Chapter 4 Techniques for calculating dimensions

A direct attempt at calculating the dimensions, in particular the Hausdorff dimension, of almost any set will convince the reader of the practical limitations of working from the definitions. Rigorous dimension calculations often involve pages of complicated manipulations and estimates that provide little intuitive enlightenment.

In this chapter we bring together some of the basic techniques that are available for dimension calculations. Other methods, that are applicable in more specific cases, will be found throughout the book.

4.1 Basic methods

As a general rule, we get upper bounds for Hausdorff measures and dimensions by finding effective coverings by small sets, and lower bounds by putting measures or mass distributions on the set. For most fractals 'obvious' upper estimates of dimension may be obtained using natural coverings by small sets.

Proposition 4.1

Suppose F can be covered by n_k sets of diameter at most δ_k with $\delta_k \to 0$ as $k \to \infty$. Then

$$\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \underline{\lim}_{k\to\infty} \frac{\log n_k}{-\log \delta_k}.$$

Moreover, if $n_k\delta_k^s$ remains bounded as $k \to \infty$, then $\mathcal{H}^s(F) < \infty$. If $\delta_k \to 0$ but $\delta_{k+1} \geqslant c\delta_k$ for some $0 < c < 1$, then

$$\overline{\dim}_{\mathrm{B}} F \leqslant \overline{\lim_{k\to\infty}} \frac{\log n_k}{-\log \delta_k}.$$

Proof. The inequalities for the box-counting dimension are immediate from the definitions and the remark at (3.14). That $\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F$ is in (3.17), and if

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$n_k\delta_k^s$ is bounded then $\mathcal{H}^s_{\delta_k}(F) \leqslant n_k\delta_k^s$, so $\mathcal{H}^s_{\delta_k}(F)$ tends to a finite limit $\mathcal{H}^s(F)$ as $k \to \infty$. □

Thus, as we have seen already (Example 2.7), in the case of the middle third Cantor set the natural coverings by 2^k intervals of length 3^{-k} give $\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F \leqslant \log 2/\log 3$.

Surprisingly often, the 'obvious' upper bound for the Hausdorff dimension of a set turns out to be the actual value. However, demonstrating this can be difficult. To obtain an upper bound it is enough to evaluate sums of the form $\sum |U_i|^s$ for *specific* coverings $\{U_i\}$ of F, whereas for a lower bound we must show that $\sum |U_i|^s$ is greater than some positive constant for *all* δ-coverings of F. Clearly an enormous number of such coverings are available. In particular, when working with Hausdorff dimension as opposed to box dimension, consideration must be given to covers where some of the U_i are very small and others have relatively large diameter—this prohibits sweeping estimates for $\sum |U_i|^s$ such as those available for upper bounds.

One way of getting around these difficulties is to show that no *individual* set U can cover too much of F compared with its size measured as $|U|^s$. Then if $\{U_i\}$ covers the whole of F the sum $\sum |U_i|^s$ cannot be too small. The usual way to do this is to concentrate a suitable mass distribution μ on F and compare the mass $\mu(U)$ covered by U with $|U|^s$ for each U. (Recall that a mass distribution on F is a measure with support contained in F such that $0 < \mu(F) < \infty$, see Section 1.3.)

Mass distribution principle 4.2

Let μ be a mass distribution on F and suppose that for some s there are numbers $c > 0$ and $\varepsilon > 0$ such that

$$\mu(U) \leqslant c|U|^s \tag{4.1}$$

for all sets U with $|U| \leqslant \varepsilon$. Then $\mathcal{H}^s(F) \geqslant \mu(F)/c$ and

$$s \leqslant \dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F.$$

Proof. If $\{U_i\}$ is any cover of F then

$$0 < \mu(F) \leqslant \mu\left(\bigcup_i U_i\right) \leqslant \sum_i \mu(U_i) \leqslant c \sum_i |U_i|^s \tag{4.2}$$

using properties of a measure and (4.1).

Taking infima, $\mathcal{H}^s_\delta(F) \geqslant \mu(F)/c$ if δ is small enough, so $\mathcal{H}^s(F) \geqslant \mu(F)/c$. Since $\mu(F) > 0$ we get $\dim_{\mathrm{H}} F \geqslant s$. □

Notice that the conclusion $\mathcal{H}^s(F) \geqslant \mu(F)/c$ remains true if μ is a mass distribution on $\mathbb{R}^n$ and F is any subset.

The Mass distribution principle 4.2 gives a quick lower estimate for the Hausdorff dimension of the middle third Cantor set F (figure 0.1). Let μ be the natural mass distribution on F, so that each of the 2^k kth level intervals of length 3^{-k} in E_k in the construction of F carry a mass 2^{-k}. (We imagine that we start with unit mass on E_0 and repeatedly divide the mass on each interval of E_k between its two subintervals in E_{k+1}; see Proposition 1.7.) Let U be a set with $|U| < 1$ and let k be the integer such that $3^{-(k+1)} \leqslant |U| < 3^{-k}$. Then U can intersect at most one of the intervals of E_k, so

$$\mu(U) \leqslant 2^{-k} = (3^{\log 2/\log 3})^{-k} = (3^{-k})^{\log 2/\log 3} \leqslant (3|U|)^{\log 2/\log 3}$$

and hence $\mathcal{H}^{\log 2/\log 3}(F) \geqslant 3^{-\log 2/\log 3} = \frac{1}{2}$ by the mass distribution principle, giving $\dim_{\mathrm{H}} F \geqslant \log 2/\log 3$.

Example 4.3

Let $F_1 = F \times [0, 1] \subset \mathbb{R}^2$ *be the product of the middle third Cantor set F and the unit interval. Then, setting* $s = 1 + \log 2/\log 3$, *we have* $\dim_{\mathrm{B}} F_1 = \dim_{\mathrm{H}} F_1 = s$, *with* $0 < \mathcal{H}^s(F_1) < \infty$.

