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Resonance oscillations in a mass-spring impact oscillator

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Abstract We investigate the presence of asymptotically stable periodic oscillations in a time-periodic impact oscillator close to an isochronous one. A new averaging method is developed to account for the position of the obstacle and for the impact restitution coefficient, which don't appear in the classical smooth situation.

Keywords Asymptotic stability · Periodic solutions · Impact oscillator · Averaging method · Perturbation approach

1 Introduction

If a linear oscillator with an eigenfrequency w is being forced by a small periodic excitation and the frequency of the excitation is close to w, then the equation of the motion reads as

$$\ddot{x} + w^2 x = \varepsilon f(t, x, \dot{x}, \varepsilon), \tag{1}$$

where $\varepsilon > 0$ is a small parameter and f is a smooth $2\pi/w$-periodic in time function. A possible way to prove the occurrence and stability of resonance oscillations in (1) is known as the *method of averaging*, see [15,29,35]. One of the conclusions of this method is that the amplitude a and phase shift ϕ of $2\pi/w$-periodic oscillations in (1) are close to the zeros of

$$\overline{F}(a,\phi) = -\int_0^{2\pi} \begin{pmatrix} \sin(\tau+\phi) \\ \frac{1}{a}\cos(\tau+\phi) \end{pmatrix} \circ \\ \circ f(\tau, a\cos(\tau+\phi), -a\sin(t+\phi), 0)d\tau,$$

known as *averaging function.* Specifically, if $\overline{F}(a_0,\phi_0) = 0$ and the real parts of the eigenvalues of $\overline{F}'(a_0,\phi_0)$ are negative, then, for all $\varepsilon > 0$ sufficiently small, equation (1) has an asymptotically stable $2\pi/w$-periodic solution that approaches

$$t \mapsto (a\cos(t+\phi), -a\sin(t+\phi)) \tag{2}$$

as $\varepsilon \to 0$. This approach is often referred to as the Van der Pol method or the second Bogolyubov's theorem [4]. The phenomenon which occurs when ε crosses 0 can be viewed as a bifurcation because the family of π-periodic cycles that corresponds to $\varepsilon = 0$ gets destroyed when ε deviates from 0. Those π-periodic that persist can gain asymptotic stability and they are termed resonance periodic solutions in such a case. In the simplest case, where equation (1) has the form

$$\ddot{x} + \varepsilon c\dot{x} + w^2 x + \varepsilon r x = \varepsilon b\cos(wt), \tag{3}$$

the method of averaging provides the existence of an asymptotically stable $2\pi/w$-periodic solution near the cycle (2) with

$$a = \frac{b}{\sqrt{w^2c^2 + r^2}}, \quad \phi = -\arccos\frac{r}{\sqrt{w^2c^2+r^2}}, \tag{4}$$

see e.g. [7]. In particular, the amplitude of $2\pi/w$-periodic resonance solutions in (3) increases infinitely when both the damping coefficient $c > 0$ and the detuning coefficient $r > 0$ approach zero.

The goal of this paper is to investigate the occurrence (bifurcation) of resonance oscillations in mechanical oscillators with impacts, where the presence of the obstacle makes the analysis more interesting. A prototypic example of an oscillator of this type is given by (see Fig. 1)

$$\ddot{x} + \varepsilon c\dot{x} + w^2 x + \varepsilon r x = \varepsilon b\cos(wt), \tag{5}$$

$$(1-\varepsilon\mu)\dot{x}(t-0) = -\dot{x}(t+0), \quad \text{if } x(t) = \varepsilon d, \tag{6}$$

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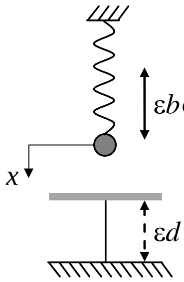

Fig. 1 A driven impact oscillator whose obstacle is εd-distant from the rest position and which is governed by equations (5)-(6).

