ precisely one time.*

Proof. First of all, it follows from Proposition 2.2 that symmetric homoclinic and heteroclinic orbits must intersect $\mathrm{Fix}(\rho)$. However, by Proposition 2.1, an orbit can intersect

Fix(ρ) only once or twice, and when it intersects Fix(ρ) twice then the orbit must be periodic. Hence, since homoclinic and heteroclinic orbits are not periodic, symmetric homoclinic and symmetric heteroclinic orbits must intersect Fix(ρ) precisely one time. □

Proposition 2.4 *Let $f^t : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a flow that is reversible with respect to a reflection ρ and possesses a ρ-symmetric heteroclinic orbit. Then, for sufficiently small perturbations that preserve the reversibility of f^t, the heteroclinic orbit will persist.*

Let $f^t : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be an area-preserving flow with a heteroclinic orbit. Then, there exist arbitrarily small area-preserving perturbations for which the heteroclinic orbit will not persist.

Proof. In case f^t is reversible, Proposition 2.2 establishes the persistence of the heteroclinic orbit. First of all, the symmetric heteroclinic orbit must intersect Fix(ρ) transversally. For sufficiently small perturbations the saddle points and their stable and unstable manifolds persist, and hence also the transversal intersections of these manifolds with Fix(ρ). The reversing symmetry ensures that the transversal intersections of the manifolds with Fix(ρ) lead to heteroclinic orbits.

In the area-preserving context, however, heteroclinic orbits can only occur when two different saddle points are on the same level sets of the Hamiltonian: i.e. two saddle points $\mathbf{x}_1$ and $\mathbf{x}_2$ can have a heteroclinic connection only if $H(\mathbf{x}_1) = H(\mathbf{x}_2)$. However, this is a degenerate situation since arbitrarily small perturbations can perturb the Hamiltonian H to $\tilde{H}$ such that the saddle points $\mathbf{x}_1$ and $\mathbf{x}_2$ perturb to saddle points $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ for which $\tilde{H}(\tilde{\mathbf{x}}_1) \neq \tilde{H}(\tilde{\mathbf{x}}_2)$. □

3 Persistently reversible and persistently non-reversible area-preserving flows

In this section we present examples of persistently reversible and persistently non-reversible area-preserving flows. The example of the non-reversible flow is due to Mather.[1] These examples will serve to prove Theorem 1.2.

3.1 Persistence of reversibility in reversible area-preserving flows

Area-preserving flows in $\mathbb{R}^2$ have a very robust structure. Generically, an area-preserving flow consists of periodic orbits, equilibrium points of centre and saddle type, homoclinic orbits, and non-periodic orbits that are not contained in any finite domain of $\mathbb{R}^2$. Typically, such a vector field is not about to undergo a bifurcation, such that sufficiently small perturbations of this flow (within the class of area-preserving flows) will preserve its structure. More precisely, the perturbed flow will be equivalent to the unperturbed one, meaning that there exists a homeomorphism $h : \mathbb{R}^2 \mapsto \mathbb{R}^2$ mapping the orbits of the perturbed flow $\tilde{f}^t$ to the orbits of the unperturbed flow f^t preserving the direction of the time-parameterization but allowing for differences of the precise periods of corresponding orbits, cf. e.g. [5].[2] This

[1]This example was reported by MacKay [21].

[2]This can be deduced from the generic robustness of the structure of the level sets of the Hamiltonian function $H : \mathbb{R}^2 \mapsto \mathbb{R}$.

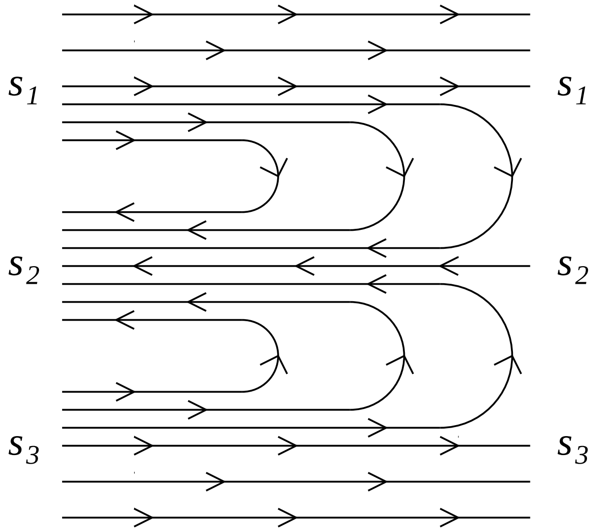

Figure 4: Phase portrait of Mather's example of a flow in $\mathbb{R}^2$ that is not reversible. The flow-lines s_1, s_2 and s_3 are separatrices.

implies that if the original flow is reversible with respect to a reversing reflection ρ, there exists a homeomorphism h such that the perturbed flow will be reversible with respect to the reflection $h^{-1} \circ \rho \circ h$.

