M2A2 Dynamics

Dr. J. Gibbons

March 25, 2003

Contents

1. Introduction.

2. Calculus of variations; constrained extrema. (5 lectures)

3. Systems of particles. Conservation laws. Virial theorem. Lagrange's equations. Examples of integrability. (7 lectures)

4. Hamilton's equations. (4 lectures)

5. Small oscillations near equilibria; normal modes. (4 lectures)

6. Rigid-body dynamics: angular velocity, angular momentum, inertia tensor, principal axes. Euler angles. Euler's equations. Motion under no forces; structure of the phase space. Lagrangian treatment of rigid body motion; the symmetric top. Stability. Examples of forced motion. (6 lectures)

Chapter 1

Introduction

This course concerns the dynamics of systems of particles. Newton's laws lead to the elegant formulations of the theory due to Lagrange and Hamilton. Applications include the approximate description of motion near equilibria, and classification of equilibria according to their stability. Rotating rigid bodies can be studied in detail - the concepts of angular velocity, angular momentum, and the inertia tensor are introduced and applied to systems such as the freely rotating body and the symmetric top or gyroscope.

The key idea is that we want to set up the equations of motion so that we obtain Newton's laws in an inertial frame; but the structure of the equations should not depend on the choice of coordinates used. The most powerful method for ensuring this is to write the equations as a variational principle. Some texts (there are many) covering much of this material are

- Kibble and Berkshire, Classical Mechanics (Longman 1996)
- Landau and Lifshitz, Mechanics (Butterworth-Heinemann 1997)
- Goldstein, Classical Mechanics (Addison-Wesley 1980).

Kibble and Berkshire was originally written for Physics students; it is at about the right level for this course. The latter two were written as graduate texts, but aim to be fully self-contained.

Chapter 2

Calculus of Variations

Many important questions in geometry can be written in variational form:

- what is the shortest path between two points in the plane?
- what is the shortest path between two points in a given surface?
- what is the shortest closed curve enclosing given area?

Further, the laws of optics could be written as a variational principle - Fermat's Principle of Least Time, that the path of a light ray minimises (or extremises) the 'optical path'

$$T = \int_{\mathbf{x_0}}^{\mathbf{x_1}} \frac{|d\mathbf{x}|}{n(\mathbf{x})}. \tag{2.1}$$

Certainly the equations of statics are a simple variational problem - a system of particles interacting via a potential $V(x_1, \ldots, x_N)$ has equilibria at those points where the potential has an extremum - a maximum, minimum or saddle.

Lagrange's idea was that dynamics - that is, Newton's laws for a particle in a potential force could also be written in variational form, as the condition for an extremum of the 'action integral':

$$S = \int_{t_0}^{t_1} \frac{m}{2}|\dot{\mathbf{x}}|^2 - V(\mathbf{x})dt. \tag{2.2}$$

We will see that this condition, the Euler-Lagrange equation, is:

$$m\ddot{\mathbf{x}} = -\nabla V. \tag{2.3}$$

This is clearly the same as Newton's 2nd law. We will see later how this approach can be extended to systems of many particles, perhaps with additional constraints. For instance a pendulum consists of a particle moving in the plane, constrained in such a way that its distance from a fixed point is constant. A 'rigid body' is a collection of many particles, subject to the constraints that the separation between each pair of particles is constant.

2.1 The Euler-Lagrange equation

To understand the Lagrangian method properly, we need to look at the ideas of the Calculus of Variations quite carefully. For simplicity let us consider the case of one dependent and one independent variable. We consider the space of all real differentiable functions $x(t)$, satisfying the two conditions $x(t_1) = x_1$ and $x(t_2) = x_2$. Let us call these functions 'paths'. We are given a differentiable function, called the Lagrangian, $f(x, \dot{x}, t)$, (we can extend to the case where f also depends on higher derivatives) and we want to extremise the action integral

$$S[x] = \int_{t_1}^{t_2} f(x, \dot{x}, t) dt. \tag{2.4}$$

That is, if we change the path by $O(\epsilon)$, keeping the end points fixed, we want the action only to change by a much smaller amount, $o(\epsilon)$. More precisely, we say a variation of a path $x(t)$ is a real differentiable function $\eta(t)$ satisfying $\eta(t_1) = \eta(t_2) = 0$. Thus $x_\epsilon(t) = x(t) + \epsilon\eta(t)$ is also a path, for any real ϵ. If we evaluate the action integral on the varied path $x_\epsilon(t)$,

$$S[x_\epsilon] = \int_{t_1}^{t_2} f(x_\epsilon, \dot{x_\epsilon}, t) dt. \tag{2.5}$$

we can differentiate it with respect to ϵ. It is convenient to treat x and $\dot{x}$ as though they were independent -

$$\frac{d}{d\epsilon} S[x_\epsilon] = \int_{t_1}^{t_2} \frac{\partial}{\partial x} f(x_\epsilon, \dot{x_\epsilon}, t)\eta + \frac{\partial}{\partial \dot{x}} f(x_\epsilon, \dot{x_\epsilon}, t)\dot{\eta} dt. \tag{2.6}$$

If this derivative vanishes for $\epsilon = 0$, for any variation $\eta(t)$, we say the path $x(t)$ is an extremum of the action $S[x]$.

Now $x(t)$ and $\dot{x(t)}$ are not really independent, as one is the derivative of the other; similarly with $\eta(t)$. To eliminate the dependence on $\dot{\eta}$ we integrate by parts:

$$\frac{d}{d\epsilon} S[x_\epsilon]|_{\epsilon=0} = \int_{t_1}^{t_2} \frac{\partial}{\partial x} f(x, \dot{x}, t)\eta + \frac{\partial}{\partial \dot{x}} f(x, \dot{x}, t)\dot{\eta} dt = \tag{2.7}$$

$$\int_{t_1}^{t_2} \frac{\partial}{\partial x} f(x, \dot{x}, t)\eta dt + [\frac{\partial}{\partial \dot{x}} f(x, \dot{x}, t)\eta]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial}{\partial \dot{x}} f(x, \dot{x}, t)\eta dt \tag{2.8}$$

and the integrated term vanishes, since $\eta(t_1) = \eta(t_2) = 0$. Hence, if $x(t)$ is an extremum of $S[x]$,

$$\int_{t_1}^{t_2} (\frac{\partial}{\partial x} f(x, \dot{x}, t) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}} f(x, \dot{x}, t))\eta dt = 0, \tag{2.9}$$

for all $\eta(t)$. The only way we can achieve this is if the expression in brackets vanishes:

$$\frac{\partial}{\partial x} f(x, \dot{x}, t) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}} f(x, \dot{x}, t) = 0. \tag{2.10}$$

This is called the Euler-Lagrange equation for this variational problem, We see that in general it will be a second-order ordinary differential equation for the path $x(t)$.

2.1.1 Example

For example if

$$S = \int_{t_1}^{t_2} \frac{1}{2}(m\dot{x}^2 - V(x))dt, \tag{2.11}$$

the Euler-Lagrange equation reads

$$-m\ddot{x} - V'(x) = 0, \tag{2.12}$$

Newton's 2nd law for a particle of mass m in potential V.

2.1.2 Example - the Brachistochrone problem

One of the earliest mechanical problems to be posed in a variational formulation was the 'Brachistochrone problem' (Greek - shortest time).

A particle slides without friction under gravity along a curve in a vertical plane, $z = Z(x)$. It is released from rest at the point x_1.

For what function $Z(x)$ is the time taken to travel from x_1 to x_2 minimised?

Choose coordinates so $x_1 = Z(x_1) = 0$. By conservation of energy, the speed of the particle at point x is $\sqrt{-2gZ(x)}$, so the total time taken is

$$T = \int_{x_1}^{x_2} \sqrt{dx^2 + dz^2}/\sqrt{-2gZ(x)} \tag{2.13}$$

$$= \int_0^{x_2} \frac{\sqrt{1+Z'^2}dx}{\sqrt{-2gZ(x)}}. \tag{2.14}$$

Now, dropping the irrelevant factor of $2g$, put $F(Z, Z', x) = \sqrt{1+Z'^2}/\sqrt{-Z(x)}$. We have:

$$\frac{\partial}{\partial Z}F(Z, Z', x) = \frac{\sqrt{1+Z'^2}}{2(-Z)^{3/2}}, \tag{2.15}$$

$$\frac{\partial}{\partial Z'}F(Z, Z', x) = \frac{Z'}{\sqrt{(1+Z'^2)(-Z}}, \tag{2.16}$$

so the Euler-Lagrange equation is:

$$\frac{d}{dx}\frac{Z'}{\sqrt{(1+Z'^2)(-Z}} = \frac{\sqrt{1+Z'^2}}{2(-Z)^{3/2}}. \tag{2.17}$$

Expanding, we get:

$$\frac{Z''}{\sqrt{(1+Z'^2)(-Z)}} - \frac{Z'^2 Z''}{(1+Z'^2)^{3/2}\sqrt{-Z}} + \frac{Z'^2}{2\sqrt{1+Z'^2}(-Z)^{3/2}} = \frac{\sqrt{1+Z'^2}}{2(-Z)^{3/2}}. \tag{2.18}$$

Rearranging terms over a common denominator, and simplifying, we get

$$2ZZ'' + Z'^2 + 1 = 0. \tag{2.19}$$

Solutions of this nonlinear 2nd order equation may not be easy to find in general, but we may use a general result to integrate the equation once, getting a first-order equation instead.

This variational problem, which is specified by the function $F(Z, Z', x) = \sqrt{1+Z'^2}/\sqrt{-Z(x)}$, has a **symmetry** - F is independent of x. Physically this means that if we translate the whole curve in the x-direction, without changing its shape, the time of descent is unchanged.

We may verify directly (see problem sheet 1, question 1) that if Z satisfies the Euler-Lagrange equation:

$$\frac{d}{dx}\frac{\partial}{\partial Z'}F(Z, Z', x) - \frac{\partial}{\partial Z}F(Z, Z', x) = 0, \tag{2.20}$$

then

$$\frac{d}{dx}(Z'\frac{\partial}{\partial Z'}F(Z, Z', x) - F(Z, Z', x)) = -\frac{\partial}{\partial x}F(Z, Z', x) \tag{2.21}$$

In this example the right-hand side vanishes, and we can integrate at once:

$$Z'\frac{\partial}{\partial Z'}F(Z, Z', x) - F(Z, Z', x) = K, \tag{2.22}$$

a constant. We will see later how **Noether's theorem** gives a more general construction of such constants of motion, whenever a variational problem has a symmetry.

In this example,

$$\frac{Z'^2}{\sqrt{(1+Z'^2)(-Z)}} - \frac{\sqrt{1+Z'^2}}{\sqrt{-Z(x)}} = K \tag{2.23}$$

giving

$$\frac{-1}{\sqrt{(1+Z'^2)(-Z)}} = K. \tag{2.24}$$

or

$$Z(1+Z'^2) = -1/K^2 = k. \tag{2.25}$$

Rearranging and separating,

$$dx = \frac{dZ}{\sqrt{k/Z - 1}} \tag{2.26}$$

Substituting $Z = k\cos^2(\theta)$, this becomes

$$dx = \frac{-2k\cos(\theta)\sin(\theta)d\theta}{\tan(\theta)} \tag{2.27}$$

$$= -2k\cos^2(\theta)d\theta. \tag{2.28}$$

Integrating,

$$x = -k(\theta + \sin(2\theta)/2), \tag{2.29}$$

which, with

$$Z = k\cos^2(\theta) = k/2(1 + \cos(2\theta)), \tag{2.30}$$

is the parametric form of the solution. This curve is a *cycloid*, the curve described by a point on the circumference of a circle as the circle is rolled along a line without slipping.