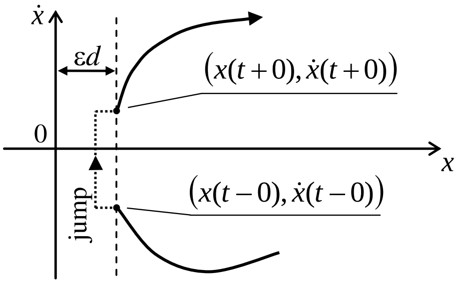

Fig. 2 A trajectory of the impact oscillator (5)-(6) depicted at Fig. 1.

which is equation (3) coupled with the Newton's impact law. Each trajectory x of oscillator (5)-(6) is governed by equation (5) until the trajectory hits the obstacle $x = \varepsilon d$ at time t, when the impact law (6) applies. The impact law sends the trajectory from the point $(x(t-0), \dot{x}(t-0))$ to $(x(t+0), \dot{x}(t+0)) = (x(t-0), -(1-\varepsilon\mu)\dot{x}(t-0))$ instantaneously and the motion along (5) continues, see Fig. 2. The collisions with the obstacle located at $x = \varepsilon d$ are absolutely elastic, if the restitution coefficient $1 - \varepsilon\mu$ equals to 1. It is natural to expect that the amplitude of resonance oscillations of the impact oscillator (5)-(6) is now proportional to $\frac{1}{|wc|+|r|+|\mu|}$. But what is the influence of d? In this paper we answer this question by deriving an analogue of the averaging function $\overline{F}$ for the second-order impact oscillator of the following general form

$$\begin{aligned} &\ddot{x} + w^2 x = \varepsilon f(t, x, \dot{x}, \varepsilon), \\ &(1-\varepsilon\mu)\dot{x}(t-0) = -\dot{x}(t+0), \quad \text{if } x(t) = \varepsilon d. \end{aligned} \tag{7}$$

The averaging of the impact oscillator (7) is discussed in Zhuravlev-Klimov [39, § 27] under the assumption that $\mu = 0$. The method employed in [39] is the discontinuous transformation of Zhuravlev-Ivanov (see Brogliato [5], Pilipchuk-Ibrahim [32]). The symmetry property $\mu = 0$ lies in heart of this method, so it is not straightforward to extend the method of [39] to (7) with $\mu > 0$, which is our main interest. Important results on averaging of impact systems are obtained by Samoylenko and Perestyuk, see [31]. However, the impacts in [31] are deemed as collisions with a surface $t = \tau(x, \dot{x})$ in the extended phase space rather than collisions with an obstacle $x = const$, that we work with here. In the autonomous case, an averaging approach to the dynamics of coupled impact oscillators is implemented in Sartorelli-Lacarbonara [33], where the small parameter ε comes from a suitable scaling. A fundamental technique to justify averaging of periodic impact oscillators is developed in Burd [9], but the focus of [9] is on the slow-fast time scales.

The impact oscillator (7) is a fundamental model of mechanics. It describes the dynamics of such important industrial systems as gear pairs [28,25], pressure relief valves [16], ocean systems [18], robot locomotion [34,36], cutting [13] and drilling [10] setups, etc. Impact oscillators are also used in neuroscience to model integrate-and-fire and resonate-and-fire neurons, see Coombes-Thul-Wedgwood [12]. Similar impact systems arise in the context of population models with impulsive feedback control [38,37,24] or state-dependent impulsive harvesting [17]. Many other applications are surveyed in [26]. The books by Babitskiy [1] and Babitsky-Krupenin [2] provide a general framework to identify resonances in impact oscillators using the so-called first Bogolyubov's theorem, which provides an approximation of the dynamics of (7) on time-intervals of order $1/\varepsilon$ (see also [20]). In particular, we refer the reader to [2, §6.5], where the authors derive the averaging functions for the impact oscillator (7) expressed in slightly different terms. The result of this paper is complementary, as we provide a way to rigorously prove asymptotic stability of the resonances. The literature on impact oscillators features several results about stability of equilibria (see e.g. Leine-Heimsch [23]) and about bifurcation of resonance homoclinic solutions (see e.g. Battelli-Feckan [3]). But, apart from the situations where the solution can be found in the closed form (see e.g. Okninski-Radziszewski [30]), the intermediate problem about bifurcation of resonance periodic solutions lacked rigorous description so far. An important step to fill in this gap has been recently made in Feckan-Pospisil [14], where the persistence of periodic orbits in oscillators of form (7) (of any dimension) is addressed in the case where the obstacle is fixed. Our paper is a somewhat parallel step where the main focus is on stability.