Calculation. For each k, there is a covering of F by 2^k intervals of length 3^{-k}. A column of 3^k squares of side 3^{-k} (diameter $3^{-k}\sqrt{2}$) covers the part of F_1 above each such interval, so taking these all together, F_1 may be covered by $2^k 3^k$ squares of side 3^{-k}. Thus $\mathcal{H}^s_{3^{-k}\sqrt{2}}(F_1) \leqslant 3^k 2^k (3^{-k}\sqrt{2})^s = (3 \cdot 2 \cdot 3^{-1-\log 2/\log 3})^k 2^{s/2} = 2^{s/2}$, so $\mathcal{H}^s(F_1) \leqslant 2^{s/2}$ and $\dim_{\mathrm{H}} F_1 \leqslant \underline{\dim}_{\mathrm{B}} F_1 \leqslant \overline{\dim}_{\mathrm{B}} F_1 \leqslant s$.

We define a mass distribution μ on F_1 by taking the natural mass distribution on F described above (each kth level interval of F of side 3^{-k} having mass 2^{-k}) and 'spreading it' uniformly along the intervals above F. Thus if U is a rectangle, with sides parallel to the coordinate axes, of height $h \leqslant 1$, above a kth level interval of F, then $\mu(U) = h2^{-k}$. Any set U is contained in a square of side $|U|$ with sides parallel to the coordinate axes. If $3^{-(k+1)} \leqslant |U| < 3^{-k}$ then U lies above at most one kth level interval of F of side 3^{-k}, so

$$\mu(U) \leqslant |U| 2^{-k} \leqslant |U| 3^{-k \log 2/\log 3} \leqslant |U|(3|U|)^{\log 2/\log 3} = 3^{\log 2/\log 3}|U|^s = 2|U|^s.$$

By the Mass distribution principle 4.2, $\mathcal{H}^s(F_1) > \frac{1}{2}$. □

Note that the method of Examples 4.2 and 4.3 extends to a wide variety of self-similar sets. Indeed, Theorem 9.3 may be regarded as a generalization of this calculation.

Notice that in this example the dimension of the product of two sets equals the sum of the dimensions of the sets. We study this in greater depth in Chapter 7.

The following *general construction* of a subset of $\mathbb{R}$ may be thought of as a generalization of the Cantor set construction. Let $[0, 1] = E_0 \supset E_1 \supset E_2 \supset \dots$ be a decreasing sequence of sets, with each E_k a union of a finite number of disjoint closed intervals (called kth *level basic intervals*), with each interval of E_k containing at least two intervals of E_{k+1}, and the maximum length of kth level intervals tending to 0 as $k \to \infty$. Then the set

$$F = \bigcap_{k=0}^{\infty} E_k \tag{4.3}$$

is a totally disconnected subset of [0, 1] which is generally a fractal (figure 4.1).

Obvious upper bounds for the dimension of F are available by taking the intervals of E_k as covering intervals, for each k, but, as usual, lower bounds are harder to find. Note that, in the following examples, the upper estimates for $\dim_{\mathrm{H}} F$ depend on the number and size of the basic intervals, whilst the lower estimates depend on their spacing. For these to be equal, the $(k+1)$th level intervals must be 'nearly uniformly distributed' inside the kth level intervals.

Example 4.4

Let s be a number strictly between 0 *and* 1. *Assume that in the general construction* (4.3) *for each kth level interval I, the* $(k+1)$*th level intervals* $I_1, \dots, I_m$ $(m \geqslant 2)$ *contained in I are of equal length and equally spaced, the lengths being given by*

$$|I_i|^s = \frac{1}{m}|I|^s \qquad (1 \leqslant i \leqslant m) \tag{4.4}$$

with the left-hand ends of I_1 *and I coinciding, and the right-hand ends of* I_m *and I coinciding. Then* $\dim_{\mathrm{H}} F = s$ *and* $0 < \mathcal{H}^s(F) < \infty$. (*Notice that m may be different for different intervals I in the construction, so that the kth level intervals may have widely differing lengths.*)

Calculation. With I, I_i, as above,

$$|I|^s = \sum_{i=1}^{m} |I_i|^s. \tag{4.5}$$

Applying this inductively to the kth level intervals for successive k, we have, for each k, that $1 = \sum |I_i|^s$, where the sum is over all the kth level intervals I_i.

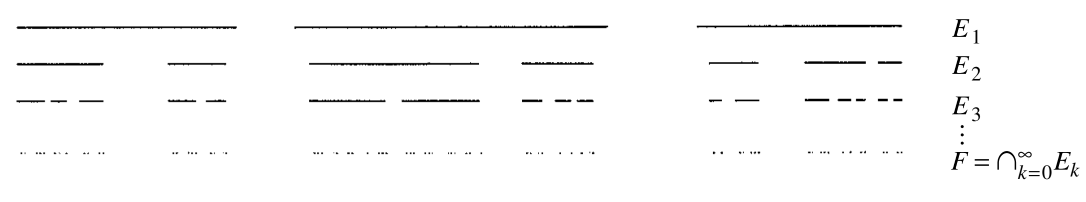

Figure 4.1 An example of the general construction of a subset of $\mathbb{R}$

The kth level intervals cover F; since the maximum interval length tends to 0 as $k \to \infty$, we have $\mathcal{H}^s_\delta(F) \leqslant 1$ for sufficiently small δ, giving $\mathcal{H}^s(F) \leqslant 1$.

Now distribute a mass μ on F in such a way that $\mu(I) = |I|^s$ whenever I is any level k interval. Thus, starting with unit mass on $[0, 1]$ we divide this equally between each level 1 interval, the mass on each of these intervals being divided equally between each level 2 subinterval, and so on; see Proposition 1.7. Equation (4.5) ensures that we get a mass distribution on F with $\mu(I) = |I|^s$ for every basic interval. We estimate $\mu(U)$ for an interval U with endpoints in F. Let I be the smallest basic interval that contains U; suppose that I is a kth level interval, and let $I_i, \dots, I_m$ be the $(k + 1)$th level intervals contained in I. Then U intersects a number $j \geqslant 2$ of the I_i, otherwise U would be contained in a smaller basic interval. The spacing between consecutive I_i is

$$\begin{aligned}(|I| - m|I_i|)/(m-1) &= |I|(1 - m|I_i|/|I|)/(m-1)\\ &= |I|(1 - m^{1-1/s})/(m-1)\\ &\geqslant c_s|I|/m\end{aligned}$$

using (4.4) and that $m \geqslant 2$ and $0 < s < 1$, where $c_s = (1 - 2^{1-1/s})$. Thus

$$|U| \geqslant \frac{j-1}{m}c_s|I| \geqslant \frac{j}{2m}c_s|I|.$$

By (4.4)

$$\begin{aligned}\mu(U) &\leqslant j\mu(I_i) = j|I_i|^s = \frac{j}{m}|I|^s\\ &\leqslant 2^s c_s^{-s}\left(\frac{j}{m}\right)^{1-s}|U|^s \leqslant 2^s c_s^{-s}|U|^s. \qquad (4.6)\end{aligned}$$

This is true for any interval U with endpoints in F, and so for any set U (by applying (4.6) to the smallest interval containing $U \cap F$). By the Mass distribution principle 4.2, $\mathcal{H}^s(F) > 0$. □

A more careful estimate of $\mu(U)$ in Example 4.4 leads to $\mathcal{H}^s(F) = 1$.

