Chapter 3 Alternative definitions of dimension

Hausdorff dimension, discussed in the last chapter, is the principal definition of dimension that we shall work with. However, other definitions are in widespread use, and it is appropriate to examine some of these and their inter-relationship. Not all definitions are generally applicable—some only describe particular classes of set, such as curves.

Fundamental to most definitions of dimension is the idea of 'measurement at scale δ'. For each δ, we measure a set in a way that ignores irregularities of size less than δ, and we see how these measurements behave as $\delta \to 0$. For example, if F is a plane curve, then our measurement, $M_\delta(F)$, might be the number of steps required by a pair of dividers set at length δ to traverse F. A dimension of F is then determined by the power law (if any) obeyed by $M_\delta(F)$ as $\delta \to 0$. If

$$M_\delta(F) \sim c\delta^{-s} \tag{3.1}$$

for constants c and s, we might say that F has 'divider dimension' s, with c regarded as the 's-dimensional length' of F. Taking logarithms

$$\log M_\delta(F) \simeq \log c - s \log \delta \tag{3.2}$$

in the sense that the difference of the two sides tends to 0 with δ, and

$$s = \lim_{\delta \to 0} \frac{\log M_\delta(F)}{-\log \delta}. \tag{3.3}$$

These formulae are appealing for computational or experimental purposes, since s can be estimated as minus the gradient of a log–log graph plotted over a suitable range of δ; see figure 3.1. Of course, for real phenomena, we can only work with a finite range of δ; theory and experiment diverge before an atomic scale is

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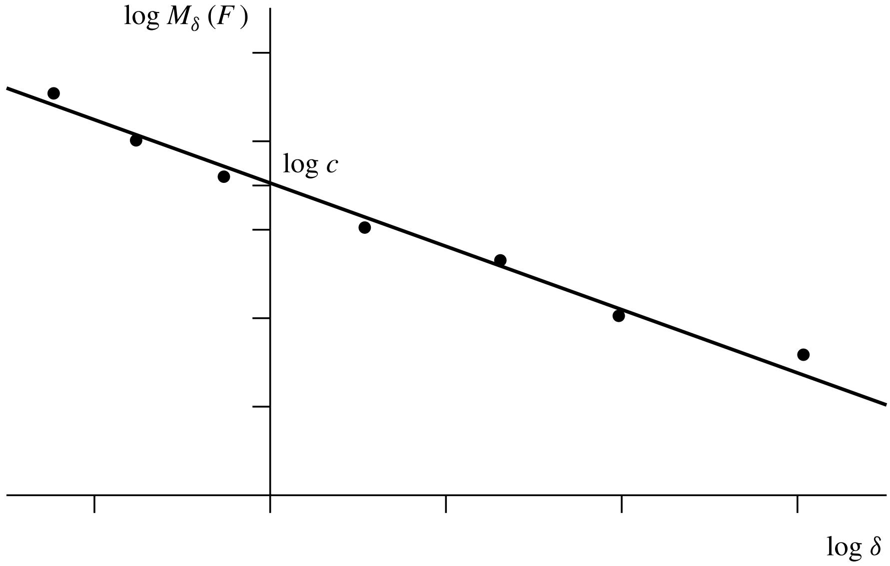

Figure 3.1 Empirical estimation of a dimension of a set F, on the power-law assumption $M_\delta(F) \sim c\delta^{-s}$

reached. For example, if F is the coastline of Britain, plotting a log–log graph for δ between 20 m and 200 km gives the divider dimension of F about 1.2.

There may be no exact power law for $M_\delta(F)$, and the closest we can get to (3.3) are the lower and upper limits.

For the value of s given by (3.1) to behave like a dimension, the method of measurement needs to scale with the set, so that doubling the size of F and at the same time doubling the scale at which measurement takes place does not affect the answer; that is, we require $M_\delta(\delta F) = M_1(F)$ for all δ. If we modify our example and redefine $M_\delta(F)$ to be the sum of the divider step lengths then $M_\delta(F)$ is homogeneous of degree 1, i.e. $M_\delta(\delta F) = \delta^1 M_1(F)$ for $\delta > 0$, and this must be taken into account when defining the dimension. In general, if $M_\delta(F)$ is homogeneous of degree d, that is $M_\delta(\delta F) = \delta^d M_1(F)$, then a power law of the form $M_\delta(F) \sim c\delta^{d-s}$ corresponds to a dimension s.

There are no hard and fast rules for deciding whether a quantity may reasonably be regarded as a dimension. There are many definitions that do not fit exactly into the above, rather simplified, scenario. The factors that determine the acceptability of a definition of a dimension are recognized largely by experience and intuition. In general one looks for some sort of scaling behaviour, a naturalness of the definition in the particular context and properties typical of dimensions such as those discussed below.

A word of warning: as we shall see, apparently similar definitions of dimension can have widely differing properties. It should not be assumed that different definitions give the same value of dimension, even for ‘nice’ sets. Such assumptions have led to major misconceptions and confusion in the past. It is necessary to derive the properties of any ‘dimension’ from its definition. The properties of Hausdorff dimension (on which we shall largely concentrate in the later chapters of this book) do not necessarily all hold for other definitions.

What are the desirable properties of a 'dimension'? Those derived in the last chapter for Hausdorff dimension are fairly typical.

Monotonicity. If $E \subset F$ then $\dim_H E \leqslant \dim_H F$.

Stability. $\dim_H(E \cup F) = \max(\dim_H E, \dim_H F)$.

Countable stability. $\dim_H \left(\bigcup_{i=1}^{\infty} F_i\right) = \sup_{1 \leqslant i < \infty} \dim_H F_i$.

Geometric invariance. $\dim_H f(F) = \dim_H F$ if f is a transformation of $\mathbb{R}^n$ such as a translation, rotation, similarity or affinity.

Lipschitz invariance. $\dim_H f(F) = \dim_H F$ if f is a bi-Lipschitz transformation.

Countable sets. $\dim_H F = 0$ if F is finite or countable.

Open sets. If F is an open subset of $\mathbb{R}^n$ then $\dim_H F = n$.

Smooth manifolds. $\dim_H F = m$ if F is a smooth m-dimensional manifold (curve, surface, etc.).

All definitions of dimension are monotonic, most are stable, but, as we shall see, some common definitions fail to exhibit countable stability and may have countable sets of positive dimension. All the usual dimensions are Lipschitz invariant, and, therefore, geometrically invariant. The 'open sets' and 'smooth manifolds' properties ensure that the dimension is an extension of the classical definition. Note that different definitions of dimension can provide different information about which sets are Lipschitz equivalent.

3.1 Box-counting dimensions

Box-counting or box dimension is one of the most widely used dimensions. Its popularity is largely due to its relative ease of mathematical calculation and empirical estimation. The definition goes back at least to the 1930s and it has been variously termed Kolmogorov entropy, entropy dimension, capacity dimension (a term best avoided in view of potential theoretic associations), metric dimension, logarithmic density and information dimension. We shall always refer to box or box-counting dimension to avoid confusion.