The paper is organized as follows. Next section is devoted to the main result of the paper (theorem 1) that links asymptotically stable periodic solutions in (7) to a suitable averaging function. An application of theorem 1 to the prototypic impact oscillator (5)-(6) is given in section 3 (theorem 2), where we derive formulas for the amplitudes and phase shifts of above-mentioned stable periodic solutions. The conclusions are summarized in section 4, followed by an acknowledgment section.

2 The main result

In this section we derive condition for the occurrence of resonance periodic solutions in the impact oscillator (7). Without loss of generality one can take $w = 1$ and rewrite (7) as

$$\begin{aligned} &\ddot{x} + x = \varepsilon f(t, x, \dot{x}, \varepsilon), \\ &(1-\varepsilon\mu)\dot{x}(t-0) = -\dot{x}(t+0), \quad \text{if } x(t) = \varepsilon d. \end{aligned} \tag{8}$$

Our main result is that the asymptotically stable π-periodic solutions in (8) correspond to zeros of the following bifurcation function

$$\overline{F}(a,\phi) = \begin{pmatrix} -\mu a \\ 2d/a \end{pmatrix} + \int_0^{\pi/2-\phi} F\left(\tau, \begin{pmatrix} a \\ \phi \end{pmatrix}, 0\right) d\tau + \\ + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \int_{\pi/2-\phi}^{\pi} F\left(\tau, \begin{pmatrix} -a \\ \phi \end{pmatrix}, 0\right) d\tau,$$

where

$$F\left(t, \begin{pmatrix} a \\ \phi \end{pmatrix}, \varepsilon\right) = \\ = -\begin{pmatrix} \sin(t+\phi) f(t, a\cos(t+\phi), -a\sin(t+\phi), \varepsilon) \\ \frac{1}{a}\cos(t+\phi) f(t, a\cos(t+\phi), -a\sin(t+\phi), \varepsilon) \end{pmatrix}.$$

Specifically, the following theorem holds.

Theorem 1 *Let $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function, such that $\overline{F}(a_0, \phi_0) = 0$ for some $a_0 > 0$ and $-\pi/2 < \phi_0 < \pi/2$. Assume that the real parts of the eigenvalues of $\overline{F}'(a_0, \phi_0)$ are negative. Then, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the impact oscillator (7) has a unique asymptotically stable π-periodic solution x_ε that satisfies*

$$(x_\varepsilon(0), \dot{x}_\varepsilon(0)) \to (a_0\cos\phi_0, -a_0\sin\phi_0) \quad \text{as} \quad \varepsilon \to 0. \tag{9}$$

Proof For convenience of the reader the proof is split into 3 steps.

<u>**Step 1:**</u> *Transforming (8) to the standard form of averaging.* This transformation will be a special form of the angle-action change of coordinates. Introduce $S : (0,\infty)\times(-\pi/2, \pi/2) \to \mathbb{R}^2$ as

$$S\begin{pmatrix} a \\ \theta \end{pmatrix} = \begin{pmatrix} a\cos\theta \\ -a\sin\theta \end{pmatrix}$$

and fix such an open neighborhood V of $S\begin{pmatrix} a_0 \\ \phi_0 \end{pmatrix}$ that $V \cap \{(x,y) : x = 0\} = \emptyset$. We will take the values of $\varepsilon > 0$ so small that V doesn't intersect $x = \varepsilon d$ either. Let $K \in V$ and consider a solution $t \mapsto u(t) = (u_1(t), u_2(t))$ of

$$\begin{cases} \dot{u}_1 = u_2, \\ \dot{u}_2 = -u_1 + \varepsilon f(t, u_1, u_2, \varepsilon), \\ u_2(t+0) = -(1-\varepsilon\mu)u_2(t-0), \text{ if } u_1(t) = \varepsilon d \end{cases} \tag{10}$$

that originates at K. Our first step consists in introducing the action-angle-like change of the variables

$$u(t) = S\begin{pmatrix} a(t) \\ t+\phi(t) \end{pmatrix}, \text{ if } t \in \mathbb{R} \text{ and } u_1(t) \neq \varepsilon d, \tag{11}$$