As an example, consider the reversible area-preserving flow sketched in Fig. 2(a). The flow is a rotation and $\mathbb{R}^2$ is filled with periodic orbits surrounding the equilibrium point of centre type at $\mathbf{0}$. This flow admits a reversing reflection intersecting each periodic orbit twice (e.g. the reflection in the x-axis). If the flow is not about to undergo a bifurcation, then for sufficiently small area-preserving perturbations the perturbed flow will be equivalent to the unperturbed one, implying the existence of a reversing reflection for the perturbed flow.

In fact, persistence of reversibility under area-preserving perturbations is a typical property of reversible area-preserving flows in $\mathbb{R}^2$ that do not have heteroclinic orbits. One needs to require that the reversible flow has no heteroclinic orbits, as heteroclinic orbits are persistent in the reversible context, but not in the area-preserving context, cf. Proposition 2.4.

3.2 Mather's example of a non-reversible area-preserving flow

In Fig. 4 we depict Mather's example of a non-reversible area-preserving flow in $\mathbb{R}^2$. One easily verifies that such a flow can be realized by an arbitrarily smooth Hamiltonian flow. It should be noted that the flow has no periodic orbits or equilibrium points.

The flow contains three separatrices. In Fig. 4 they are labeled s_1, s_2 and s_3. They divide the plane $\mathbb{R}^2$ into four components. In the top and bottom components the flow is monotonous, and in the two components in the middle the flow makes a half-way turn.

From the phase portrait it can be shown that if the flow were reversible (with respect to a reflection), all three separatrices should be symmetric. Namely, first of all, as there are precisely three separatrices, at least one of them should be symmetric. Because of the orientation of the flow on the separatrices, s_1 and s_3 cannot be mapped onto each other by ρ. Hence, suppose that ρ maps s_1 to s_2 and vice-versa. Then $\mathrm{Fix}(\rho)$ must be contained in the region between s_1 and s_2. However, then s_3 cannot be symmetric, because it cannot be intersected by $\mathrm{Fix}(\rho)$. Analogously, it follows that ρ cannot map s_3 to s_2, and thus that s_1, s_2 and s_3 must be symmetric.

If s_1, s_2 and s_3 all are symmetric, then $\mathrm{Fix}(\rho)$ should intersect s_1, s_2 and s_3. However, in the regions between s_1 and s_2 and s_2 and s_3 all orbits run from minus infinity to minus infinity making somewhere a turn-around. If there would exist a reflection ρ that is a reversing symmetry of this flow, then it should intersect the region between s_1 and s_2. However, the flow in this region is such that every curve connecting s_1 to s_2 must intersect certain orbits twice. This would imply that there are periodic orbits in the region between s_1 and s_2 (cf. Proposition 2.1). However, the flow was constructed as to have no periodic orbits. This contradiction establishes the non-reversibility of this flow.

It should be noted that in this example the obstructions against reversibility arise due to boundary conditions at infinity, and not from local considerations.

One easily verifies that the example can be realized in such a way that sufficiently small area-preserving perturbations will preserve the obstructions against reversibility: no periodic orbits, no equilibrium points, and three separatrices dividing $\mathbb{R}^2$ into four regions containing dynamics that is incompatible with reversibility.

3.3 Proof of Theorem 1.2

The proof of Theorem 1.2 consists of the examples presented in Section 3.1 and Section 3.2. The persistently reversible examples prove the fact that being not reversible is a non-generic property for area-preserving flows, and Mather's example establishes the fact that reversibility is also a non-generic property. □

Remark 3.1 Although we focus on reversibility with respect to reflections, one easily verifies that Mather's example also cannot be reversible with respect to a reversing symmetry that is topologically conjugate to a k-fold rotation. This property is also persistent.