Let F be any non-empty bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. The *lower* and *upper box-counting dimensions* of F respectively are defined as

$$\underline{\dim}_B F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \tag{3.4}$$

$$\overline{\dim}_B F = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}. \tag{3.5}$$

If these are equal we refer to the common value as the *box-counting dimension* or *box dimension* of F

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}. \tag{3.6}$$

Here, and throughout the book, we assume that $\delta > 0$ is sufficiently small to ensure that $-\log\delta$ and similar quantities are strictly positive. To avoid problems with '$\log 0$' or '$\log\infty$' we generally consider box dimension only for non-empty bounded sets. In developing the general theory of box dimensions we assume that sets considered are non-empty and bounded.

There are several equivalent definitions of box dimension that are sometimes more convenient to use. Consider the collection of cubes in the δ-coordinate mesh of $\mathbb{R}^n$, i.e. cubes of the form

$$[m_1\delta, (m_1+1)\delta] \times \cdots \times [m_n\delta, (m_n+1)\delta]$$

where $m_1, \ldots, m_n$ are integers. (Recall that a 'cube' is an interval in $\mathbb{R}^1$ and a square in $\mathbb{R}^2$.) Let $N'_\delta(F)$ be the number of δ-mesh cubes that intersect F. They obviously provide a collection of $N'_\delta(F)$ sets of diameter $\delta\sqrt{n}$ that cover F, so

$$N_{\delta\sqrt{n}}(F) \leqslant N'_\delta(F).$$

If $\delta\sqrt{n} < 1$ then

$$\frac{\log N_{\delta\sqrt{n}}(F)}{-\log(\delta\sqrt{n})} \leqslant \frac{\log N'_\delta(F)}{-\log\sqrt{n} - \log\delta}$$

so taking limits as $\delta \to 0$

$$\underline{\dim}_{\mathrm{B}} F \leqslant \varliminf_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta} \tag{3.7}$$

and

$$\overline{\dim}_{\mathrm{B}} F \leqslant \varlimsup_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta}. \tag{3.8}$$

On the other hand, any set of diameter at most δ is contained in 3^n mesh cubes of side δ (by choosing a cube containing some point of the set together with its neighbouring cubes). Thus

$$N'_\delta(F) \leqslant 3^n N_\delta(F)$$

and taking logarithms and limits as $\delta \to 0$ leads to the opposite inequalities to (3.7) and (3.8). Hence to find the box dimensions (3.4)–(3.6), we can equally well take $N_\delta(F)$ to be the number of mesh cubes of side δ that intersect F.

This version of the definitions is widely used empirically. To find the box dimension of a plane set F we draw a mesh of squares or boxes of side δ and count the number $N_\delta(F)$ that overlap the set for various small δ (hence the name 'box-counting'). The dimension is the logarithmic rate at which $N_\delta(F)$ increases as $\delta \to 0$, and may be estimated by the gradient of the graph of $\log N_\delta(F)$ against $-\log\delta$.

This definition gives an interpretation of the meaning of box dimension. The number of mesh cubes of side δ that intersect a set is an indication of how spread out or irregular the set is when examined at scale δ. The dimension reflects how rapidly the irregularities develop as $\delta \to 0$.

Another frequently used definition of box dimension is obtained by taking $N_\delta(F)$ in (3.4)–(3.6) to be the smallest number of *arbitrary* cubes of side δ required to cover F. The equivalence of this definition follows as in the mesh cube case, noting that any cube of side δ has diameter $\delta\sqrt{n}$, and that any set of diameter of at most δ is contained in a cube of side δ.

Similarly, we get exactly the same values if in (3.4)–(3.6) we take $N_\delta(F)$ as the smallest number of closed balls of radius δ that cover F.

A less obviously equivalent formulation of box dimension has the *largest* number of *disjoint* balls of radius δ with centres in F. Let this number be $N'_\delta(F)$, and let $B_1, \dots, B_{N'_\delta(F)}$ be disjoint balls centred in F and of radius δ. If x belongs to F then x must be within distance δ of one of the B_i, otherwise the ball of centre x and radius δ can be added to form a larger collection of disjoint balls. Thus the $N'_\delta(F)$ balls concentric with the B_i but of radius 2δ (diameter 4δ) cover F, giving

$$N_{4\delta}(F) \leqslant N'_\delta(F). \tag{3.9}$$

Suppose also that $B_1, \dots, B_{N'_\delta(F)}$ are disjoint balls of radii δ with centres in F. Let $U_1, \dots, U_k$ be any collection of sets of diameter at most δ which cover F. Since the U_j must cover the centres of the B_i, each B_i must contain at least one of the U_j. As the B_i are disjoint there are at least as many U_j as B_i. Hence

$$N'_\delta(F) \leqslant N_\delta(F). \tag{3.10}$$

Taking logarithms and limits of (3.9) and (3.10) shows that the values of (3.4)–(3.6) are unaltered if $N_\delta(F)$ is replaced by this $N'_\delta(F)$.

These various definitions are summarized below and in figure 3.2.

Equivalent definitions 3.1

The lower and upper box-counting dimensions of a subset F of $\mathbb{R}^n$ are given by

$$\underline{\dim}_{\mathrm{B}} F = \underline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \tag{3.11}$$

$$\overline{\dim}_{\mathrm{B}} F = \overline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \tag{3.12}$$

and the box-counting dimension of F by

$$\dim_{\mathrm{B}} F = \lim_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \tag{3.13}$$

(*if this limit exists*), *where $N_\delta(F)$ is any of the following*:

(i) *the smallest number of closed balls of radius δ that cover F*;
(ii) *the smallest number of cubes of side δ that cover F*;
(iii) *the number of δ-mesh cubes that intersect F*;
(iv) *the smallest number of sets of diameter at most δ that cover F*;
(v) *the largest number of disjoint balls of radius δ with centres in F.*

