Symmetric homoclinic tangles in reversible systems

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February 22, 2006

Abstract

We study the dynamics near transverse intersections of stable and unstable manifolds of sheets of symmetric periodic orbits in reversible systems. We prove that the dynamics near such homoclinic and heteroclinic intersections is not C^1 structurally stable. This is in marked contrast to the dynamics near transverse intersections in both general and conservative systems, which can be C^1 structurally stable.

We further show that there are infinitely many sheets of symmetric periodic orbits near the homoclinic or heteroclinic orbits. We establish the robust occurrence of heterodimensional cycles, that is heteroclinic cycles between hyperbolic periodic orbits of different index, near the transverse intersections. This is shown to imply the existence of hyperbolic horseshoes and infinitely many periodic orbits of different index, all near the transverse intersections.

1 Introduction

It is a classical result, originating with the work of S. Smale [Sma65], that a differential equation possesses horseshoes when it has a hyperbolic periodic orbit with transversally intersecting stable and unstable manifolds. In this paper we consider the dynamical consequences of homoclinic and heteroclinic tangles, i.e. transverse intersections of stable and unstable manifolds, of one-parameter families of symmetric periodic solutions in reversible vector fields.

Definition 1.1. *A differential equation $\dot{x} = X(x)$ on a manifold M is reversible if there exists an involution R on M, that is $R^2(x) = x$, so that*

$$R_* X = -X,$$

or in other words $R(x(-t))$ is an orbit if and only if $x(t)$ is an orbit.

We illustrate the background of this notion by describing the occurrence of reversibility in two contexts.

First, reversibility is an important theme in mechanics; Hamiltonian systems

$$\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}$$

with $q, p \in \mathbb{R}^n$ are reversible with involution $R(q,p) = (q,-p)$ if $H(q,p)$ is even in the momentum p.

Second, reversible systems show up in reduced differential equations for standing and traveling waves in partial differential equations with a spatial symmetry [Ioo98, LamRob98, Cha98]. Such systems may, but need not be Hamiltonian, see [ChaHär00], and thus need not possess a first integral. For instance, consider a reaction-diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u)$$

for $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$, with D a diagonal matrix with positive entries. A standing wave $v(x) = u(x,t)$ satisfies the ordinary differential equation

$$v' = w, w' = -D^{-1}f(v),$$

which is reversible with involution $R(v,w) = (v,-w)$. It is Hamiltonian if f is a gradient.

We focus in this paper on systems $\dot{x} = X(x)$ on manifolds M^{2n} of even dimension $2n$, that are reversible with respect to an involution R whose action is such that the fixed point space $\mathrm{Fix}(R) = \{x \mid R(x) = x\}$ is an embedded manifold of half the dimension of the ambient space,

(H 1) $\qquad \dim \mathrm{Fix}(R) = n.$

Hypothesis (H 1) will be a standing assumption throughout this paper.

Our aim is to study reversible vector fields with a fixed (given) choice of time-reversal symmetry R. The particular choice of the type of time-reversal symmetry is motivated by the fact that a vast majority of published examples of reversible systems (that we are aware of), often with a mechanical background or from standing wave reductions for reaction-diffusion equations, has a time-reversal symmetry of this type. We refer to [LamRob98] for an overview, as well as for a substantial bibliography, of reversible systems. Other references of general interest include [Arn84, Sev86, Cha98].

A particular property of reversible systems satisfying hypothesis (H 1), is that their symmetric periodic orbits are typically not isolated, but form two dimensional sheets. To make this precise, write $\mathcal{X}(M^{2n})$ for the space of smooth vector fields on M^{2n}, reversible with respect to the action of a given involution R, equipped with the C^1 topology. Recall that we call a periodic orbit for $X \in \mathcal{X}(M^{2n})$ symmetric if it is set-wise invariant under R. Observe that such a periodic orbit must intersect $\mathrm{Fix}(R)$ exactly twice. A symmetric periodic solution with period T thus contains a point from the intersection $\mathrm{Fix}(R) \cap X_{T/2}(\mathrm{Fix}(R))$, where X_t denotes the time-t evolution of the vector field. Let I be a small time-interval containing $T/2$. Counting dimensions shows that a transverse intersection of $\{t \in I \mid X_t(\mathrm{Fix}(R))\}$ with $\mathrm{Fix}(R)$ is one-dimensional. Each point in this intersection lies on a (different) symmetric periodic orbit. Hence, under this generic transversality assumption, symmetric periodic orbits form two dimensional sheets. Note that also in Hamiltonian vector fields periodic solutions typically form sheets, parameterized by the level of the Hamiltonian.

We consider a reversible vector field X with a family of symmetric periodic orbits $\gamma_a : t \in \mathbb{R} \mapsto \gamma_a(t) \in M^{2n}$, parameterized by a real parameter a. We consider parameter values close to a single parameter value a_0. The flow near the sheet of symmetric periodic orbits is assumed to satisfy the following property.

(H 2) Apart from one multiplier of γ_a which equals one (coming from the family), the multipliers of γ_a lie off the unit circle.

Note that this is an open, though not generic, condition. The exception arises when there are multipliers on the unit circle, away from ± 1. This may occur persistently in reversible systems. In this case the periodic solutions are (partially) elliptic, see e.g. [Dev76a]. As a consequence of (H 2), the sheet of symmetric periodic orbits forms a normally hyperbolic manifold and thus possesses stable and unstable manifolds (see the next section). Reversibility implies that the stable and unstable manifolds of the sheet of periodic orbits have equal dimension $n+1$.

Write $W^s(\{\gamma_a\})$ for the stable manifold of the sheet of periodic orbits $\{\gamma_a\}$ and $W^u(\{\gamma_a\})$ for its unstable manifold. We are interested in the situation where $W^s(\{\gamma_a\})$ and $W^u(\{\gamma_a\})$ intersect transversally and where this intersection takes place inside $\mathrm{Fix}(R)$, see Figure 1 for an illustration. This results in the persistent occurrence of a two-dimensional sheet of solutions $\{\rho_a\}$ in $W^u(\{\gamma_a\}) \cap W^s(\{\gamma_a\})$. In general, if $W^u(p)$ intersects $\mathrm{Fix}(R)$ in a point q then by reversibility so does $W^s(R(p))$, so that q lies on a heteroclinic orbit from p to $R(p)$, which is homoclinic when p is symmetric. In our case, each solution ρ_a is symmetric and homoclinic to the symmetric periodic solution γ_a with the same value of a, i.e. $\rho_a(t)$ converges to the orbit γ_a for $t \to \pm\infty$. For the moment we restrict to symmetric homoclinic solutions. Transverse intersections of $W^u(\{\gamma_a\})$ and $W^s(\{\gamma_a\})$ in non-symmetric solutions will be treated in Section 6.

We note that our situation is reminiscent of the one described by Smale [Sma65] in the case of general systems, concerning the transversal intersection of stable and unstable manifolds of hyperbolic periodic solutions. A main difference with our case is that in general systems periodic orbits typically arise isolated. Smale showed that the dynamics resulting from such transversal intersections yields nontrivial uniformly hyperbolic dynamics (a *horseshoe*). The aim of our study is to describe the corresponding dynamics in the reversible setting, where we have non-isolated homoclinic orbits to non-isolated periodic solutions.

Before discussing the reversible situation in more detail, we mention another, better known, example where non-isolated homoclinic orbits to non-isolated periodic solutions arise. In Hamiltonian vector fields, such situations occur naturally because of the existence of a conserved quantity (or first integral, the Hamiltonian function). The level sets of the Hamiltonian foliate the phase space, and within a given level set we may find an isolated periodic solution with an isolated transversal homoclinic orbit, giving rise to horseshoe dynamics. Under the assumption that the corresponding level set is regular, due to hyperbolicity such dynamics continues to nearby level sets, thus yielding parameterized families of periodic solutions, homoclinic solutions and horseshoes [HirPugShu77]. More generally this occurs in conservative vector fields, i.e. vector fields that possess a first integral.

Since in the reversible context we may not have conserved quantities, the question thus arises whether similar conclusions hold as in the Hamiltonian setting. To

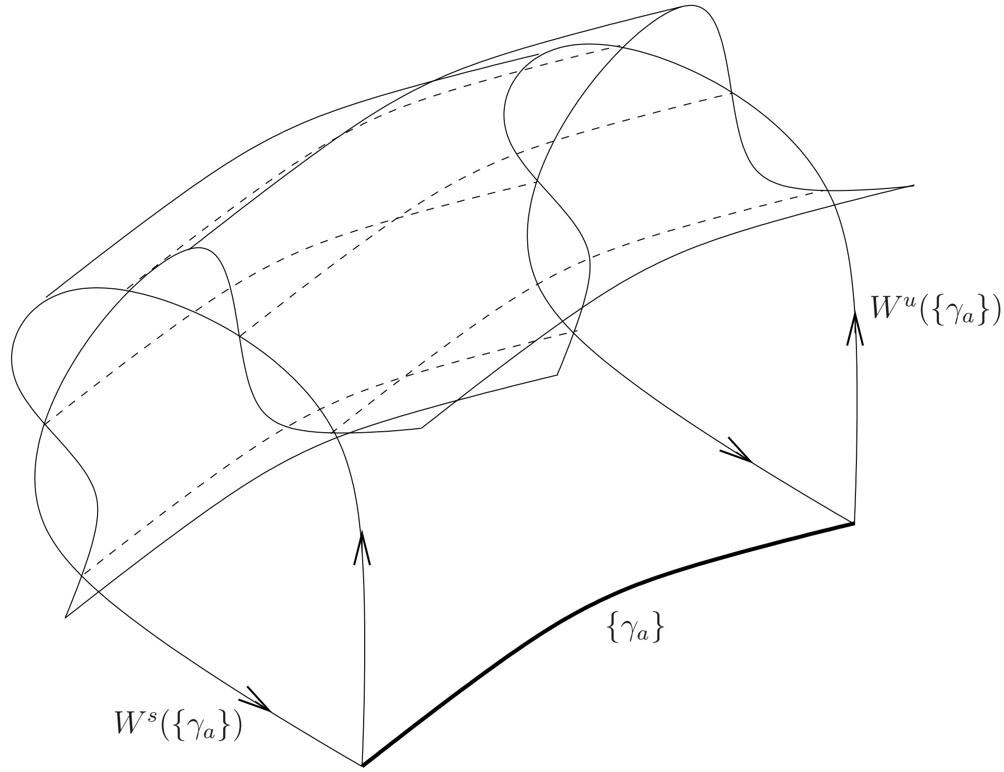

Figure 1: Homoclinic tangles to a family $\{\gamma_a\}$ of symmetric periodic orbits. The picture indicates the manifolds for a return map on a global cross-section; in general only local cross-sections near $\{\gamma_a\}$ and $\{\rho_a\}$ would be considered.

illustrate the problem, it may be useful to consider the following *Gedankenexperiment*. Take a reversible Hamiltonian system containing one-parameter families of symmetric periodic solutions, symmetric homoclinic solutions and symmetric horseshoes, and consider a (generic) small non-Hamiltonian but reversible perturbation. The question is what remains of the one-parameter family of horseshoes.