2.2 Many dependent variables

The extension to more than one dependent variable x_i,$i = 1, \ldots, N$, is straightforward. Now the Lagrangian depends on N functions and their derivatives $\dot{x}_i$. As before we look at differentiable paths, and keep the end-points fixed. Writing the N variables x_i as an N-component vector, we have

$$S[\mathbf{x}] = \int_{t_1}^{t_2} f(\mathbf{x}, \dot{\mathbf{x}}, t) dt. \tag{2.31}$$

and our variational equation is

$$\frac{d}{d\epsilon} S[\mathbf{x}_\epsilon] = \int_{t_1}^{t_2} \sum_{i=1}^{N} (\frac{\partial}{\partial x_i} f(\mathbf{x}_\epsilon, \dot{\mathbf{x}}_\epsilon, t)\eta_i + \frac{\partial}{\partial \dot{x}_i} f(\mathbf{x}_\epsilon, \dot{\mathbf{x}}_\epsilon, t)\dot{\eta}_i) dt. \tag{2.32}$$

We integrate by parts as before, using $\eta(t_1) = \eta(t_2) = 0$, getting

$$\int_{t_1}^{t_2} \sum_{i=1}^{N} (\frac{\partial}{\partial x_i} f(\mathbf{x}, \dot{\mathbf{x}}, t) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i} f(\mathbf{x}, \dot{\mathbf{x}}, t))\eta_i dt = 0, \tag{2.33}$$

and this can only vanish if each coefficient of η_i vanishes separately. We thus get N separate Euler-Lagrange equations, one for each independent variation η_i:

$$\frac{\partial}{\partial x_i} f(\mathbf{x}, \dot{\mathbf{x}}, t) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i} f(\mathbf{x}, \dot{\mathbf{x}}, t) = 0. \tag{2.34}$$

2.2.1 Example - Motion in the plane: changing coordinates

Take

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x, y)),$$

giving

$$m\ddot{x} = -\frac{\partial V}{\partial x}, \tag{2.35}$$

$$m\ddot{y} = -\frac{\partial V}{\partial y}. \tag{2.36}$$

But if we write the same Lagrangian in polar coordinates,setting $V(x,y) = \tilde{V}(r,\theta)$, we get

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \tilde{V}(r,\theta)).$$

Now

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \tag{2.37}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{2.38}$$

and

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial \tilde{V}}{\partial r}, \tag{2.39}$$

$$\frac{\partial L}{\partial \theta} = -\frac{\partial \tilde{V}}{\partial \theta}. \tag{2.40}$$

Hence the Euler-Lagrange equations are

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial \tilde{V}}{\partial r}, \tag{2.41}$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -\frac{\partial \tilde{V}}{\partial \theta}. \tag{2.42}$$

We can now easily transform to a rotating frame with $\phi = \theta - \omega t,$, giving

$$L = \frac{m}{2}(\dot{r}^2 + r^2(\dot{\phi} + \omega)^2) - \tilde{V}(r, \phi + \omega t)).$$

and you can verify that this gives the correct equations of motion in a rotating frame. Transformation to other coordinate systems is similarly straightforward. We are no longer restricted to Cartesian coordinates in inertial frames. The advantage of the Lagrangian approach is that the Euler-Lagrange equations always have the same form, so we are free to transform coordinates arbitrarily.

2.2.2 Example - Newton's laws

The general rule for constructing the Lagrangian of a system of N particles is to construct the kinetic energy - in Cartesian coordinates this is:

$$T = \sum_{i=1}^{N} \frac{m_i}{2} |\dot{\mathbf{x}}_\mathbf{i}|^2$$

and the potential energy $V(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$, then take

$$L = T - V.$$

The Euler-Lagrange equations are then

$$m_i \ddot{\mathbf{x}}_i = -\nabla_i V, \tag{2.43}$$

where ∇_i denotes the gradient with respect to the coordinate $\mathbf{x}_i$. These are exactly Newton's 2nd law for this system.

2.2.3 Example - Coupled Harmonic Oscillators

Take

$$L = \sum_{i=1}^{N} \frac{m_i}{2} \dot{x_i}^2 - \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{2} x_i K_{ij} x_j$$

where K is a symmetric matrix, and the action integral is

$$S = \int_{t_1}^{t_2} L dt.$$

The Euler-Lagrange equations then read:

$$-m_i \ddot{x}_i - \sum_{j=1}^{N} K_{ij} x_j = 0. \tag{2.44}$$

We will see later how any Lagrangian near an equilibrium point can be modelled by a Lagrangian of this form, and how the equations of these coupled oscillators can be separated.

2.3 Several independent variables

Let $y(\mathbf{x})$ be a function from R^n to R. Given some Lagrangian density

$$f(y, \nabla y, \mathbf{x})$$

we want to find extrema of the action

$$S[y] = \int_\Omega f d^n\mathbf{x}$$

where the integral is taken over some finite, simply connected (no holes) domain Ω. We denote the i-th component of ∇y by $y_{,i}$ for short, and as before we treat these and y as being independent. Again as before, we fix y on the boundary of Ω, which we denote $\partial\Omega$. This will be some $(n-1)$ dimensional 'surface' in R^n. A variation of y is some differentiable function $\eta(\mathbf{x})$ from Ω to R, which vanishes on $\partial\Omega$. Then if we evaluate $S[y+\epsilon\eta]$, and differentiate with respect to ϵ at $\epsilon = 0$, we get:

$$\int_\Omega \frac{\partial f}{\partial y}\eta + \sum_{i=1}^{n} \frac{\partial f}{\partial y_{,i}} \eta_{,i} d^n\mathbf{x}. \tag{2.45}$$

We now use the divergence theorem, and the identity $\mathrm{div}(\eta\mathbf{v}) = \eta \mathrm{div}\mathbf{v} + (\nabla\eta).\mathbf{v}$, to integrate the second term by parts, getting

$$\int_\Omega [\frac{\partial f}{\partial y} - \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}x_i} \frac{\partial f}{\partial y_{,i}}]\eta d^n\mathbf{x} + \sum_{i=1}^{n} \int_{\partial\Omega} \eta \ldots . \tag{2.46}$$

The last term is linear in $\eta|_{\partial\Omega}$ which vanishes, hence it is zero. If the variation of S vanishes for all η, we have the Euler-Lagrange equation:

$$\frac{\partial f}{\partial y} - \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}x_i} \frac{\partial f}{\partial y_{,i}} = 0, \tag{2.47}$$

or with the obvious shorthand notation,

$$\frac{\partial f}{\partial y} - \operatorname{div} \frac{\partial f}{\partial \nabla y} = 0.$$

2.3.1 Example - The wave equation

If we take

$$f = u_t^2/2 - u_x^2/2$$

then we get the Euler-Lagrange equation:

$$u_{xx} - u_{tt} = 0. \tag{2.48}$$

2.3.2 Example - Minimal surfaces

The area of a surface given by $z = z(x, y)$, with $(x, y) \in \Omega$ is:

$$S = \int\int_{\Omega} \sqrt{1 + z_x^2 + z_y^2} dx dy.$$

It is worthwhile to calculate the boundary term explicitly here when calculating the variation - (exercise); the Euler-Lagrange equation is:

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0. \tag{2.49}$$

This is the equation satisfied by soap films. Exercise: compare the equation obtained by putting $z = z(r)$ in the integral S and finding the EL equation of the restricted expression, with direct substitution of $z = z(r)$ in the full EL equation. The more general problem, of a soap bubble - a constant, rather than zero mean curvature surface - is described by minimising

$$S = \int\int_{\Omega} \sqrt{1 + z_x^2 + z_y^2} - pz dx dy.$$

Here p is proportional to the pressure in the bubble. Can you show that there are spherical bubbles? What is p for a spherical bubble of radius R?

2.4 Constrained systems

Here we are given a variational problem together with certain side conditions.

As in an ordinary constrained optimisation problem, the most powerful approach is to use Lagrange multipliers.

2.4.1 Lagrange multipliers

Suppose we are given a function $V(\mathbf{x})$ depending on N variablesx_i. These x_i satisfy $M < N$ independent constraints $F_i(\mathbf{x}) = 0$. The set of $\mathbf{x}$ satisfying all the constraints, supposed non-empty, is denoted C. We suppose that at any point of C, the gradients ∇F_i are all linearly independent. A vector $\mathbf{u} \in R^N$ is said to be tangent to C at $\mathbf{x}$ if the derivative of each F_i in the direction $\mathbf{u}$ vanishes: $\mathbf{u}.\nabla\mathbf{F_i}(\mathbf{x}) = 0$, for $i = 1, \ldots, M$. These tangent vectors form a vector space of dimension $N - M$. Any other vector, not satisfying these conditions, is called transverse to C. We can split R^N into the direct sum of two subspaces, the space tangent to C, of dimension $N - M$, and a space of vectors transverse to C, of dimension M.

Now we want $V(\mathbf{x})$ to be stationary as $\mathbf{x}$ moves in C, so that $\mathrm{d}/\mathrm{d}\epsilon V(\mathbf{x}+\epsilon\mathbf{u}) = 0$, for $\mathbf{u}$ tangent to C at $\mathbf{x}$. Thus at the stationary point $\mathbf{x}$, we have

$$\mathbf{u}.\nabla(V - \sum_{i=1}^{M} \lambda_i F_i) = 0,$$

for all constants λ_i, and all $\mathbf{u}$ tangent to C at $\mathbf{x}$.

However, for any other $\mathbf{u}$, transverse to C, we may still impose

$$\mathbf{u}.\nabla(V - \sum_{i=1}^{M} \lambda_i F_i) = 0,$$

giving M independent conditions, which we can solve for the M unknowns λ_i. The $N + M$ variables $x_i, \quad i = 1, \ldots, N$, and $\lambda_i, \quad i = 1, \ldots, M$, then satisfy $N + M$ equations:

$$\frac{\mathrm{d}}{\mathrm{d}x_j}(V - \sum_{i=1}^{M} \lambda_i F_i) = 0, \quad j = 1, \ldots, N, \tag{2.50}$$

$$F_i = 0, \quad i = 1, \ldots M. \tag{2.51}$$

These are exactly the conditions for an unconstrained extremum of the function

$$\tilde{V}(\mathbf{x}, \lambda) = V - \sum_{i=1}^{M} \lambda_i F_i$$

with respect to the $N + M$ variables $(\mathbf{x}, \lambda)$.

We may apply the same method to variational problems.

2.4.2 Example - The isoperimetric problem

This problem is to find the curve between two points (x_1, y_1) and (x_2, y_2), of specified length, which maximises the area integral $\int_{x_1}^{x_2} y dx$.

In this example the length of the curve is

$$L[y] = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx,$$

which takes the given value l. The area is

$$A[y] = \int_{x_1}^{x_2} y dx.$$

We look for extrema of the modified functional

$$S[y] = \int_{x_1}^{x_2} y dx - \lambda (\int_{x_1}^{x_2} \sqrt{1 + y'^2} dx - l),$$

where λ is some scalar constant, to be determined. The Euler-Lagrange equation is

$$\lambda \frac{\mathrm{d}}{\mathrm{d}x} (\frac{y'}{\sqrt{1 + y'^2}}) + 1 = 0 \tag{2.52}$$

Hence $(\frac{y'}{\sqrt{1+y'^2}}) = -(x - x_0)/\lambda$, giving the parametric solution, after a little work,

$$x = x_0 + \lambda \sin(\theta) \tag{2.53}$$

$$y = y_0 + \lambda \cos(\theta), \tag{2.54}$$

so the extremum is a circular arc of radius λ. The variational problem satisfied by a soap bubble is another isoperimetric problem, in which the surface area is extremised, holding the volume integral constant. The Lagrange multiplier is p.

2.5 Problems 1

1. *Conservation of 'energy'.*

   The Euler-Lagrange equation corresponding to a functional $F(y, y', x)$ is

   $$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y'} = 0.$$

   Show that

   $$\frac{\mathrm{d}}{\mathrm{d}x} (F - y' \frac{\partial F}{\partial y'}) = \frac{\partial F}{\partial x}.$$

   Hence, in the case that F is independent of x, show that

   $$F - y' \frac{\partial F}{\partial y'} = \text{constant}.$$

2. *The hanging rope.*

   A rope hangs between the two points $(x, y) = (\pm a, 0)$ in a curve $y = y(x)$, so as to minimise its potential energy

   $$\int_{-a}^{a} mgy \sqrt{1 + y'^2} \mathrm{d}x$$

while keeping its length constant:

$$\int_{-a}^{a} \sqrt{1+y'^2}\mathrm{d}x = L$$

Of course $L > 2a$. Find and solve the Euler-Lagrange equation.

3. *The relativistic particle* A particle moving with speed near c, the speed of light, has Lagrangian

$$L = -m_0 c^2 \sqrt{1 - \frac{\dot{\mathbf{x}}^2}{c^2}} - U(\mathbf{x}).$$

Show that the equation of motion can be interpreted as Newton's 2nd law, but with a mass depending on the speed of the particle-

$$m = \frac{m_0}{\sqrt{1 - \frac{\dot{\mathbf{x}}^2}{c^2}}}.$$

The constant m_0 is called the 'rest mass' of the particle. Use the result of question 1 to find a conserved quantity - the relativistic energy of the particle. Find the leading approximation to this Lagrangian in the case

$$\frac{\dot{\mathbf{x}}^2}{c^2} << 1.$$

4. *A nonlinear Laplace equation.*

Find the Euler-Lagrange equation for the function $u(x, y)$ which minimises (extremises):

$$\int_\Omega F(u_x^2 + u_y^2) + f(x, y)u\mathrm{d}x\mathrm{d}y,$$

where the domain of integration is a simply connected finite region Ω. Suppose u takes a specified value on the boundary $\partial\Omega$ of Ω; these are called Dirichlet boundary conditions.

5. *Geodesics on a cylinder.*

Consider a circular cylinder of radius a, whose axis is the z-axis. The metric - the element of arc length - is given in cylindrical polars by:

$$\mathrm{d}s^2 = a^2\mathrm{d}\phi^2 + \mathrm{d}z^2.$$

Write down the length of a curve on the cylinder joining the points (z_1, ϕ_1) and (z_2, ϕ_2). Find the curve which minimises this length. Is there more than one solution?

Repeat this calculation in Cartesian coordinates, using a Lagrange multiplier and the constraint $x^2 + y^2 = a^2$.

Chapter 3

Lagrange's equations

We may use the ideas of constrained Calculus of Variations to construct the equations of motion for a system of N bodies subject to potential forces, and satisfying a certain kind of constraint.