Remark 3.2 Theorem 1.2 illustrates the fact that integrable Hamiltonian systems need not be reversible. Namely, every area-preserving flow is integrable: it is a one degree of freedom Hamiltonian system having one conserved quantity, being the Hamiltonian function itself. Indeed, in the case of an integrable system there exist transformations to coordinate frames in which the flow *locally* in the neighborhood of an equilibrium point of centre type can be written in terms of so-called action-angle variables: $\dot{I} = 0$ and $\dot{\phi} = \omega(I)$. The latter equations of motion are evidently reversible with respect to the transformation $(I, \phi) \mapsto (I, -\phi)$. However, a transformation that globally transforms the vector field into action-angle variables need not exist.

3.4 Local reversibility of area-preserving flows

Mather's example of a non-reversible area-preserving flow relies on boundary conditions at infinity, and may therefore seem somewhat peculiar. However, it turns out to be difficult – if not impossible – to construct persistent local obstructions against reversibility in area-preserving flows. Namely, area-preserving flows are generically (locally) reversible inside bounded invariant sets:[3]

[3] We use the terminology "local reversibility" here in the meaning of "reversibility in a neighbourhood", cf. also [21]. Note that in [20, 27, 31] the same terminology is used slightly differently, with reference to a *formal* reversibility property of Taylor expansions and normal forms instead of a reversibility property in an open neighbourhood of an equilibrium point.

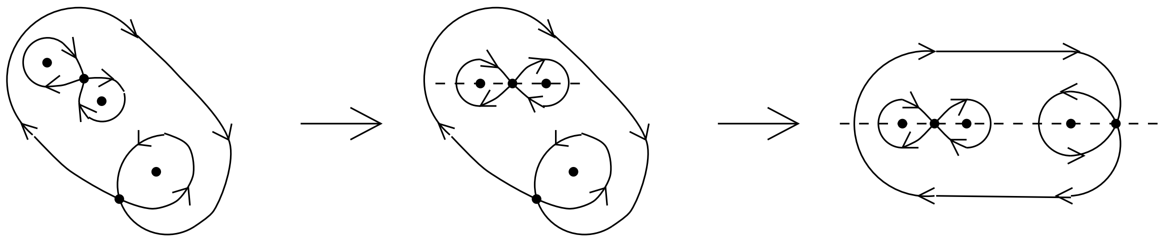

Figure 5: Local transformations acting on nested *pairs of eyes*, establishing local reversibility.

Proposition 3.3 *Let B be a connected and bounded subset of $\mathbb{R}^2$ and $f^t : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be an area-preserving flow. Suppose that B is an invariant domain for f^t (i.e. $f^t : B \mapsto B$), then, generically, f^t is reversible within B, i.e. there exists a reflection $\rho : B \mapsto B$ such that $f^t \circ \rho = \rho \circ f^{-t}$ inside B.*

Proof. As $f^t : B \mapsto B$ and f^t is continuous, f^t must also preserve the boundary ∂B of its closure. Generically, ∂B will contain either no equilibrium point or precisely one equilibrium point. Namely, if there would be more than one equilibrium point, then generically these points should be saddle points (centre points cannot occur on ∂B). However, because ∂B is invariant under f^t, the stable or unstable manifold of a saddle point in ∂B has to lie entirely inside ∂B. Hence, if ∂B contains more than one equilibrium point, the stable and unstable manifolds of the saddle points on ∂B cannot form homoclinic orbits. However, from Proposition 2.4 we know that heteroclinic orbits generically do not occur in area-preserving flows. Hence, generically ∂B will be a periodic orbit or a saddle point and one or both of its homoclinic orbits.

Inside B, generically f^t contains only a finite number of equilibrium points, cf. e.g. [25], that are of saddle and centre type. Generically, they form nested *pairs of eyes*. With the term *pair of eyes* we denote a local flow consisting of a saddle point and two centre points surrounded by homoclinic orbits of the saddle point. It should be noted that there are two types of pairs of eyes, cf. Fig. 5. The spaces between the centre points and homoclinic orbits (eyes) are filled with periodic orbits. A nested family of pairs of eyes is constructed by repeatedly substituting centre points by a pair of eyes.

Thus inside B we have nested pairs of eyes. It is easily verified that a single pair of eyes is (topologically) conjugate to a flow that is reversible with respect to the reflection in a straight line, intersecting every periodic orbit twice and every homoclinic orbit and saddle point once: there exists a homeomorphism transforming the flow locally[4] to a flow that is reversible with respect to the reflection in a straight line. If this locally reversible flow is contained in some other pair of eyes one can do such a transformation twice, resulting in two nested pairs of eyes that are locally reversible with respect to a linear reversing reflection. This is illustrated in Fig. 5. Because generically there are only a finite number of equilibrium points in B, this procedure can be repeated a finite number of times after which it follows that the whole flow inside B is reversible with respect to a reversing reflection. The last step involves connecting the reflection line to ∂B. When ∂B is a periodic orbit of f^t this can be done in a way that is compatible with reversibility, such that the periodic orbit on ∂B is symmetric with respect to the reversing reflection. If ∂B consists of a saddle point and its

[4]This transformation can be made to act non-trivially inside the pair of eyes, (smoothly) isotoped to the identity transformation just outside this region.