or, equivalently, in finding differential equations for two functions $t \mapsto (a(t), \phi(t))$ that solve (11) and satisfy $\begin{pmatrix} a(0) \\ \phi(0) \end{pmatrix} = S^{-1}(K)$. Similarly to how (11) works in the classical method of averaging, one gets

$$\begin{pmatrix} \dot{a} \\ \dot{\phi} \end{pmatrix} = \varepsilon F\left(t, \begin{pmatrix} a \\ \phi \end{pmatrix}, \varepsilon\right), \quad \text{if } a(t)\cos(t+\phi(t)) \neq \varepsilon d. \tag{12}$$

Equation (12) governs $(a(t), \phi(t))$ until $S\begin{pmatrix} a(t) \\ \phi(t) \end{pmatrix}$ reaches $x = \varepsilon d$, which moment of time we denote by $T(a, \phi, \varepsilon)$.

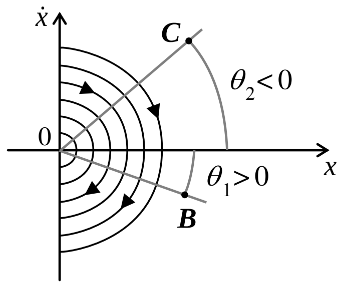

Fig. 3 The directions of the flow of (8) at $\varepsilon = 0$ and two sample points $B = (a\cos\theta_1, -a\sin\theta_1)$ with $\theta_1 \in (0, \pi/2)$ and $C = (a\cos\theta_2, -a\sin\theta_2)$ with $\theta_2 \in (-\pi/2, 0)$.

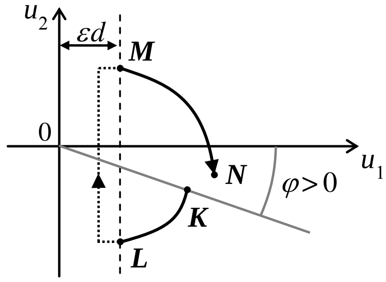

Fig. 4 Four important points on a sample trajectory $t \mapsto (u_1(t), u_2(t)) = (x(t), \dot{x}(t))$ of (8) and (10) during the time interval $[0, \pi]$ (the case of positive initial angle):
$K = (u_1(0), u_2(0)) = (a\cos\phi, -a\sin\phi)$,
$L = (u_1, u_2)(T(a,\phi,\varepsilon) - 0) =$
$= (a_L\cos(T(a,\phi,\varepsilon) + \phi_L), -a_L\sin(T(a,\phi,\varepsilon) + \phi_L))$,
$M = (u_1, u_2)(T(a,\phi,\varepsilon) + 0) =$
$= (a_M\cos(T(a,\phi,\varepsilon) + \phi_M), -a_M\sin(T(a,\phi,\varepsilon) + \phi_M))$,
$N = (u_1, u_2)(\pi) = (a_N\cos(\pi + \phi_N), -a_N\sin(\pi + \phi_N))$.

Let $\begin{pmatrix} a_L \\ \phi_L \end{pmatrix} = \begin{pmatrix} a \\ \phi \end{pmatrix}(T(a,\phi,\varepsilon))$. To comply with the jump in (10) (i.e. in order that (11) hold for all t from a neighborhood of $T(a,\phi,\varepsilon)$ except of $T(a,\phi,\varepsilon)$ itself), the trajectory $t \mapsto (a(t),\phi(t))$ must undergo a jump from $\begin{pmatrix} a_L \\ \phi_L \end{pmatrix}$ to a new point $\begin{pmatrix} a_M \\ \phi_M \end{pmatrix}$ at $t = T(a,\phi,\varepsilon)$ (see Fig. 4) and

$$\begin{aligned} &a_M\cos(T(a,\phi,\varepsilon)+\phi_M) = a_L\cos(T(a,\phi,\varepsilon)+\phi_L), \\ &a_M\sin(T(a,\phi,\varepsilon)+\phi_M) = \\ &\qquad = -(1-\varepsilon\mu)a_L\sin(T(a,\phi,\varepsilon)+\phi_L). \end{aligned}$$