We call the sets obtained when m is kept constant throughout the construction of Example 4.4 *uniform Cantor sets*; see figure 4.2. These provide a natural generalization of the middle third Cantor set.

Example 4.5. Uniform Cantor sets

Let $m \geqslant 2$ *be an integer and* $0 < r < 1/m$. *Let* F *be the set obtained by the construction in which each basic interval I is replaced by m equally spaced subintervals of lengths* $r|I|$, *the ends of I coinciding with the ends of the extreme subintervals. Then* $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \log m/-\log r$, *and* $0 < \mathcal{H}^{\log m/-\log r}(F) < \infty$.

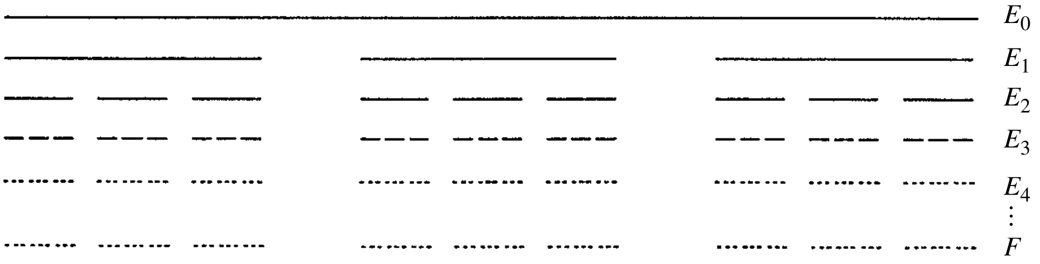

Figure 4.2 A uniform Cantor set (Example 4.5) with $m = 3, r = \frac{4}{15}$, $\dim_H F = \dim_B F = \log 3/-\log \frac{4}{15} = 0.831\ldots$

Calculation. The set F is obtained on taking m constant and $s = \log m/(-\log r)$ in Example 4.4. Equation (4.4) becomes $(r|I|)^s = (1/m)|I|^s$, which is satisfied identically, so $\dim_H F = s$. For the box dimension, note that F is covered by the m^k kth level intervals of length r^{-k} for each k, leading to $\overline{\dim}_B F \leqslant \log m/-\log r$ in the usual way. □

The *middle λ Cantor set* is obtained by repeatedly removing a proportion $0 < \lambda < 1$ from the middle of intervals, starting with the unit interval. This is a special case of a uniform Cantor set, having $m = 2$ and $r = \frac{1}{2}(1-\lambda)$ and thus Hausdorff and box dimensions $\log 2/\log(2/(1-\lambda))$.

The next example is another case of the general construction.

Example 4.6

Suppose in the general construction (4.3) *each* $(k-1)$*th level interval contains at least* $m_k \geqslant 2$ k*th level intervals* $(k = 1, 2, \ldots)$ *which are separated by gaps of at least* ε_k*, where* $0 < \varepsilon_{k+1} < \varepsilon_k$ *for each k. Then*

$$\dim_H F \geqslant \varliminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k)}. \tag{4.7}$$

Calculation. We may assume that the right hand side of (4.7) is positive, otherwise (4.7) is obvious. We may assume that each $(k-1)$th level interval contains exactly m_k kth level intervals; if not we may throw out excess intervals to get smaller sets E_k and F for which this is so. We may define a mass distribution μ on F by assigning a mass of $(m_1 \cdots m_k)^{-1}$ to each of the $m_1 \cdots m_k$ kth level intervals.

Let U be an interval with $0 < |U| < \varepsilon_1$; we estimate $\mu(U)$. Let k be the integer such that $\varepsilon_k \leqslant |U| < \varepsilon_{k-1}$. The number of kth level intervals that intersect U is

(i) at most m_k since U intersects at most one $(k-1)$th level interval
(ii) at most $(|U|/\varepsilon_k) + 1 \leqslant 2|U|/\varepsilon_k$ since the kth level intervals have gaps of at least ε_k between them.

Each kth level interval supports mass $(m_1 \cdots m_k)^{-1}$ so that

$$\begin{aligned}\mu(U) &\leqslant (m_1 \cdots m_k)^{-1} \min\{2|U|/\varepsilon_k, m_k\}\\ &\leqslant (m_1 \cdots m_k)^{-1}(2|U|/\varepsilon_k)^s m_k^{1-s}\end{aligned}$$

for every $0 \leqslant s \leqslant 1$.

Hence

$$\frac{\mu(U)}{|U|^s} \leqslant \frac{2^s}{(m_1 \cdots m_{k-1}) m_k^s \varepsilon_k^s}.$$

If

$$s < \varliminf_{k\to\infty} \log(m_1 \cdots m_{k-1})/-\log(m_k\varepsilon_k)$$

then $(m_1 \cdots m_{k-1}) m_k^s \varepsilon_k^s > 1$ for large k, so $\mu(U) \leqslant 2^s|U|^s$, and $\dim_{\mathrm{H}} F \geqslant s$ by Principle 4.2, giving (4.7). □

Now suppose that in Example 4.6 the kth level intervals are all of length δ_k, and that each $(k-1)$th level interval contains exactly m_k kth level intervals, which are ‘roughly equally spaced’ in the sense that $m_k\varepsilon_k \geqslant c\delta_{k-1}$, where $c > 0$ is a constant. Then (4.7) becomes

$$\dim_{\mathrm{H}} F \geqslant \varliminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log c - \log \delta_{k-1}} = \varliminf_{k\to\infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log \delta_{k-1}}.$$

But E_{k-1} comprises $m_1 \cdots m_{k-1}$ intervals of length δ_{k-1}, so this expression equals the upper bound for $\dim_{\mathrm{H}} F$ given by Proposition 4.1. Thus in the situation where the intervals are well spaced, we get equality in (4.7).