We identify the following particular issues concerning the dynamics near symmetric homoclinic tangles in reversible systems:

- Are there (uniformly hyperbolic) horseshoes?
- Is the nonwandering set structurally stable?

These questions have their roots in the properties of the corresponding tangles in Hamiltonian systems: when considering the dynamics restricted to one level set of the Hamiltonian we find horseshoes and the nonwandering set near the tangles is (assuming transversality conditions) structurally stable.

Below we state our main results, answering in particular the above questions.

1.1 Horseshoes and bifurcations

First we provide some more details concerning our setting. Recall that we consider a reversible vector field X with a family of symmetric periodic orbits $\gamma_a : t \in \mathbb{R} \mapsto \gamma_a(t) \in M^{2n}$, parameterized by a single parameter a, close to some parameter value a_0. Hypothesis (H 2) implies that the family $\{\gamma_a\}$, a from a small interval containing

a_0, gives a normally hyperbolic sheet of symmetric periodic orbits. That is, along $\{\gamma_a\}$, there is a DX_t-invariant splitting

$$TM^{2n} = E^s \oplus T\{\gamma_a\} \oplus E^u,$$

where $T\{\gamma_a\}$ is the tangent bundle of the family of periodic orbits. The bundles E^s, E^u form the stable and unstable directions: there are $C > 0, \lambda > 0$ so that

$$\begin{aligned} \|DX_t(\gamma_a)v\| &\leq Ce^{-\lambda t}\|v\|, \\ \|DX_t(\gamma_a)w\| &\geq \frac{1}{C}e^{\lambda t}\|w\|, \end{aligned}$$

for $v \in E^s_{\gamma_a}$, $w \in E^u_{\gamma_a}$, $t > 0$ (see [HirPugShu77]). By reversibility, the bundles E^s, E^u consist both of $(n-1)$ dimensional planes.

It is well known that the normal hyperbolicity of $\{\gamma_a\}$ implies that it possesses a $(n+1)$-dimensional stable manifold $W^s(\{\gamma_a\})$ and a $(n+1)$-dimensional unstable manifold $W^u(\{\gamma_a\})$. The stable manifold $W^s(\{\gamma_a\})$ consists of all orbits for X_t that converge to $\{\gamma_a\}$ as $t \to \infty$. The unstable manifold $W^u(\{\gamma_a\})$ likewise consists of all orbits for X_t that converge to $\{\gamma_a\}$ as $t \to -\infty$. For a near a_0, $W^s(\{\gamma_a\})$ is foliated with stable manifolds $W^s(\gamma_a)$ of single periodic orbits. Likewise for $W^u(\{\gamma_a\})$.

Suppose that at $a = a_0$, the stable manifold $W^s(\gamma_{a_0})$ intersects the unstable manifold $W^u(\gamma_{a_0})$ in a symmetric homoclinic orbit ρ_{a_0}. Since ρ_{a_0} is symmetric, it intersects $\mathrm{Fix}(R)$ in a single point r_{a_0}. We assume a transversality condition:

(H 3) $\qquad W^s(\gamma_{a_0}) \pitchfork_{\rho_{a_0}} W^u(\{\gamma_a\}), \qquad W^u(\gamma_{a_0}) \pitchfork_{\rho_{a_0}} W^s(\{\gamma_a\}).$

By reversibility, the two conditions imply each other. It follows from (H 3) that there is a sheet of symmetric homoclinic orbits ρ_a for a near a_0, where ρ_a is homoclinic to γ_a. See Figure 1 for an illustration.

In order to study the dynamics near ρ_{a_0} we construct an appropriate return map, details of which we postpone to Section 2.

Theorem 1.2. *Let X be a smooth vector field as above, satisfying (H 2), (H 3). Then, the non-wandering set of the return map describing the dynamics near ρ_{a_0} is contained in a set with a lamination of one-dimensional leaves parameterized by a subshift of finite type.*

A lamination of one-dimensional leaves of a set is a disjoint decomposition of the set in smooth curves, where the tangent spaces of the curves depend continuously on the base point. See also [HirPugShu77]. The statement means that there is a Cantor set of smooth curves containing the non-wandering set of the return map. As a consequence of this result, the dynamics of the return map restricted to the Cantor set of smooth curves can be written as a skew-product of interval maps over a subshift of finite type and is thus of the form

$$(\nu, x) \mapsto (\sigma\nu, f_\nu(x)),$$

where ν is a symbolic sequence, σ the shift operator and f_ν the interval map. Theorem 1.2 can be seen as giving a dimensional reduction for the non-wandering dynamics; it suffices to study the skew-product system to reveal the structure of the non-wandering set. In the case of a Hamiltonian vector field, f_ν is the identity map as a is a constant of motion. The lamination is then given by the one-parameter families of horseshoes that exist near the homoclinic orbit.

We continue with more specific descriptions of dynamics and bifurcations near the homoclinic tangles. It turns out that both symmetric and non-symmetric periodic orbits occur near the symmetric homoclinic orbit. As a consequence of a Kupka-Smale theorem for reversible vector fields [Dev76a], generically symmetric periodic orbits form sheets and non-symmetric periodic orbits are isolated. If γ is a hyperbolic non-symmetric periodic orbit, then γ and $R(\gamma)$ have different index, i.e. their stable and unstable manifolds have different dimension. If the stable and unstable manifold of γ intersect the fixed space of R, there are heteroclinic connections from γ to $R(\gamma)$ and vice versa. Following [Dia95] we refer to these heteroclinic cycles between hyperbolic periodic orbits of different index as *heterodimensional cycles*. However, the reader should note that our cycles are robust (i.e. persistent under small perturbations) due to the transversal intersections of the (un)stable manifolds with $\mathrm{Fix}(R)$, whereas in [Dia95] in the context of general systems heterodimensional cycles are not robust and in fact associated to heteroclinic bifurcations.

Denote by $\mathcal{U} \subset \mathcal{X}(M^{2n})$ the open set of vector fields with symmetric homoclinic tangles to a sheet of symmetric periodic orbits satisfying (H 2), (H 3), equipped with the C^1 topology. The following theorem describes aspects of the dynamics of vector fields in $\mathcal{U}$. Recall that a basic set is a transitive compact invariant set with a dense set of periodic trajectories. We call a basic set nontrivial if it contains infinitely many periodic orbits.

Theorem 1.3. *A vector field $X \in \mathcal{U}$ has infinitely many sheets of symmetric periodic orbits accumulating onto the sheet of homoclinic solutions. Moreover, each of the following classes of vector fields form a dense subset of $\mathcal{U}$:*

- *Vector fields with a saddle-node bifurcation of a non-symmetric periodic orbit.*
- *Vector fields with infinitely many heterodimensional cycles, accumulating onto the sheet of homoclinic solutions.*
- *Vector fields with infinitely many nontrivial hyperbolic basic sets, accumulating onto the sheet of homoclinic solutions.*

The density of saddle-node bifurcations of periodic orbits implies the following corollary.

Corollary 1.4. *Vector fields from $\mathcal{U}$ are not C^1 structurally stable.*

Theorem 1.3 and its corollary provide answers to the questions posed above: horseshoes exist for vector fields from an open and dense subset of $\mathcal{U}$ and the non-wandering dynamics is not structurally stable. This is notably different from homoclinic tangles in general vector fields and conservative or Hamiltonian vector fields, where the dynamics near the homoclinic tangle is C^1 structurally stable. In the class of general vector fields, structurally unstable ones can be found near vector fields with a homoclinic tangency in the C^2 topology; from the substantial literature we mention [PalTak93, GonShi95, New04]. Near vector fields with heterodimensional cycles within the class of general vector fields, one can find structurally unstable vector fields in the C^1 topology [Dia95]. We emphasize that these mechanisms of occurrence of instability are more complex and different from the one we describe in this paper for reversible vector fields. In particular the geometric reduction provided by Theorem 1.2 has no analog in these bifurcation problems.

Our strategy to obtain Theorem 1.3 relies on the following proposition.

Proposition 1.5. *Let $X \in \mathcal{U}$. There exists, arbitrary C^1 close to X, a vector field which is conservative in some neighborhood of ρ_{a_0}.*

We use this proposition to prove the density of saddle-node bifurcations in Theorem 1.3. In conservative systems the non-symmetric periodic orbits arise as sheets, while by the reversible Kupka-Smale theorem, in reversible vector fields they typically arise isolated. Hence C^1 small perturbations can create new non-symmetric periodic orbits by means of saddle-node bifurcations.

Different types of homoclinic dynamics arise from homoclinic loops, i.e. homoclinic orbits to equilibria. In the reaction-diffusion context, homoclinic loops correspond to pulses. Sheets of symmetric periodic orbits are found in the vicinity of symmetric homoclinic loops. Also homoclinic tangles to sheets of symmetric homoclinic orbits can occur near symmetric homoclinic loops. This is in particular true near homoclinic loops to a saddle-focus [Dev76b, Dev77, Har98] and, under some conditions, near multiple homoclinic loops to a saddle [HomKno06]. Our results are applicable to symmetric homoclinic orbits to sheets of symmetric periodic solutions arising in these settings.

The organization of the paper is as follows. In Section 2 we provide dimension reductions for the nonwandering set near symmetric homoclinic tangles and prove Theorem 1.2. In Section 3 we show that the symmetric homoclinic orbits are accumulated by sheets of symmetric periodic orbits. In Section 4 we prove Proposition 1.5. In Section 5 we show that heterodimensional cycles can be created by arbitrarily C^1 small perturbations. We consider heterodimensional cycles in their own right and show how they are accumulated by sheets of symmetric periodic orbits, by further hyperbolic periodic orbits of different index, and by nontrivial hyperbolic basic sets. At the end of Section 5 we combine our results obtained earlier in the paper to provide a proof of Theorem 1.3. In Section 6 we consider transverse heteroclinic connections between two families of symmetric periodic orbits (possibly belonging to the same global family). The reversibility implies the existence of heteroclinic cycles. The results are analogous to the ones obtained for homoclinic orbits.