This gives

$$\begin{pmatrix} a_M \\ \phi_M \end{pmatrix} = S^{-1}\left[\begin{pmatrix} 1 & 0 \\ 0 & -(1-\varepsilon\mu) \end{pmatrix} \circ \right.$$
$$\left. \circ\, S\left(\begin{pmatrix} 0 \\ T(a,\phi,\varepsilon) \end{pmatrix} + \begin{pmatrix} a_L \\ \phi_L \end{pmatrix}\right)\right] - \begin{pmatrix} 0 \\ T(a,\phi,\varepsilon) \end{pmatrix}$$

and, therefore, the equation (12) must be complemented by the following impact law

$$\begin{pmatrix} a \\ \phi \end{pmatrix}(t+0) = S^{-1}\left[\begin{pmatrix} 1 & 0 \\ 0 & -(1-\varepsilon\mu) \end{pmatrix} \circ \right.$$
$$\left. \circ S\begin{pmatrix} a(t-0) \\ t+\phi(t-0) \end{pmatrix}\right] - \begin{pmatrix} 0 \\ t \end{pmatrix}, \qquad (13)$$
if $a(t)\cos(t+\phi(t)) = \varepsilon d$.

A function $t \mapsto (a(t),\phi(t))$ is now a solution of (12)-(13) if and only if $(u_1(t),u_2(t)) = (a(t)\cos(t+\phi(t)), -a(t)\sin(t+\phi(t)))$ is a solution of (10). Denote by P_ε the Poincaré map of (12)-(13) over period π, in particular $P_\varepsilon(K) = N$ (see Fig. 4). If $u_1 > 0$, then

$$Q_\varepsilon\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = S\left(P_\varepsilon\left(S^{-1}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) + \begin{pmatrix} 0 \\ \pi \end{pmatrix}\right)$$

for all $\varepsilon > 0$ sufficiently small. Furthermore, the eigenvalues of $(Q_\varepsilon)'\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $(P_\varepsilon)'\left(S^{-1}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right)$ coincide.

Step 2: *Expanding the Poincaré map P_ε in powers of ε and deriving a closed form of the leading term.* Denote by

$$t \mapsto \begin{pmatrix} A \\ \Phi \end{pmatrix}\left(t, t_0, \begin{pmatrix} a \\ \phi \end{pmatrix}, \varepsilon\right)$$

the solution $t \mapsto \begin{pmatrix} a(t) \\ \phi(t) \end{pmatrix}$ of (12) with the initial conditions $a(t_0) = a$, $\phi(t_0) = \phi$. Then

$$P_\varepsilon\begin{pmatrix} a \\ \phi \end{pmatrix} = \begin{pmatrix} A \\ \Phi \end{pmatrix}\left(\pi, T(a,\phi,\varepsilon), \begin{pmatrix} a_M \\ \phi_M \end{pmatrix}, \varepsilon\right) = \begin{pmatrix} a_M \\ \phi_M \end{pmatrix} +$$
$$+\varepsilon \int\limits_{T(a,\phi,\varepsilon)}^{\pi} F\left(\tau, \begin{pmatrix} A \\ \Phi \end{pmatrix}\left(\pi, T(a,\phi,\varepsilon), \begin{pmatrix} a_M \\ \phi_M \end{pmatrix}, \varepsilon\right), \varepsilon\right) d\tau$$

To extract the leading term in the expansion of P_ε in powers of ε, we observe the following