Examples of the following form occur in number theory; see Section 10.3.

Example 4.7

Fix $0 < s < 1$ *and let* $n_0, n_1, n_2, \ldots$ *be a rapidly increasing sequence of integers, say* $n_{k+1} \geqslant \max\{n_k^k, 4n_k^{1/s}\}$ *for each k. For each k let* $H_k \subset \mathbb{R}$ *consist of equally spaced equal intervals of lengths* $n_k^{-1/s}$ *with the midpoints of consecutive intervals distance* n_k^{-1} *apart. Then writing* $F = \bigcap_{k=1}^{\infty} H_k$, *we have* $\dim_{\mathrm{H}} F = s$.

Calculation. Since $F \subset H_k$ for each k, the set $F \cap [0,1]$ is contained in at most $n_k + 1$ intervals of length $n_k^{-1/s}$, so Proposition 4.1 gives $\dim_{\mathrm{H}}(F \cap [0,1]) \leqslant \varliminf_{k\to\infty} \log(n_k+1)/-\log n_k^{-1/s} = s$. Similarly, $\dim_{\mathrm{H}}(F \cap [n, n+1]) \leqslant s$ for all $n \in \mathbb{Z}$, so $\dim_{\mathrm{H}} F \leqslant s$ as a countable union of such sets.

Now let $E_0 = [0,1]$ and, for $k \geqslant 1$, let E_k consist of the intervals of H_k that are completely contained in E_{k-1}. Then each interval I of E_{k-1} contains at least $n_k|I| - 2 \geqslant n_k n_{k-1}^{-1/s} - 2 \geqslant 2$ intervals of E_k, which are separated by gaps of at least $n_k^{-1} - n_k^{-1/s} \geqslant \frac{1}{2}n_k^{-1}$ if k is large enough. Using Example 4.6, and noting that setting $m_k = n_k n_{k-1}^{-1/s}$ rather than $m_k = n_k n_{k-1}^{-1/s} - 2$ does not affect the limit,

$$\begin{aligned}\dim_{\mathrm{H}}(F \cap [0,1]) \geqslant \dim_{\mathrm{H}} \bigcap_{k=1}^{\infty} E_k &\geqslant \underline{\lim}_{k\to\infty} \frac{\log((n_1 \cdots n_{k-2})^{1-1/s} n_{k-1})}{-\log(n_k n_{k-1}^{-1/s} \frac{1}{2} n_k^{-1})} \\ &= \underline{\lim}_{k\to\infty} \frac{\log(n_1 \cdots n_{k-2})^{1-1/s} + \log n_{k-1}}{\log 2 + (\log n_{k-1})/s}.\end{aligned}$$

Provided that n_k is sufficiently rapidly increasing, the terms in $\log n_{k-1}$ in the numerator and denominator of this expression are dominant, so that $\dim_{\mathrm{H}} F \geqslant \dim_{\mathrm{H}}(F \cap [0,1]) \geqslant s$, as required. □

Although the Mass distribution principle 4.2 is based on a simple idea, we have seen that it can be very useful in finding Hausdorff and box dimensions. We now develop some important variations of the method.

It is enough for condition (4.1) to hold just for sufficiently small balls centred at each point of F. This is expressed in Proposition 4.9(a). Although mass distribution methods for upper bounds are required far less frequently, we include part (b) because it is, in a sense, dual to (a). Note that density expressions, such as $\lim_{r\to 0} \mu(B(x,r))/r^s$ play a major role in the study of local properties of fractals—see Chapter 5. (Recall that $B(x,r)$ is the closed ball of centre x and radius r.)

We require the following covering lemma in the proof of Proposition 4.9(b).

Covering lemma 4.8

Let $\mathcal{C}$ be a family of balls contained in some bounded region of $\mathbb{R}^n$. Then there is a (finite or countable) disjoint subcollection $\{B_i\}$ such that

$$\bigcup_{B\in\mathcal{C}} B \subset \bigcup_i \tilde{B}_i \tag{4.8}$$

where $\tilde{B}_i$ is the closed ball concentric with B_i and of four times the radius.

Proof. For simplicity, we give the proof when $\mathcal{C}$ is a finite family; the basic idea is the same in the general case. We select the $\{B_i\}$ inductively. Let B_1 be a ball in $\mathcal{C}$ of maximum radius. Suppose that $B_1, \ldots, B_{k-1}$ have been chosen. We take B_k to be the largest ball in $\mathcal{C}$ (or one of the largest) that does not intersect $B_1, \ldots, B_{k-1}$. The process terminates when no such ball remains. Clearly the balls selected are disjoint; we must check that (4.8) holds. If $B \in \mathcal{C}$, then either $B = B_i$ for some i, or B intersects one of the B_i with $|B_i| \geqslant |B|$; if this were not the case, then B would have been chosen instead of the first ball B_k with $|B_k| < |B|$. Either way, $B \subset \tilde{B}_i$, so we have (4.8). (It is easy to see that the result remains true taking $\tilde{B}_i$ as the ball concentric with B_i and of $3+\varepsilon$ times the radius, for any $\varepsilon > 0$; if $\mathcal{C}$ is finite we may take $\varepsilon = 0$.) □

Proposition 4.9

Let μ be a mass distribution on $\mathbb{R}^n$, let $F \subset \mathbb{R}^n$ be a Borel set and let $0 < c < \infty$ be a constant.