2 Dimension reductions

We will derive reduction theorems for the dynamics near symmetric homoclinic connections, reducing the dynamics to a skew product system of interval maps over a subshift of finite type.

Consider a vector field $X \in \mathcal{U}$ with a sheet of symmetric periodic orbits $\{\gamma_a\}$ and symmetric homoclinic connections $\{\rho_a\}$, as in the introduction. We will write $x \mapsto X_t(x)$ for the time t flow of X. The sheet of periodic orbits $\{\gamma_a\}$ gives a normally hyperbolic manifold. Write $TM^{2n} = E^s \oplus T\{\gamma_a\} \oplus E^u$ for the corresponding DX_t-invariant splitting along $\{\gamma_a\}$, as before. The splitting in stable, center, and unstable directions along the periodic orbits $\{\gamma_a\}$ extends to a continuous splitting $E^s \oplus T\{\rho_a\} \oplus E^u$ along the homoclinic orbits $\{\rho_a\}$.

We will only take orbits into account that are in the vicinity of the periodic family $\{\gamma_a\}$ and the family $\{\rho_a\}$ of homoclinic connections. Take a small cross-section Σ_0

transverse to γ_{a_0}. Recall that r_{a_0} denotes the intersection of ρ_{a_0} with $\mathrm{Fix}(R)$. A small neighborhood of $\{\gamma_a\}$ is stretched along $W^u(\{\gamma_a\})$ by the flow X_t. For some $t > 0$ it covers a neighborhood of r_{a_0}. Consider a second small cross-section Σ_1 inside this neighborhood and transverse to ρ_{a_0} at $r_{a_0} \in \mathrm{Fix}(R)$. We may choose Σ_0 and Σ_1 to be R-invariant; $R(\Sigma_0) = \Sigma_0$ and $R(\Sigma_1) = \Sigma_1$. By the fact that R is a time-reversal symmetry it follows that the intersection of $\mathrm{Fix}(R)$ with Σ_0, Σ_1 is n-dimensional. Write

$$\Psi : \Sigma_0 \cup \Sigma_1 \to \Sigma_0 \cup \Sigma_1$$

for the first return map defined on a subset of $\Sigma_0 \cup \Sigma_1$ (considering orbits only as long as they are near $\{\gamma_a\}$ and $\{\rho_a\}$). Observe that Ψ maps a point in Σ_1 necessarily to Σ_0. The return map Ψ is reversible;

$$\Psi \circ R = R \circ \Psi^{-1}.$$

Associated to an orbit $x = \{x(i)\}$, $x(i+1) = \Psi(x(i))$ for $i \in \mathbb{Z}$, in the nonwandering set Ω of Ψ, there is an itinerary $\Upsilon(x) : \mathbb{Z} \to \{0, 1\}$ defined by

$$\Upsilon(x)(i) = j, \text{ if } x(i) \in \Sigma_j.$$

In an itinerary of an orbit for Ψ, the symbol 1 is always followed by a 0. Let $\mathcal{B}$ be the subshift of finite type consisting of the subset of sequences $\mathbb{Z} \mapsto \{0, 1\}$ with transition matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, equipped with the product topology. The shift operator $\sigma : \mathcal{B} \to \mathcal{B}$ is, as usual, given by $\sigma(y)(k) = y(k + 1)$. The next proposition provides a lamination of normally hyperbolic center manifolds in the vicinity of (and including) $\{\gamma_a\} \cap \Sigma_0$ and $\{\rho_a\} \cap \Sigma_1$. The nonwandering set of Ψ is contained in the center manifolds; the theorem provides a reduction of the dynamics to a skew product system of interval maps over a subshift of finite type. Observe that this does not imply that corresponding to each symbolic sequence in the subshift, there exists a nonwandering orbit with this sequence as its itinerary.

Proposition 2.1. *For each $\eta \in \mathcal{B}$, there is a one-dimensional center manifold W^c_η for Ψ, so that any orbit x with itinerary $\Upsilon(x) = \eta$, satisfies $x \in W^c_\eta$. The curve W^c_η is smooth and depends continuously on η. Moreover, $W^c_{\sigma(\eta)} = \Psi(W^c_\eta)$.*

Proof. The invariant curves are obtained as intersections of center stable with center unstable manifolds. We will construct invariant center stable manifolds. Center unstable manifolds are constructed analogously. The context is reminiscent of the construction of a Cantor bouquet of center manifolds near multiple homoclinic orbits in [HomKno06]. In the present context, the return map Ψ is smooth. The technical machinery needed to cope with non-smooth transition maps near equilibria, see [HomKno06], is therefore not needed.

The method of proof is classical: we extend well known constructions for local center stable manifolds [Irw80b, GilVan87] (originating from Perron's method of proof for local stable manifolds [Per29]) to similar constructions near general points in hyperbolic basic sets as in [Irw80a, HomVilSan03]. The techniques differ from graph transform techniques applied in [HirPugShu77], which would provide an alternative tool.

The family $\{\gamma_a\}$ yields a curve $\{p_a\}$ of fixed points $p_a = \gamma_a \cap \Sigma_0$ for Ψ in Σ_0. Likewise, $\{\rho_a\}$ yields a curve $\{r_a\}$ of homoclinic points $r_a = \rho_a \cap \Sigma_1$ for Ψ in Σ_1. Take coordinates $x = (x_s, x_c, x_u)$ in $\mathbb{R}^{2n-1}$ on Σ_0 such that, for values of a near a_0,

1. the fixed points p_a are contained in the x_c-axis $\{x_s, x_u = 0\}$,

2. $\{x_c, x_u\} = 0$ is tangent to the intersection of the stable manifold $W^s(\gamma_a)$ with $\mathrm{Fix}(R)$,

3. $\{x_s, x_c\} = 0$ is tangent to the intersection of the unstable manifold $W^u(\gamma_a)$ with $\mathrm{Fix}(R)$.

Take similar coordinates on Σ_1, which we also denote by $x = (x_s, x_c, x_u)$. The x_c-axis contains the homoclinic points r_a for a near a_0.

We may assume that the involution R acts linearly in the x-coordinates on Σ_0 and Σ_1 by

$$R(x_s, x_c, x_u) = (x_u, x_c, x_s). \tag{2.1}$$

For $x = (x_s, x_c, x_u)$, introduce coordinate projections $\Pi_{s,c}(x) = (x_s, x_c)$ and $\Pi_u(x) = x_u$.

The map Ψ consists of maps $\Psi_{0,0} : \Sigma_0 \cap \Psi^{-1}(\Sigma_0) \to \Sigma_0$, $\Psi_{0,1} : \Sigma_0 \cap \Psi^{-1}(\Sigma_1) \to \Sigma_1$ and $\Psi_{1,0} : \Sigma_1 \cap \Psi^{-1}(\Sigma_0) \to \Sigma_0$. Using the coordinate systems on Σ_0 and Σ_1, we extend each of the maps $\Psi_{0,0}, \Psi_{0,1}, \Psi_{1,0}$ to a map $\mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1}$ as follows. Let $\theta : \mathbb{R}^{2n-1} \to \mathbb{R}$ be a nonnegative testfunction, with $\theta \equiv 1$ near the origin and $\theta \equiv 0$ outside a neighborhood of the origin. For $\epsilon > 0$, let $\theta_\epsilon(x) = \theta(x/\epsilon)$ be a test function vanishing outside an $\mathcal{O}(\epsilon)$ neighborhood of the origin. Replace $\Psi_{0,0}$ by

$$x \mapsto \theta_\epsilon(x)\Psi_{0,0}(x) + (1 - \theta_\epsilon(x))D\Psi_{0,0}(0)x,$$

Replace $\Psi_{0,1}$ by

$$x \mapsto \theta_\epsilon(x)\Psi_{0,1}(x) + (1 - \theta_\epsilon(x))(\Psi_{0,1}(r) + D\Psi_{0,1}(r)(x - r)),$$

where $r = \rho_{a_0} \cap \Sigma_0 \cap \Psi^{-1}(\Sigma_1)$ is the homoclinic point in $\rho_{a_0} \cap \Sigma_0$ that gets mapped to Σ_1. Finally, replace $\Psi_{1,0}$ by

$$x \mapsto \theta_\epsilon(x)\Psi_{1,0}(x) + (1 - \theta_\epsilon(x))(\Psi_{1,0}(r_{a_0}) + D\Psi_{1,0}(r_{a_0})(x - r_{a_0})).$$

For ϵ small, the maps $\Psi_{0,0}, \Psi_{0,1}, \Psi_{1,1}$ are globally close to affine maps. We may take θ symmetric in the sense $\theta \circ R = \theta$. The symmetric choice of the test function θ ensures that the extended maps are reversible with reverser R given by (2.1).

Now fix an element $\eta \in \mathcal{B}$. We will construct a center stable manifold $W^{s,c}_\eta$ so that $\xi(0) \in W^{s,c}_\eta$ for $\Upsilon(\xi) = \eta$. Following the construction we discuss dependence on η. The manifold $W^{s,c}_\eta$ is obtained as a union of orbits ζ with varying initial data $(\zeta_s(0), \zeta_c(0)) = (x_s, x_c)$. Such orbits are found as fixed points of a contraction depending on parameters (x_s, x_c).

Denote by $\mathcal{C}(\mathbb{N}, \mathbb{R}^{2n-1})$ the space of sequences $\xi : \mathbb{N} \to \mathbb{R}^{2n-1}$ with $\Upsilon(\xi) = \eta$. For $(x_s, x_c) \in \mathbb{R}^{n+1}$ given, define $\mathcal{H} : \mathcal{C}(\mathbb{N}, \mathbb{R}^{2n-1}) \to \mathcal{C}(\mathbb{N}, \mathbb{R}^{2n-1})$ by

$$\mathcal{H}(\xi)(k) = \begin{cases} (x_s, x_c, \Pi_u \Psi^{-1}_{\eta(0),\eta(1)}(\xi(1))), & k = 0, \\ (\Pi_{s,c}\Psi_{\eta(k-1),\eta(k)}(\xi(k-1)), \Pi_u \Psi^{-1}_{\eta(k),\eta(k+1)}(\xi(k+1))), & k > 0. \end{cases}$$

Orbits of Ψ are fixed points of $\mathcal{H}$. As $\mathcal{H}$ is not a contraction, we modify $\mathcal{H}$. Write $\mathcal{D}_\alpha(\mathbb{N}, \mathbb{R}^{2n-1})$ for the set of sequences $\mathbb{N} \to \mathbb{R}^{2n-1}$, equipped with the norm

$$\|x\|_\alpha = \sup_{k \in \mathbb{N}} \alpha^k \|x(k)\|.$$

Let $A : \mathcal{D}_\alpha(\mathbb{N}, \mathbb{R}^{2n-1}) \to \mathcal{D}_1(\mathbb{N}, \mathbb{R}^{2n-1})$ be given by

$$Ax(k) = \alpha^k x(k).$$

Let $\alpha < 1$ be fixed and close to 1. Define $\mathcal{J} = A \circ \mathcal{H} \circ A^{-1}$.