$$\begin{aligned} &T(a,\phi,0) = \pi/2 - \phi, \\ &\begin{pmatrix} A'_t \\ \Phi'_t \end{pmatrix}(\pi/2-\phi, 0, a, \phi, 0) = 0, \\ &\begin{pmatrix} A'_\varepsilon \\ \Phi'_\varepsilon \end{pmatrix}(\pi/2-\phi, 0, a, \phi, 0) = \int\limits_0^{\pi/2-\phi} F\left(\tau, \begin{pmatrix} a \\ \phi \end{pmatrix}, 0\right) d\tau, \\ &T'_\varepsilon(a,\phi,0) = -\int\limits_0^{\pi/2-\phi} F_2\left(\tau, \begin{pmatrix} a \\ \phi \end{pmatrix}, 0\right) d\tau - \tfrac{d}{a}, \\ &S\begin{pmatrix} a \\ \pi/2 \end{pmatrix} = \begin{pmatrix} 0 \\ -a \end{pmatrix}, \qquad S'\begin{pmatrix} a \\ \pi/2 \end{pmatrix} = \begin{pmatrix} 0 & -a \\ -1 & 0 \end{pmatrix}, \\ &S^{-1}\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} a \\ -\pi/2 \end{pmatrix}, \quad (S^{-1})'\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1/a & 0 \end{pmatrix}, \\ &\begin{pmatrix} a_L \\ \phi_L \end{pmatrix}\Big|_{\varepsilon=0} = \begin{pmatrix} a \\ \phi \end{pmatrix}, \qquad \begin{pmatrix} a_M \\ \phi_M \end{pmatrix}\Big|_{\varepsilon=0} = \begin{pmatrix} a \\ \phi-\pi \end{pmatrix}. \end{aligned}$$

These formulas allow us to conclude

$$\begin{aligned} P_\varepsilon\begin{pmatrix} a \\ \phi \end{pmatrix} &= P_0\begin{pmatrix} a \\ \phi \end{pmatrix} + \varepsilon\left(\tfrac{\partial}{\partial\varepsilon}P_\varepsilon(a,\phi)\big|_{\varepsilon=0}\right) + \\ &\quad +\varepsilon^2 R(a,\phi,\varepsilon) = \\ &= \begin{pmatrix} a \\ \phi \end{pmatrix} + \varepsilon\left(\frac{\partial}{\partial\varepsilon}\begin{pmatrix} a_M \\ \phi_M \end{pmatrix}\Big|_{\varepsilon=0}\right) + \\ &\quad +\varepsilon\int\limits_{\pi/2-\phi}^{\pi} F\left(\tau, \begin{pmatrix} a_M \\ \phi_M \end{pmatrix}\Big|_{\varepsilon=0}, 0\right) d\tau + \varepsilon^2 R(a,\phi,\varepsilon) \\ &= \begin{pmatrix} a \\ \phi \end{pmatrix} + \varepsilon\left(\begin{pmatrix} -\mu a \\ 2d/a \end{pmatrix} + \int\limits_0^{\pi/2-\phi} F\left(\tau, \begin{pmatrix} a \\ \phi \end{pmatrix}, 0\right) d\tau\right) + \\ &\quad +\varepsilon\int\limits_{\pi/2-\phi}^{\pi} F\left(\tau, \begin{pmatrix} a \\ \phi-\pi \end{pmatrix}, 0\right) d\tau + \varepsilon^2 R(a,\phi,\varepsilon) = \\ &= \begin{pmatrix} a \\ \phi \end{pmatrix} + \varepsilon\overline{F}(a,\phi) + \varepsilon^2 R(a,\phi,\varepsilon). \end{aligned}$$

We of course refer to the Implicit Function Theorem and to the differentiability of the implicit function (Krantz-Parks [22] or Kolmogorov-Fomin [21]) to ensure the correctness of the above expansion.

Step 3: *Making conclusions about the fixed points of P_ε based on the properties of the leading term.* Since $\overline{F}(a_0,\phi_0) = 0$ and $\overline{F}'(a_0,\phi_0)$ is invertible, there exists $\varepsilon_0 > 0$ (that can be chosen as small as possible) such that for any $\varepsilon \in (0,\varepsilon_0)$ the map

$$(a,\phi) \mapsto \overline{F}(a,\phi) + \varepsilon R(a,\phi,\varepsilon) \qquad (14)$$

has a unique zero $(a_\varepsilon,\phi_\varepsilon)$ in the ε_0-neighborhood of (a_0,ϕ_0) and $(a_\varepsilon,\phi_\varepsilon) \to (a_0,\phi_0)$ as $\varepsilon \to 0$. Zeros of (14) coincide with the fixed points of P_ε and it remains to show that the absolute values of the eigenvalues of $(P_\varepsilon)'\begin{pmatrix} a_\varepsilon \\ \phi_\varepsilon \end{pmatrix}$ don't exceed 1. Now we use the following algebraic fact: *if λ is an eigenvalue of the square matrix A, then $1+\varepsilon\lambda$ is an eigenvalue of $I+\varepsilon A$.* Therefore, since the absolute values of the eigenvalues of $\overline{F}'(a_0,\phi_0)$