3.1 Example - The simple pendulum

To illustrate our approach, we take the Lagrangian of a particle of mass m in the (x, y) plane with potential $mgy(t)$, but we impose the constraint $\sqrt{(}x(t)^2 + y(t)^2) = l$. This gives the constrained Lagrangian:

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 - 2gy) - \lambda(t)(\sqrt{(}x(t)^2 + y(t)^2) - l).$$

Here the multiplier λ is a function of t, for now the constraint $\sqrt{(}x(t)^2+y(t)^2) = l$ has to hold separately for each t.

We get

$$m\ddot{x} = -\lambda\frac{x}{\sqrt{x(t)^2 + y(t)^2}}, \tag{3.1}$$

$$m\ddot{y} = -mg - \lambda\frac{y}{\sqrt{x(t)^2 + y(t)^2}}. \tag{3.2}$$

We can simplify this by choosing different coordinates.

If we write $x = r\sin(\theta)$, $y = -r\cos(\theta)$, we can solve the constraint explicitly. Here we get

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + 2gr\cos(\theta)) - \lambda(t)(r - l).$$

This gives two Euler-Lagrange equations, and the constraints:

$$m\ddot{r} = mr\dot{\theta}^2 + mg\cos(\theta) - \lambda, \tag{3.3}$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -mgr\sin(\theta) \tag{3.4}$$

$$r = l \tag{3.5}$$

$$\dot{r} = 0. \tag{3.6}$$

Thus, the equation tangential to the constraint set is the usual pendulum equation, and the other, transverse, equation yields an explicit formula for λ. In this problem λ is the tension in the string of the pendulum.

3.2 Holonomic and non-holonomic constraints

To extend this approach, we need to define our idea of a constraint more carefully.

A physical system may often be modelled by one with fewer independent variables. For instance, two masses moving in R^3 joined by a spring have 6 independent coordinates; but if the spring is very strong, we model the system by treating it as a rigid rod - then the system has only 5 independent coordinates; the 6 coordinates $(x_1, y_1, z_1; x_2, y_2, z_2)$ satisfy $(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2 = L^2$, say. This kind of constraint only involves the coordinates. Such contraints are called **holomomic**.

We sometimes have constraints involving the velocities - suppose a particle at position $\mathbf{x}$ can only move perpendicular to a specified vector field $\mathbf{A}(\mathbf{x})$. Then $\mathbf{A}(\mathbf{x}).\dot{\mathbf{x}} = 0$. Some such constraints can be integrated to give constraints on the coordinates alone; if $\mathbf{A}(\mathbf{x}) = \nabla F(\mathbf{x})$, then we integrate to get $F(\mathbf{x})$ =constant. Such contraints are also called **holomomic**. However if curl$\mathbf{A} \neq 0$, the constraint can not be integrated; we call these constraints **non-holomomic**. Such non-holonomic constraints are much harder to treat in general.

3.2.1 Example - the ice skate

The coordinates of an ice skate are given by its position in the plane (x, y), and the direction it is pointing, θ, measured from the x-axis say. The ice skate can only move in the direction it points, so the rate of change of the coordinates satisfies $\sin(\theta)\dot{x} - \cos\theta\dot{y} = 0$. But the curl

$$(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}) \wedge (\sin(\theta), -\cos\theta, 0) = -(\sin(\theta), -\cos\theta, 0)$$

which does not vanish. There is no function $F(x, y, \theta)$ which remains constant throughout the motion because of the constraint. Hence, although $\dot{x}, \dot{y}$, and $\dot{\theta}$ are linearly related, x, y and θ are independent.

This course will only deal with holonomic constraints.

3.2.2 Lagrangian for motion with constraints

We may apply this approach much more generally. Suppose we have a system of N particles whose unconstrained Lagrangian is

$$L = \sum_{i=1}^{N} \frac{m_i}{2}\dot{\mathbf{x}}_i^2 - V(\mathbf{x}_1, \ldots, \mathbf{x}_N).$$

We now require the configuration of the system to satisfy a set of $M < 3N$ independent constraints

$$F_\alpha(\mathbf{x}_1, \ldots, \mathbf{x}_N) = 0, \quad i = 1, \ldots, M.$$

The extended Lagrangian is thus

$$\tilde{L} = L - \sum_{\alpha=1}^{M} \lambda_\alpha F_\alpha.$$

We can obtain the equations of motion either directly, or better, after a change of variables to new coordinates $q_i, \quad i = 1, \ldots, 3N - M$ and $Q_\alpha = F_\alpha, \quad \alpha = 1, \ldots, M$. The only condition on the q_i is that they should be differentiable functions of the $\mathbf{x}_i$, and that the gradients of the q_i and Q_α should all be linearly independent. Then the original coordinates $\mathbf{x}_i$ will be differentiable functions of the q_i and Q_α.

The number of independent coordinates q_i on the set satisfying all the constraints is called the number of **degrees of freedom** of the system.

Then the kinetic energy is

$$\sum_{i=1}^{N} \frac{m_i}{2} \dot{\mathbf{x}}_i^2 = \sum_{i=1}^{N} \frac{m_i}{2} (\sum_{j=1}^{3N-M} \sum_{k=1}^{3N-M} \frac{\partial \mathbf{x}_i}{\partial q_j} . \frac{\partial \mathbf{x}_i}{\partial q_k} \dot{q}_j \dot{q}_k$$

$$+2 \sum_{j=1}^{3N-M} \sum_{\alpha=1}^{M} \frac{\partial \mathbf{x}_i}{\partial q_j} . \frac{\partial \mathbf{x}_i}{\partial Q_\alpha} \dot{q}_j \dot{Q}_\alpha$$

$$+ \sum_{\alpha=1}^{M} \sum_{\beta=1}^{M} \frac{\partial \mathbf{x}_i}{\partial Q_\alpha} . \frac{\partial \mathbf{x}_i}{\partial Q_\beta} \dot{Q}_\alpha \dot{Q}_\beta).$$

It is still quadratic in the components of the velocity. The potential energy will be some function of the new coordinates, $\tilde{V}(\mathbf{q}, \mathbf{Q})$, so our extended Lagrangian becomes:

$$\tilde{L} = \sum_{i=1}^{N} \frac{m_i}{2} (\sum_{j=1}^{3N-M} \sum_{k=1}^{3N-M} \frac{\partial \mathbf{x}_i}{\partial q_j} . \frac{\partial \mathbf{x}_i}{\partial q_k} \dot{q}_j \dot{q}_k + 2 \sum_{j=1}^{3N-M} \sum_{\alpha=1}^{M} \frac{\partial \mathbf{x}_i}{\partial q_j} . \frac{\partial \mathbf{x}_i}{\partial Q_\alpha} \dot{q}_j \dot{Q}_\alpha$$

$$+ \sum_{\alpha=1}^{M} \sum_{\beta=1}^{M} \frac{\partial \mathbf{x}_i}{\partial Q_\alpha} . \frac{\partial \mathbf{x}_i}{\partial Q_\beta} \dot{Q}_\alpha \dot{Q}_\beta) - \tilde{V}(\mathbf{q}, \mathbf{Q}) - \sum_{\alpha=1}^{M} \lambda_\alpha Q_\alpha.$$

We can now calculate the equations of motion from this in the usual way; we first calculate:

$$\frac{\partial \tilde{L}}{\partial \dot{q}_j} = \sum_{i=1}^{N} m_i (\sum_{k=1}^{3N-M} \frac{\partial \mathbf{x}_i}{\partial q_j} . \frac{\partial \mathbf{x}_i}{\partial q_k} \dot{q}_k + \sum_{\alpha=1}^{M} \frac{\partial \mathbf{x}_i}{\partial q_j} . \frac{\partial \mathbf{x}_i}{\partial Q_\alpha} \dot{Q}_\alpha),$$

$$\frac{\partial \tilde{L}}{\partial \dot{Q}_\alpha} = \sum_{i=1}^{N} m_i (\sum_{j=1}^{3N-M} \frac{\partial \mathbf{x}_i}{\partial q_j} . \frac{\partial \mathbf{x}_i}{\partial Q_\alpha} \dot{q}_j + \sum_{\beta=1}^{M} \frac{\partial \mathbf{x}_i}{\partial Q_\alpha} . \frac{\partial \mathbf{x}_i}{\partial Q_\beta} \dot{Q}_\beta).$$

The constaints are given by

$$\frac{\partial \tilde{L}}{\partial \lambda_\alpha} = -Q_\alpha = 0,$$

so $\dot{Q}_\alpha = 0$ as well, since the constraints hold for all t. The equations of motion split into a set of $3N - M$ equations for the q_j, reading:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{q}_j}\right) = \frac{\partial \tilde{L}}{\partial q_j}, \tag{3.7}$$

that is, substituting the constraints,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} m_i \sum_{k=1}^{3N-M} \frac{\partial \mathbf{x}_i}{\partial q_j} . \frac{\partial \mathbf{x}_i}{\partial q_k} \dot{q}_k\right) = \frac{\partial \tilde{L}}{\partial q_j}|_{\mathbf{Q}=0}, \tag{3.8}$$

and a set of M equations for the constraint variables:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} m_i \sum_{j=1}^{3N-M} \frac{\partial \mathbf{x}_i}{\partial q_j} . \frac{\partial \mathbf{x}_i}{\partial Q_\alpha} \dot{q}_j\right) = \frac{\partial \tilde{L}}{\partial Q_\alpha}|_{\mathbf{Q}=0}, \tag{3.9}$$

which can always be satisfied by choosing the λ_α appropriately. The $3N - M$ equations for the q_j are *identical* to those we would obtain by substituting the constraints $Q_\alpha = 0$ and $\dot{Q}_\alpha = 0$ into $\tilde{L}$, and calculating the Euler-Lagrange equations directly from this expression.

Thus it is possible to calculate the equations of motion more directly; first we choose coordinates q_j on the set C satisfying the constraints, next we calculate the kinetic energy T and potential energy V on C, in terms of the q_j and $\dot{q}_j$; the Lagrangian $L = T - V$ then gives the correct constrained equations of motion.

3.2.3 Example - a simple mass-pulley system.

A light inextensible rope hangs vertically from a fixed support. It passes round a light pulley which supports a mass M, then goes up, over a second pulley, which is fixed. A second mass m is suspended on the end of the rope. Find the Lagrangian of the system, and describe its motion.

Let the mass M have vertical coordinate Z, measured downwards. The mass m has vertical coordinate z. As the string is inextensible, $2Z + z$ =constant. The kinetic energy is

$$T = \frac{M}{2}\dot{Z}^2 + \frac{m}{2}\dot{z}^2,$$

or, using the constraint to eliminate z,

$$T = \left(\frac{M}{2} + 2m\right)\dot{Z}^2.$$

The potential energy is:

$$V = -MgZ - mgz = -(M - 2m)gZ,$$

where we have neglected an irrelevant constant.

Then

$$L = (\frac{M}{2} + 2m)\dot{Z}^2 + (M - 2m)gZ.$$

Hence the Euler-Lagrange equation is:

$$(M + 4m)\ddot{Z} = (M - 2m)g.$$

Note that we did not need to calculate the tension in the rope.

3.3 Example - The compound pendulum

To illustrate our approach, we take the Lagrangian of two particle of mass m_1, m_2 in the (x, y) plane with potential $m_1 g y_1 + m_2 g y_2$, but we impose the constraints $\sqrt(x_1^2 + y_1^2) = l_1$, and$\sqrt((x_2 - x_1)^2 + (y_2 - y_1)^2) = l_2$. Thus particle 1 is attached to a fixed point by an inextensible string of length l_1, and particle 2 is attached to particle 1 by a string of length l_2.

We have the unconstrained Lagrangian:

$$L = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2 - 2gy_1) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2 - 2gy_2).$$

We can forget the constraints by choosing appropriate coordinates.

If we write $x_1 = l_1 \sin(\theta_1)$, $y_1 = -l_1 \cos(\theta_1)$, and $x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2)$, $y_2 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2)$, we get

$$L = \frac{m_1}{2} l_1^2 \dot{\theta_1}^2 + \frac{m_2}{2}(l_1^2 \dot{\theta_1}^2 + 2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + l_2^2 \dot{\theta_2}^2) + m_1 g l_1 \cos(\theta_1) + m_2 g(l_1 \cos(\theta_1) + l_2 \cos(\theta_2)).$$

This gives two Euler-Lagrange equations, which are the same as we could have found by substituting these coordinates in Newton's laws, and eliminating the unknown tensions in the strings.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_2) &= -m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 \\
&\quad -m_1 g l_1 \sin(\theta_1) - m_2 g l_1 \sin(\theta_1) \\
\frac{\mathrm{d}}{\mathrm{d}t}(m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 + m_2 l_2^2 \dot{\theta}_2) &= m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 \\
&\quad -m_2 g l_2 \sin(\theta_2).
\end{aligned}$$

3.3.1 Exercise

Repeat the above calculation using Cartesian coordinates, and Lagrange multipliers for the two constraints.