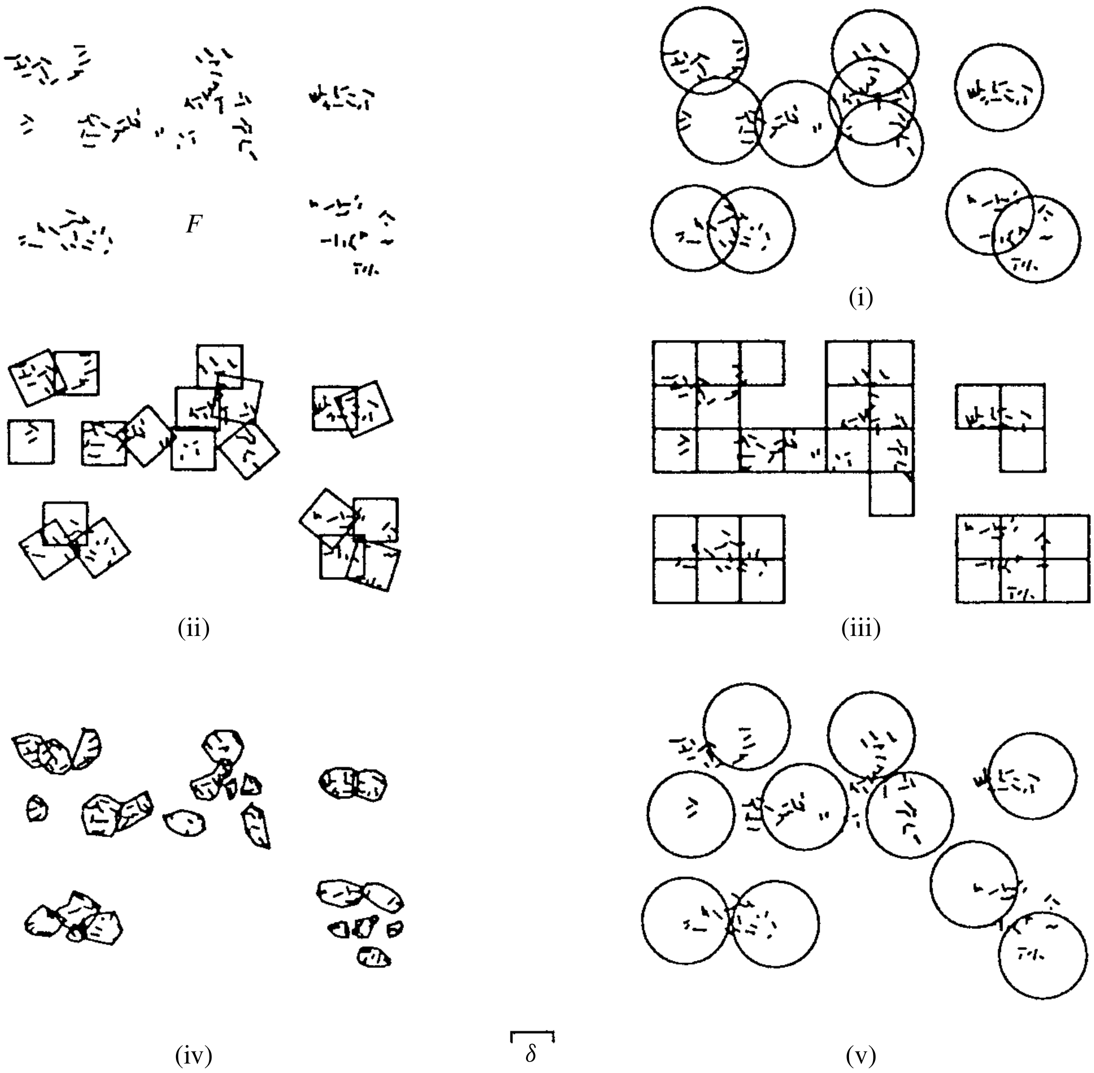

Figure 3.2 Five ways of finding the box dimension of F; see Equivalent definitions 3.1. The number $N_\delta(F)$ is taken to be: (i) the least number of closed balls of radius δ that cover F; (ii) the least number of cubes of side δ that cover F; (iii) the number of δ-mesh cubes that intersect F; (iv) the least number of sets of diameter at most δ that cover F; (v) the greatest number of disjoint balls of radius δ with centres in F

This list could be extended further; in practice one adopts the definition most convenient for a particular application.

It is worth noting that, in (3.11)–(3.13), it is enough to consider limits as δ tends to 0 through any decreasing sequence δ_k such that $\delta_{k+1} \geqslant c\delta_k$ for some constant $0 < c < 1$; in particular for $\delta_k = c^k$. To see this, note that if $\delta_{k+1} \leqslant \delta < \delta_k$, then, with $N_\delta(F)$ the least number of sets in a δ-cover of F,

$$\frac{\log N_\delta(F)}{-\log \delta} \leqslant \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_k} = \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_{k+1} + \log(\delta_{k+1}/\delta_k)} \leqslant \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_{k+1} + \log c}$$

and so

$$\overline{\lim_{\delta\to 0}}\frac{\log N_\delta(F)}{-\log\delta} \leqslant \overline{\lim_{k\to\infty}}\frac{\log N_{\delta_k}(F)}{-\log\delta_k}. \tag{3.14}$$

The opposite inequality is trivial; the case of lower limits may be dealt with in the same way.

There is an equivalent definition of box dimension of a rather different form that is worth mentioning. Recall that the *δ-neighbourhood* F_δ of a subset F of $\mathbb{R}^n$ is

$$F_\delta = \{x \in \mathbb{R}^n : |x-y| \leqslant \delta \text{ for some } y \in F\} \tag{3.15}$$

i.e. the set of points within distance δ of F. We consider the rate at which the n-dimensional volume of F_δ shrinks as $\delta \to 0$. In $\mathbb{R}^3$, if F is a single point then F_δ is a ball with $\text{vol}(F_\delta) = \frac{4}{3}\pi\delta^3$, if F is a segment of length l then F_δ is 'sausage-like' with $\text{vol}(F_\delta) \sim \pi l\delta^2$, and if F is a flat set of area a then F_δ is essentially a thickening of F with $\text{vol}(F_\delta) \sim 2a\delta$. In each case, $\text{vol}(F_\delta) \sim c\delta^{3-s}$ where the integer s is the dimension of F, so that exponent of δ is indicative of the dimension. The coefficient c of δ^{3-s}, known as the *Minkowski content* of F, is a measure of the length, area or volume of the set as appropriate.

This idea extends to fractional dimensions. If F is a subset of $\mathbb{R}^n$ and, for some s, $\text{vol}^n(F_\delta)/\delta^{n-s}$ tends to a positive finite limit as $\delta \to 0$ where vol^n denotes n-dimensional volume, then it makes sense to regard F as s-dimensional. The limiting value is called the *s-dimensional content* of F—a concept of slightly restricted use since it is not necessarily additive on disjoint subsets, i.e. is not a measure. Even if this limit does not exist, we may be able to extract the critical exponent of δ and this turns out to be related to the box dimension.

Proposition 3.2

If F is a subset of $\mathbb{R}^n$, then

$$\underline{\dim}_B F = n - \overline{\lim_{\delta\to 0}}\frac{\log \text{vol}^n(F_\delta)}{\log\delta}$$

$$\overline{\dim}_B F = n - \underline{\lim_{\delta\to 0}}\frac{\log \text{vol}^n(F_\delta)}{\log\delta}$$

where F_δ is the δ-neighbourhood of F.