are negative, the constant $\varepsilon_0 > 0$ can be chosen so small, that the absolute values of the eigenvalues of

$$I + \varepsilon\left(\overline{F}'(a_\varepsilon, \phi_\varepsilon) + \varepsilon R'_{(a,\phi)}(a_\varepsilon, \phi_\varepsilon, \varepsilon)\right)$$

don't exceed 1. □

3 Application to the mass-spring impact oscillator

The change of the variables $\tilde{x} = x(wt)$ brings (5)-(6) to the form

$$\begin{aligned} &\ddot{x} + \varepsilon c\dot{x} + x + \varepsilon r x = \varepsilon b\cos(t), \\ &(1 - \varepsilon\mu)\dot{x}(t-0) = -\dot{x}(t+0), \quad \text{if } x(t) = \varepsilon d. \end{aligned} \tag{15}$$

We now apply theorem 1 in order to locate asymptotically stable π-periodic solutions (resonances) in (15).

Theorem 2 *Let $b > 3d \geq 0$, $\mu \geq 0$, $c \geq 0$ and $\mu + c \neq 0$. Then there exists $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ the impact oscillator (15) has a unique asymptotically stable π-periodic solution x_ε satisfying*

$$(x_\varepsilon(0), \dot{x}_\varepsilon(0)) \to (a_0\cos\phi_0, -a_0\sin\phi_0) \quad \text{as} \quad \varepsilon \to 0, \tag{16}$$

where a_0 is the unique real positive root of the quadratic polynomial

$$16b^2 = 9a^2\left(-\mu - (\pi/2)c\right)^2 + (3\pi r a + 12d)^2 \tag{17}$$

and $\phi_0 \in (-\pi/2, \pi/2)$ is the unique solution of

$$\cos^2 2\phi = \frac{16b^2 - 9a^2\left(-\mu - (\pi/2)c\right)^2}{16b^2} \tag{18}$$

which verifies $\cos 2\phi > 0$ and $\sin 2\phi < 0$.

Proof Computing $\overline{F}$ gives

$$\overline{F}(a,\phi) = \frac{1}{2}\begin{pmatrix} -2\mu a - \pi a c \\ 4d/a + \pi r \end{pmatrix} - \frac{2b}{3a}\begin{pmatrix} 2a\sin 2\phi \\ \cos 2\phi \end{pmatrix} \tag{19}$$

and

$$\overline{F}'(a,\phi) = \begin{pmatrix} -\mu - \frac{\pi}{2}c & -\frac{8}{3}b\cos 2\phi \\ -\frac{2d}{a^2} + \frac{2b}{3a^2}\cos 2\phi & \frac{4b}{3a}\sin 2\phi \end{pmatrix}.$$

Equaling $\overline{F}(a,\phi)$ to zero one obtains

$$4b\sin 2\phi = -3\mu a - (3/2)\pi a c, \tag{20}$$

$$4b\cos 2\phi = 3a\pi r + 12d. \tag{21}$$

Observe, that any solution (a,ϕ) of (17)-(18) such that $\sin 2\phi < 0$ and $\cos 2\phi > 0$ verifies (20)-(21). In order to see that (17) has an unique real positive solution, we rewrite (17) as

$$\alpha a^2 + \beta a + \gamma = 0, \tag{22}$$

where $\alpha = 9(\pi^2 r^2 + (\mu + (\pi/2)c)^2)$, $\beta = 72\pi r d$, $\gamma = 12^2 d^2 - 16b^2$. Since $\alpha > 0$ and $\beta > 0$, (22) has a real positive root if and only if $\gamma < 0$, which follows from the hypothesis $b > 3d$ of the theorem. The second root of (22) is always negative.