(*a*) *If $\overline{\lim}_{r\to 0}\mu(B(x,r))/r^s < c$ for all $x \in F$ then $\mathcal{H}^s(F) \geqslant \mu(F)/c$*
(*b*) *If $\overline{\lim}_{r\to 0}\mu(B(x,r))/r^s > c$ for all $x \in F$ then $\mathcal{H}^s(F) \leqslant 2^s\mu(\mathbb{R}^n)/c$.*

Proof

(*a*) For each $\delta > 0$ let

$$F_\delta = \{x \in F : \mu(B(x,r)) < cr^s \text{ for all } 0 < r \leqslant \delta\}.$$

Let $\{U_i\}$ be a δ-cover of F and thus of F_δ. For each U_i containing a point x of F_δ, the ball B with centre x and radius $|U_i|$ certainly contains U_i. By definition of F_δ,

$$\mu(U_i) \leqslant \mu(B) < c|U_i|^s$$

so that

$$\mu(F_\delta) \leqslant \sum_i \{\mu(U_i) : U_i \text{ intersects } F_\delta\} \leqslant c\sum_i |U_i|^s.$$

Since $\{U_i\}$ is any δ-cover of F, it follows that $\mu(F_\delta) \leqslant c\mathcal{H}^s_\delta(F) \leqslant c\mathcal{H}^s(F)$. But F_δ increases to F as δ decreases to 0, so $\mu(F) \leqslant c\mathcal{H}^s(F)$ by (1.7).

(*b*) For simplicity, we prove a weaker version of (*b*) with 2^s replaced by 8^s, but the basic idea is similar. Suppose first that F is bounded. Fix $\delta > 0$ and let $\mathcal{C}$ be the collection of balls

$$\{B(x,r) : x \in F, 0 < r \leqslant \delta \text{ and } \mu(B(x,r)) > cr^s\}.$$

Then by the hypothesis of (*b*) $F \subset \bigcup_{B\in\mathcal{C}} B$. Applying the Covering lemma 4.8 to the collection $\mathcal{C}$, there is a sequence of disjoint balls $B_i \in \mathcal{C}$ such that $\bigcup_{B\in\mathcal{C}} B \subset \bigcup_i \tilde{B}_i$ where $\tilde{B}_i$ is the ball concentric with B_i but of four times the radius. Thus $\{\tilde{B}_i\}$ is an 8δ-cover of F, so

$$\begin{aligned}\mathcal{H}^s_{8\delta}(F) &\leqslant \sum_i |\tilde{B}_i|^s \leqslant 4^s \sum_i |B_i|^s \\ &\leqslant 8^s c^{-1} \sum_i \mu(B_i) \leqslant 8^s c^{-1}\mu(\mathbb{R}^n).\end{aligned}$$

Letting $\delta \to 0$, we get $\mathcal{H}^s(F) \leqslant 8^s c^{-1}\mu(\mathbb{R}^n) < \infty$. Finally, if F is unbounded and $\mathcal{H}^s(F) > 8^s c^{-1}\mu(\mathbb{R}^n)$, the $\mathcal{H}^s$-measure of some bounded subset of F will also exceed this value, contrary to the above. □

Note that it is immediate from Proposition 4.9 that if $\lim_{r\to 0}\log\mu(B(x,r))/\log r = s$ for all $x \in F$ then $\dim_{\mathrm{H}} F = s$.

Applications of Proposition 4.9 will occur throughout the book.

We conclude this section by a reminder that these calculations can be used in conjunction with the basic properties of dimensions discussed in Chapters 2 and 3. For example, since $f(x) = x^2$ is Lipschitz on $[0, 1]$ and bi-Lipschitz on $[\frac{2}{3}, 1]$, it follows that $\dim_{\mathrm{H}}\{x^2 : x \in C\} = \dim_{\mathrm{H}} f(C) = \log 2/\log 3$, where C is the middle third Cantor set.

4.2 Subsets of finite measure

This section may seem out of place in a chapter about finding dimensions. However, Theorem 4.10 is required for the important potential theoretic methods developed in the following section. Sets of infinite measure can be awkward to work with, and reducing them to sets of positive finite measure can be a very useful simplification.

Theorem 4.10 guarantees that any (Borel) set F with $\mathcal{H}^s(F) = \infty$ contains a subset E with $0 < \mathcal{H}^s(E) < \infty$ (i.e. with E an s-set). At first, this might seem obvious—just shave pieces off F until what remains has positive finite measure. Unfortunately it is not quite this simple—it is possible to jump from infinite measure to zero measure without passing through any intermediate value. Stating this in mathematical terms, it is possible to have a decreasing sequence of sets $E_1 \supset E_2 \supset \dots$ with $\mathcal{H}^s(E_k) = \infty$ for all k, but with $\mathcal{H}^s(\bigcap_{k=1}^{\infty} E_k) = 0$. (For a simple example, take $E_k = [0, 1/k] \subset \mathbb{R}$ and $0 < s < 1$.) To prove the theorem we need to look rather more closely at the structure of Hausdorff measures. Readers mainly concerned with applications may prefer to omit the proof!

Theorem 4.10

Let F be a Borel subset of $\mathbb{R}^n$ with $0 < \mathcal{H}^s(F) \leqslant \infty$. Then there is a compact set $E \subset F$ such that $0 < \mathcal{H}^s(E) < \infty$.

****Sketch of proof.*** The complete proof of this is complicated. We indicate the ideas involved in the case where F is a compact subset of $[0, 1) \subset \mathbb{R}$ and $0 < s < 1$.

We work with the net measures $\mathcal{M}^s$ which are defined in (2.17)–(2.18) using the binary intervals $[r2^{-k}, (r+1)2^{-k})$ and are related to Hausdorff measure by (2.19). We define inductively a decreasing sequence $E_0 \supset E_1 \supset E_2 \supset \dots$ of compact subsets of F. Let $E_0 = F$. For $k \geqslant 0$ we define E_{k+1} by specifying its intersection with each binary interval I of length 2^{-k}. If $\mathcal{M}^s_{2^{-(k+1)}}(E_k \cap I) \leqslant 2^{-sk}$ we let $E_{k+1} \cap I = E_k \cap I$. Then

$$\mathcal{M}^s_{2^{-(k+1)}}(E_{k+1} \cap I) = \mathcal{M}^s_{2^{-k}}(E_k \cap I) \tag{4.9}$$

since using I itself as a covering interval in calculating $\mathcal{M}^s_{2^{-k}}$ gives an estimate at least as large as using shorter binary intervals. On the other hand, if $\mathcal{M}^s_{2^{-(k+1)}}(E_{k+1} \cap I) > 2^{-sk}$ we take $E_{k+1} \cap I$ to be a compact subset of $E_k \cap I$ with $\mathcal{M}^s_{2^{-(k+1)}}(E_{k+1} \cap I) = 2^{-sk}$. Such a subset exists since $\mathcal{M}^s_{2^{-(k+1)}}(E_k \cap I \cap [0, u])$ is finite and continuous in u. (This is why we need to work with the $\mathcal{M}^s_\delta$ rather than $\mathcal{M}^s$.) Since $\mathcal{M}^s_{2^{-k}}(E_k \cap I) = 2^{-sk}$, (4.9) again holds. Summing (4.9) over all binary intervals of length 2^{-k} we get

$$\mathcal{M}^s_{2^{-(k+1)}}(E_{k+1}) = \mathcal{M}^s_{2^{-k}}(E_k). \tag{4.10}$$