Proof. If F can be covered by $N_\delta(F)$ balls of radius $\delta < 1$ then F_δ can be covered by the concentric balls of radius 2δ. Hence

$$\text{vol}^n(F_\delta) \leqslant N_\delta(F)c_n(2\delta)^n$$

where c_n is the volume of the unit ball in $\mathbb{R}^n$. Taking logarithms,

$$\frac{\log \mathrm{vol}^n(F_\delta)}{-\log\delta} \leqslant \frac{\log 2^n c_n + n\log\delta + \log N_\delta(F)}{-\log\delta},$$

so

$$\varliminf_{\delta\to 0} \frac{\log \mathrm{vol}^n(F_\delta)}{-\log\delta} \leqslant -n + \underline{\dim}_{\mathrm{B}} F \tag{3.16}$$

with a similar inequality for the upper limits. On the other hand if there are $N_\delta(F)$ disjoint balls of radius δ with centres in F, then by adding their volumes,

$$N_\delta(F) c_n \delta^n \leqslant \mathrm{vol}^n(F_\delta).$$

Taking logarithms and letting $\delta \to 0$ gives the opposite inequality to (3.16), using Equivalent definition 3.1(v). $\square$

In the context of Proposition 3.2, box dimension is sometimes referred to as *Minkowski dimension* or *Minkowski–Bouligand dimension.*

It is important to understand the relationship between box-counting dimension and Hausdorff dimension. If F can be covered by $N_\delta(F)$ sets of diameter δ, then, from definition (2.1),

$$\mathcal{H}^s_\delta(F) \leqslant N_\delta(F)\delta^s.$$

If $1 < \mathcal{H}^s(F) = \lim_{\delta\to 0}\mathcal{H}^s_\delta(F)$ then $\log N_\delta(F) + s\log\delta > 0$ if δ is sufficiently small. Thus $s \leqslant \varliminf_{\delta\to 0} \log N_\delta(F)/-\log\delta$ so

$$\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F \tag{3.17}$$

for every $F \subset \mathbb{R}^n$. We do *not* in general get equality here. Although Hausdorff and box dimensions are equal for many 'reasonably regular' sets, there are plenty of examples where this inequality is strict.

Roughly speaking (3.6) says that $N_\delta(F) \simeq \delta^{-s}$ for small δ, where $s = \dim_{\mathrm{B}} F$. More precisely, it says that

$$N_\delta(F)\delta^s \to \infty \qquad \text{if } s < \dim_{\mathrm{B}} F$$

and

$$N_\delta(F)\delta^s \to 0 \qquad \text{if } s > \dim_{\mathrm{B}} F.$$

But

$$N_\delta(F)\delta^s = \inf\left\{\sum_i \delta^s : \{U_i\} \text{ is a (finite) } \delta\text{-cover of } F\right\},$$

which should be compared with

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_i |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\},$$

which occurs in the definitions of Hausdorff measure and dimension. In calculating Hausdorff dimension, we assign different weights $|U_i|^s$ to the covering sets U_i, whereas for the box dimensions we use the same weight δ^s for each covering set. Box dimensions may be thought of as indicating the efficiency with which a set may be covered by small sets of equal size, whereas Hausdorff dimension involves coverings by sets of small but perhaps widely varying size.

There is a temptation to introduce the quantity $v(F) = \underline{\lim}_{\delta\to 0} N_\delta(F)\delta^s$, but this does *not* give a measure on subsets of $\mathbb{R}^n$. As we shall see, one consequence of this is that box dimensions have a number of unfortunate properties, and can be awkward to handle mathematically.

Since box dimensions are determined by coverings by sets of equal size they tend to be easier to calculate than Hausdorff dimensions. Just as with Hausdorff dimension, calculations of box dimension usually involve finding a lower bound and an upper bound separately, each bound depending on a geometric observation followed by an algebraic estimate.

Example 3.3

Let F be the middle third Cantor set (figure 0.1). Then $\underline{\dim}_B F = \overline{\dim}_B F = \log 2/\log 3$.

Calculation. The obvious covering by the 2^k level-k intervals of E_k of length 3^{-k} gives that $N_\delta(F) \leqslant 2^k$ if $3^{-k} < \delta \leqslant 3^{-k+1}$. From (3.5)

$$\overline{\dim}_B F = \overline{\lim_{\delta\to 0}} \frac{\log N_\delta(F)}{-\log\delta} \leqslant \overline{\lim_{k\to\infty}} \frac{\log 2^k}{\log 3^{k-1}} = \frac{\log 2}{\log 3}.$$

On the other hand, any interval of length δ with $3^{-k-1} \leqslant \delta < 3^{-k}$ intersects at most one of the level-k intervals of length 3^{-k} used in the construction of F. There are 2^k such intervals so at least 2^k intervals of length δ are required to cover F. Hence $N_\delta(F) \geqslant 2^k$ leading to $\underline{\dim}_B F \geqslant \log 2/\log 3$. □

Thus, at least for the Cantor set, $\dim_H F = \dim_B F$.

3.2 Properties and problems of box-counting dimension

The following elementary properties of box dimension mirror those of Hausdorff dimension, and may be verified in much the same way.

(i) A smooth m-dimensional submanifold of $\mathbb{R}^n$ has $\dim_B F = m$.
(ii) $\underline{\dim}_B$ and $\overline{\dim}_B$ are monotonic.
(iii) $\overline{\dim}_B$ is *finitely* stable, i.e.

$$\overline{\dim}_B(E \cup F) = \max\{\overline{\dim}_B E, \overline{\dim}_B F\};$$

the corresponding identity does *not* hold for $\underline{\dim}_B$.
(iv) $\underline{\dim}_B$ and $\overline{\dim}_B$ are bi-Lipschitz invariant. This is so because, if $|f(x) - f(y)| \leqslant c|x - y|$ and F can be covered by $N_\delta(F)$ sets of diameter at most δ, then the $N_\delta(F)$ images of these sets under f form a cover of $f(F)$ by sets of diameter at most $c\delta$, thus $\dim_B f(F) \leqslant \dim_B F$. Similarly, box dimensions behave just like Hausdorff dimensions under bi-Lipschitz and Hölder transformations.

We now start to encounter the disadvantages of box-counting dimension. The next proposition is at first appealing, but has undesirable consequences.

Proposition 3.4

Let $\overline{F}$ denote the closure of F (i.e. the smallest closed subset of $\mathbb{R}^n$ containing F). Then

$$\underline{\dim}_B \overline{F} = \underline{\dim}_B F$$

and

$$\overline{\dim}_B \overline{F} = \overline{\dim}_B F.$$

Proof. Let $B_1, \ldots, B_k$ be a finite collection of closed balls of radii δ. If the closed set $\bigcup_{i=1}^k B_i$ contains F, it also contains $\overline{F}$. Hence the smallest number of closed balls of radius δ that cover F equals the smallest number required to cover the larger set $\overline{F}$. The result follows. □

An immediate consequence of this is that if F is a dense subset of an open region of $\mathbb{R}^n$ then $\underline{\dim}_B F = \overline{\dim}_B F = n$. For example, let F be the (countable) set of rational numbers between 0 and 1. Then $\overline{F}$ is the entire interval $[0, 1]$, so that $\underline{\dim}_B F = \overline{\dim}_B F = 1$. Thus countable sets can have non-zero box dimension. Moreover, the box-counting dimension of each rational number regarded as a one-point set is clearly zero, but the countable union of these singleton sets has dimension 1. Consequently, it is not generally true that $\dim_B \bigcup_{i=1}^\infty F_i = \sup_i \dim_B F_i$.

This severely limits the usefulness of box dimension—introducing a small, i.e. countable, set of points can play havoc with the dimension. We might hope to salvage something by restricting attention to closed sets, but difficulties still remain.