For ϵ small enough, some iterate of $\mathcal{J}$ is a contraction on $\mathcal{C}_1$:

$$\|\mathcal{J}^n(\xi_1 - \xi_0)\| \leq C\lambda^n \|x_1 - x_0\|.$$

for some $C > 0, 0 < \lambda < 1$. The verification of this estimate is straightforward if $\Psi_{0,0}, \Psi_{0,1}, \Psi_{1,0}$ are replaced by their first order Taylor expansions, and therefore holds for ϵ small. The map $\mathcal{J}$ therefore possesses a unique fixed point ζ.

We claim that ζ satisfies $\zeta(k+1) = \alpha^{(k+1)}\Psi_{\eta(k),\eta(k+1)}(\alpha^{-k}\zeta(k))$. The claim implies that $\alpha^{-k}\zeta(k)$ is an orbit for Ψ. For $k \geq 0$ the fixed point equation $\zeta = \mathcal{J}(\zeta)$ yields

$$\begin{aligned} \Pi_{s,c}\alpha^{-(k+1)}\zeta(k+1) &= \Pi_{s,c}\Psi_{\eta(k),\eta(k+1)}\alpha^{-k}\zeta(k), \\ \Pi_u\alpha^{-k}\zeta(k) &= \Pi_u\Psi^{-1}_{\eta(k),\eta(k+1)}\alpha^{-(k+1)}\zeta(k+1). \end{aligned}$$

Given $\Pi_{s,c}\zeta(k)$ and $\Pi_u\zeta(k+1)$ (which are determined by the other equations), these equations are uniquely solvable for $\Pi_{s,c}\zeta(k+1)$ and $\Pi_u\zeta(k)$. By uniqueness, $\zeta(k), \zeta(k+1)$ satisfies $\zeta(k+1) = \alpha^{(k+1)}\Psi_{\eta(k),\eta(k+1)}(\alpha^{-k}\zeta(k))$.

Write $w(x_s, x_c) = \zeta(0)$ and define

$$W^{s,c}_\eta = \bigcup_{(x_s,x_c)} w(x_s, x_c).$$

It follows from [Irw80b, GilVan87] that w is smooth; $W^{s,c}_\eta$ is the sought for center stable manifold.

It remains to see that w depends continuously on η. If $l(j) = \eta(j)$ for $-N \leq j \leq N$, then the same maps $\Psi_{0,0}, \Psi_{1,0}, \Psi_{0,1}$ are applied for the symbolic sequences η and l for $-N+1 \leq j \leq N-1$. It follows that the fixed points lie close if N is large and hence that w depends continuously on η. ∎

As stated in the following result, the center stable manifolds constructed in the above proof are foliated by stable manifolds of points in W^c_η. Analogously, center unstable manifolds are foliated by unstable manifolds.

Proposition 2.2. *For $\eta \in \mathcal{B}$, there is an invariant foliation $\mathcal{F}^s_\eta$ of $W^{s,c}_\eta$, whose leaves are stable manifolds of points in W^c_η. The foliations $\mathcal{F}^s_\eta$ depend continuously on η.*

Proof. The foliation $\mathcal{F}^s_\eta$ will consist of $(n-1)$ dimensional leaves. A foliation is determined by its tangent bundle, which in this case consists of $(n-1)$ dimensional planes in $\mathbb{R}^{2n-1}$. Denote by $G^{n-1}(\mathbb{R}^{2n-1})$ the Grassmannian manifold of $(n-1)$ dimensional linear subspaces in $\mathbb{R}^{2n-1}$. Extend Ψ to $\Psi^{(1)}$ on the fiber bundle $\mathbb{R}^{2n-1} \times G^{n-1}(\mathbb{R}^{2n-1})$ by

$$\Psi^{(1)}(x, \alpha) = (\Psi(x), D\Psi(x)\alpha).$$

Observe that the bundle E^s of stable directions along $\{p_a\}$ is fixed under $\Psi^{(1)}$.

In order to compute the spectrum of $D\Psi^{(1)}$ at a fixed point $(p_a, E^s_{p_a})$, it is convenient to do the computations in a coordinate chart. Take coordinates $x =$

(x_s, x_c, x_u) as in the proof of Proposition 2.1. Take a local coordinate chart near $E^s_{p_{a_0}} \in G^{n-1}(\mathbb{R}^{2n-1})$ with values in $\mathcal{L}(E^s_{p_{a_0}}, E^c_{p_{a_0}} \times E^u_{p_{a_0}})$, such that $v \in G^{n-1}(\mathbb{R}^{2n-1})$ is represented by the element of $\mathcal{L}(E^s_{p_{a_0}}, E^c_{p_{a_0}} \times E^u_{p_{a_0}})$ whose graph equals v. Using this chart for the fibers, $\Psi^{(1)}$ takes the expression

$$(x, v) \mapsto (\Psi(x), w),$$

with

$$\text{graph } w = D\Psi(x)\text{graph } v.$$

Write $\mathbb{R}^{2n-1}$ as a product of $E^s_{p_{a_0}}$ and $E^c_{p_{a_0}} \times E^u_{p_{a_0}}$ and in this product write $D\Psi(x) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Here A, B, C, D depend on x. We find

$$w = (Av + B)(Cv + D)^{-1} \tag{2.2}$$

Note that in $x = p_{a_0}$, this formula reads $w = AvD^{-1}$.

Since $\Psi^{(1)}$ maps fibers to fibers, the spectrum of $D\Psi^{(1)}(p_a, E^s_a)$ is the union of the spectrum of $D\Psi(p_a)$ and the spectrum of $D\Psi^{(1)}(p_a, E^s_a)$ restricted to the fiber $\{0\} \times \mathcal{L}(E^s_{p_{a_0}}, E^c_{p_{a_0}} \times E^u_{p_{a_0}})$. From (2.2) it follows that (p_a, E^s_a) is attracting within the fibers $G^{n-1}(\mathbb{R}^{2n-1})$.

Therefore iteration of a suitable trial foliation on $\cup_{i \in \mathbb{Z}} W^{s,c}_{\sigma^i(\eta)}$ converges to an invariant stable foliation. Continuity with η follows from the construction. ■

Note that for $\bar{\eta} = 0^\infty$, the itinerary of the sheet of fixed points $\{p_a\}$, the foliation $\mathcal{F}^s_{\bar{\eta}}$ is formed by the stable manifolds of the individual fixed points p_a.

3 Symmetric periodic orbits

The dimension reductions from Section 2 show how the dynamics near symmetric homoclinic connections is described by skew products systems of interval maps over subshifts of finite type. In a general context, the dynamical complexities of skew product systems of interval or circle maps over shifts or subshifts are investigated in [GorIly00].

In this section we start exploring the consequences of the dimension reduction. We will show in Theorem 3.1 that symmetric homoclinic tangles are accumulated by families of symmetric periodic orbits. A similar result holds true near heteroclinic tangles, see Section 6. Where in this section we establish the robust occurrence of families of symmetric periodic orbits, the following section will discuss bifurcations taking place near the symmetric homoclinic tangles.

Recall that Ψ is the return map on cross sections $\Sigma_0 \cup \Sigma_1$. Write $\mathcal{V}_\varepsilon$ for a ε-neighborhood of $p_{a_0} \cup r_{a_0}$ in $\Sigma_0 \cup \Sigma_1$. Take coordinates $x = (x_s, x_c, x_u)$ near p_{a_0} as before. Consider $x \in \Sigma_0$ with $\Psi^k(x) \in \Sigma_0$ and write $\Psi^k(x) = (x_s(k), x_c(k), x_u(k))$. Even if $x_s(k)$ and $x_u(k)$ lie within distance ε of 0, $|x_c(k)|$ can be larger then ε so that $\Psi^k(x)$ does not lie in $\mathcal{V}_\varepsilon$. This possible drift in the central direction makes it delicate to describe those orbits that remain in $\mathcal{V}_\varepsilon$.

Recall that the return map $\Psi : \Sigma_0 \cup \Sigma_1 \to \Sigma_0 \cup \Sigma_1$ is reversible; $\Psi \circ R = R \circ \Psi^{-1}$. This implies that $\Psi^n \circ R = R \circ \Psi^{-n}$. Therefore $\Psi^n \circ R$ is an involution for each n and Ψ is reversible with respect to it;

$$\Psi \circ (\Psi^n \circ R) = (\Psi^n \circ R) \circ \Psi^{-1},$$

see also [DeV58, Dev76a]. We will define symmetric itineraries which are related to symmetric orbits of the vector field. Define an involution $\mathcal{R}$ on $\mathcal{B}$, as follows:

$$\mathcal{R}\eta(k) = \eta(-k).$$

For each orbit x, $\Upsilon(Rx) = \mathcal{R}\Upsilon(x)$. The construction of the center manifolds W^c_η implies

$$W^c_{\mathcal{R}\eta} = RW^c_\eta. \tag{3.1}$$

Note that the shift σ on $\mathcal{B}$ is reversible with respect to the involution $\mathcal{R}$. We call an itinerary η symmetric if there exists $s \in \mathbb{Z}$ such that

$$\mathcal{R}\eta = \sigma^s \eta.$$

The set of symmetric itineraries comprises the set of those whose σ-orbit is symmetric. Moreover, if η is a symmetric N-periodic orbit, then $\mathcal{R}\eta = \sigma^s\eta$ for some $0 < s \leq N$.