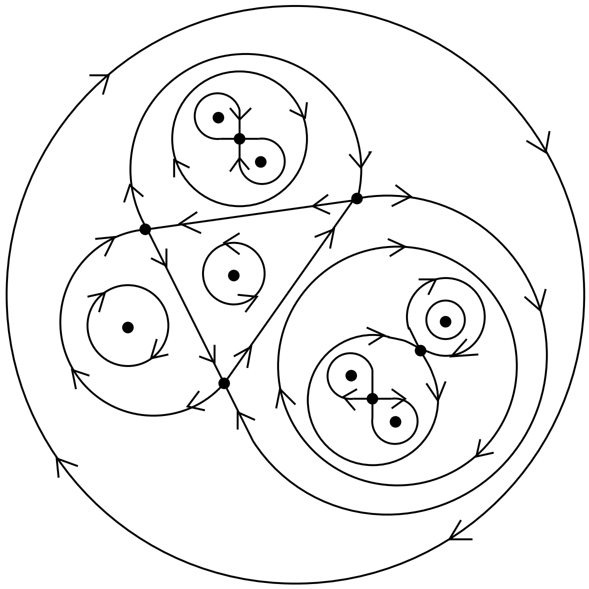

Figure 6: Phase portrait of a non-reversible area-preserving flow in $\mathbb{R}^2$. This flow possesses no reversing reflection symmetry.

homoclinic orbit(s), then the reflection can be made to intersect the saddle point and each homoclinic orbit once. $\square$

4 Persistently non-reversible area-preserving diffeomorphisms: an example

In this section we will construct an example of a persistently non-reversible area-preserving diffeomorphism in $\mathbb{R}^2$. From the previous section it may be natural to start from Mather's example, but we will follow a different trail here, illustrating the existence of persistent local obstructions against reversibility.

Our example is constructed by perturbing a non-reversible area-preserving flow as displayed in Fig. 6. This flow can be realized by arbitrarily smooth vector fields. This example will provide as the backbone for proving Theorem 1.3 concerning the non-genericity of reversibility in area-preserving diffeomorphisms.

4.1 Demonstration of the non-reversibility of the flow in Fig. 6

We investigate the possibility of a reversing symmetry in the phase portrait sketched in Fig. 6 and will show that there are obstructions against the existence of a reversing reflection symmetry.

Considering the flow in Fig. 6, there are three equilibrium points of saddle type that are connected to each other via heteroclinic orbits. These heteroclinic orbits enclose four invariant regions.

This situation is depicted schematically in Fig.7(a). Importantly, the three outer invariant components I, II, and III contain different types of flows, in particular different numbers