Example 3.5

$F = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ *is a compact set with* $\dim_B F = \frac{1}{2}$.

Calculation. Let $0 < \delta < \frac{1}{2}$ and let k be the integer satisfying $1/(k-1)k > \delta \geqslant 1/k(k+1)$. If $|U| \leqslant \delta$, then U can cover at most one of the points $\{1, \frac{1}{2}, \dots, 1/k\}$ since $1/(k-1) - 1/k = 1/(k-1)k > \delta$. Thus at least k sets of diameter δ are required to cover F, so $N_\delta(F) \geqslant k$ giving

$$\frac{\log N_\delta(F)}{-\log\delta} \geqslant \frac{\log k}{\log k(k+1)}.$$

Letting $\delta \to 0$ so $k \to \infty$ gives $\underline{\dim}_B F \geqslant \frac{1}{2}$. On the other hand, if $\frac{1}{2} > \delta > 0$, take k such that $1/(k-1)k > \delta \geqslant 1/k(k+1)$. Then $(k+1)$ intervals of length δ cover $[0, 1/k]$, leaving $k-1$ points of F which can be covered by another $k-1$ intervals. Thus $N_\delta(F) \leqslant 2k$, so

$$\frac{\log N_\delta(F)}{-\log\delta} \leqslant \frac{\log(2k)}{\log k(k-1)}$$

giving

$$\overline{\dim}_B F \leqslant \tfrac{1}{2}. \qquad \square$$

No-one would regard this set, with all but one of its points isolated, as a fractal, yet it has large box dimension.

Nevertheless, as well as being convenient in practice, box dimensions are very useful in theory. If, as often happens, it can be shown that a set has equal box and Hausdorff dimensions, the interplay between these definitions can be used to powerful effect.

*3.3 Modified box-counting dimensions

There are ways of overcoming the difficulties of box dimension outlined in the last section. However, they may not at first seem appealing since they re-introduce all the difficulties of calculation associated with Hausdorff dimension and more.

For F a subset of $\mathbb{R}^n$ we can try to decompose F into a countable number of pieces $F_1, F_2, \dots$ in such a way that the largest piece has as small a dimension as possible. This idea leads to the following *modified box-counting dimensions*:

$$\underline{\dim}_{MB} F = \inf\left\{\sup_i \underline{\dim}_B F_i : F \subset \bigcup_{i=1}^{\infty} F_i\right\} \tag{3.18}$$

$$\overline{\dim}_{MB} F = \inf\left\{\sup_i \overline{\dim}_B F_i : F \subset \bigcup_{i=1}^{\infty} F_i\right\}. \tag{3.19}$$

(In both cases the infimum is over all possible countable covers $\{F_i\}$ of F.) Clearly $\underline{\dim}_{MB} F \leqslant \underline{\dim}_B F$ and $\overline{\dim}_{MB} F \leqslant \overline{\dim}_B F$. However, we now have that

$\underline{\dim}_{\rm MB} F = \overline{\dim}_{\rm MB} F = 0$ if F is countable—just take the F_i to be one-point sets. Moreover, for any subset F of $\mathbb{R}^n$,

$$0 \leqslant \dim_{\rm H} F \leqslant \underline{\dim}_{\rm MB} F \leqslant \overline{\dim}_{\rm MB} F \leqslant \overline{\dim}_{\rm B} F \leqslant n. \tag{3.20}$$

It is easy to see that $\underline{\dim}_{\rm MB}$ and $\overline{\dim}_{\rm MB}$ recover all the desirable properties of a dimension, but they can be hard to calculate. However, there is a useful test for compact sets to have equal box and modified box dimensions. It applies to sets that might be described as 'dimensionally homogeneous'.

Proposition 3.6

Let $F \subset \mathbb{R}^n$ be compact. Suppose that

$$\overline{\dim}_{\rm B}(F \cap V) = \overline{\dim}_{\rm B} F \tag{3.21}$$

for all open sets V that intersect F. Then $\overline{\dim}_{\rm B} F = \overline{\dim}_{\rm MB} F$. A similar result holds for lower box-counting dimensions.

Proof. Let $F \subset \bigcup_{i=1}^{\infty} F_i$ with each F_i closed. A version of Baire's category theorem (which may be found in any text on basic general topology, and which we quote without proof) states that there is an index i and an open set $V \subset \mathbb{R}^n$ such that $F \cap V \subset F_i$. For this i, $\overline{\dim}_{\rm B} F_i = \overline{\dim}_{\rm B} F$. Using (3.19) and Proposition 3.4

$$\begin{aligned}\overline{\dim}_{\rm MB} F &= \inf\left\{\sup \overline{\dim}_{\rm B} F_i : F \subset \bigcup_{i=1}^{\infty} F_i \text{ where the } F_i \text{ are closed sets}\right\}\\ &\geqslant \overline{\dim}_{\rm B} F.\end{aligned}$$

The opposite inequality is contained in (3.20). A similar argument deals with the lower dimensions. □

For an application, let F be a compact set with a high degree of self-similarity, for instance the middle third Cantor set or von Koch curve. If V is any open set that intersects F, then $F \cap V$ contains a geometrically similar copy of F which must have upper box dimension equal to that of F, so that (3.21) holds, leading to equal box and modified box dimensions.

*3.4 Packing measures and dimensions

Unlike Hausdorff dimension, neither the box dimensions or modified box dimensions are defined in terms of measures, and this can present difficulties in their theoretical development. Nevertheless, the circle of ideas in the last section may be completed in a way that is, at least mathematically, elegant. Recall that Hausdorff dimension may be defined using economical coverings by small balls (2.16) whilst $\underline{\dim}_B$ may be defined using economical coverings by small balls of equal radius (Equivalent definition 3.1(i)). On the other hand $\overline{\dim}_B$ may be thought of as a dimension that depends on packings by disjoint balls of equal radius that are as dense as possible (Equivalent definition 3.1(v)). Coverings and packings play a dual role in many areas of mathematics and it is therefore natural to try to look for a dimension that is defined in terms of dense packings by disjoint balls of differing small radii.