Theorem 3.1. *For each $\varepsilon > 0$, there are symmetric periodic itineraries $\eta \in \mathcal{B}$, for which there is a one parameter family of periodic points $x_\lambda \in W^c_\eta$, with $\Upsilon(x_\lambda) = \eta$ and whose orbits are contained in $\mathcal{V}_\varepsilon$.*

Proof. We start with a computation bounding the drift in the central direction, when iterating Ψ. Consider an orbit piece $\zeta(0), \ldots, \zeta(k)$ near p_{a_0}, with $\zeta(0)$ close to $W^s(\{p_a\})$ and $\zeta(k)$ close to $W^u(\{p_a\})$. In the coordinate system $x = (x_s, x_c, x_u)$, write $\zeta(i) = (\zeta_s(i), \zeta_c(i), \zeta_u(i))$. Differentiability of Ψ implies that there exists $C > 0$ such that

$$|\zeta_c(k) - \zeta_c(0)| \leq C \sum_{i=0}^{k} d(\zeta(i), W^s(\{p_a\}) \cup W^u(\{p_a\}).$$

Here $d(\zeta_i, W^s(\{p_a\}) \cup W^u(\{p_a\})$ stands for the distance between $\zeta(i)$ and $W^s(\{p_a\}) \cup W^u(\{p_a\})$. For some $K > 0, 0 < \lambda < 1$,

$$\begin{aligned} d(\zeta(i), W^s(\{p_a\}) &\leq K\lambda^{k-i}, \\ d(\zeta(i), W^u(\{p_a\}) &\leq K\lambda^{i}. \end{aligned}$$

Because $d(\zeta(i), W^u(\{p_a\} \cup W^u(\{p_a\})) \leq K \min\{\lambda^i, \lambda^{k-i}\}$, the maximal distance between $\zeta(i)$, $0 \leq i \leq k$, and $W^s(\{p_a\}) \cup W^u(\{p_a\})$ goes to 0 exponentially fast in k. Hence, $|\zeta_c(k) - \zeta_c(0)|$ goes to zero as $k \to \infty$.

The above calculation implies the following. Consider periodic itineraries $\eta = (0^{n_1}1 \ldots 0^{n_m}1)^\infty$ (or shifts of such itineraries) with $n_i \geq N$. Write $S = n_1 + \cdots + n_m + m$. Then for each $\delta > 0, m > 0$, one can choose $N > 0$ so that $|\Psi^S(x) - x| < \delta$ for $x \in W^c_\eta$. The whole orbit $\mathcal{O}(x)$ of x is therefore close to $p_{a_0} \cup \mathcal{O}(r_{a_0})$.

We now prove the occurrence of symmetric periodic orbits along families. Consider a periodic itinerary η of period S as above, which is moreover symmetric. By (3.1), there is an $0 \leq s < S$ such that $W^c_{\mathcal{R}\eta} = W^c_{\sigma^s\eta}$. Hence $R(W^c_\eta)$ equals $\Psi^s(W^c_\eta)$. It follows that $\Psi^k \circ R(W^c_\eta) = W^c_\eta$ with $k = S - s$. Because of the vicinity to the homoclinic tangle, $U_k = \Psi^k \circ R$ is order preserving on W^c_η. The involution U_k can only leave the curve W^c_η invariant, if it fixes each point on W^c_η. Thus $W^c_\eta \subset \mathrm{Fix}(U_k)$. Because Ψ^S maps W^c_η into itself and Ψ^S is reversible with respect to U_k, the center manifold W^c_η consists of periodic points: orbits that intersect the fixed space of an involution twice are symmetric periodic orbits. The bounds derived above, show that the periodic orbits are as close as desired to $p_{a_0} \cup \mathcal{O}(r_{a_0})$ by taking N large enough. Moreover, these are all periodic points of period S, if S is the minimal period of η. ■

4 Structural stability

Write $\mathcal{X}^1(M^{2n})$ for the space of vector fields on M^{2n}, equipped with the C^1 topology. We will show that a vector field with a symmetric homoclinic tangle is not C^1 structurally stable. We do this by showing that by C^1 small perturbations one obtains vector fields that are conservative close to the homoclinic connection. The arguments will make clear that for instance saddle-node bifurcations of periodic orbits are found in arbitrary small perturbations from a vector field in $\mathcal{X}^1(M^{2n})$ with a symmetric homoclinic cycle. Although we formulate the result in the context of reversible systems, the arguments depend only on the persistent occurrence of a sheet of periodic orbits with homoclinic tangles, and make no essential use of the reversibility. We conclude the section by indicating that our arguments to prove Proposition 1.5 below cannot be generalized to smoother topologies.

Let $\mathcal{U} \subset \mathcal{X}^1(M^{2n})$ be the open set of vector fields with symmetric homoclinic tangles as described in Section 2. Let $X \in \mathcal{U}$ be a vector field with a family of periodic orbits $\{\gamma_a\}$ and symmetric homoclinic orbits $\{\rho_a\}$, a near a_0.

Proof of Proposition 1.5. Take cross sections Σ_0, Σ_1 as before, and write Ψ for the return map on $\Sigma_0 \cup \Sigma_1$. Denote $p_a = \gamma_a \cap \Sigma_0$ and $r_a = \rho_a \cap \Sigma_1$. Recall from Section 2 that there exists a $D\Psi$ invariant bundle $E^{s,u}$ of normal directions along $\cup_{i\in\mathbb{Z}}\Psi^i(r_{a_0}) \cup p_{a_0}$; $E^{s,u}$ is the direct sum of the tangent spaces of stable and unstable manifolds of p_{a_0}.

A vector field $\tilde{X}$ close to X in the C^1 topology has periodic orbits $\tilde{\gamma}_a$ near γ_a and homoclinic orbits $\tilde{\rho}_a$ near ρ_a. Its return map $\tilde{\Psi}$ on $\Sigma_0 \cup \Sigma$ is C^1 close to Ψ. The perturbed periodic and homoclinic points for $\tilde{\Psi}$ will be denoted by $\tilde{p}_a$ and $\tilde{r}_a$ respectively. There exists a $D\tilde{\Psi}$ invariant bundle $\tilde{E}^{s,u}$ along $\cup_{i\in\mathbb{Z}}\tilde{\Psi}^i(\tilde{r}_{a_0}) \cup \tilde{p}_{a_0}$, close to $E^{s,u}$.

We will construct a perturbed vector field $\tilde{X}$, arbitrarily C^1 close to X, together with a function H on $\Sigma_0 \cup \Sigma_1$ which is $\tilde{\Psi}$-invariant near the closure of the orbit of $\tilde{r}_{a_0}$. The existence of the $\tilde{\Psi}$-invariant function H implies that $\tilde{X}$ is conservative near $\tilde{\rho}_{a_0} \cup \tilde{\gamma}_{a_0}$. The constructions in Section 2 imply the existence of sheets of non-symmetric periodic orbits close to $\tilde{\rho}_{a_0} \cup \tilde{\gamma}_{a_0}$. Clearly, $\tilde{X}$ cannot be C^1 structurally stable as an arbitrary small perturbation makes non-symmetric periodic orbits hyperbolic (compare the Kupka-Smale theorem for reversible vector fields in [Dev76a]).

The level surface of an invariant function through $\tilde{p}_{a_0}$ contains the homoclinic orbit $\mathcal{O}(\tilde{r}_{a_0})$. Such a level surface is therefore tangent to $\tilde{E}^{s,u}$ at points of $\mathcal{O}(\tilde{r}_{a_0})$. It follows that H must be constructed with this condition satisfied. We start with a perturbation of X near the periodic orbit γ_{a_0} and find a suitable invariant function for $\tilde{\Psi}$ close to $\tilde{p}_{a_0}$.

Take coordinates $x = (x_s, x_c, x_u)$ on the cross section Σ_0 so that γ_{a_0} is the origin,

$\{\gamma_a\} = \{x_s, x_u = 0\}$,

$$D\tilde{\Psi}(0, x_c, 0) = \begin{pmatrix} A(x_c) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A^{-1}(x_c) \end{pmatrix}$$

and Ψ is reversible with involution $R(x_s, x_c, x_u) = (x_u, x_c, x_s)$. The coordinate axes are tangent to the directions of the splitting $E^s \oplus E^c \oplus E^u$ at $\tilde{p}_a$. The matrix A has its spectrum within the unit circle for all small x_c. By an initial C^1 small perturbation, we get a vector field $\tilde{X}$ with normally hyperbolic linear return map near p_{a_0}:

$$\tilde{\Psi}(x_s, x_c, x_u) = (A(x_c)x_s, x_c, A^{-1}(x_c)x_u), \tag{4.1}$$

for small x. Write $\mathcal{V} \subset \Sigma_0$ for the neighborhood of p_{a_0} on which (4.1) holds. By a further C^1 small perturbation we may assume that A is constant for $x \in \mathcal{V}$. We can moreover obtain that A is complex diagonalizable with eigenvalues of algebraic multiplicity 1. The perturbations can be chosen such that the perturbed system remains reversible with involution $R(x_s, x_c, x_u) = (x_u, x_c, x_s)$. Replacing the coordinates (x_s, x_u) by coordinates of the form $(Ux_s, U^{-1}x_u)$ for a suitable matrix U brings A to normal form, while retaining the reversibility with respect to R. Note that one can take $\tilde{X}$ closer to X in the C^1 topology by taking $\mathcal{V}$ smaller.

We will consider functions H on $\mathcal{V}$ of the form

$$H(x_s, x_c, x_u) = x_c + P_2(x_s, x_u)$$

for quadratic polynomials P_2, which are symmetric ($H \circ R = H$) and $\tilde{\Psi}$-invariant ($H \circ \tilde{\Psi} = H$) on $\mathcal{V}$. That is, $P_2(x_s, x_u) = P_2(x_u, x_s)$ and $P_2(Ax_s, A^{-1}x_u) = P_2(x_s, x_u)$. For example, on a four dimensional manifold M^4 where x_s and x_u are one dimensional coordinates, such polynomials are of the form

$$P_2(x_s, x_u) = kx_sx_u$$

for some $k \in \mathbb{R}$. To present another example, on a six dimensional manifold M^6 with $A = \lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $0 < \lambda < 1$, one has

$$P_2(x_s, x_u) = \left\langle x_s, \begin{pmatrix} a & b \\ b & -a \end{pmatrix} x_u \right\rangle$$

for $a, b \in \mathbb{R}$. Elementary considerations that build on these two examples show that in general

$$P_2(x_s, x_u) = \sum_{i=1}^{n-1} \langle x_s, C_i x_u \rangle \tag{4.2}$$

for linearly independent matrices C_i, $0 \leq i \leq n-1$, with $C_i^T = C_i$ and $(C_iA)^T = C_iA$ (C_i and C_iA are symmetric).