Repeated application of (4.10) gives $\mathcal{M}^s_{2^{-k}}(E_k) = \mathcal{M}^s_1(E_0)$ for all k. Let E be the compact set $\bigcap_{k=0}^{\infty} E_k$. Taking the limit as $k \to \infty$ gives $\mathcal{M}^s(E) = \mathcal{M}^s_1(E_0)$ (this step needs some justification). The covering of $E_0 = F$ by the single interval $[0, 1)$ gives $\mathcal{M}^s(E) = \mathcal{M}^s_1(E_0) \leqslant 1$. Since $\mathcal{M}^s(E_0) \geqslant \mathcal{H}^s(E_0) > 0$ we have $\mathcal{M}^s_{2^{-k}}(E_0) > 0$ if k is large enough. Thus either $\mathcal{M}^s(E) = \mathcal{M}^s_1(E_0) \geqslant 2^{-ks}$, or $\mathcal{M}^s_1(E_0) < 2^{-ks}$ in which case $\mathcal{M}^s(E) = \mathcal{M}^s_1(E_0) = \mathcal{M}^s_{2^{-k}}(E_0) > 0$. Thus $0 < \mathcal{M}^s(E) < \infty$, and the theorem follows from (2.19). □

A number of results, for example those in Chapter 5, apply only to s-sets, i.e. sets with $0 < \mathcal{H}^s(F) < \infty$. One way of approaching s-dimensional sets with $\mathcal{H}^s(F) = \infty$ is to use Theorem 4.10 to extract a subset of positive finite measure, to study its properties as an s-set, and then to interpret these properties in the context of the larger set F. Similarly, if $0 < s < t$, any set F of Hausdorff dimension t has $\mathcal{H}^s(F) = \infty$ and so contains an s-set.

The following proposition which follows from Proposition 4.9, leads to an extension of Theorem 4.10.

Proposition 4.11

Let F be a Borel set satisfying $0 < \mathcal{H}^s(F) < \infty$. There is a constant b and a compact set $E \subset F$ with $\mathcal{H}^s(E) > 0$ such that

$$\mathcal{H}^s(E \cap B(x, r)) \leqslant br^s \tag{4.11}$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

Proof. In Proposition 4.9(b) take μ as the restriction of $\mathcal{H}^s$ to F, i.e. $\mu(A) = \mathcal{H}^s(F \cap A)$. Then, if

$$F_1 = \left\{ x \in \mathbb{R}^n : \overline{\lim_{r\to 0}}\, \mathcal{H}^s(F \cap B(x, r))/r^s > 2^{1+s} \right\}$$

it follows that $\mathcal{H}^s(F_1) \leqslant 2^s 2^{-(1+s)} \mu(F) = \frac{1}{2}\mathcal{H}^s(F)$. Thus $\mathcal{H}^s(F \backslash F_1) \geqslant \frac{1}{2}\mathcal{H}^s(F) > 0$, so if $E_1 = F \backslash F_1$ then $\mathcal{H}^s(E_1) > 0$ and $\overline{\lim}_{r\to 0} \mathcal{H}^s(F \cap B(x, r))/r^s \leqslant 2^{1+s}$ for $x \in E_1$. By Egoroff's theorem (see also Section 1.3) it follows that there is a compact set $E \subset E_1$ with $\mathcal{H}^s(E) > 0$ and a number $r_0 > 0$ such that

$\mathcal{H}^s(F \cap B(x,r))/r^s \leqslant 2^{2+s}$ for all $x \in E$ and all $0 < r \leqslant r_0$. However, we have that $\mathcal{H}^s(F \cap B(x,r))/r^s \leqslant \mathcal{H}^s(F)/r_0^s$ if $r \geqslant r_0$ so (4.11) holds for all $r > 0$. □

Corollary 4.12

Let F be a Borel subset of $\mathbb{R}^n$ *with* $0 < \mathcal{H}^s(F) \leqslant \infty$. *Then there is a compact set* $E \subset F$ *such that* $0 < \mathcal{H}^s(E) < \infty$ *and a constant b such that*

$$\mathcal{H}^s(E \cap B(x,r)) \leqslant br^s$$

for all $x \in \mathbb{R}^n$ *and* $r > 0$.

Proof. Theorem 4.10 provides us with a compact subset F_1 of F of positive finite measure, and applying Proposition 4.11 to F_1 gives the result. □

Corollary 4.12, which may be regarded as a converse of the Mass distribution principle 4.2, is often called ‘Frostman’s lemma’.

4.3 Potential theoretic methods

In this section we introduce a technique for calculating Hausdorff dimensions that is widely used both in theory and in practice. This replaces the need for estimating the mass of a large number of small sets, as in the Mass distribution principle, by a single check for the convergence of a certain integral.

The ideas of potential and energy will be familiar to readers with a knowledge of gravitation or electrostatics. For $s \geqslant 0$ the *s-potential* at a point x of $\mathbb{R}^n$ due to the mass distribution μ on $\mathbb{R}^n$ is defined as

$$\phi_s(x) = \int \frac{\mathrm{d}\mu(y)}{|x-y|^s}. \tag{4.12}$$

(If we are working in $\mathbb{R}^3$ and $s = 1$ then this is essentially the familiar Newtonian gravitational potential.) The *s-energy* of μ is

$$I_s(\mu) = \int \phi_s(x)\mathrm{d}\mu(x) = \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^s}. \tag{4.13}$$

The following theorem relates Hausdorff dimension to seemingly unconnected potential theoretic ideas. Particularly useful is part (*a*): if there is a mass distribution on a set F which has finite s-energy, then F has dimension at least s.