It now remains to examine the eigenvalues of $\overline{F}'(a,\phi)$. The real parts of these eigenvalues are negative, if

$$\begin{aligned} \text{trace}: &\left(-\mu - \frac{\pi}{2}c\right) + \frac{4b}{3a}\sin 2\phi < 0, \\ \text{determinant}: &\frac{16b^2}{9a^2}\cos^2 2\phi - \frac{16bd}{3a^2}\cos 2\phi + \\ &+\left(-\mu - \frac{\pi}{2}c\right)\frac{4b}{3a}\sin 2\phi > 0, \end{aligned}$$

which holds true, provided that $b > 0$, $\mu + (\pi/2)c > 0$, $\sin 2\phi < 0$, $\cos 2\phi > 0$ and $b\cos 2\phi - 3d > 0$. All these inequalities are just the assumptions of the theorem and $b\cos 2\phi - 3d > 0$ follows from (21).

□

Remark 1 The positive root of (17) is given by

$$a = \frac{-12\pi r d + \sqrt{D}}{3(\pi^2 r^2 + (\mu + (\pi/2)c)^2)}, \tag{23}$$

where $D = (12\pi r d)^2 - (\pi^2 r^2 + (\mu + (\pi/2)c)^2) \cdot (12^2 d^2 - 4^2 b^2)$. Therefore, (23) is an ε-approximation of the amplitude of π-periodic asymptotically stable oscillations of (15) provided that the assumptions of theorem 2 hold.

Remark 2 If the collisions are purely symmetric (i.e. $\mu = d = 0$), (23) reduces to

$$a = \frac{4b}{3\sqrt{\pi^2 r^2 + (\pi/2)^2 c^2}}. \tag{24}$$

Formulas (23) and (24) allow to compare the properties of the resonance solutions of the smooth oscillator (3) with those of the impact one (5)-(6).

4 Conclusion

In this paper we provided sufficient conditions for the occurrence of asymptotically stable π/w-periodic solutions (1:1-resonances) in a periodically driven mass-spring impact oscillator which is close to the following reduced system

$$\begin{aligned} &\ddot{x} + w^2 x = 0, \\ &\dot{x}(t-0) = -\dot{x}(t+0), \quad \text{if } x(t) = 0; \end{aligned} \tag{25}$$

see Fig. 3 for the phase portrait of (25). The obstacle is plugged at the position $x = 0$ in order to make the periods of all the periodic solutions of (25) equal. Such an oscillator is also known as isochronous. Similar to the classical approach that is used in nonlinear dynamics to study resonances in ε-perturbed isochronous systems (see [15, 29, 35]), the ideas of the method of averaging are used in our paper. However, the presence of the impact obstacle suggested two more natural parameters, which are the deviation of the Newton restitution coefficient $\widetilde{\mu}$ from 1 and the deviation of the position $\widetilde{d}$ of the obstacle from $x = 0$. Assuming that the smallness of both these coefficients is of the order of the perturbation (i.e. $\widetilde{\mu} = \varepsilon\mu$ and $\widetilde{d} = \varepsilon d$), we derived an averaging function, whose zeros correspond to those periodic solutions of (25) that produce asymptotically stable π/w-periodic solutions in the respective full system. The conclusion of our paper is that the constant μ plays a role similar to viscous friction as far as 1:1-resonance solutions are concerned. The role of the constant d is new, but (23) suggests that this coefficient diminishes the influence of the external excitation. Finally, when the impact law changes the coordinates of all trajectories symmetrically, the formula for the amplitude (24) of 1:1-resonance solutions is similar to the smooth case (4).

Along the lines of Burd [8] and Chicone [11], our work can be further extended to the cases where the position εd of the obstacle is fixed at some $x = \widetilde{d}$, or where the unperturbed Hamiltonian system is mutlidimensional and nonlinear, or where the perturbation f is almost periodic. Kamenskii-Makarenkov-Nistri [19] provides a dimension reduction scheme that can be used to examine the situations where a perturbed Hamiltonian system with impacts is given in a part of the phase space only. Following the ideas of Buica-Llibre-Makarenkov [6], the asymptotic stability of resonance solutions x_ε can be replaced by attractivity of x_ε in such a neighborhood that doesn't depend on ε (uniform attractivity). The proof of the absence of 1:1 resonances other than that given by theorems 1 and 2 is similar to Makarenkov-Ortega [27, lemma 2].

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