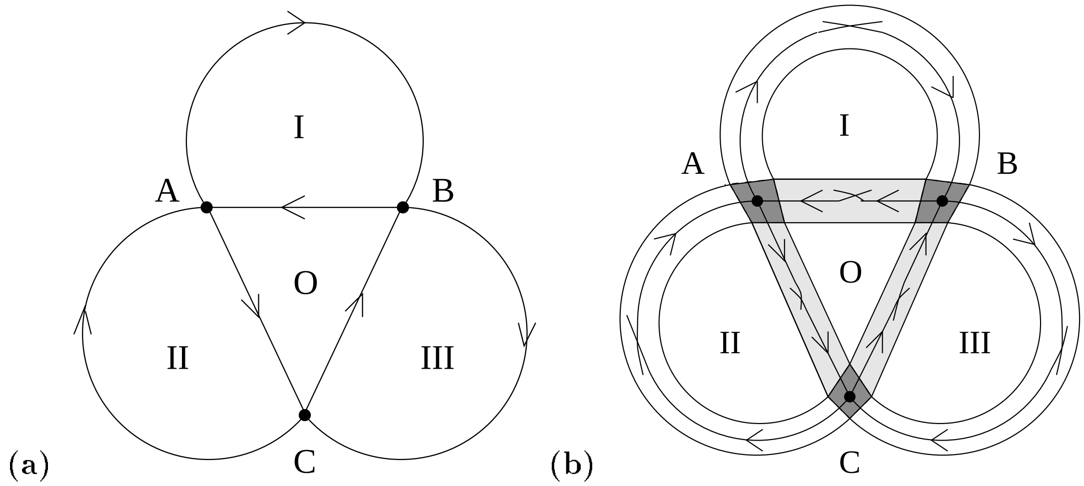

Figure 7: **(a)** Schematic presentation of the area-preserving flow in Fig. 6. The flow contains four flow-invariant regions 0, I, II, III (where I, II and III have different interiors), and three saddle points A, B, C. **(b)** Schematic presentation of the dynamics of an area-preserving diffeomorphism obtained by adding a small area-preserving perturbation to the time-one map of the flow in Fig. 6. There are four invariant regions 0, I, II, III and three saddle points A, B, C. The stable and unstable manifolds of adjacent saddle points intersect transversally and constitute channels with a monotonic drift between the neighbourhoods of adjacent saddle points. Of the stable and unstable manifolds only some part is depicted near to the saddle points they belong to.

of singularities, and so cannot be mapped between themselves by a possible reversing symmetry. Moreover, because the orientation of the flow at the boundary of component 0 is opposite to the orientation of the flow at the boundaries of I, II and III, it follows that a reversing reflection symmetry can never map component 0 to component I, II, or III. Hence, when the flow would possess a reversing reflection, then each of the four invariant components must be symmetric (i.e. ρ-invariant). We will show that this is impossible.

First of all, by Proposition 2.3 each heteroclinic orbit can intersect $\mathrm{Fix}(\rho)$ only once. Furthermore, if $\mathrm{Fix}(\rho)$ intersects one of the saddle points A, B, or C, then it cannot intersect its stable and unstable manifolds. Namely, such an intersection would imply the existence of a homoclinic orbit, but the flow contains no homoclinic orbits attached to these saddle points. As $\mathrm{Fix}(\rho)$ has no self-intersections it can intersect a saddle point only once.

Because the flow has precisely three saddle points with heteroclinic orbits, at least one of them must be symmetric. Suppose, with reference to Fig. 7(a), that C is a symmetric saddle. Now there are two ways $\mathrm{Fix}(\rho)$ can intersect the saddle point: either $\mathrm{Fix}(\rho)$ runs horizontally from region II to region III, or it runs vertically from the region outside the invariant regions to region 0. Only these two possibilities have to be considered, as ρ must send the stable manifold of C to the unstable manifold of C. Note that if $\mathrm{Fix}(\rho)$ intersects the saddle point C then it cannot intersect its stable and unstable manifolds.

Suppose, $\mathrm{Fix}(\rho)$ intersects the saddle point C running from region II to region III, then it must also intersect the saddle points A and B because it cannot intersect the stable and unstable manifolds of C. But in that case it cannot intersect any of the heteroclinic orbits and hence $\mathrm{Fix}(\rho)$ cannot intersect region 0. This is in contradiction with the fact that region

0 is symmetric.

Alternatively, suppose that $\mathrm{Fix}(\rho)$ runs vertically from outside the invariant regions to region 0. Then, $\mathrm{Fix}(\rho)$ cannot intersect the regions II and III. Namely, because $\mathrm{Fix}(\rho)$ cannot intersect the heteroclinic orbits connecting C with A, region II can only be entered and left by $\mathrm{Fix}(\rho)$ via the saddle point A. But $\mathrm{Fix}(\rho)$ can intersect A only once. Thus it cannot intersect region II. Similarly on finds that $\mathrm{Fix}(\rho)$ cannot intersect region III. This is in contradiction that regions II and III are symmetric.

Hence, the flow sketched in Fig. 6 cannot be reversible with respect to a reversing reflection.

4.2 Construction of a persistently non-reversible area-preserving diffeomorphism

We will now construct a persistently non-reversible diffeomorphism. We start from a non-reversible diffeomorphism f that is precisely the time-one map of the flow in Fig. 6. As f is the time-one map of a non-reversible flow, obviously f is not reversible. However, from Proposition 2.4 it follows that f is not persistently non-reversible. Namely, arbitrarily small area-preserving perturbations may perturb f to the time-one map of a reversible flow possessing no heteroclinic orbits.