Theorem 4.13

Let F be a subset of $\mathbb{R}^n$.

(*a*) *If there is a mass distribution* μ *on F with* $I_s(\mu) < \infty$ *then* $\mathcal{H}^s(F) = \infty$ *and* $\dim_{\mathrm{H}} F \geqslant s$.

(b) *If F is a Borel set with $\mathcal{H}^s(F) > 0$ then there exists a mass distribution μ on F with $I_t(\mu) < \infty$ for all $0 < t < s$.*

Proof

(a) Suppose that $I_s(\mu) < \infty$ for some mass distribution μ with support contained in F. Define

$$F_1 = \left\{ x \in F : \overline{\lim_{r\to 0}}\, \mu(B(x,r))/r^s > 0 \right\}.$$

If $x \in F_1$ we may find $\varepsilon > 0$ and a sequence of numbers $\{r_i\}$ decreasing to 0 such that $\mu(B(x,r_i)) \geqslant \varepsilon r_i^s$. Since $\mu(\{x\}) = 0$ (otherwise $I_s(\mu) = \infty$) it follows from the continuity of μ that, by taking q_i $(0 < q_i < r_i)$ small enough, we get $\mu(A_i) \geqslant \frac{1}{4}\varepsilon r_i^s$ $(i = 1, 2, \ldots)$, where A_i is the annulus $B(x,r_i)\backslash B(x,q_i)$. Taking subsequences if necessary, we may assume that $r_{i+1} < q_i$ for all i, so that the A_i are disjoint annuli centred on x. Hence for all $x \in F_1$

$$\begin{aligned}\phi_s(x) = \int \frac{\mathrm{d}\mu(y)}{|x-y|^s} &\geqslant \sum_{i=1}^{\infty} \int_{A_i} \frac{\mathrm{d}\mu(y)}{|x-y|^s}\\ &\geqslant \sum_{i=1}^{\infty} \tfrac{1}{4}\varepsilon r_i^s r_i^{-s} = \infty\end{aligned}$$

since $|x-y|^{-s} \geqslant r_i^{-s}$ on A_i. But $I_s(\mu) = \int \phi_s(x)\mathrm{d}\mu(x) < \infty$, so $\phi_s(x) < \infty$ for μ-almost all x. We conclude that $\mu(F_1) = 0$. Since $\overline{\lim}_{r\to 0}\mu(B(x,r))/r^s = 0$ if $x \in F\backslash F_1$, Proposition 4.9(a) tells us that, for all $c > 0$, we have

$$\mathcal{H}^s(F) \geqslant \mathcal{H}^s(F\backslash F_1) \geqslant \mu(F\backslash F_1)/c \geqslant (\mu(F) - \mu(F_1))/c = \mu(F)/c.$$

Hence $\mathcal{H}^s(F) = \infty$.

(b) Suppose that $\mathcal{H}^s(F) > 0$. We use $\mathcal{H}^s$ to construct a mass distribution μ on F with $I_t(\mu) < \infty$ for every $t < s$.

By Corollary 4.12 there exist a compact set $E \subset F$ with $0 < \mathcal{H}^s(E) < \infty$ and a constant b such that

$$\mathcal{H}^s(E \cap B(x,r)) \leqslant br^s$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Let μ be the restriction of $\mathcal{H}^s$ to E, so that $\mu(A) = \mathcal{H}^s(E \cap A)$; then μ is a mass distribution on F. Fix $x \in \mathbb{R}^n$ and write

$$m(r) = \mu(B(x,r)) = \mathcal{H}^s(E \cap B(x,r)) \leqslant br^s. \tag{4.14}$$

Then, if $0 < t < s$

$$\begin{aligned}\phi_t(x) &= \int_{|x-y|\leqslant 1} \frac{\mathrm{d}\mu(y)}{|x-y|^t} + \int_{|x-y|>1} \frac{\mathrm{d}\mu(y)}{|x-y|^t}\\ &\leqslant \int_0^1 r^{-t}\mathrm{d}m(r) + \mu(\mathbb{R}^n)\\ &= [r^{-t}m(r)]_{0+}^1 + t\int_0^1 r^{-(t+1)}m(r)\mathrm{d}r + \mu(\mathbb{R}^n)\\ &\leqslant b + bt\int_0^1 r^{s-t-1}\mathrm{d}r + \mu(\mathbb{R}^n)\\ &= b\left(1 + \frac{t}{s-t}\right) + \mathcal{H}^s(F) = c,\end{aligned}$$

say, after integrating by parts and using (4.14). Thus $\phi_t(x) \leqslant c$ for all $x \in \mathbb{R}^n$, so that $I_t(\mu) = \int \phi_t(x)\mathrm{d}\mu(x) \leqslant c\mu(\mathbb{R}^n) < \infty$. □

Important applications of Theorem 4.13 will be given later in the book, for example, in the proof of the projection theorems in Chapter 6 and in the determination of the dimension of Brownian paths in Chapter 16. The theorem is often used to find the dimension of fractals F_θ which depend on a parameter θ. There may be a natural way to define a mass distribution μ_θ on F_θ for each θ. If we can show, that for some s,

$$\int I_s(\mu_\theta)\mathrm{d}\theta = \iiint \frac{\mathrm{d}\mu_\theta(x)\mathrm{d}\mu_\theta(y)\mathrm{d}\theta}{|x-y|^s} < \infty,$$

then $I_s(\mu_\theta) < \infty$ for almost all θ, so that $\dim_{\mathrm{H}} F_\theta \geqslant s$ for almost all θ.

Readers familiar with potential theory will have encountered the definition of the *s-capacity* of a set F:

$$C_s(F) = \sup_\mu\{1/I_s(\mu) : \mu \text{ is a mass distribution on } F \text{ with } \mu(F) = 1\}$$

(with the convention that $1/\infty = 0$). Thus another way of expressing Theorem 4.13 is

$$\dim_{\mathrm{H}} F = \inf\{s \geqslant 0 : C_s(F) = 0\} = \sup\{s \geqslant 0 : C_s(F) > 0\}.$$

Whilst this is reminiscent of the definition (2.11) of Hausdorff dimension in terms of Hausdorff measures, it should be noted that capacities behave very differently from measures. In particular, they are not generally additive.