We try to follow the pattern of definition of Hausdorff measure and dimension. For $s \geqslant 0$ and $\delta > 0$, let

$$\mathcal{P}_\delta^s(F) = \sup\left\{\sum_i |B_i|^s : \{B_i\} \text{ is a collection of disjoint balls of radii at most } \delta \text{ with centres in } F\right\}. \tag{3.22}$$

Since $\mathcal{P}_\delta^s(F)$ decreases with δ, the limit

$$\mathcal{P}_0^s(F) = \lim_{\delta\to 0}\mathcal{P}_\delta^s(F) \tag{3.23}$$

exists. At this point we meet the problems encountered with box-counting dimensions. By considering countable dense sets it is easy to see that $\mathcal{P}_0^s(F)$ is not a measure. Hence we modify the definition to

$$\mathcal{P}^s(F) = \inf\left\{\sum_i \mathcal{P}_0^s(F_i) : F \subset \bigcup_{i=1}^{\infty} F_i\right\}. \tag{3.24}$$

It may be shown that $\mathcal{P}^s(F)$ is a measure on $\mathbb{R}^n$, known as the *s-dimensional packing measure*. We may define the *packing dimension* in the natural way:

$$\dim_P F = \sup\{s : \mathcal{P}^s(F) = \infty\} = \inf\{s : \mathcal{P}^s(F) = 0\}. \tag{3.25}$$

The underlying measure structure immediately implies monotonicity: that $\dim_P E \leqslant \dim_P F$ if $E \subset F$. Moreover, for a countable collection of sets $\{F_i\}$,

$$\dim_P\left(\bigcup_{i=1}^{\infty} F_i\right) = \sup_i \dim_P F_i, \tag{3.26}$$

since if $s > \dim_P F_i$ for all i, then $\mathcal{P}^s(\bigcup_i F_i) \leqslant \sum_i \mathcal{P}^s(F_i) = 0$ implying $\dim_P\left(\bigcup_i F_i\right) \leqslant s$.

We now investigate the relationship of packing dimension with other definitions of dimension and verify the surprising fact that packing dimension is just the same as the modified upper box dimension.

Lemma 3.7

$$\dim_{\mathrm{P}} F \leqslant \overline{\dim}_{\mathrm{B}} F. \tag{3.27}$$

Proof. If $\dim_{\mathrm{P}} F = 0$, the result is obvious. Otherwise choose any t and s with $0 < t < s < \dim_{\mathrm{P}} F$. Then $\mathcal{P}^s(F) = \infty$, so $\mathcal{P}_0^s(F) = \infty$. Thus, given $0 < \delta \leqslant 1$, there are disjoint balls $\{B_i\}$, of radii at most δ with centres in F, such that $1 < \Sigma_{i=1}^{\infty}|B_i|^s$. Suppose that, for each k, exactly n_k of these balls satisfy $2^{-k-1} < |B_i| \leqslant 2^{-k}$; then

$$1 < \sum_{k=0}^{\infty} n_k 2^{-ks}. \tag{3.28}$$

There must be some k with $n_k > 2^{kt}(1 - 2^{t-s})$, otherwise the sum in (3.28) is at most $\Sigma_{k=0}^{\infty} 2^{kt-ks}(1 - 2^{t-s}) = 1$, by summing the geometric series. These n_k balls all contain balls of radii $2^{-k-2} \leqslant \delta$ centred in F. Hence if $N_\delta(F)$ denotes the greatest number of disjoint balls of radius δ with centres in F, then

$$N_{2^{-k-2}}(F)(2^{-k-2})^t \geqslant n_k(2^{-k-2})^t > 2^{-2t}(1 - 2^{t-s})$$

where $2^{-k-2} < \delta$. It follows that $\overline{\lim}_{\delta\to 0} N_\delta(F)\delta^t > 0$, so that $\overline{\dim}_{\mathrm{B}} F \geqslant t$ using Equivalent definition 3.1(v). This is true for any $0 < t < \dim_{\mathrm{P}} F$ so (3.27) follows. □

Proposition 3.8

If $F \subset \mathbb{R}^n$ *then* $\dim_{\mathrm{P}} F = \overline{\dim}_{\mathrm{MB}} F$.

Proof. If $F \subset \bigcup_{i=1}^{\infty} F_i$ then, by (3.26) and (3.27),

$$\dim_{\mathrm{P}} F \leqslant \sup_i \dim_{\mathrm{P}} F_i \leqslant \sup_i \overline{\dim}_{\mathrm{B}} F_i.$$

Definition (3.19) now gives that $\dim_{\mathrm{P}} F \leqslant \overline{\dim}_{\mathrm{MB}} F$.

Conversely, if $s > \dim_{\mathrm{P}} F$ then $\mathcal{P}^s(F) = 0$, so that $F \subset \bigcup_i F_i$ for a collection of sets F_i with $\mathcal{P}_0^s(F_i) < \infty$ for each i, by (3.24). Hence, for each i, if δ is small enough, then $\mathcal{P}_\delta^s(F_i) < \infty$, so by (3.22) $N_\delta(F_i)\delta^s$ is bounded as $\delta \to 0$, where $N_\delta(F_i)$ is the largest number of disjoint balls of radius δ with centres in F_i. By Equivalent definition 3.1(v) $\overline{\dim}_{\mathrm{B}} F_i \leqslant s$ for each i, giving that $\overline{\dim}_{\mathrm{MB}} F \leqslant s$ by (3.19), as required. □

We have established the following relations:

$$\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{MB}} F \leqslant \overline{\dim}_{\mathrm{MB}} F = \dim_{\mathrm{P}} F \leqslant \overline{\dim}_{\mathrm{B}} F. \tag{3.29}$$

Suitable examples show that none of the inequalities can be replaced by equality.

As with Hausdorff dimension, packing dimension permits the use of powerful measure theoretic techniques in its study. The introduction of packing measures (remarkably some 60 years after Hausdorff measures) has led to a greater understanding of the geometric measure theory of fractals, with packing measures behaving in a way that is 'dual' to Hausdorff measures in many respects. Indeed corresponding results for Hausdorff and packing measures are often presented side by side. Nevertheless, one cannot pretend that packing measures and dimensions are easy to work with or to calculate; the extra step (3.24) in their definition makes them more awkward to use than the Hausdorff analogues.

This situation is improved slightly by the equality of packing dimension and the modified upper box dimension. It is improved considerably for compact sets with 'local' dimension constant throughout—a situation that occurs frequently in practice, in particular in sets with some kind of self-similarity.

Corollary 3.9

Let $F \subset \mathbb{R}^n$ *be compact and such that*

$$\overline{\dim}_B(F \cap V) = \overline{\dim}_B F \tag{3.30}$$

for all open sets V that intersect F. Then $\dim_P F = \overline{\dim}_B F$.

Proof. This is immediate from Propositions 3.6 and 3.8. □

The nicest case, of course, is of fractals with equal Hausdorff and upper box dimensions, in which case equality holds throughout (3.29)—we shall see many such examples later on. However, even the much weaker condition $\dim_H F = \dim_P F$, though sometimes hard to prove, eases analysis of F.

3.5 Some other definitions of dimension

A wide variety of other definitions of dimension have been introduced, many of them only of limited applicability, but nonetheless useful in their context.

The special form of curves gives rise to the several definitions of dimension. We define a *curve* or *Jordan curve* C to be the image of an interval $[a, b]$ under a continuous bijection $f : [a, b] \to \mathbb{R}^n$. (Thus, we restrict attention to curves that are non-self-intersecting.) If C is a curve and $\delta > 0$, we define $M_\delta(C)$ to be the maximum number of points $x_0, x_1, \ldots, x_m$, on the curve C, in that order, such that $|x_k - x_{k-1}| = \delta$ for $k = 1, 2, \ldots, m$. Thus $(M_\delta(C) - 1)\delta$ may be thought of as the 'length' of the curve C measured using a pair of dividers with points set at a distance δ apart. The *divider dimension* is defined as

$$\lim_{\delta \to 0} \frac{\log M_\delta(C)}{-\log \delta} \tag{3.31}$$

assuming the limit exists (otherwise we may define upper and lower divider dimensions using upper and lower limits). It is easy to see that the divider dimension of a curve is at least equal to the box dimension (assuming that they both exist) and in simple self-similar examples, such as the von Koch curve, they are equal. The assertion that the coastline of Britain has dimension 1.2 is usually made with the divider dimension in mind—this empirical value comes from estimating the ratio in (3.31) for values of δ between about 20 m and 200 km.