We will now show that with a small perturbation we can construct from the time-one map f of the flow in Fig. 6 an area-preserving diffeomorphism that is persistently non-reversible. We choose our perturbation such as to destroy the integrability of the map, and to create transversal intersections of the stable and unstable manifolds of adjacent saddle points. This can be realized by a very small area-preserving perturbation under which the saddle points persist. By KAM-theory (cf. e.g. [23]) it follows that we can also preserve the invariance of large parts of the former invariant components 0, I, II, III. Furthermore, we can make sure that the dynamics inside the new invariant regions I,II, and III remain different from each other such that again all these regions must be symmetric if the map is reversible.

As a result, taking the perturbation sufficiently small we arrive at the situation as sketched in Fig. 7(b): there are four invariant regions 0, I, II, and III, and three saddle points A, B, and C. The situation is comparable to the flow case, except from the more complicated dynamics in the neighbourhoods of the saddle points A, B, C, and the heteroclinic orbits between them. Between the invariant regions, the heteroclinic orbits of the flow are replaced by small channels constituted by heteroclinic orbits of the map. For sufficiently small perturbations the neighbourhoods and channels can be chosen in such a way that the channels support a monotonic drift between the neighbourhoods U_A, U_B, U_C of the saddle points A, B, C, such that e.g. for each point $\mathbf{x}$ in the channel C_{AC} between A and C there exists $N_+ > 0$ and $N_- < 0$ such that $f^{N_+}(\mathbf{x}) \in U_C$, $f^{N_-}(\mathbf{x}) \in U_A$, and $f^n(\mathbf{x}) \in C_{AC}$ for all $N_- < n < N_+$.[5] The neighbourhoods U_A, U_B and U_C act as junctions between the various channels.[6] Importantly, we choose the perturbation such that the stable and unstable manifolds of adjacent saddle points have transversal intersections inside the channels, e.g. in channel C_{AC} there is a transversal intersection of W^s_C and W^u_A.

[5] Monotonicity of the drift in a channel can be realized with respect an appropriately chosen foliation.

[6] It should be noted that the schematic picture of the dynamics in Fig. 7(b) is also valid in the unperturbed integrable case, apart from the transversal intersections of stable and unstable manifolds. By continuity, this schematic picture remains valid for sufficiently small perturbations that preserve heteroclinic orbits between adjacent saddle points, as is the case in the perturbation discussed here.

We will focus on the first parts of the branches of the stable and unstable manifolds of the saddle points A, B, and C, until they intersect each other inside the first channel they enter, as indicated in Fig. 7(b). For small perturbations these parts of the branches lie very near to the heteroclinic orbit in the integrable case. We will now show that in a similar way as in the flow case, these manifolds form obstructions against reversibility.

We first note that for similar reasons as in the flow case, if the map is reversible, at least one of the three saddle points A, B, C should be symmetric and thus be intersected by $\mathrm{Fix}(\rho)$.

Suppose $\mathrm{Fix}(\rho)$ intersects the saddle point C, then inside U_C we may take the perturbation such that $\mathrm{Fix}(\rho)$ does not intersect the first part of the branches of W^u_C and W^s_C, as such an intersection would imply a homoclinic intersection of these two branches before they leave U_C. For sufficiently small perturbations this can be avoided.

When $\mathrm{Fix}(\rho)$ intersects the first parts of the stable or unstable manifold of C inside one of the channels attached to U_C e.g. W^s_C inside C_{AC}, then ρ must map this part of W^s_C to the first part of a branch of W^u_C. This would imply that, as the first part of W^s_C intersects U_C and only one channel, also the image of this part of the stable manifold must have intersected at most one channel attached to U_C. This would in turn imply that there must exist a channel connecting U_C to itself, which is not the case. Hence, $\mathrm{Fix}(\rho)$ cannot intersect the first parts of the branches of the stable and unstable manifolds of C.

We will now show that $\mathrm{Fix}(\rho)$ also cannot intersect the first parts of the stable and unstable manifolds of the saddle point A. Namely, suppose that $\mathrm{Fix}(\rho)$ intersects the first part of W^u_A inside U_A. Then this would imply that at this intersection point there is a transversal intersection of W^u_A with $W^s_{\rho(A)}$. Since, W^u_A did not leave U_A before having this intersection, accordingly $W^s_{\rho(A)}$ cannot have left $U_{\rho(A)}$. Hence, we must have $\rho(A) = A$, because $U_{\rho(A)}$ and U_A are either identical of strictly separated by channels and in the latter case the stable manifold of $\rho(A)$ must intersect a channel before entering U_A. However, we may assume our perturbation to be such that the first part of W^u_A and W^s_A do not intersect inside U_A. For similar reasons we may rule out that $\mathrm{Fix}(\rho)$ intersects the first part of W^s_A inside U_A.