3.4 Example - a particle moves on a surface of revolution

A surface of revolution is given in cylindrical coordinates by

$$z = f(r).$$

Now the Lagrangian of a particle moving under gravity on this surface is

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz - \lambda(z - f(r)),$$

or imposing the constraint and its derivative, $\dot{z} = f'(r)\dot{r}$,

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + f'(r)^2\dot{r}^2) - mgf(r).$$

This yields the EL equations in the usual way:

$$\begin{aligned}
\frac{\partial L}{\partial \dot{r}} &= m(1 + f'(r)^2)\dot{r}, \\
\frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - mgf'(r), \\
\frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta}, \\
\frac{\partial L}{\partial \theta} &= 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} m(1 + f'(r)^2)\dot{r} &= mr\dot{\theta}^2 - mgf'(r), \\
\frac{\mathrm{d}}{\mathrm{d}t} mr^2\dot{\theta} &= 0.
\end{aligned}$$

If $f(r) = -\sqrt{l^2 - z^2}$, this gives the equation for the sphercal pendulum.

3.4.1 Exercise - The Spherical Pendulum

Alternatively, calculate the Lagrangian and equations of motion for the spherical pendulum in spherical polar coordinates.

3.4.2 Example - A free particle on an ellipsoid

A particle moves in R^3, with kinetic energy

$$T = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/2,$$

and the constraint

$$K = x^2/a + y^2/b + z^2/c = 1.$$

We form the Lagrangian

$$L = T - \lambda K.$$

The equations of motion are then

$$\begin{aligned}\ddot{x} &= -2\lambda x/a,\\ \ddot{y} &= -2\lambda y/b,\\ \ddot{z} &= -2\lambda z/c.\end{aligned}$$

Using the constraint $K = 1$ and its first and second time derivatives, we can calculate λ. We find

$$x\dot{x}/a + y\dot{y}/b + z\dot{z}/c = 0,$$

$$x\ddot{x}/a + y\ddot{y}/b + z\ddot{z}/c + \dot{x}^2/a + \dot{y}^2/b + \dot{z}^2/c = 0,$$

so that, after a little manipulation,

$$-2\lambda(x^2/a^2 + y^2/b^2 + z^2/c^2) + \dot{x}^2/a + \dot{y}^2/b + \dot{z}^2/c = 0.$$

Hence

$$\begin{aligned}\ddot{x} &= -\frac{\dot{x}^2/a + \dot{y}^2/b + \dot{z}^2/c}{x^2/a^2 + y^2/b^2 + z^2/c^2}x/a,\\ \ddot{y} &= -\frac{\dot{x}^2/a + \dot{y}^2/b + \dot{z}^2/c}{x^2/a^2 + y^2/b^2 + z^2/c^2}y/b,\\ \ddot{z} &= -\frac{\dot{x}^2/a + \dot{y}^2/b + \dot{z}^2/c}{x^2/a^2 + y^2/b^2 + z^2/c^2}z/c.\end{aligned}$$

It is easier to find the equation of motion using 2 coordinates on the ellipse and one other which is the constraint K itself. The best choice is the confocal coordinates; there are three roots of the equation

$$K(u) = x^2/(a-u) + y^2/(b-u) + z^2/(c-u) = 1$$

Evidently one of these vanishes on our ellipse. The other two are the coordinates we need. In fact the equation of motion can be solved in these variables, though the methods needed go well beyond this course.

3.5 Conservation laws

We saw early on an example where the Lagrangian of a variational problem had a symmetry, and this led to a conserved quantity. The Euler-Lagrange equations, for a Lagrangian with N degrees of freedom, are:

$$\frac{\partial}{\partial q_i} L(\mathbf{q}, \dot{\mathbf{q}}, t) - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i} L(\mathbf{q}, \dot{\mathbf{q}}, t) = 0, \quad i = 1, \ldots, N. \tag{3.10}$$

We need to consider examples of what we mean by a **symmetry**. The simplest example is seen at once if L is independent of one or more of the coordinates, q_i say. Then

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i} L(\mathbf{q}, \dot{\mathbf{q}}, t) = 0,$$

so that

$$\frac{\partial}{\partial \dot{q}_i} L(\mathbf{q}, \dot{\mathbf{q}}, t) = \text{constant}.$$

It is convenient to have a name for this quantity, conserved or not; we denote the derivative $\frac{\partial}{\partial \dot{q}_i} L$ by p_i. It is called the **conjugate momentum** to q_i.

3.5.1 *N* particles on the line

Consider a system of N particles moving along a line, interacting by pairwise potentials $V_{ij}(q_i - q_j)$. The Lagrangian is then:

$$L = \sum_{i=1}^{N} \frac{m_i}{2} \dot{q}_i^2 - \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} V_{ij}(q_i - q_j).$$

Evidently this is invariant under translations $q_i \to q_i + x$. If we rewrite $q_i = q_1 + Q_i$, for $i > 1$, then

$$L = \frac{m_1}{2}\dot{q}_1^2 + \sum_{i=2}^{N} \frac{m_i}{2}(\dot{q}_1 + \dot{Q}_i)^2 - \sum_{i=2}^{N-1} \sum_{j=i+1}^{N} V_{ij}(Q_i - Q_j) - \sum_{j=2}^{N} V_{1j}(-Q_j).$$

Now we can see that in these coordinates L is independent of q_1, so that the momentum conjugate to q_1 is conserved

$$m_1 \dot{q}_1 + \sum_{i=2}^{N} m_i(\dot{q}_1 + \dot{Q}_i) = \text{constant}.$$

This is called the total momentum of the system. In the original coordinates, this is

$$P = \sum_{i=1}^{N} m_i \dot{q}_i.$$

Here the independence of L from a coordinate was only achieved after a change of variables. We would like to be able to identify symmetries and calculate the conserved quantities corresponding to them without having to choose a particular set of coordinates.

3.5.2 Conservation of energy

We saw in the section on the Brachistochrone problem, and in exercise 1, that the Euler-Lagrange equation can be integrated once if the lagrangian does not depend explicitly on the independent variable. We see directly that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\sum_{i=1}^{N}\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}-L)= \\
&(\frac{\partial}{\partial t}+\sum_{i=1}^{N}\dot{q}_i\frac{\partial}{\partial q_i}+\sum_{i=1}^{N}\ddot{q}_i\frac{\partial}{\partial \dot{q}_i})(\sum_{i=1}^{N}\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}-L)= \\
&\sum_{i=1}^{N}\ddot{q}_i\frac{\partial L}{\partial \dot{q}_i}+\sum_{i=1}^{N}\dot{q}_i\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial t}-\sum_{i=1}^{N}\dot{q}_i\frac{\partial L}{\partial q_i}-\sum_{i=1}^{N}\ddot{q}_i\frac{\partial L}{\partial \dot{q}_i}= \\
&-\frac{\partial L}{\partial t},
\end{aligned}
$$

if the Euler-Lagrange equation is satisfied. Thus a symmetry, the time-independence of L leads to the conservation of the quantity

$$E(\mathbf{q},\dot{\mathbf{q}})=\sum_{i=1}^{N}\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}-L.$$

This quantity is known as the **energy**.

We will see later that the momentum variables p_i can be used instead of the velocities $\dot{q}_i$; the energy written in the new set of variables is then called the **Hamiltonian**. This leads to a very elegant re-formulation of the equations of Lagrangian mechanics.

3.6 Noether's Theorem

To generalise the above results, we return to the variational problem; we had

$$S[\mathbf{x}]=\int_{t_1}^{t_2}L(\mathbf{q},\dot{\mathbf{q}},t)dt. \tag{3.11}$$

and our variational equation was

$$\frac{d}{d\epsilon}S[\mathbf{q}]=\int_{t_1}^{t_2}\sum_{i=1}^{N}(\frac{\partial}{\partial q_i}L(\mathbf{q},\dot{\mathbf{q}},t)\eta_i+\frac{\partial}{\partial \dot{q}_i}L(\mathbf{q},\dot{\mathbf{q}},t)\dot{\eta}_i)dt. \tag{3.12}$$

Now let us consider a special class of variations, explicitly given as functions of the coordinates and velocities, $\eta(\mathbf{q},\dot{\mathbf{q}},\mathbf{t})$, satisfying:

$$\sum_{i=1}^{N}\frac{\partial}{\partial q_i}L(\mathbf{q},\dot{\mathbf{q}},t)\eta_i+\frac{\partial}{\partial \dot{q}_i}L(\mathbf{q},\dot{\mathbf{q}},t)\frac{\mathrm{d}\eta_i}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}G(\mathbf{q},\dot{\mathbf{q}},t).$$

We drop the condition that $\eta_i(t_1) = \eta_i(t_2)$.

Such sets of functions, $(\eta(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{t}), G(\mathbf{q}, \dot{\mathbf{q}}, t))$, will be called **symmetries** of the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. The idea is that if we replace $\mathbf{q} \to \mathbf{q} + \epsilon\eta$, then L is changed only by an exact t-derivative $\frac{\mathrm{d}}{\mathrm{d}t} G(\mathbf{q}, \dot{\mathbf{q}}, t)$.

It is important to realise that any Lagrangian of this form gives an Euler-Lagrange equation which is identically zero.

Exercise Prove this. Why does the result make sense?

Hence the replacement $\mathbf{q} \to \mathbf{q} + \epsilon\eta$, takes solutions of the Euler-Lagrange equations into nearby solutions of the same equations.

Now let us look at the right-hand side of (**??**areq). We have, if η is a symmetry,

$$\int_{t_1}^{t_2} \sum_{i=1}^{N} (\frac{\partial}{\partial q_i} L(\mathbf{q}, \dot{\mathbf{q}}, t)\eta_i + \frac{\partial}{\partial \dot{q}_i} L(\mathbf{q}, \dot{\mathbf{q}}, t)\dot{\eta}_i) dt = \tag{3.13}$$

$$\int_{t_1}^{t_2} \frac{\mathrm{d}G}{\mathrm{d}t} \mathrm{d}t = \tag{3.14}$$

$$G(\mathbf{q}, \dot{\mathbf{q}}, t)|_{t_1}^{t_2}. \tag{3.15}$$

On the other hand, if $\mathbf{q}$ satisfies the Euler-Lagrange equation, we have

$$\int_{t_1}^{t_2} \sum_{i=1}^{N} (\frac{\partial}{\partial q_i} L(\mathbf{q}, \dot{\mathbf{q}}, t)\eta_i + \frac{\partial}{\partial \dot{q}_i} L(\mathbf{q}, \dot{\mathbf{q}}, t)\dot{\eta}_i) dt = \tag{3.16}$$

$$\int_{t_1}^{t_2} \sum_{i=1}^{N} \eta_i (\frac{\partial}{\partial q_i} L - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}_i} L) dt + (\sum_{i=1}^{N} \eta_i \frac{\partial L}{\partial \dot{q}_i})|_{t_1}^{t_2} = \tag{3.17}$$

$$(\sum_{i=1}^{N} \eta_i \frac{\partial L}{\partial \dot{q}_i})|_{t_1}^{t_2}. \tag{3.18}$$

Hence, comparing the two expressions, we find that there is a function which is constant between the times t_1 and t_2:

$$(\sum_{i=1}^{N} \eta_i \frac{\partial L}{\partial \dot{q}_i} - G)|_{t_1}^{t_2} = 0.$$

Such a function, constant along any path which solves the Euler-Lagrange equations, is called a **constant of motion** or **integral of motion**. This result, which gives an explicit formula for an integral of motion given a symmetry of the Lagrangian, is called **Noether's Theorem**.

3.6.1 Example - Angular momentum

Consider the Lagrangian for a particle in a central potential,

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2),$$

with the symmetry

$$x \to x + \epsilon y$$
$$y \to y - \epsilon x.$$

This leaves L unchanged to first order in ϵ. Thus $G = 0$ here. The Noether integral corresponding to this symmetry is seen to be

$$y(m\dot{x}) - x(m\dot{y}) = m(y\dot{x} - x\dot{y}).$$

This is the angular momentum.

3.6.2 Example - Galilean invariance

Consider the Lagrangian for a pair of particles interacting pairwise:

$$L = (\frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2) - V(x_2 - x_1).$$

This has a symmetry that we may add the same constant to each of the $\dot{x}_i$, - effectively transforming to a moving frame of reference. In detail we have:

$$\eta_1 = t$$
$$\eta_2 = t$$
$$\dot{\eta}_1 = 1$$
$$\dot{\eta}_2 = 1,$$

Hence the derivative along the symmetry of L,

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} L(\mathbf{x} + \epsilon\eta) =$$
$$\sum_{i=1}^{2} (\frac{\partial L}{\partial x_i}\eta_i + \frac{\partial L}{\partial \dot{x}_i}\dot{\eta}_i) = m_1\dot{x}_1 + m_2\dot{x}_2 =$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(m_1x_1 + m_2x_2).$$

Hence G is here given by $m_1x_1 + m_2x_2$, the centre of mass of the system. Thus the Noether integral here is

$$K = \sum_{i=1}^{2} \eta_i \frac{\partial L}{\partial \dot{x}_i} - (m_1x_1 + m_2x_2) =$$
$$t(m_1\dot{x}_1 + m_2\dot{x}_2) - (m_1x_1 + m_2x_2).$$

The fact that this quantity is conserved means that the centre of mass moves with constant speed, for the total momentum $m_1\dot{x}_1 + m_2\dot{x}_2$ is also constant.