Write $\mathcal{C}$ for the linear space spanned by the matrices C_i. There is a positive integer N so that $\tilde{\Psi}^n(\tilde{r}_{a_0}) \in \mathcal{V}$ for $|n| \geq N$. By a small perturbation of $\tilde{X}$ if necessary, we may assume that each coefficient of the x_u coordinate of $\tilde{\Psi}^{-N}(\tilde{r}_{a_0})$ differs from 0 (the perturbation can be restricted to a domain away from $\tilde{\rho}_{a_0}$). The map

$$\mathcal{C} \ni C \mapsto D_{x_s}P_2(\tilde{\Psi}^{-N}(\tilde{r}_{a_0})) \in \mathcal{L}(E^s, \mathbb{R})$$

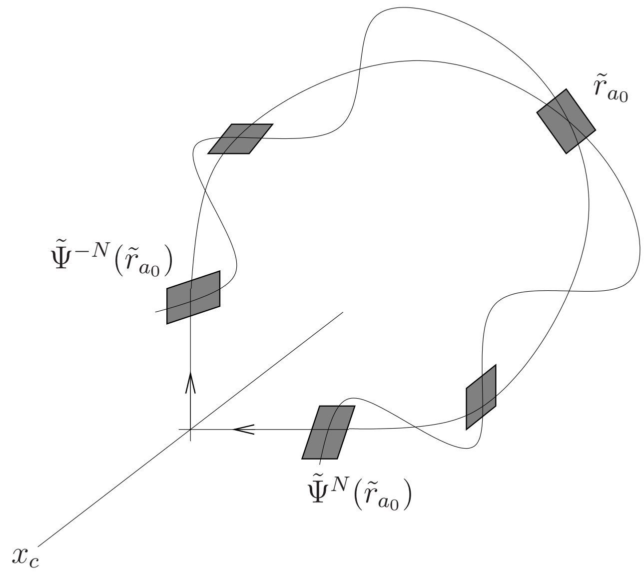

Figure 2: Level surfaces of an invariant function H are tangent to the subspaces $\tilde{E}^{s,u}$ along the orbit $\mathcal{O}(r_a)$. The derivative DH thus vanishes on $\tilde{E}^{s,u}$ along $\mathcal{O}(r_a)$.

is then surjective. Therefore, we can fix $C \in \mathcal{C}$ so that DH vanishes on $\tilde{E}^{s,u}$ at $\tilde{\Psi}^{-N}(\tilde{r}_{a_0})$, compare Figure 2. By symmetry of H and the homoclinic orbit $\mathcal{O}(\tilde{r}_{a_0})$, DH vanishes on $\tilde{E}^{s,u}$ along $\mathcal{O}(\tilde{r}_{a_0}) \cap \mathcal{V}$. It remains to define H near $\cup_{-N<i<N} \tilde{\Psi}^i(\tilde{r}_{a_0})$ and alter $\tilde{\Psi}$ so that H is invariant close to these points. The remaining perturbations of $\tilde{\Psi}$ will leave the homoclinic orbit $\mathcal{O}(\tilde{r}_{a_0})$ and the bundle $\tilde{E}^{s,u}$ along $\mathcal{O}(\tilde{r}_{a_0})$ unaltered.

Assume for now that H has been defined on Σ_0 and $\tilde{\Psi}$ has been changed so that H close enough to points of $\mathcal{O}(\tilde{r}_{a_0}) \cap \Sigma_0$ is left invariant by $\tilde{\Psi}$. We will show how to define H on Σ_1 as a symmetric function with level sets tangent to $\tilde{E}^{s,u}$ at $\tilde{r}_{a_0}$, and change $\tilde{\Psi}$ so that it leaves H invariant. Note that an invariant function H on Σ_1 is forced to be symmetric: for x and $R(x)$ in Σ_1,

$$H(x) = H(\tilde{\Psi}^{-1}(x)) = H(R \circ \tilde{\Psi}^{-1}(x)) = H(\tilde{\Psi} \circ R(x)) = H(R(x)),$$

where the steps use the invariance of H, the symmetry of H on Σ_0, the reversibility of $\tilde{\Psi}$ and again the invariance of H.

Take coordinates $x = (x_s, x_c, x_u)$ on Σ_1 with $\tilde{r}_{a_0} = (0, 0, 0)$ and coordinate axes tangent to the directions of the splitting $E^s \oplus E^c \oplus E^u$ at $\tilde{r}_{a_0}$. With H given near $\tilde{\Psi}^{-1}(\tilde{r}_{a_0})$, we define H near $\tilde{r}_{a_0}$ by

$$H(x) = D\tilde{\Psi}(\tilde{\Psi}^{-1}(\tilde{r}_{a_0}))DH(\tilde{\Psi}^{-1}(\tilde{r}_{a_0}))x.$$

Observe that H is symmetric on Σ_1. Moreover, DH vanishes on $\tilde{E}^{s,u}$ at $\tilde{r}_{a_0}$. With $\tilde{r}_{a_0} = \tilde{X}_\tau(\tilde{\Psi}^{-1}(\tilde{r}_{a_0}))$, write $O = \{X_t(\tilde{\Psi}^{-1}(\tilde{r}_{a_0}))\}, 0 < t < \tau$, for the orbit piece between $\tilde{\Psi}^{-1}(\tilde{r}_{a_0})$ and $\tilde{r}_{a_0}$. It is clear that there exists a C^1 small perturbation of the flow near a compact part of O so that the resulting return map $\tilde{\Psi}$, considered near $\Psi^{-1}(\tilde{r}_{a_0})$, leaves H invariant. Reversibility defines the perturbation near a compact part of $R(O)$. For x near $\tilde{r}_{a_0}$, $H(\tilde{\Psi}(x)) = H(x)$ as $\tilde{\Psi} = R \circ \tilde{\Psi}^{-1} \circ R$ and $H(R \circ \tilde{\Psi}^{-1} \circ R(x)) = H(x)$ by construction.

With an analogous reasoning one defines H near $\tilde{\Psi}^i(\tilde{r}_{a_0})$, $-N < i < 0$, given H near $\tilde{\Psi}^{i-1}(\tilde{r}_{a_0})$, and alters $\tilde{\Psi}$ near $\tilde{\Psi}^{i-1}(\tilde{r}_{a_0})$ by perturbing the flow near the orbit piece between $\tilde{\Psi}^{i-1}(\tilde{r}_{a_0})$ and $\tilde{\Psi}^i(\tilde{r}_{a_0})$. Reversibility gives similar perturbations near $\tilde{\Psi}^i(\tilde{r}_{a_0})$ for $0 < i < N$. ■

The arguments in the above proof cannot be generalized to smoother topologies. The strategy of the proof of Proposition 1.5 consists of constructing a perturbation in the C^1 topology of the original vector field X, so that the perturbed vector field $\tilde{X}$ has a first integral close to the symmetric homoclinic orbit. For k high enough, it is in general not possible to find a C^k nearby vector field with a C^k first integral close to the symmetric homoclinic orbit. We will not further pursue this, but briefly indicate the reasoning. Notation will be as in the proof of Proposition 1.5. In the C^k topology, one can perturb X so that the resulting perturbed return map $\tilde{\Psi}$ in suitable coordinates on Σ_0 takes the normal form

$$\tilde{\Psi}(x_s, x_c, x_u) = \begin{pmatrix} A(x_c)(I + F(x_c, x_s, x_u))x_s \\ x_c \\ (A(x_c)(I + F(x_c, x_s, x_u)))^{-1} x_u \end{pmatrix}$$

for small x. Here F is a polynomial function, starting with terms of second order, of x_c and the symmetric monomials $\langle x_s, C_i x_u \rangle$, compare (4.2). The map $\tilde{\Psi}$ is reversible, $\tilde{\Psi} = R \circ \tilde{\Psi}^{-1} \circ R$, with involution $R(x_s, x_c, x_u) = (x_u, x_c, x_s)$.

As in the proof of Proposition 1.5, $\tilde{\Psi}^{-N}(\tilde{r}_{a_0}) \in \Sigma_0$ denotes a point in the orbit of $\tilde{r}_{a_0}$ that lies in the domain of validity of the truncated normal form. Write $\mathcal{H}_k$ for the invariant polynomials of order k, i.e. the polynomial functions in $\langle x_s, C_i x_u \rangle$ and x_c of order k. For $H \in \mathcal{H}_k$, write $J^k H(x)$ for the k-jet of H at $x \in \Sigma_0$. The pull-back by $\tilde{\Psi}^{-N}$ yields k-jets $J^k(H \circ \tilde{\Psi}^{-N})$ at points in Σ_1.

Recall from the proof of Proposition 1.5 that an invariant function is necessarily symmetric on Σ_1. In particular the k-jet at $\tilde{r}_{a_0}$ of an invariant function is R-symmetric. Write $\mathcal{J}_k$ for the collection of k-jets of pulled back functions $H \circ \tilde{\Psi}^{-N}$, $H \in \mathcal{H}_k$, calculated at $\tilde{r}_{a_0}$. We need to find $H \in \mathcal{H}_k$ so that $J^k(H \circ \tilde{\Psi}^{-N}) \in \mathcal{J}_k$ is symmetric. Observe that $\tilde{\Psi}^N$ considered near $\tilde{\Psi}^{-N}(\tilde{r}_{a_0})$ can be perturbed arbitrarily since the orbit piece connecting $\tilde{\Psi}^{-N}(\tilde{r}_{a_0})$ to $\tilde{r}_{a_0}$ is not symmetric. A dimension count shows that for k large enough and for general $\tilde{\Psi}$, $\mathcal{J}_k$ will miss the set of R-symmetric k-jets of functions at $\tilde{r}_{a_0}$: the sum of dimensions of $\mathcal{J}_k$ and the space of k-jets of R-symmetric functions on Σ_1 is less then the dimension of the space of k-jets of all smooth functions on Σ_1. There are therefore open sets in the C^k topology of reversible vector fields without smooth first integrals near the homoclinic orbit.

5 Heterodimensional cycles

Heterodimensional cycles appear near symmetric homoclinic orbits in reversible vector fields, as asserted by the following lemma.