A variant of Hausdorff dimension may be defined for curves by using intervals of the curves themselves as covering sets. Thus we look at $\inf\left\{\sum_{i=1}^{m} |f[t_{i-1}, t_i]|^s\right\}$ where the infimum is over all dissections $a = t_0 < t_1 < \cdots < t_m = b$ such that the diameters $|f([t_{i-1}, t_i])|$ are all at most δ. We let δ tend to 0 and deem the value of s at which this limit jumps from ∞ to 0 to be the dimension. For self-similar examples such as the von Koch curve, this equals the Hausdorff dimension, but for 'squeezed' curves, such as graphs of certain functions (see Chapter 11) we may get a somewhat larger value.

Sometimes, we are interested in the dimension of a fractal F that is the boundary of a set A. We can define the box dimension of F in the usual way, but sometimes it is useful to take special account of the distinction between A and its complement. Thus the following variation of the 's-dimensional content' definition of box dimension, in which we take the volume of the set of points within distance δ of F that are contained in A is sometimes useful. We define the *one-sided dimension* of the boundary F of a set A in $\mathbb{R}^n$ as

$$n - \lim_{\delta \to 0} \frac{\log \mathrm{vol}^n (F_\delta \cap A)}{\log \delta} \tag{3.32}$$

where F_δ is the δ-neighbourhood of F (compare Proposition 3.2). This definition has applications to the surface physics of solids where it is the volume very close to the surface that is important and also to partial differential equations in domains with fractal boundaries.

It is sometimes possible to define dimension in terms of the complement of a set. Suppose F is obtained by removal of a sequence of intervals $I_1, I_2, \ldots$ from, say, the unit interval [0, 1], as, for example, in the Cantor set construction. We may define a dimension as the number s_0 such that the series

$$\sum_{j=1}^{\infty} |I_j|^s \text{ converges if } s < s_0 \text{ and diverges if } s > s_0; \tag{3.33}$$

the number s_0 is called the *critical exponent* of the series. For the middle third Cantor set, this series is $\sum_{k=1}^{\infty} 2^{k-1} 3^{-ks}$, giving $s_0 = \log 2/\log 3$, equal to the Hausdorff and box dimensions in this case. In general, s_0 equals the upper box dimension of F.

Dimension prints provide an interesting variation on Hausdorff dimension of a rather different nature. Dimension prints may be thought of as a sort of 'fingerprint' that enables sets with differing characteristics to be distinguished, even though they may have the same Hausdorff dimension. In particular they reflect non-isotropic features of a set.

We restrict attention to subsets of the plane, in which case the dimension print will also be planar. The definition of dimension prints is very similar to that of Hausdorff dimension but coverings by rectangles are used with side lengths replacing diameters. Let U be a rectangle (the sides need not be parallel to the coordinate axes) and let $a(U) \geqslant b(U)$ be the lengths of the sides of U. Let s, t be non-negative numbers. For F a subset of $\mathbb{R}^2$, let

$$\mathcal{H}_\delta^{s,t}(F) = \inf\left\{\sum_i a(U_i)^s b(U_i)^t : \{U_i\} \text{ is a } \delta\text{-cover of } F \text{ by rectangles}\right\}.$$

In the usual way, we get measures of 'Hausdorff type', $\mathcal{H}^{s,t}$, by letting $\delta \to 0$:

$$\mathcal{H}^{s,t}(F) = \lim_{\delta\to 0} \mathcal{H}_\delta^{s,t}(F).$$

(Note that $\mathcal{H}^{s,0}$ is just a minor variant of s-dimensional Hausdorff measure where only rectangles are allowed in the δ-covers.) The *dimension print*, print F, of F is defined to be the set of non-negative pairs (s, t) for which $\mathcal{H}^{s,t}(F) > 0$.

Using standard properties of measures, it is easy to see that we have monotonicity

$$\text{print } F_1 \subset \text{print } F_2 \quad \text{if } F_1 \subset F_2 \tag{3.34}$$

and countable stability

$$\text{print}\left(\bigcup_{i=1}^{\infty} F_i\right) = \bigcup_{i=1}^{\infty} \text{print } F_i. \tag{3.35}$$

Moreover, if (s, t) is a point in print F and (s', t') satisfies

$$\begin{aligned} s' + t' &\leqslant s + t \\ t' &\leqslant t \end{aligned} \tag{3.36}$$

then (s', t') is also in print F.

Unfortunately, dimension prints are not particularly easy to calculate. We display a few known examples in figure 3.3. Notice that the Hausdorff dimension of a set is given by the point where the edge of its print intersects the x-axis.

Dimension prints are a useful and appealing extension of the idea of Hausdorff dimension. Notice how the prints in the last two cases distinguish between two sets of Hausdorff (or box) dimension $1\frac{1}{2}$, one of which is dust-like, the other stratified.

One disadvantage of dimension prints defined in this way is that they are *not* Lipschitz invariants. The straight line segment and smooth convex curve are bi-Lipschitz equivalent, but their prints are different. In the latter case the dimension