3.6.3 Exercise

A system has Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - V(x^2 - y^2).$$

Find a symmetry of this Lagrangian, and hence construct the corresponding Noether integral. Verify directly that this is conserved if x and y satisfy the Euler-Lagrange equations.

3.7 Homogeneous functions

A function $f(\mathbf{x})$ is called homogeneous if it has a simple scaling property - specifically, if:

$$f(t\mathbf{x}) = t^n f(\mathbf{x}), \tag{3.19}$$

for all $t \in R$, we say that $f(\mathbf{x})$ is **homogeneous of degree** n. For instance the usual form of the kinetic energy of a many-particle system

$$T = \sum_{i=1}^{N} \frac{m_i}{2} \dot{\mathbf{x}}_{\mathbf{i}}{}^2,$$

is a homogeneous function of $\dot{\mathbf{x}}$ of degree 2.

Euler established the following theorem on homogeneous functions. If $f(\mathbf{x})$ is homogeneous of degree n, we may differentiate both sides of (3.19) with respect to t:

$$\mathbf{x}.\nabla f(t\mathbf{x}) = nt^{n-1} f(\mathbf{x}),$$

so that if we set $t = 1$, we get:

$$\mathbf{x}.\nabla f((\mathbf{x}) = nf(\mathbf{x}). \tag{3.20}$$

Now this result, though simple, has a lot of applications. For instance, many Lagrangians in non-relativistic classical mechanics have the form

$$L = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}),$$

and T is homogeneous in the velocities $\dot{\mathbf{q}}$, with degree 2. Now the Energy integral is given in all cases by:

$$E \quad = \quad \sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

which by Euler's theorem is, in these cases:

$$\begin{aligned} &= \quad 2T - (T - V) \\ &= \quad T + V, \end{aligned}$$

by the homogeneity of T.

3.7.1 Example - A particle in a magnetic field - 1

The Lagrangian for a particle in a time-independent magnetic field with vector potential $\mathbf{A}(\mathbf{x})$ is given by:

$$L = \frac{m}{2}|\dot{\mathbf{x}}|^2 + e\mathbf{A}(\mathbf{x}).\dot{\mathbf{x}}, \tag{3.21}$$

so that it is the sum of two homogeneous functions of the $\dot{\mathbf{x}}$, L_2 and L_1, say, of degrees 2 and 1 respectively. The energy is then

$$E = \sum_{i=1}^{3} \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L$$

which by Euler's theorem is

$$\begin{aligned} &= (2L_2 + L_1) - (L_2 + L_1) \\ &= L_2. \end{aligned}$$

For any time-independent Lagrangian, the energy is constant along the solutions of the equations of motion, so we find that $|\dot{\mathbf{x}}|$ =constant.

3.7.2 Exercise - A particle in a magnetic field - 2

SKIP - use as coursework Calculate the Euler-Lagrange equations for the Lagrangian (7.1), and verify directly that $|\dot{\mathbf{x}}|$ =constant. For the more general Lagrangian

$$L = \frac{m}{2}\dot{\mathbf{x}}^2 + e\mathbf{A}(\mathbf{x}, t).\dot{\mathbf{x}} + e\phi(\mathbf{x}, t),$$

find the Euler-Lagrange equations. Denoting $\mathbf{B} = \mathrm{curl}\mathbf{A}$, and $\mathbf{E} = \mathrm{grad}\phi - \frac{\partial \mathbf{A}}{\partial t}$, write out the equations of motion. What happens if $\mathbf{A} = \mathrm{grad}\psi$, and $\phi = \frac{\partial \psi}{\partial t}$?

3.8 The Virial Theorem

Suppose a system has Lagrangian

$$L = \sum_{i=1}^{N} \frac{m_i}{2}|\dot{\mathbf{x}}_\mathbf{i}|^2 - V(\mathbf{x_1}, \ldots, \mathbf{x_N}), \tag{3.22}$$

where V is homogeneous of degree d, say. For instance, a system of coupled harmonic oscillators has $d = 2$, a system of gravitating particles has $d = -1$.

We consider the quantity:

$$I = \sum_{i=1}^{N} \frac{m_i}{2}|\mathbf{x_i}|^{\mathbf{2}}.$$

and suppose that the motion is bounded, so all the $\mathbf{x_i}$ and $\dot{\mathbf{x}}_\mathbf{i}$ are bounded functions of time. In particular I is bounded. We define the **time average** $\langle f \rangle$ of a function f by $\langle f \rangle = \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} f(t')dt'$.

Differentiating I twice with respect to time,

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \sum_{i=1}^{N} m_i(\mathbf{x_i}.\ddot{\mathbf{x}}_\mathbf{i} + \dot{\mathbf{x}}_\mathbf{i}.\dot{\mathbf{x}}_\mathbf{i}).$$

The time average of the right-hand side is zero, or else $\frac{\mathrm{d}I}{\mathrm{d}t}$ will grow as $t \to \pm\infty$. The second term is twice the kinetic energy $T = \sum_{i=1}^{N} \frac{m_i}{2}|\dot{\mathbf{x}}_\mathbf{i}|^2$, the first term is:

$$\sum_{i=1}^{N} m_i(\mathbf{x_i}.\ddot{\mathbf{x}}_\mathbf{i} =$$
$$-\sum_{i=1}^{N} \mathbf{x_i}.\frac{\partial V}{\partial \mathbf{x_i}} =$$
$$-dV,$$

by Euler's theorem. Hence

$$\langle 2T - dV \rangle = 0.$$

This result is known as the **Virial Theorem**.

For instance the time averages of the potential energy V and kinetic energy T are equal for a system of coupled harmonic oscillators, with $d = 2$. For a gravitating system, with $d = -1$, we find, similarly, $\langle 2T + V \rangle = 0$. This result gives us a way of estimating the mass of a distant astronomical object, such as a globular cluster. By measuring the red-shift in the spectrum of a star, we can find the component of its velocity in the line of sight. We can do this for different stars in a cluster, so we can estimate the total kinetic energy in the centre of mass frame, up to a factor of m, the unknown total mass. We can also estimate the radius R of the object, so the potential energy is something like Gm^2/R. The total mass can thus be estimated in terms of measurable quantities. Surprisingly these estimates are often significantly higher than the amount of mass we can observe, for instance by multiplying the number of stars by their average mass. This is known as the 'missing mass' problem. The conjectured solution was that there was a lot of invisible mass in there too, perhaps black holes, see

http://www.sciencenews.org/20020921/fob3.asp

http://hubblesite.org/newscenter/archive/2002/18/text

for evidence confirming this.

3.9 Integrable systems- Lagrangian description

Suppose a Lagrangian system has D degrees of freedom. Every symmetry it possesses will correspond to a conserved quantity. We are particularly interested in the extreme case in which the system has D independent conserved quantities, and a symmetry corresponding to each. We will also require that these symmetries satisfy a compatibility condition; each of the conserved quantities should be unchanged under the action of each of the symmetries. We will see how to check this when we look at Hamiltonian mechanics. Then a powerful theorem due to Liouville, and in a more modern version to Arnol'd, states that the system can be solved by quadratures - evaluating and inverting integrals. We can see what this implies by looking at a couple of examples.

3.9.1 Example - a pair of harmonic oscillators

We take

$$L = \frac{m_1}{2}(\dot{x}_1^2 - \omega_1^2 x_1^2) + \frac{m_2}{2}(\dot{x}_2^2 - \omega_2^2 x_2^2),$$

and this has one obvious conserved quantity, the energy:

$$E = \frac{m_1}{2}(\dot{x}_1^2 + \omega_1^2 x_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \omega_2^2 x_2^2).$$

However, the Euler-Lagrange equations are:

$$\ddot{x}_1 = -\omega_1^2 x_1,$$

$$\ddot{x}_2 = -\omega_2^2 x_2,$$

so the dynamics of the two particles at x_1 and x_2 are totally decoupled. Hence we may write the energy as a sum $E = E_1 + E_2$, with

$$E_1 = \frac{m_1}{2}(\dot{x}_1^2 + \omega_1^2 x_1^2),$$

$$E_2 = \frac{m_2}{2}(\dot{x}_2^2 + \omega_2^2 x_2^2),$$

There is a symmetry corresponding to each of these. The symmetry corresponding to E_1 (what is it?) only involves x_1, so does not affect E_2. We can read the equations for E_1 and E_2 as first order ode for x_1, x_2, so

$$\int^t \omega_1 \mathrm{d}t' = \int^{x_1} \frac{\mathrm{d}x'}{\sqrt{X_1^2 - x_1^2}},$$

$$\int^t \omega_2 \mathrm{d}t' = \int^{x_2} \frac{\mathrm{d}x'}{\sqrt{X_2^2 - x'^2}},$$

The solution of the two degree of freedom system can be found explicitly by inverting these integrals:

$$x_1 = a_1 \cos(\omega_1 t - \phi_1)$$

$$x_2 = a_2 \cos(\omega_2 t - \phi_2).$$

Such systems, which possibly after a change of coordinates can be split into uncoupled systems, are called **separable**. We will see later that any system of N linearly coupled harmonic oscillators is in fact a separable system.

3.9.2 Example - a particle in a central potential

In polar coordinates our Lagrangian is:

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r).$$

Since θ is ignorable,

$$h = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant}.$$

Because L is independent of t,

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \text{constant}.$$

We can eliminate $\dot{\theta}$, getting

$$E = \frac{m}{2}\dot{r}^2 + \frac{h^2}{2mr^2} + V(r) = \text{constant}.$$

Now we can read this as a first order differential equation for r, and we get

$$m\dot{r}^2 = 2E - \frac{h^2}{mr^2} - 2V(r),$$

giving

$$\int^t \mathrm{d}t' = \int^r \frac{\sqrt{m}}{2E - \frac{h^2}{mr'^2} - 2V(r')}\mathrm{d}r'.$$

Once $r(t)$ has been found, the equation $mr^2\dot{\theta} = h$ can be integrated to find $\theta(t)$.

These are both examples of **completely integrable systems**. Although most mechanical systems with more than one degree of freedom are not integrable, very (infinitely) many examples of this type are known.

3.9.3 A heavy particle on a surface of revolution

We saw the Lagrangian for this system was:

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + f'(r)^2\dot{r}^2) - mgf(r).$$

This yields the EL equations in the usual way:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}m(1 + f'(r)^2)\dot{r} &= mr\dot{\theta}^2 - mgf'(r), \\ \frac{\mathrm{d}}{\mathrm{d}t}mr^2\dot{\theta} &= 0. \end{aligned}$$

The symmetry – that the system is unchanged under rotations in θ – leads to the conservation of angular momentum:

$$mr^2\dot{\theta} = h = \text{constant}.$$

Thus we can use the symmetry to eliminate not one but two variables, both θ, which is irrelevant to the motion, and $\dot{\theta}$, which is determined once $r(t)$ is found. We may replace the energy

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + f'(r)^2\dot{r}^2) + mgf(r),$$

by the function of r and $\dot{r}$, taking the same constant value,

$$E' = \frac{m}{2}(\dot{r}^2 + f'(r)^2\dot{r}^2) + \frac{h^2}{2mr^2} + mgf(r).$$

The equations of motion reduce to a single first order equation for $r(t)$,

$$\dot{r}^2 = \frac{2E'}{m} - f'(r)^2\dot{r}^2 - 2gf(r) - \frac{h^2}{m^2r^2},$$

giving

$$\int_{t_0}^{t} \mathrm{d}t' = \int_{r_0}^{r} \frac{1}{\sqrt{\frac{2E'}{m} - f'(r)^2\dot{r}^2 - 2gf(r) - \frac{h^2}{m^2r^2}}}\mathrm{d}r'.$$

This gives the solution in terms of (the inverse function of) a definite integral.

3.10 Problem Sheet 2 - Lagrangian Mechanics

1. *Particle in a central potential.*

   A particle of mass m moves in R^3 under a central force
   $$F(r) = -\frac{\mathrm{d}V}{\mathrm{d}r},$$
   in spherical coordinates, so
   $$(x, y, z) = (r\cos(\phi)\sin(\theta), r\sin(\phi)\sin(\theta), r\cos(\theta)).$$
   Find the Lagrangian from first principles, in terms of (r, θ, ϕ) and their time derivatives.

   Hence

   (a) show that h, defined by $h = mr^2\dot{\phi}\sin^2(\theta)$ is a constant of the motion.

   (b) derive the other two equations of motion.

2. *The spherical pendulum.*

   An inextensible string of length l is fixed at one end, and has a bob of mass m attached at the other. The bob swings freely in R^3 under gravity, and the string remains taut, so the system is a spherical pendulum. Find the Lagrangian in an appropriate coordinate system, and identify a conserved quantity. Write down both equations of motion.

3. *Horizontal Atwood machine.*

   An inextensible taut string of length l has a mass m at each end. It passes through a hole in a smooth horizontal plane, and the lower mass hangs vertically, while the upper is free to move in the plane. Write down the Lagrangian, in terms of the two coordinates of the upper particle, and find the equations of motion. Identify two conserved quantities, and hence reduce the equations of motion to a single first-order equation.