Lemma 5.1. *Let $X \in \mathcal{U}$. By an arbitrarily C^1 small perturbation, hyperbolic periodic orbits and a heterodimensional cycle connecting them is created.*

Proof. Perturbations from a conservative vector field can create hyperbolic periodic orbits arbitrarily close to ρ_{a_0}. Therefore, as shown in the previous section, hyperbolic

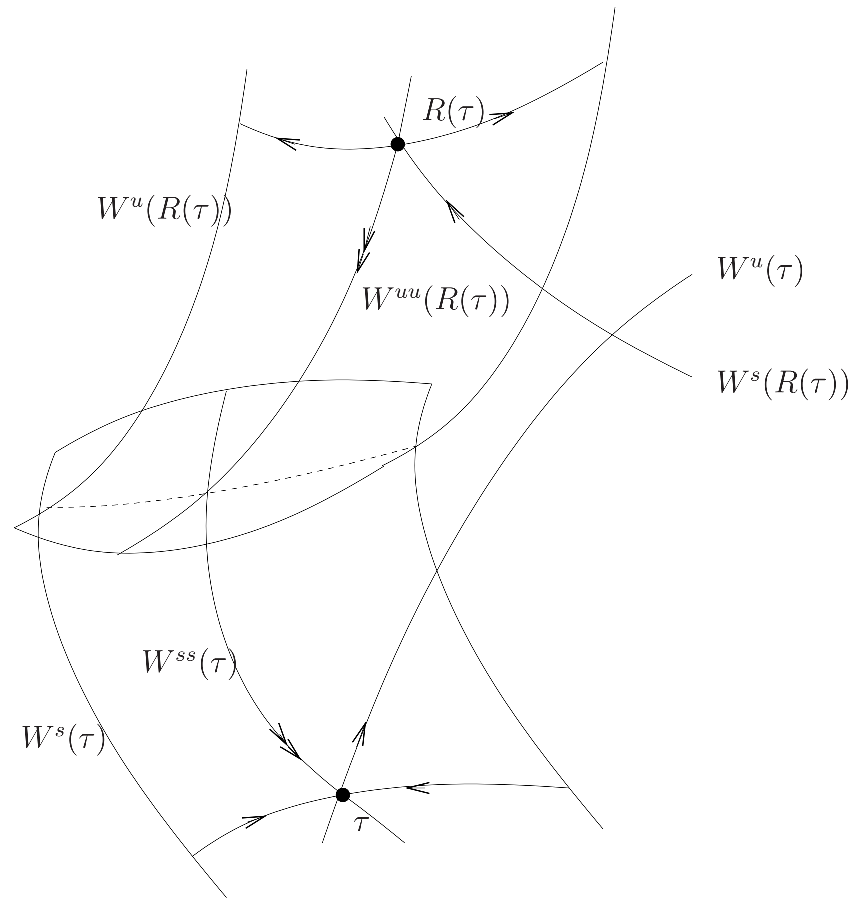

Figure 3: A heterodimensional cycle containing a hyperbolic periodic orbit τ and its symmetric image $R(\tau)$.

periodic orbits arbitrarily close to ρ_{a_0} can be created through C^1 small perturbations. Suppose now there is a hyperbolic periodic orbit τ near the homoclinic cycle. For definiteness, assume that $\dim W^u(\tau) = n$ and $\dim W^s(\tau) = n + 1$. The proof of Proposition 2.1 gives that $W^s(\tau)$ is near $W^s(\{\gamma_a\})$. There is therefore a transverse intersection of $W^s(\tau)$ with $\mathrm{Fix}(R)$. Proposition 2.2 implies that also $W^u(\tau)$ has a transverse intersection with $\mathrm{Fix}(R)$. It follows that there is a heteroclinic cycle between the two symmetrically related hyperbolic periodic orbits τ and $R(\tau)$. As the indices of τ and $R(\tau)$ differ, this is a heterodimensional cycle. ■

We study heterodimensional cycles as an object in itself, but we only consider the geometry with which heterodimensional cycles occur near symmetric homoclinic connections to a sheet of periodic solutions. The contents of this section are otherwise independent from the previous sections.

We gather the conditions we will assume. Let γ be a hyperbolic periodic orbit of index $\dim W^u(\gamma) = n$. Then $R(\gamma)$ is a hyperbolic periodic orbit of index $n+1$. Assume that both $W^s(\gamma)$ and $W^u(\gamma)$ intersect $\mathrm{Fix}(R)$ transversally; by reversibility this implies the existence of a heterodimensional cycle between γ and $R(\gamma)$. Suppose that $W^s(\gamma)$ contains a strong stable manifold $W^{ss}(\gamma)$ of codimension one. There is thus a DX_t invariant bundle of lines along γ, forming the principal or weak stable directions. The strong stable manifold extends to a strong stable foliation $\mathcal{F}^s(\gamma)$

of $W^s(\gamma)$. Leaves through a point $x \in W^s(\gamma)$ are denoted by $\mathcal{F}^s_x(\gamma)$. There are center unstable manifolds $W^{s,u}(\gamma)$ of dimension $n+1$ whose tangent space at γ is the direct sum of the unstable directions and the principal stable directions. The tangent bundle of $W^{s,u}(\gamma)$ along $W^u(\gamma)$ is a unique smooth bundle, see e.g. [Hom96]. By reversibility, $W^u(R(\gamma))$ contains a strong unstable manifold $W^{uu}(R(\gamma))$ of codimension one, and a strong unstable foliation $\mathcal{F}^u(R(\gamma))$. The image $R(W^{s,u}(\gamma))$ is a center stable manifold $W^{s,u}(R(\gamma))$ of $R(\gamma)$.

Assume there exists a heterodimensional cycle consisting of the periodic orbits $\gamma, R(\gamma)$, a symmetric heteroclinic orbit ρ^1 in $W^u(\gamma) \cap W^s(R(\gamma))$, and a symmetric heteroclinic orbit ρ^2 in $W^{ss}(\gamma) \cap W^{uu}(R(\gamma))$. The transversality condition (H 3) in Section 2 is replaced by analogous conditions

(H 4) $\qquad W^{s,u}(\gamma) \pitchfork_{\rho^1} W^s(R(\gamma)) \qquad W^{s,u}(R(\gamma)) \pitchfork_{\rho^1} W^u(\gamma),$

(H 5) $\qquad W^{ss}(\gamma) \pitchfork_{\rho^2} W^u(R(\gamma)), \qquad W^{uu}(R(\gamma)) \pitchfork_{\rho^2} W^s(\gamma).$

For both (H 4) and (H 5), the two conditions imply each other by reversibility.

Consider a small neighborhood of the periodic orbits $\gamma, R(\gamma)$ and the heteroclinic orbits ρ^1, ρ^2. Take small cross-sections Σ_0 and $\Sigma_2 = R(\Sigma_0)$ near γ and $R(\gamma)$ respectively. Take small symmetric cross-sections Σ_1, Σ_3 near $\rho^1 \cap \mathrm{Fix}(R)$ and $\rho^2 \cap \mathrm{Fix}(R)$ respectively. Consider the first return map Ψ on the union of these four cross-sections, following orbits only as long as they are near the heterodimensional cycle.

Associated to an orbit $x = \{x(i)\}$, $x(i+1) = \Psi(x(i))$ for $i \in \mathbb{Z}$, in the nonwandering set Ω of Ψ, there is an itinerary $\Upsilon(x) : \mathbb{Z} \to \{0,1,2,3\}$ defined by

$$\Upsilon(x)(i) = j, \text{ if } x(i) \in \Sigma_j.$$

Obvious restrictions exist for itineraries of orbits for Ψ, e.g. the symbol 1 is always followed by a 2. Let $\mathcal{B}$ be the subshift of finite type consisting of the subset of sequences $\mathbb{Z} \mapsto \{0,1,2,3\}$ with transition matrix $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, equipped with the product topology. As in Section 2, the dynamics of Π can be reduced to a skew product of interval maps.

Proposition 5.2. *For each $\eta \in \mathcal{B}$, there is a one dimensional center manifold W^c_η for Ψ, so that any orbit x with itinerary $\Upsilon(x) = \eta$, satisfies $x \in W^c_\eta$. The curve W^c_η is smooth and depends continuously on η. Moreover, $W^c_{\sigma(\eta)} = \Psi(W^c_\eta)$.*

The proof is analogous to the proof of Proposition 2.1 and is therefore not included. The transversality conditions (H 4) and (H 5), guarantee that appropriate coordinate systems (x_s, x_c, x_u) on the cross-sections can be chosen. The center manifolds are transverse intersections of center stable with center unstable manifolds. The center stable manifolds are foliated by stable manifolds of points in W^c_η. The proof of this result, formulated in the proposition below, follows the proof of Proposition 2.2. Analogously, center unstable manifolds are foliated by unstable manifolds.

Proposition 5.3. *For $\eta \in \mathcal{B}$, there is an invariant foliation $\mathcal{F}^s_\eta$ of $W^{s,c}_\eta$, whose leaves are stable manifolds of points in W^c_η. The foliations $\mathcal{F}^s_\eta$ depend continuously on η.*

As before we define symmetric itineraries which are related to symmetric orbits of the vector field. Define an involution $\mathcal{R}$ on $\mathcal{B}$ by

$$\mathcal{R}\eta(k) = \bar{\eta}(-k),$$

where $\bar{\eta}$ is obtained from η by changing every symbol 0 into the symbol 2 and vice versa. We call an itinerary η symmetric if there exists $s \in \mathbb{Z}$ such that

$$\mathcal{R}\eta = \sigma^s \eta.$$

We get the following description of the nonwandering set near the heterodimensional cycle.