*4.4 Fourier transform methods

In this section, we do no more than indicate that Fourier transforms can be a powerful tool for analysing dimensions.

The n-dimensional Fourier transforms of an integrable function f and a mass distribution μ on $\mathbb{R}^n$ are defined by

$$\hat{f}(u) = \int_{\mathbb{R}^n} f(x)\exp(\mathrm{i}x \cdot u)\mathrm{d}x \qquad (u \in \mathbb{R}^n) \tag{4.15}$$

$$\hat{\mu}(u) = \int_{\mathbb{R}^n} \exp(\mathrm{i}x \cdot u)\mathrm{d}\mu(x) \qquad (u \in \mathbb{R}^n) \tag{4.16}$$

where $x \cdot u$ represents the usual scalar product. (Fourier transformation extends to a much wider class of function using the theory of distributions.)

The s-potential (4.12) of a mass distribution μ is just the convolution

$$\phi_s(x) = (|\cdot|^{-s} * \mu)(x) \equiv \int |x - y|^{-s}\mathrm{d}\mu(y).$$

Formally, the transform of $|x|^{-s}$ may be shown to be $c|u|^{s-n}$, where c depends on n and s, so the convolution theorem, which states that the transform of the convolution of two functions equals the product of the transforms of the functions, gives

$$\hat{\phi}_s(u) = c|u|^{s-n}\hat{\mu}(u).$$

Parseval's theorem tells us that

$$\int \phi_s(x)\mathrm{d}\mu(x) = (2\pi)^n \int \hat{\phi}_s(u)\overline{\hat{\mu}(u)}\mathrm{d}u$$

where the bar denotes complex conjugation, so

$$I_s(\mu) = (2\pi)^n c \int |u|^{s-n}|\hat{\mu}(u)|^2\mathrm{d}u. \tag{4.17}$$

This expression for $I_s(\mu)$, which may be established rather more rigorously, is sometimes a convenient way of expressing the energy (4.13) required in Theorem 4.13. Thus if there is a mass distribution μ on a set F for which the integral (4.17) is finite, then $\dim_{\mathrm{H}}F \geqslant s$. In particular, if

$$|\hat{\mu}(u)| \leqslant b|u|^{-t/2} \tag{4.18}$$

for some constant b, then, noting that, by (4.16), $|\hat{\mu}(u)| \leqslant \mu(\mathbb{R}^n)$ for all u, we have from (4.17) that

$$I_s(\mu) \leqslant c_1 \int_{|u| \leqslant 1} |u|^{s-n}\mathrm{d}u + c_2 \int_{|u|>1} |u|^{s-n}|u|^{-t}\mathrm{d}u$$

which is finite if $0 < s < t$. Thus if (4.18) holds, any set F which supports μ has Hausdorff dimension at least t. The greatest value of t for which there is a mass distribution μ on F satisfying (4.18) is sometimes called the *Fourier dimension* of F, which never exceeds the Hausdorff dimension.

4.5 Notes and references

Many papers are devoted to calculating dimensions of various classes of fractal, for example the papers of Eggleston (1952), Beardon (1965) and Peyrière (1977) discuss fairly general constructions.

The potential theoretic approach was, essentially, due to Frostman (1935); see Taylor (1961), Hayman and Kennedy (1976), Carleson (1967) or Kahane (1985) for more recent accounts. For an introduction to Fourier transforms see Papoulis (1962).

The work on subsets of finite measure originates from Besicovitch (1952) and a very general treatment is given in Rogers (1998). Complete proofs of Theorem 4.10 may be found in Falconer (1985a) and Mattila (1995).

Subsets of finite positive packing measure are investigated by Joyce and Preiss (1995).

Exercises

4.1 What is the Hausdorff dimension of the 'Cantor tartan' given by $\{(x, y) \in \mathbb{R}^2 :$ either $x \in F$ or $y \in F\}$ where F is the middle third Cantor set?

4.2 Use the mass distribution principle and a natural upper bound to show that the set of numbers in $[0, 1]$ containing only even digits has Hausdorff dimension $\log 5/\log 10$.

4.3 Use the mass distribution method to show that the 'Cantor dust' depicted in figure 0.4 has Hausdorff dimension 1. (Hint: note that, taking the square E_0 to have side 1, any two squares in the set E_k of the construction are separated by a distance of at least 4^{-k}.)

4.4 Fix $0 < \lambda \leqslant \frac{1}{2}$, and let F be the set of real numbers

$$F = \left\{ \sum_{k=1}^{\infty} a_k \lambda^k : a_k = \ 0 \text{ or } 1 \text{ for } \ k = 1, 2, \ldots \right\}.$$

Find the Hausdorff and box dimensions of F.

4.5 What is the Hausdorff dimension of $F \times F \subset \mathbb{R}^2$, where F is the middle third Cantor set?

4.6 Let F be the middle third Cantor set. What is the Hausdorff dimension of the plane set given by $\{(x, y) \in \mathbb{R}^2 : x \in F \text{ and } 0 \leqslant y \leqslant x^2\}$?

4.7 Use a mass distribution method to obtain the result of Example 4.5 directly rather than via Example 4.4.

4.8 Show that every number $x \geqslant 0$ may be expressed in the form

$$x = m + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots$$

where $m \geqslant 0$ is an integer and a_k is an integer with $0 \leqslant a_k \leqslant k - 1$ for each k. Let $F = \{x \geqslant 0 : m = 0 \text{ and } a_k \text{ is even for } k = 2, 3, \ldots\}$. Find $\dim_{\mathrm{H}} F$.

4.9 Show that there is a compact subset F of $[0, 1]$ of Hausdorff dimension 1 but with $\mathcal{H}^1(F) = 0$. (Hint: try a 'Cantor set' construction, but reduce the proportion of intervals removed at each stage.)

4.10 Deduce from Theorem 4.10 that if F is a Borel subset of $\mathbb{R}^n$ with $\mathcal{H}^s(F) = \infty$ and c is a positive number, then there is a Borel subset E of F with $\mathcal{H}^s(E) = c$.

4.11 Let μ be the natural mass distribution on the middle third Cantor set F (see after Principle 4.2). Estimate the s-energy of μ for $s < \log 2/\log 3$ and deduce from Theorem 4.13 that $\dim_{\mathrm{H}} F \geqslant \log 2/\log 3$.