$\mathrm{Fix}(\rho)$ also cannot intersect the first part of the stable and unstable manifolds of the saddle point A inside the channels C_{AC} and C_{CA}. If $\mathrm{Fix}(\rho)$ intersects the first part of W^u_A inside C_{AC} then $\mathrm{Fix}(\rho)$ should also intersect at the same point also the first part of $W^s_\rho(A)$ inside $C_{C\rho(A)}$, because $\mathrm{Fix}(\rho)$ maps the branch of W^s_C constituting C_{AC} to the branch of W^u_C constituting $C_{C\rho(A)}$. However, as the channels C_{AC} and $C_{C\rho(A)}$ cannot intersect for any choice of $\rho(A)$, it follows that $\mathrm{Fix}(\rho)$ cannot intersect the first part of $W^u(A)$ inside C_{AC}. For similar reasons we may rule out that $\mathrm{Fix}(\rho)$ intersects the first part of W^s_A inside C_{CA}.

The above arguments show that $\mathrm{Fix}(\rho)$ intersecting a saddle point prohibits intersections with any of the first parts of the stable and unstable manifolds that constitute the channels attached to the neighbourhood of this saddle point. Note that a similar restriction applied in the flow case: $\mathrm{Fix}(\rho)$ cannot intersect one of the saddle points A, B, C, and the heteroclinic orbits attached to it at the same time. In fact, from here, it is straightforward to show that the diffeomorphism constructed by applying the small perturbation to the non-reversible flow of Fig. 6, cannot be reversible. The argumentation is similar to the one presented in Section 4.1.

Unlike the flow in Fig. 6, for the diffeomorphism just constructed, the obstructions against reversibility are persistent with respect to sufficiently small area-preserving perturbations. Since the heteroclinic orbits arise from transversal intersections of stable and unstable manifolds, they cannot be perturbed away by arbitrarily small perturbations. It is not difficult to check that the other aspects of the dynamics used in the argumentation also persist under sufficiently small perturbations. It should be noted that the persistent obstructions against reversibility are of a local nature, in contrast to the obstructions in Mather's example of a persistently non-reversible flow that has been discussed in Section 3.2.

4.3 Proof of Theorem 1.3.

The example of a persistently non-reversible area-preserving diffeomorphism as given in Section 4.2 provides the proof of non-genericity of reversibility in the class of area-preserving diffeomorphisms in $\mathbb{R}^2$. □

Remark 4.1 Although we focus on reversibility with respect to reflections, one easily verifies that the example constructed in Section 4.2 also cannot be reversible with respect to a reversing symmetry that is topologically conjugate to a k-fold rotation.

When a diffeomorphism f is not reversible, it may happen that f^k is reversible, for some value of k. In [19] this phenomenon has been called *reversing k-symmetry*. It should be noted that the obstructions used to construct the example of a persistently non-reversible area-preserving diffeomorphism in Section 4.2, also rule out the possibility of f^k being reversible for any value of k.

5 Concluding remarks

In this paper it has been shown that reversibility is not a generic property in area-preserving dynamical systems. We discussed the case of flows and diffeomorphisms separately. One might argue that this result is precisely what one would expect, as symmetry properties usually refer to special situations. However, it may happen that properties which are conceptually different in general, become equivalent in a low-dimensional setting. For instance, the flow of a vector field in $\mathbb{R}^2$ is Hamiltonian and integrable if and only if it is volume-preserving . In $\mathbb{R}^{2n}$, $n \geq 2$ this equivalence does not hold. With respect to reversibility, it turns out that the situation is rather delicate. On the one hand Mather's example of a persistently non-reversible area-preserving flow shows that reversibility is not a generic property of area-preserving flows, but on the other hand Proposition 3.3 establishes the generic occurrence of local reversibility in flow-invariant subsets of the plane. Interestingly, it turns out that being not reversible is also a non-generic property of area-preserving flows. The persistence of the reversibility one finds in certain reversible area-preserving flows appears to be related to the integrability of these flows, which gives them a very robust structure.