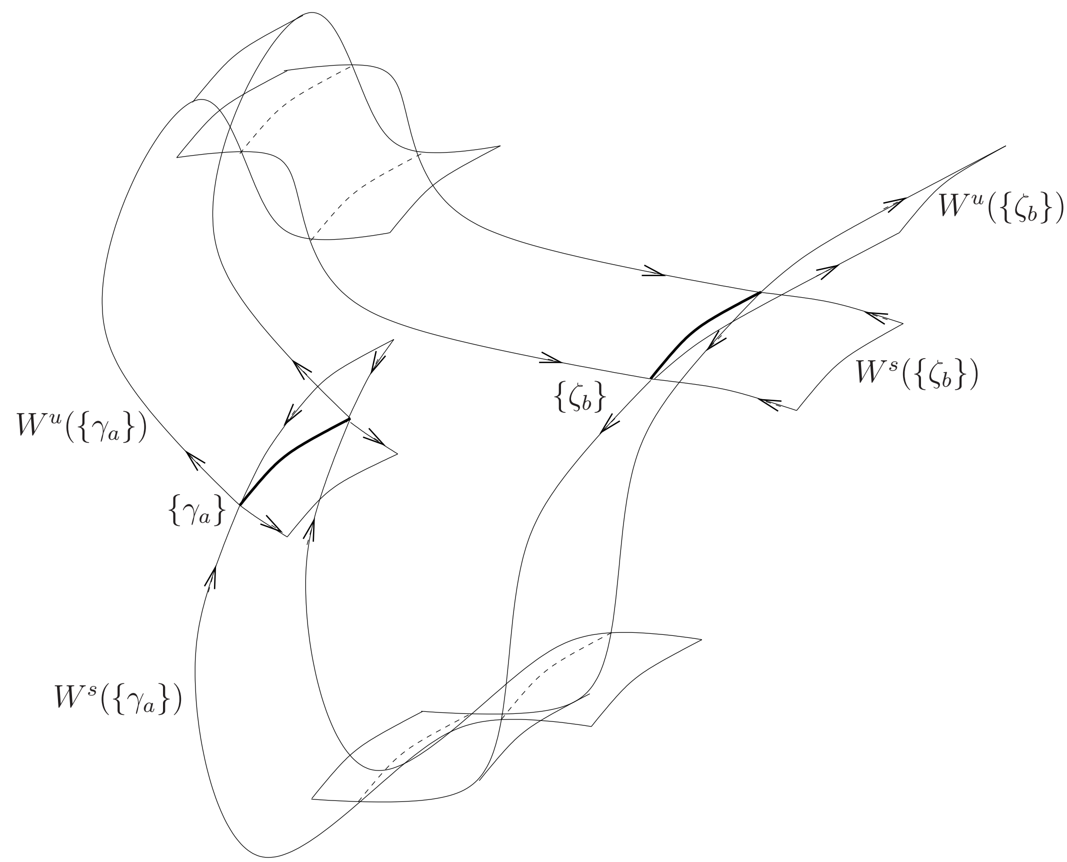

Figure 4: Heteroclinic tangles to families $\{\gamma_a\}, \{\zeta_b\}$ of symmetric periodic orbits. The picture indicates the manifolds for a return map on a global cross-section.

Theorem 5.4. *In any neighborhood of the heterodimensional cycle, there are sheets of symmetric periodic orbits, hyperbolic periodic orbits of index n and of index $n+1$, as well as nontrivial hyperbolic basic sets.*

Proof. We will demonstrate the existence of a homoclinic connection to γ. The existence of a hyperbolic basic set follows from this. The strong λ-lemma [Den89] implies that $W^u(\gamma)$ accumulates onto $W^{uu}(R(\gamma))$. Hence $W^u(\gamma)$ intersects $W^s(\gamma)$ transversally, the intersection being a homoclinic connection.

By reversibility, there is also a homoclinic connection to $R(\gamma)$ with a nearby hyperbolic basic set. Periodic orbits in hyperbolic basic sets Λ and $R(\Lambda)$ have different indices n and $n+1$.

Consider symmetric itineraries $\eta = (2^n 3 0^n 1)^\infty$. Observe that Ψ, restricted to a center manifold in Σ_2, expands distances between points. It follows that the iterate Ψ^n maps an interval I_n in W^c_η, which is exponentially small in n, onto $W^c(\sigma^n \eta)$. Iterating further, $\Psi^{2n+2}(I_n)$ is again an exponentially small interval in W^c_η. The arguments in the proof of Theorem 3.1 can be applied to show that I_n consists of periodic points. This proves the occurrence of sheets of symmetric periodic orbits arbitrarily close to the heterodimensional cycle. ∎

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. The statement on families of symmetric periodic orbits is contained in Theorem 3.1. The argument to establish density of saddle-node bifurcations is contained in the introduction to prove Corollary 1.4. Lemma 5.1 proves the dense occurrence of heterodimensional cycles. From this, by Theorem 5.4 the dense occurrence of nontrivial hyperbolic basic sets is obtained. ∎

6 Heteroclinic connections

In this section we consider heteroclinic connections. Strategy and results closely follow the above sections, so that we can be brief. Let $\{\gamma_a\}$ be a family of symmetric periodic orbits, as before. Consider, in addition to the family $\{\gamma_a\}$, a second family $\{\zeta_b\}$ of symmetric periodic orbits parameterized by a single parameter b. The families $\{\gamma_a\}$ and $\{\zeta_b\}$ can belong to the same global family of periodic orbits. We will assume that there exists a heteroclinic connection ρ_{a_0,b_0} from γ_{a_0} to ζ_{b_0}. The image under R yields a second heteroclinic connection from ζ_{b_0} to γ_{a_0}. Figure 4 gives an impression.

Assume transversality conditions

(H 6) $\qquad W^u(\gamma_{a_0}) \pitchfork_{\rho_{a_0,b_0}} W^s(\{\zeta_b\}), \qquad W^u(\{\gamma_a\}) \pitchfork_{\rho_{a_0,b_0}} W^s(\zeta_{b_0}),$

(H 7) $\qquad W^s(\gamma_{a_0}) \pitchfork_{R(\rho_{a_0,b_0})} W^u(\{\zeta_b\}), \qquad W^s(\{\gamma_a\}) \pitchfork_{R(\rho_{a_0,b_0})} W^u(\zeta_{b_0}).$

By reversibility, (H 6) and (H 7) follow from each other. These conditions replace Hypothesis (H 3) in Section 2. The immediate analogs of Propositions 2.1 and 2.2 hold true. We leave details to the reader.

Theorem 3.1 is true near heteroclinic tangles as well, for a suitable subshift of finite type. The arguments in the proof can be followed mutatis mutandis. Reasoning as in the proof of Proposition 1.5 shows that the dynamics near heteroclinic tangles is not C^1 structurally stable. The material on heterodimensional cycles can likewise be applied near heteroclinic tangles.

References

[Arn84] V.I. Arnol'd, Reversible systems, in: *Nonlinear and turbulent processes in physics, Vol. 3 (Kiev, 1983)* Harwood Academic Publ., 1984.

[Cha98] A.R. Champneys, Homoclinic orbits in reversible systems and their applications in mechanics, fluids and optics, *Phys. D* **112** (1998), 158–186.

[ChaHär00] A.R. Champneys, J. Härterich, Cascades of homoclinic orbits to a saddle-centre for reversible and perturbed Hamiltonian systems, *Dyn. Stab. Syst.* **15** (2000), 231–252.

[Den89] B. Deng, The Šil'nikov problem, exponential expansion, strong λ-lemma, C^1-linearization, and homoclinic bifurcation, *J. Differential Equations* **79** (1989), 189–231.

[Dev76a] R.L. Devaney, Reversible diffeomorphisms and flows, *Trans. Amer. Math. Soc.* **218** (1976), 89–113.

[Dev76b] R.L. Devaney, Homoclinic orbits in Hamiltonian systems, *J. Differential Equations* **21** (1976), 431–438.

[Dev77] R.L. Devaney, Blue sky catastrophes in reversible and Hamiltonian systems, *Indiana Univ. Math. Journal* **26** (1977), 247–263.

[DeV58] R. DeVogelaere, On the structure of symmetric periodic solutions of conservative systems, with applications, in: *Contributions to the theory of nonlinear oscillations, vol. IV*, Annals of Mathematics Studies no. 41, Princeton Univ. Press. 1958.

[Dia95] L.J. Díaz, Robust nonhyperbolic dynamics and heterodimensional cycles, *Ergodic Theory Dynam. Systems* **15** (1995), 291–315.

[GilVan87] S.A. van Gils, A. Vanderbauwhede, Center manifolds and contractions on a scale of Banach spaces, *J. of Functional Analysis* **72** (1987), 209–224.

[GonShi95] V.S. Gonchenko, L.P. Shil'nikov, On geometrical properties of two-dimensional diffeomorphisms with homoclinic tangencies, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **5** (1995), 819-829.

[GorIly00] A.S. Gorodetskiĭ, Yu. S. Il'yashenko, Some properties of skew products over a horseshoe and a solenoid, *Proc. Steklov Inst. Math.* **231** (2000), 90–112.

[Har98] J. Härterich, Cascades of reversible homoclinic orbits to a saddle-focus equilibrium, *Phys. D* **112** (1998), 187–200.

[HirPugShu77] M.W. Hirsch, C.C. Pugh, M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics **583**, Springer Verlag, 1977.

[Hom96] A.J. Homburg, *Global aspects of homoclinic bifurcations of vector fields*, Memoirs Amer. Math. Soc. **578**, 1996.

[HomKno06] A.J. Homburg, J. Knobloch, Multiple homoclinic orbits in conservative and reversible systems, *Transactions Amer. Math. Soc.* **358** (2006), 1715–1740.

[HomVilSan03] A.J. Homburg, R. de Vilder, D. Sands, Computing invariant sets, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **13** (2003), 497–504.

[Ioo98] G. Iooss, Travelling water-waves, as a paradigm for bifurcations in reversible infinite-dimensional "dynamical" systems. Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998). *Doc. Math.* (1998), Extra Vol. III, 611–622.

[Irw80a] M.C. Irwin, *Smooth dynamical systems*, Academic Press, 1980.

[Irw80b] M.C. Irwin, A new proof of the pseudo-stable manifold theorem, *J. London Math. Soc.* **21** (1980), 557–566.

[LamRob98] J.S.W. Lamb, J.A.G. Roberts, Time-reversal symmetry in dynamical systems: a survey, *Phys. D* **112** (1998), 1–39.

[New04] S.E. Newhouse, New phenomena associated with homoclinic tangencies, *Ergodic Theory Dynam. Systems* **24** (2004), 1725–1738.

[Pal00] J. Palis, A global view of dynamics and a conjecture on the denseness of finitude of attractors, *Astérisque* **261** (2000), 335–347.

[PalTak93] J. Palis, F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations. Fractal dimensions and infinitely many attractors*, Cambridge University Press, 1993.

[Per29] O. Perron, Über Stabilität und asymptotisches Verhalten der Lösungen eines Systems endlicher Differentialgleichungen, *J. Reine Angew. Math.* **161** (1929), 41–64.

[RobLam95] J.A.G. Roberts, J.S.W. Lamb, Self-similarity of period-doubling branching in 3-D reversible mappings, *Phys. D* **82** (1995), 317–332.

[Sev86] M.B. Sevryuk, *Reversible systems*, Lecture Notes in Mathematics **1211**, Springer-Verlag, 1986.

[Sma65] S. Smale, Diffeomorphisms with many periodic points, in: *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, Princeton Univ. Press, 1965.