   Alternatively, treat the particles as though they moved independently, but subject to the constraint that the string is of constant length. Construct the appropriate Lagrangian, with a Lagrange multiplier λ, multiplying the length of the string. Calculate this λ, which is the tension in the string.

4. *The guitar string.*

   Suppose a string is tied between two fixed end points $x = 0, y = 0$ and $x = l, y = 0$. Let $y(x,t)$ be the small transverse displacement of the string from its equilibrium at position $x \in (0,l)$ and $t > 0$. The string has mass μ per unit length, and constant tension F.

   Show that the kinetic and potential energies are given by:
   $$\begin{aligned} T &= \frac{\mu}{2}\int_0^l y_t^2 \mathrm{d}x, \\ V &= F\int_0^l (\sqrt{1+y_x^2} - 1)\mathrm{d}x. \end{aligned}$$

Here subscripts denote partial derivatives. You may neglect the effect of gravity. If the displacement y is small, so that $|y_x| << 1$, show that the Lagrangian can be approximated by an expression quadratic in y, and find the Euler-Lagrange equation for the approximate Lagrangian.

5. *The diatomic molecule.*

   Two atoms of masses m_1, m_2 move freely in the plane, with the constraint that the distance between them $|\mathbf{x}_1 - \mathbf{x}_2| - l = 0$, where l is a constant.

   (a) Write down the kinetic energy and the constrained Lagrangian in Cartesian coordinates, and find the the Lagrange multiplier of the constraint, which is the force in the bond between the two atoms.

   (b) Rewrite the Lagrangian in new coordinates $(\mathbf{X}, \mathbf{r})$, where $\mathbf{X}$ is the centre of mass, and $(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{r}$.

   Identify six symmetries of the system and write down the corresponding Noether integrals.

   - If the particles are subjected to a gravitational potential $\sum_{i=1}^{2} m_i \mathbf{x}_i.\mathbf{j}g$, where $\mathbf{j}$ is the unit vector $(0, 1)$, write down the modified Lagrangian.
   - If the particles are subjected instead to a harmonic potential $\sum_{i=1}^{2} m_i \omega^2 |\mathbf{x}_i|^2$, write down the modified Lagrangian.
   - If the particles are no longer subjected to the constraint, but instead there is a force between them due to a potential $V(|\mathbf{x}_1 - \mathbf{x}_2|)$, write down the Lagrangian.

   State which of the symmetries and conservation laws survive in each of these cases.

M2A2 Problem Sheet 2 - Lagrangian Mechanics

Solutions

1. *Particle in a central potential.*

   A particle of mass m moves in R^3 under a central force

   $$F(r) = -\frac{\mathrm{d}V}{\mathrm{d}r},$$

   in spherical coordinates, so

   $$(x, y, z) = (r\cos(\phi)\sin(\theta), r\sin(\phi)\sin(\theta), r\cos(\theta)).$$

   Find the Lagrangian from first principles, in terms of (r, θ, ϕ) and their time derivatives.

   Hence

(a) show that h, defined by $h = mr^2\dot{\phi}\sin^2(\theta)$ is a constant of the motion.

(b) derive the other two equations of motion.

Solution: *The kinetic energy is* $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. *We substitute*

$$x = r\sin(\theta)\cos(\phi),$$

$$y = r\sin(\theta)\sin(\phi),$$

$$z = r\cos(\theta).$$

Differentiating these, substituting into T, *and simplifying, we find*

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2).$$

The potential energy is $V(r)$, *so our Lagrangian is:*

$$L = T - V = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2) - V(r).$$

This Lagrangian is independent of ϕ; *so the corresponding EL equation is*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} = 0.$$

Hence $h = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2(\theta)$ *is a constant of motion. The radial equation is:*

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) = mr(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2) - V'(r),$$

and the θ *equation is:*

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = mr^2\sin(\theta)\cos(\theta)\dot{\phi}^2.$$

2. *The spherical pendulum.*

An inextensible string of length l is fixed at one end, and has a bob of mass m attached at the other. The bob swings freely in R^3 under gravity, and the string remains taut, so the system is a spherical pendulum. Find the Lagrangian in an appropriate coordinate system, and identify a conserved quantity. Write down both equations of motion.

Solution: *The unconstrained Lagrangian, as in question 1, is*

$$L_{unconstrained} = T - V = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2) + mgr\cos(\theta).$$

Here we have taken the lowest *point of the sphere as the origin of θ. Now the constraint is $r - l = 0$. Hence the extended Lagrangian is*

$$L_{ext} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2) + mgr\cos(\theta) - \lambda(r - l).$$

The r equation determines λ; the other equations are the same as we would find from the simpler Lagrangian:

$$L = \frac{ml^2}{2}(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2) + mgl\cos(\theta).$$

As before, ϕ is ignorable, so

$$h = \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi}\sin^2(\theta)$$

is a constant of the motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}(ml^2\dot{\phi}\sin^2(\theta)) = 0.$$

The θ equation is:

$$\frac{\mathrm{d}}{\mathrm{d}t}(ml^2\dot{\theta}) = ml^2\sin(\theta)\cos(\theta)\dot{\phi}^2 - mgl\sin(\theta).$$

3. *Horizontal Atwood machine.*

An inextensible taut string of length l has a mass m at each end. It passes through a hole in a smooth horizontal plane, and the lower mass hangs vertically, while the upper is free to move in the plane. Write down the Lagrangian, in terms of the two coordinates of the upper particle, and find the equations of motion. Identify two conserved quantities, and hence reduce the equations of motion to a single first-order equation.

Alternatively, treat the particles as though they moved independently, but subject to the constraint that the string is of constant length. Construct the appropriate Lagrangian, with a Lagrange multiplier λ, multiplying the length of the string. Calculate this λ, which is the tension in the string.

Solution: *We use polar coordinates (r, θ) for the particle in the horizontal plane, and measure the vertical coordinate z of the other particle downwards. Then the extended constrained Lagrangian is:*

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + mgz - \lambda(r + z - l).$$

Here the r and z equations both involve λ linearly - eliminating z with the constraint $r + z - l = 0$, the Lagrangian simplifies to:

$$L = \frac{m}{2}(2\dot{r}^2 + r^2\dot{\theta}^2) + mg(l - r).$$

We see

$$h = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

is a conserved quantity, as L does not depend explicitly on θ. Since L is also independent of t, the energy is constant:

$$E = \frac{m}{2}(2\dot{r}^2 + r^2\dot{\theta}^2) + mg(r - l).$$

Returning to the extended Lagrangian, the z equation is: $m\ddot{z} = mg - \lambda$, so substituting the constraint we find:

$$\lambda = mg + m\ddot{r}.$$

4. *The guitar string.*

Suppose a string is tied between two fixed end points $x = 0, y = 0$ and $x = l, y = 0$. Let $y(x, t)$ be the small transverse displacement of the string from its equilibrium at position $x \in (0, l)$ and $t > 0$. The string has mass μ per unit length, and constant tension F.

Show that the kinetic and potential energies are given by:

$$\begin{aligned} T &= \frac{\mu}{2}\int_0^l y_t^2 \mathrm{d}x, \\ V &= F\int_0^l (\sqrt{1+y_x^2}-1)\mathrm{d}x. \end{aligned}$$

Here subscripts denote partial derivatives. You may neglect the effect of gravity. If the displacement y is small, so that $|y_x| << 1$, show that the Lagrangian can be approximated by an expression quadratic in y, and find the Euler-Lagrange equation for the approximate Lagrangian.

Solution: *The change in length of the displaced string is:*

$$\Delta l = \int_0^l (\sqrt{1+y_x^2}-1)\mathrm{d}x,$$

so the potential energy (the work done in displacing the string) is $F\Delta l +$ *times this:*

$$V = F\int_0^l (\sqrt{1+y_x^2}-1)\mathrm{d}x,$$

as required. The string is supposed to move only in the y*-direction, so its kinetic energy is, as usual, the integral of the mass density times half the square of the speed:*

$$T = \frac{\mu}{2}\int_0^l y_t^2 \mathrm{d}x.$$

Hence

$$L = \int_0^l \frac{\mu}{2}y_t^2 - F(\sqrt{1+y_x^2}-1)\mathrm{d}x.$$

Now let y_x *be small; the leading approximation to* L *is then:*

$$L^{(2)} = \int_0^l \frac{\mu}{2}y_t^2 - \frac{F}{2}y_x^2 \mathrm{d}x,$$

where terms quartic in y_x *have been neglected. The EL equation for this appproximate Lagrangian is:*

$$\mu y_{tt} = F y_{xx},$$

which is the familiar wave equation; the speed c *of these waves is given by* $c^2 = F/\mu$.

5. *The diatomic molecule.*

Two atoms of masses m_1, m_2 move freely in the plane, with the constraint that the distance between them $|\mathbf{x}_1 - \mathbf{x}_2| - l = 0$, where l is a constant.

(a) Write down the kinetic energy and the constrained Lagrangian in Cartesian coordinates, and find the the Lagrange multiplier of the constraint, which is the force in the bond between the two atoms.

(b) Rewrite the Lagrangian in new coordinates $(\mathbf{X}, \mathbf{r})$, where $\mathbf{X}$ is the centre of mass, and $(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{r}$.

Identify six symmetries of the system and write down the corresponding Noether integrals.

- If the particles are subjected to a gravitational potential $\sum_{i=1}^{2} m_i \mathbf{x}_i.\mathbf{j}g$, where $\mathbf{j}$ is the unit vector $(0, 1)$, write down the modified Lagrangian.
- If the particles are subjected instead to a harmonic potential $\sum_{i=1}^{2} m_i \omega^2 |\mathbf{x}_i|^2$, write down the modified Lagrangian.
- If the particles are no longer subjected to the constraint, but instead there is a force between them due to a potential $V(|\mathbf{x}_1 - \mathbf{x}_2|)$, write down the Lagrangian.

State which of the symmetries and conservation laws survive in each of these cases.

Solution:

(a) Let the particles have Cartesian coordinates (x_1, y_1), (x_2, y_2). *The extended Lagrangian is:*

$$L = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) - \lambda(|\mathbf{x}_1 - \mathbf{x}_2| - l).$$

The EL equations are

$$m_1 \ddot{\mathbf{x}}_1 = \lambda(\mathbf{x}_1 - \mathbf{x}_2)/l,$$

$$m_2 \ddot{\mathbf{x}}_2 = \lambda(\mathbf{x}_2 - \mathbf{x}_2)/l,$$

where we have substituted in the constraint. If we dot each equation with $(\mathbf{x}_1 - \mathbf{x}_2)$, *divide by the masses, and subtract, we find*

$$(\ddot{\mathbf{x}}_1 - \ddot{\mathbf{x}}_2) \cdot (\mathbf{x}_1 - \mathbf{x}_2) = \lambda l(1/m_1 + 1/m_2).$$

Now the second derivative of the constraint gives:

$$(\ddot{\mathbf{x}}_1 - \ddot{\mathbf{x}}_2) \cdot (\mathbf{x}_1 - \mathbf{x}_2) + (\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2) \cdot (\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2) = 0.$$

Thus

$$\lambda = -\frac{m_1 m_2}{l(m_1 + m_2)}|\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2|^2.$$

(b) *In the relative coordinates,*

$$\mathbf{X} = (m_1\mathbf{x}_1 + m_2\mathbf{x}_2)/(m_1 + m_2),$$

$$\mathbf{r} = (\mathbf{x}_1 - \mathbf{x}_2),$$

we get

$$\mathbf{x}_1 = \mathbf{X} + \frac{m_2}{m_1 + m_2}\mathbf{r},$$

$$\mathbf{x}_2 = \mathbf{X} - \frac{m_1}{m_1 + m_2}\mathbf{r}.$$

Substituting these expressions and their derivatives into L we get, after some cancellation:

$$L = \frac{m_1 + m_2}{2}\dot{\mathbf{X}}^2 + \frac{m_1 m_2}{2(m_1 + m_2)}\dot{\mathbf{r}}^2 - \lambda(|\mathbf{r}| - l).$$

The centre of mass motion and the relative motion are now decoupled.

The system has several symmetries -

- *L is independent of t, so the energy*

$$E = \frac{m_1 + m_2}{2}\dot{\mathbf{X}}^2 + \frac{m_1 m_2}{2(m_1 + m_2)}\dot{\mathbf{r}}^2$$

  is conserved.

- *L is independent of $\mathbf{X}$, so both components of the total momentum*

$$\mathbf{P} = \frac{\partial L}{\partial \dot{\mathbf{X}}} = (m_1 + m_2)\dot{\mathbf{X}},$$

  are conserved.

- *L is unchanged under rotations of the vectors $\mathbf{X}$ or of $\mathbf{r}$, separately. Thus the two quantities:*

$$J = (m_1 + m_2)\mathbf{X} \wedge \dot{\mathbf{X}},$$

  and

$$K = \frac{m_1 m_2}{m_1 + m_2}\mathbf{r} \wedge \dot{\mathbf{r}},$$

  are each conserved.