In the case of diffeomorphisms one has a little more freedom to manouvre in $\mathbb{R}^2$ which turns out to be sufficient to construct persistent local obstructions against reversibility. The arguments used in Section 4 to establish the non-reversibility of a diffeomorphism can be used to construct a wide range of other such examples. The example presented here is one of the least complicated ones. It is straightforward to construct many more examples along the same lines. One would not expect to find persistently reversible examples that easily. It remains an open question, however, whether there are any such examples or whether being non-reversible is a generic property of area-preserving diffeomorphisms in $\mathbb{R}^2$ (which would be natural to be expected). In this context, more detailed considerations on the positions of stable and unstable manifolds may turn out to be useful.

In [20, 27, 31], the idea was put forward to formulate necessary conditions for reversibility which can be used to rule out the possibility of reversibility for particular dynamical systems. In this context, one may note that from a visual inspection of a phase portrait with various heteroclinic saddle connections one could determine whether our type of local obstructions against reversibility apply. This examination of saddle connections suggests that one might test for reversibility of a diffeomorphism by trying to find a nearby integrable vector field, e.g. by using averaging methods, cf. e.g. [1]. If the averaged system (which is the flow of a vector field) displays a combination of saddle connections and invariant regions with internal dynamics that is incompatible with reversibility, then it is very likely that the original diffeomorphism can be shown to be non-reversible too. In this respect, one should note that the non-reversibility of the averaged system may be not persistent within the class of area-preserving flows, despite the fact that it is derived from a diffeomorphism for which the non-reversibility is a persistent property. This may occur because averaging methods project many diffeomorphisms to only one flow. Actually, in our construction in Section 4, the perturbation of the non-reversible flow is intended to be such that averaging will yield the non-reversible flow from which we started. Unfortunately, this method seems to be of little use to shed light on the alleged non-reversibility of Rannou's map [28].

In the literature, there have been some suggestions on how a comparison between the genericity of local bifurcations in area-preserving flows and diffeomorphisms relates to the existence (and genericity) of non-reversible area-preserving dynamical systems [21, 22]. Rimmer [29, 30] has shown that in the context of area-preserving diffeomorphisms a particular type of local bifurcation in which a centre point bifurcates into a saddle point and two centre points (a *pair of eyes*, in the terminology of Section 3.4) is non-generic in general, but generic in case the center point before the bifurcation is symmetric with respect to a reversing reflection and the centre points after the bifurcation do not lie anymore on the fixed point subspace of the reflection.

One must take care in interpreting this result in relation to the generic occurrence of reversibility or non-reversibility in the class of area-preserving diffeomorphisms. Unfortunately, the term *generic* tends to have different interpretations in different settings. In (local) bifurcation theory, a particular bifurcation is called generic of codimension one, not if the one-parameter families of diffeomorphisms displaying this bifurcation form a residual set in the space of one-parameter families of diffeomorphisms, but when this bifurcation occurs persistently in certain one-parameter families of diffeomorphisms, cf. e.g. the exposition in [11]. Hence, it may well be that a certain local bifurcation (such as Rimmer's bifurcation) is generic in reversible area-preserving diffeomorphisms, but that the particular one-parameter family of area-preserving diffeomorphisms at hand does not display this bifurcation and neither one-parameter families of diffeomorphisms that are nearby. Therefore, one cannot conclude from Rimmer's result that reversible systems are not generic in the class of area-preserving systems, with the natural interpretation of the term *generic* as used throughout this paper, in contrast to a suggestion made in [22].

Considering the examples of non-reversible flows and mappings by Arnol'd & Sevryuk [2] and Roberts & Capel [31], the question arises how their examples relate to the examples presented in this paper. Recall that their examples rely on degenerate equilibrium points that are incompatible with the existence of a revering symmetry.[7] In this respect, it seems worthwhile to note that locally non-reversible flows such as the one depicted in Fig. 6, can

[7]The term *degenerate* refers to the fact that the normal forms of the equilibrium points satisfy non-generic conditions, such as having many coefficients being equal to zero.

arise out of a local bifurcation from a degenerate equilibrium point (that is generic only with high codimension).[8] It is likely that the non-reversibility of the bifurcation product has a counterpart in the formal properties of the normal form of the point from which this local structure arises. A natural way to study this relationship is using the method of blowing-up (cf. e.g. [9]). With this technique, one could examine how, in terms of the fine structure of singularities, the occurrence of formally non-reversible normal forms relate to the possibility of local bifurcations from which non-reversible local phase portraits arise. It is tempting to conjecture that these two properties will turn out to coincide. It would be interesting to investigate this connection in further detail, but this is beyond the scope of the present paper.

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