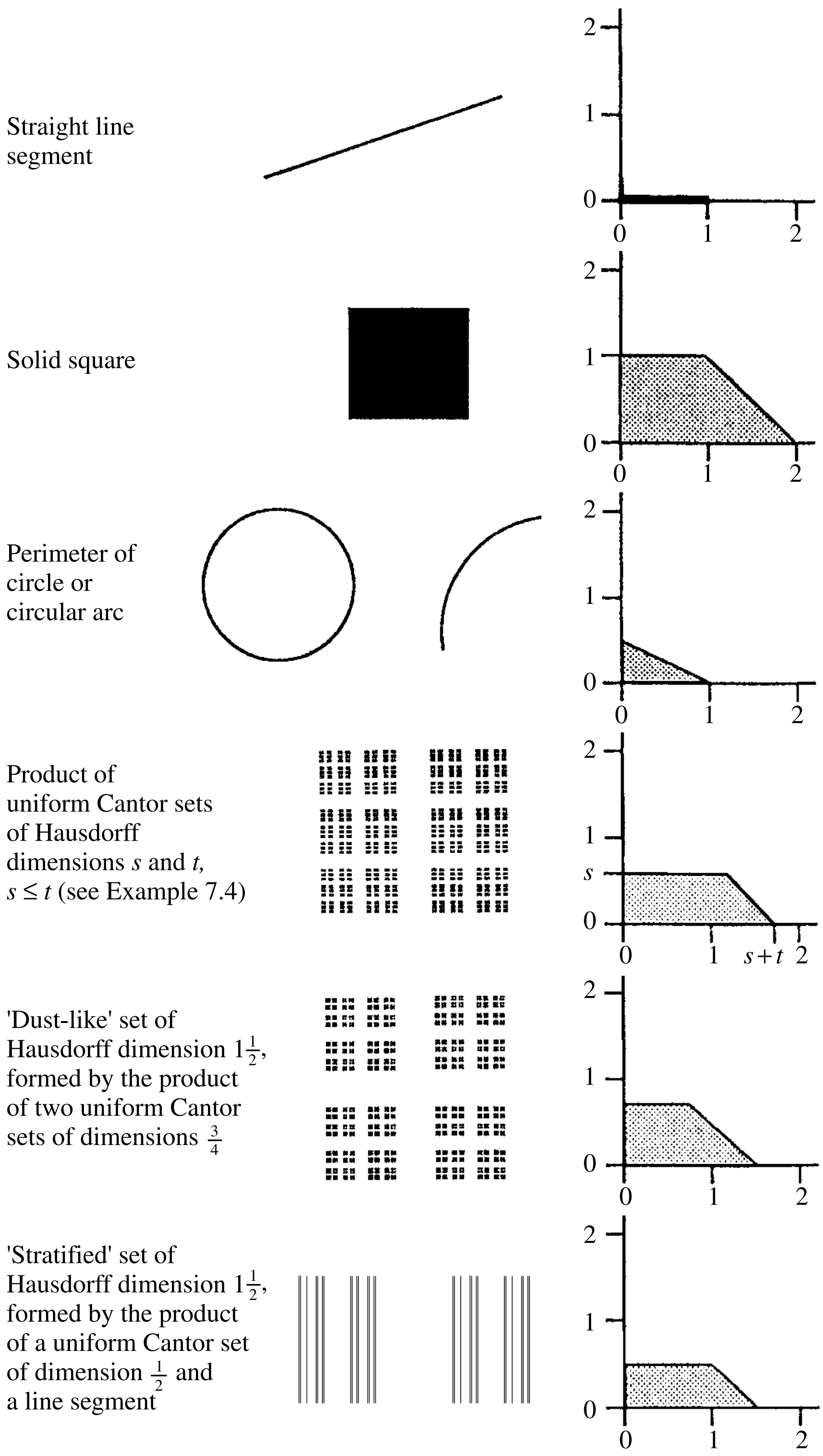

Figure 3.3 A selection of dimension prints of plane sets

print takes into account the curvature. It would be possible to avoid this difficulty by redefining print F as the set of (s, t) such that $\mathcal{H}^{s,t}(F') > 0$ for all bi-Lipschitz images F' of F. This would restore Lipschitz invariance of the prints, but would add further complications to their calculation.

Of course, it would be possible to define dimension prints by analogy with box dimensions rather than Hausdorff dimensions, using covers by equal rectangles. Calculations still seem awkward.

3.6 Notes and references

Many different definitions of 'fractal dimension' are scattered throughout the mathematical literature. The origin of box dimension seems hard to trace—it seems certain that it must have been considered by the pioneers of Hausdorff measure and dimension, and was probably rejected as being less satisfactory from a mathematical viewpoint. Bouligand adapted the Minkowski content to non-integral dimensions in 1928, and the more usual definition of box dimension was given by Pontrjagin and Schnirelman in 1932.

Packing measures and dimensions are much more recent, introduced by Tricot (1982). Their similarities and contrasts to Hausdorff measures and dimensions have proved an important theoretical tool. Packing measures and box and packing dimensions are discussed in Mattila (1995) and Edgar (1998). Dimensions of curves are considered by Tricot (1995).

Dimension prints are an innovation of Rogers (1988, 1998).

Exercises

3.1 Let $f : F \to \mathbb{R}^n$ be a Lipschitz function. Show that $\underline{\dim}_B f(F) \leqslant \underline{\dim}_B F$ and $\overline{\dim}_B f(F) \leqslant \overline{\dim}_B F$. More generally, show that if f satisfies a Hölder condition $|f(x) - f(y)| \leqslant c|x - y|^\alpha$ where $c > 0$ and $0 < \alpha \leqslant 1$ then $\underline{\dim}_B f(F) \leqslant \frac{1}{\alpha}\underline{\dim}_B f(F)$.

3.2 Verify directly from the definitions that Equivalent definitions 3.1(i) and (iii) give the same values for box dimension.

3.3 Let F consist of those numbers in $[0, 1]$ whose decimal expansions do not contain the digit 5. Find $\dim_B F$, showing that this box dimension exists.

3.4 Verify that the Cantor dust depicted in figure 0.4 has box dimension 1 (take E_0 to have side length 1).

3.5 Use Equivalent definition 3.1(iv) to check that the upper box dimension of the von Koch curve is at most $\log 4/\log 3$ and 3.1(v) to check that the lower box dimension is at least this value.

3.6 Use convenient parts of Equivalent definition 3.1 to find the box dimension of the Sierpiński triangle in figure 0.3.

3.7 Let F be the middle third Cantor set. For $0 < \delta < 1$, find the length of the δ-neighbourhood F_δ of F, and hence find the box dimension of F using Proposition 3.2.

3.8 Construct a set F for which $\underline{\dim}_B F < \overline{\dim}_B F$. (Hint: let $k_n = 10^n$, and adapt the Cantor set construction by deleting, at the kth stage, the middle $\frac{1}{3}$ of intervals if $k_{2n} < k \leqslant k_{2n+1}$, but the middle $\frac{3}{5}$ of intervals if $k_{2n-1} < k \leqslant k_{2n}$.)

3.9 Verify that $\overline{\dim}_B(E \cup F) = \max\{\overline{\dim}_B E, \overline{\dim}_B F\}$ for bounded E, $F \subset \mathbb{R}$.

3.10 Find subsets E and F of $\mathbb{R}$ such that $\underline{\dim}_B(E \cup F) > \max\{\underline{\dim}_B E, \underline{\dim}_B F\}$. (Hint: consider two sets of the form indicated in Exercise 3.8.)

3.11 What are the Hausdorff and box dimensions of the set $\{0, 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\}$?

3.12 Find two disjoint Borel subsets E and F of $\mathbb{R}$ such that $\mathcal{P}_0^s(E \cup F) \neq \mathcal{P}_0^s(E) + \mathcal{P}_0^s(F)$.

3.13 What is the packing dimension of the von Koch curve?

3.14 Find the divider dimension (3.31) of the von Koch curve.

3.15 Show that the divider dimension (3.31) of a curve is greater than or equal to its box dimension, assuming that they both exist.

3.16 Let $0 < \lambda < 1$ and let F be the 'middle λ Cantor set' obtained by repeated removal of the middle proportion λ from intervals. Show that the dimension of F defined by (3.33) in terms of removed intervals equals the Hausdorff and box dimensions of F.

3.17 Verify properties (3.34)–(3.36) of dimension prints. Given an example of a set with a non-convex dimension print.