- *The system is invariant under Galilean transformations -*

$$\mathbf{X} \to \mathbf{X} + \epsilon \mathbf{V} t,$$

$$\dot{\mathbf{X}} \to \dot{\mathbf{X}} + \epsilon \mathbf{V},$$

  for any constant vector $\mathbf{V}$. L is not unchanged under this transformation; rather

$$L \to L + \epsilon(m_1 + m_2)\dot{\mathbf{X}} \cdot \mathbf{V}$$

$$= L + \epsilon \frac{\mathrm{d}}{\mathrm{d}t}(m_1 + m_2)\mathbf{X} \cdot \mathbf{V}.$$

Hence the Noether integral is here

$$\mathbf{I} \cdot \mathbf{V} = (m_1 + m_2)\mathbf{X} \cdot \mathbf{V} - \mathbf{P} \cdot \mathbf{V}t,$$

so the centre of mass moves at constant speed.

Note that the unconstrained system has 4 degrees of freedom, so it consists of 4 second order equations. With these 7 conserved quantities, the motion is reduced to a single first order system. In fact, including the constraint means the equation for the time dependence of $|\mathbf{r}|$ *is trivial - the motion is determined completely.*

- *If the particles are subjected to a gravitational potential* $\sum_{i=1}^{2} m_i \mathbf{x}_i.\mathbf{j}g$, *where* $\mathbf{j}$ *is the unit vector* $(0, 1)$, *the modified Lagrangian is:*

  $$L = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) - \lambda(|\mathbf{x}_1 - \mathbf{x}_2| - l) - \sum_{i=1}^{2} m_i \mathbf{x}_i.\mathbf{j}g.$$

  The new potential term is not invariant under translations in the $\mathbf{j}$ *direction, or under rotations. The other symmetries survive.*

- *If the particles are subjected instead to a harmonic potential* $\sum_{i=1}^{2} m_i \omega^2 |\mathbf{x}_i|^2$, *the modified Lagrangian is:*

  $$L = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) - \lambda(|\mathbf{x}_1 - \mathbf{x}_2| - l) - \sum_{i=1}^{2} m_i \omega^2 |\mathbf{x}_i|^2.$$

  Both rotational symmetries still survive under this perturbation, so J *and* K, *defined as before, are still conserved separately. The system is no longer Galilean invariant, so* $\mathbf{I}$ *is not conserved for this system.*

- *If the particles are no longer subjected to the constraint, but instead there is a force between them due to a potential* $V(|\mathbf{x}_1 - \mathbf{x}_2|)$, *the Lagrangian is:*

  $$L = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) - V(|\mathbf{x}_1 - \mathbf{x}_2|).$$

  All 7 symmetries survive in this model.

Chapter 4

Hamiltonian mechanics

We have seen that the energy

$$E = \sum_{i=1}^{D} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L,$$

and the conjugate momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i},$$

are important quantities, being conserved if L has the corresponding symmetry. We can use these variables to rewrite Lagrange's equations in a more symmetrical form. We need to define this type of transformation more generally first.

4.1 The Legendre Transformation

Suppose we have a real function f of a vector variable, $\mathbf{x}$ in a vector space X, and suppose it is **convex**. That is we require the 'Hessian' matrix:

$$\frac{\partial^2 f}{\partial x_i \partial x_j}$$

to be positive definite. This implies a weaker property:

$$\lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2) < f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2),$$

for all $\mathbf{x}_1$, $\mathbf{x}_2$, and $0 < \lambda < 1$. The 'chord' from $\mathbf{x}_1$ to $\mathbf{x}_2$ lies above the arc between the same two points. We can see that the function $f_1(x) = x^2$ is convex. However the function $f_2(x) = x^4$ although it satisfies the second property, is not convex at $x = 0$, since $f_2''(0) = 0$; it is 'too flat'.

PICTURE

We can transform this function $f(\mathbf{x})$ into into another convex function of a new vector variable $g(\mathbf{y})$ in X^*, the dual vector space of X, by taking:

$$g(\mathbf{y}) = \max_{\mathbf{x}}(\mathbf{y} \cdot \mathbf{x} - \mathrm{f}(\mathbf{x})).$$

This maximum will exist and will be unique, and if f is smooth, the map between $\mathbf{x}$ and $\mathbf{y}$ will be smooth and invertible. This transformation is known as the **Legendre transformation**, and $g(\mathbf{y})$ is called the **Legendre dual** of $f(\mathbf{x})$. What is the Legendre dual of $g(\mathbf{y})$? Legendre duals are important in other fields - thermodynamics, economics, geometrical optics etc., wherever we need to maximise a function of many variables (entropy, profit, or whatever).

4.2 Hamilton's equations

We can use the Legendre transformation to rewrite Lagrange's equations in a more symmetrical form. Consider a function of $3D$ variables, all treated as independent:

$$\tilde{E}((p_i, q_i, \dot{q}_i)|_{i=1}^D) = \sum_{i=1}^D \dot{q}_i p_i - L((q_i, \dot{q}_i)|_{i=1}^D),$$

and we consider how it varies as all these variables change slightly:

$$\tilde{E}((p_i + \delta p_i, q_i + \delta q_i, \dot{q}_i + \delta\dot{q}_i)|_{i=1}^D) - \tilde{E}((p_i, q_i, \dot{q}_i)|_{i=1}^D)$$

$$= \sum_{i=1}^D \frac{\partial \tilde{E}}{\partial p_i}\delta p_i + \frac{\partial \tilde{E}}{\partial q_i}\delta q_i + \frac{\partial \tilde{E}}{\partial \dot{q}_i}\delta \dot{q}_i,$$

to first order. Now

$$\frac{\partial \tilde{E}}{\partial p_i} = \dot{q}_i,$$

$$\frac{\partial \tilde{E}}{\partial \dot{q}_i,} = p_i - \frac{\partial L}{\partial \dot{q}_i},$$

and

$$\frac{\partial \tilde{E}}{\partial q_i,} = -\frac{\partial L}{\partial q_i}.$$

We need to consider L which are convex functions of the velocities $\dot{q}_i$. We then evaluate $\tilde{E}$ at its unique extremum (a maximum), with respect to all the $\dot{q}_i$, so we get

$$p_i - \frac{\partial L}{\partial \dot{q}_i} = 0.$$

These equations should be solved for the $\dot{q}_i$ in terms of the p_i and q_i.

This maximum, $H = \max_{\dot{\mathbf{q}}} \mathrm{E}(\mathbf{p}, \mathbf{q}, \dot{\mathbf{q}})$, is equal to the energy; it should be considered as a function only of the $2D$ variables $(\mathbf{p}, \mathbf{q})$. It will also be a convex function, but of the new variables $\mathbf{p}$.

Denote

$$H(p_i, q_i)|_{i=1}^D = \tilde{E}((p_i, q_i, \dot{q}_i)|_{i=1}^D)|_{p_i = \frac{\partial L}{\partial \dot{q}_i}}.$$

This function $H(p_i, q_i)|_{i=1}^D)$ is called the **Hamiltonian**. We have

$$\frac{\partial H}{\partial p_i} = \dot{q}_i,$$

and

$$
\begin{aligned}
\frac{\partial H}{\partial q_i} &= -\frac{\partial L}{\partial q_i} = \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} &= -\dot{p}_i.
\end{aligned}
$$

Hence the equations of motion are rewritten as a set of $2D$ first order equations,

$$
\begin{aligned}
\dot{p}_i &= -\frac{\partial H}{\partial q_i}, \\
\dot{q}_i &= \frac{\partial H}{\partial p_i}.
\end{aligned}
$$

This system is known as **Hamilton's equations**.

4.2.1 Example - A particle in a central potential.

A point mass m moves in a central potential $V(r)$ in R^3. In spherical polars, its Lagrangian is

$$L = \frac{m}{2}(\dot{r}^2 + r^2(\dot{\phi}^2 \sin^2(\theta) + \dot{\theta}^2) - V(r).$$

The conjugate momenta are:

$$
\begin{aligned}
p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \\
p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \\
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2(\theta)\dot{\phi}.
\end{aligned}
$$

Now because L is quadratic in the velocity components, we know that E is the sum of the kinetic and potential terms;

$$E = \frac{m}{2}(\dot{r}^2 + r^2(\dot{\phi}^2 \sin^2(\theta) + \dot{\theta}^2) + V(r).$$

Rewriting this in terms of the momenta, we get:

$$H = \frac{1}{2m}(p_r^2 + \frac{1}{r^2\sin^2(\theta)}p_\phi^2 + \frac{1}{r^2}p_\theta^2) + V(r).$$

Hence Hamilton's equations are:

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}t} &= \tfrac{\partial H}{\partial p_r} = \frac{p_r}{m}, \\
\frac{\mathrm{d}p_r}{\mathrm{d}t} &= -\tfrac{\partial H}{\partial r} = \frac{1}{mr^3\sin^2(\theta)}p_\phi^2 + \frac{1}{mr^3}p_\theta^2 - \frac{\partial V}{\partial r}, \\
\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \tfrac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}p_\theta}{\mathrm{d}t} &= -\tfrac{\partial H}{\partial \theta} &&= -\frac{\cos(\theta)}{mr^2\sin^3(\theta)}p_\phi^2, \\
\frac{\mathrm{d}\phi}{\mathrm{d}t} &= \tfrac{\partial H}{\partial p_\phi} &&= \frac{p_\phi}{mr^2\sin^2(\theta)}, \\
\frac{\mathrm{d}p_\phi}{\mathrm{d}t} &= \tfrac{\partial H}{\partial \phi} &&= 0.
\end{aligned}
$$

Note that ϕ is an ignorable coordinate, and p_ϕ is conserved.

4.2.2 Example - A particle moving freely on a surface.

A particle of mass m moves on a surface $\mathcal{S}$, given parametrically by

$$x_i = x_i(q_1, q_2).$$

Thus the coordinates x_i in R^3 vary according to:

$$\dot{x}_i = M_{i\alpha}\dot{q}_\alpha,$$

where the 2×3 matrix $M_{i\alpha}$ is given by:

$$M_{i\alpha} = \frac{\partial x_i}{\partial q_\alpha}.$$

We suppose this matrix has rank 2. The kinetic energy T is thus

$$\frac{m}{2}\sum_{i=1}^{3}\dot{x}_i^2$$

$$
\begin{aligned}
&= \frac{m}{2}\sum_{\alpha=1}^{2}\sum_{j=1}^{3}\sum_{\beta=1}^{2}\dot{q}_\alpha (M^T)_{\alpha j} M_{j\beta}\dot{q}_\beta \\
&= \frac{m}{2}\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\dot{q}_\alpha g_{\alpha\beta}\dot{q}_\beta.
\end{aligned}
$$

The 2×2 matrix $g_{\alpha\beta}$, called the metric tensor in the surface, is symmetric and invertible. *Why?*.

Then the momentum conjugate to q_α is

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} = m\sum_{\beta=1}^{2} g_{\alpha\beta}\dot{q}_\beta,$$

and inverting this, we have

$$\dot{q}_\beta = \frac{1}{m}\sum_{\alpha=1}^{2}(g^{-1})_{\beta\alpha}p_\alpha.$$

Thus our Hamiltonian is

$$H = \frac{1}{2m} \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} p_\beta (g^{-1})_{\beta\alpha} p_\alpha.$$

Hamilton's equations follow in the usual way; we get

$$\begin{aligned}
\dot{q}_\beta &= \tfrac{\partial H}{\partial p_\beta} &= \frac{1}{m} \sum_{\alpha=1}^{2} (g^{-1})_{\beta\alpha} p_\alpha, \\
\dot{p}_\beta &= -\tfrac{\partial H}{\partial q_\beta} &= \frac{1}{2m} \sum_{\gamma=1}^{2} \sum_{\delta=1}^{2} p_\gamma \frac{\partial}{\partial q_\beta} (g^{-1})_{\gamma\delta} \quad p_\delta.
\end{aligned}$$

These are the equivalent, up to reparametrisation in time, as the equations of a geodesic on the surface $\mathcal{S}$.

4.2.3 Exercise

What is the Euler Lagrange equation corresponding to the path length on a surface $\mathcal{S}$, giving

$$L = \sqrt{\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \frac{\mathrm{d}q_\alpha}{\mathrm{d}s} g_{\alpha\beta} \frac{\mathrm{d}q_\beta}{\mathrm{d}s}}$$

where s is any parameter on the curve? What is the Hamiltonian corresponding to this Lagrangian L? Why is the result strange?

4.3 The Poisson Bracket

Suppose we have a Hamiltonian system with Hamiltonian $H(\mathbf{p}, \mathbf{q}, t)$. Then the equations of motion are

$$\begin{aligned}
\dot{\mathbf{q}} &= \frac{\partial H}{\partial \mathbf{p}}, \\
\dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{q}}.
\end{aligned}$$

If we have a function $K(\mathbf{p}, \mathbf{q}, t)$, its time evolution is given by

$$\begin{aligned}
\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{\partial K}{\partial \mathbf{q}}.\dot{\mathbf{q}} + \frac{\partial K}{\partial \mathbf{p}}.\dot{\mathbf{p}} + \frac{\partial K}{\partial t} &= \\
\frac{\partial K}{\partial \mathbf{q}}.\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial K}{\partial \mathbf{p}}.\frac{\partial H}{\partial \mathbf{q}} + \frac{\partial K}{\partial t} &= \\
\{K, H\} + \frac{\partial K}{\partial t}.
\end{aligned}$$

The expression

$$\{K, H\} = \frac{\partial K}{\partial \mathbf{q}}.\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial K}{\partial \mathbf{p}}.\frac{\partial H}{\partial \mathbf{q}}$$

is called the **Poisson bracket** of H with K. Hence, if we have a time-independent function $K(\mathbf{p},\mathbf{q})$, its Poisson bracket with $H(\mathbf{p},\mathbf{q},t)$ vanishes if and only if K is constant on solutions of Hamilton's equations with Hamiltonian H.

The Poisson bracket operation has the following fundamental properies:

$$\begin{aligned}
\{H,K\} &= -\{K,H\}, && \text{(antisymmetry)},\\
\{H,JK\} &= J\{H,K\}+\{H,J\}K, && \text{(derivation property)},\\
\{H,\alpha J+\beta K\} &= \alpha\{H,J\}+\beta\{H,K\}, && \text{(linearity)},\\
\{\{H,J\},K\}+\{\{J,K\},H\}+\{\{K,H\},J\} &= 0, && \text{(Jacobi identity)}.
\end{aligned}$$

The first three of these are elementary, but the fourth is deeper. If the left hand side is expanded in full, every term will contain one of the second derivatives of H, J, and K. Also every term, for instance

$$\frac{\partial^2 H}{\partial p_i \partial p_j}\frac{\partial J}{\partial q_i}\frac{\partial K}{\partial q_j},$$

will appear once with a plus sign and once with a minus sign, but with the indices i and j exchanged. Since the matrix of second derivatives of any continuously differentiable function is symmetric, all the terms cancel in pairs.

We can now construct a converse of Noether's theorem - we recall that gave an explicit construction of an integral of motion for a Lagrangian system, given a symmetry of the Lagrangian - here we construct a symmetry of a Hamiltonian system, given an integral of motion.

If $J(\mathbf{p},\mathbf{q})$ is an integral of motion for the Hamiltonian $H(\mathbf{p},\mathbf{q})$, that is $\{J(\mathbf{p},\mathbf{q}),H(\mathbf{p},\mathbf{q})\}=0$, then, by antisymmetry $H(\mathbf{p},\mathbf{q})$ Poisson commutes with $J(\mathbf{p},\mathbf{q})$, so $H(\mathbf{p},\mathbf{q})$ is an integral of motion for the Hamiltonian $J(\mathbf{p},\mathbf{q})$. Thus Hamilton's equations with Hamiltonian J are a **symmetry** of the Hamiltonian H, as required.

We can use Jacobi's identity to prove:

Poisson's theorem If $J(\mathbf{p},\mathbf{q})$ and $K(\mathbf{p},\mathbf{q})$ are both constants of the motion for Hamilton's equations with Hamiltonian $H(\mathbf{p},\mathbf{q},t)$, then their Poisson bracket $\{J,K\}$ is another constant of the motion for H.

Proof: Jacobi's identity can be rewritten:

$$\{\{J,K\},H\}=\{\{J,H\},K\}-\{J,\{K,H\}\}.$$

Both terms on the right hand side vanish if J and K are both constants of the motion, so $\{J,K\}$ is another constant of the motion for H.

We note that this does not always generate new constants of motion; sometimes two such functions satisfy $\{J,K\}=0$. In this case we say that J and K are in involution, or that they Poisson commute.

4.3.1 Angular momentum - worked exercise.

1. Consider the two functions L_1, L_2 for a system with three degrees of freedom:
$$L_1 = p_2 q_3 - p_3 q_2,$$
$$L_2 = p_3 q_1 - p_1 q_3.$$

2. Show directly that they both Poisson commute with the Hamiltonian for a particle in a central potential:
$$H = \frac{|\mathbf{p}|^2}{2m} + V(|\mathbf{q}|).$$

3. Show that L_3 defined by $L_3 = -\{L_1, L_2\}$ is not identically zero and verify directly that $\{L_3, H\} = 0$. Verify directly that L_1, L_2 and L_3 satisfy the Jacobi identity.

4. Show that L_3 and $K = L_1^2 + L_2^2 + L_3^2$ do Poisson commute.

5. Find Hamilton's equations, when the Hamiltonian is $\omega_1 L_1 + \omega_2 + \omega_3 L_3$, and you may assume ω is a unit vector. What is the geometrical meaning of these equations? That is, what is the corresponding symmetry of H?

4.3.2 Solution

1. Direct calculation:
$$\begin{aligned}\{L_1, H\} = \frac{\partial L_1}{\partial \mathbf{q}}.\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial L_1}{\partial \mathbf{p}}.\frac{\partial H}{\partial \mathbf{q}} &= \\ p_2 \frac{\partial H}{\partial p_3} - p_3 \frac{\partial H}{\partial p_2} + q_2 \frac{\partial H}{\partial q_3} - q_3 \frac{\partial H}{\partial q_2} &= \\ p_2 \frac{p_3}{m} - p_3 \frac{p_2}{m} + (q_2 \frac{q_3}{|\mathbf{q}|} - q_3 \frac{q_2}{|\mathbf{q}|}) V'(|\mathbf{q}|) & \\ &= 0.\end{aligned}$$

   Also $\{L_2, H\} = 0$, similarly.

2. Direct calculation:
$$\begin{aligned}\{L_1, L_2\} = \frac{\partial L_1}{\partial \mathbf{q}}.\frac{\partial L_2}{\partial \mathbf{p}} - \frac{\partial L_1}{\partial \mathbf{p}}.\frac{\partial L_2}{\partial \mathbf{q}} &= \\ p_2 \frac{\partial L_2}{\partial p_3} - p_3 \frac{\partial L_2}{\partial p_2} + q_2 \frac{\partial L_2}{\partial q_3} - q_3 \frac{\partial L_2}{\partial q_2} &= \\ p_2 q_1 - p_3.0 + q_2(-p_1) - q_3.0 &= \\ -(p_1 q_2 - p_2 q_3),\end{aligned}$$

   so that
$$L_3 = p_1 q_2 - p_2 q_1.$$

Similarly to $\{L_2, H\} = 0$, we get $\{L_3, H\} = 0$. By cyclic permutation of indices, we get all the Poisson brackets:

$$\{L_1, L_2\} = -L_3$$

$$\{L_2, L_3\} = -L_1$$

$$\{L_3, L_1\} = -L_2$$

Then, for instance,

$$\{\{L_1, L_2\}, L_1\} = -\{L_3, L_1\} = L_2,$$

$$\{\{L_1, L_1\}, L_2\} = -\{0, L_2\} = 0,$$

$$\{\{L_2, L_1\}, L_1\} = \{L_3, L_1\} = -L_2,$$

which add to zero as required. Other cases are similar or easier.

3. We see, by the chain rule,

$$\{L_3, K\} = 2L_1\{L_3, L_1\} + 2L_2\{L_3, L_2\} = -2L_1L_2 + 2L_2L_1 = 0.$$

4. The Hamiltonian is $\omega.(\mathbf{p} \wedge \mathbf{q})$.

   Thus Hamilton's equations, with parameter θ are

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{q} = \mathbf{q} \wedge \omega,$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{p} = \mathbf{p} \wedge \omega,$$

   which describe rigid rotation about the axis ω, by an angle θ, if $|\omega| = 1$.

4.3.3 Integrable systems- Hamiltonian description

If a Hamiltonian system with N degrees of freedom has N integrals of motion $I_i, \quad i = 1, \ldots, N$, which are all in involution, $\{I_i, I_j\} = 0$, then it can be solved by quadratures; the condition that all the I_i are in involution is the concise statement, in terms of Poisson brackets, of the compatibility condition mentioned before. This result was proved by Liouville, while Arnol'd strengthened the result - if the motion of such an integrable system is bounded, then the N-dimensional level set $I_i = c_i$ is an N-torus, the product of N circles.

In the last example, the motion of a particle in a central potential is seen to be integrable; a set of three Poisson commuting integrals is: $I_1 = H$,$I_2 = L_3$ and $I_3 = K$.

Chapter 5

Small oscillations and normal modes

This section is concerned with equilibria and their stability properties. We need to understand:

- What do we mean by an equilibrium for a Lagrangian system?
- What do we mean by stability?
- We would like to *classify* equilibria by their stability properties.
- We want to find approximate equations of motion valid near an equilibrium.
- We would like to solve these approximate equations.

We start with a natural definition:

An equilibrium of a time-independent Lagrangian system is a configuration which is independent of time.

Now the Euler-Lagrange equations read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{q}_j}\right) = \frac{\partial \tilde{L}}{\partial q_j}, \tag{5.1}$$

so that equilibrium configurations satisfy

$$\frac{\partial \tilde{L}}{\partial q_j} = 0, \tag{5.2}$$

as well as

$$\dot{q}_j = 0. \tag{5.3}$$

We wish to Taylor expand the Lagrangian about some equilibrium configuration $\mathbf{q}_0$, so we put $\mathbf{q} = \mathbf{q}^{(0)} + \epsilon \mathbf{q}^{(1)}$, and $\dot{\mathbf{q}} = \epsilon \dot{\mathbf{q}}^{(1)}$,where we suppose $0 < \epsilon \ll 1$.

We get

$$L = L^{(0)} + \epsilon L^{(1)} + \epsilon^2 L^{(2)} + \mathrm{O}(\epsilon^3),$$

where

$$L^{(0)} = L(\mathbf{q}^{(0)}, \mathbf{0}), \tag{5.4}$$

$$L^{(1)} = \sum_{i=1}^{D} \frac{\partial L}{\partial \dot{q}_i}(\mathbf{q}^{(0)}, \mathbf{0})\dot{q}_i^{(1)}, \tag{5.5}$$

$$L^{(2)} = \sum_{i=1}^{D}\sum_{j=1}^{D} \frac{1}{2}\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}(\mathbf{q}^{(0)}, \mathbf{0})\dot{q}_i^{(1)}\dot{q}_j^{(1)} + \tag{5.6}$$

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}(\mathbf{q}^{(0)}, \mathbf{0})\dot{q}_i^{(1)}q_j^{(1)} + \frac{1}{2}\frac{\partial^2 L}{\partial q_i \partial q_j}(\mathbf{q}^{(0)}, \mathbf{0})q_i^{(1)}q_j^{(1)}. \tag{5.7}$$

Here we note that the equilibrium position $\mathbf{q}^{(0)}$ is a constant, so our coordinates are the displacements $\mathbf{q}^{(1)}$. The Euler-Lagrange equation is linear in L, so it is the sum of the Euler-Lagrange equations due to the separate terms $\epsilon^n L^{(n)}$. Now $L^{(0)}$ is a constant, and gives no contribution. Further $\epsilon L^{(1)}$ is clearly a time derivative - the coefficient of $\dot{q}_i^{(1)}$ is a constant - and we can verify that the Euler-Lagrange equation generated by *any* exact time derivative is zero. Why is this? The leading term in the equation of motion is thus the Euler-Lagrange equation for $\epsilon^2 L^{(2)}$, and any corrections will be of order $\mathrm{O}(\epsilon^3)$. We may *ignore* these correction terms so long as ϵ can be treated as *small*, that is, that the system is *near* the equilibrium.

We rewrite $L^{(2)}$ as

$$L^{(2)} = \frac{1}{2}(\dot{\mathbf{q}}^{(1)})^T A\dot{\mathbf{q}}^{(1)} + (\dot{\mathbf{q}}^{(1)})^T B\mathbf{q}^{(1)} - \frac{1}{2}(\mathbf{q}^{(1)})^T C\mathbf{q}^{(1)}.$$

Here A, B and C are the matrices of the second derivatives of L at the equilibrium configuration:

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}(\mathbf{q}^{(0)}, \mathbf{0}), \tag{5.8}$$

$$B_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}(\mathbf{q}^{(0)}, \mathbf{0}), \tag{5.9}$$

$$C_{ij} = -\frac{\partial^2 L}{\partial q_i \partial q_j}(\mathbf{q}^{(0)}, \mathbf{0}). \tag{5.10}$$

The minus sign in our definition of C is for convenience later on. We see, by the usual symmetry of the second derivatives, that A and C are symmetric matrices. Further we know that in most 'physical' Lagrangians, A corresponds to the kinetic energy, a sum of squares. Hence A must be a positive definite matrix in such cases.

The Euler-Lagrange equation for $L^{(2)}$ is:

$$A\ddot{\mathbf{q}}^{(1)} = (B^T - B)\dot{\mathbf{q}}^{(1)} - C\mathbf{q}^{(1)}.$$

We note that this is a constant coefficient linear system of ordinary differential equations, which we will solve in the usual way below. Note also that if B is symmetric, then it does not contribute to the equation of motion at all - in this case the term

$$(\mathbf{q}^{(1)})^T B\dot{\mathbf{q}}^{(1)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{q}^{(1)})^T B\mathbf{q}^{(1)},$$

is an exact time derivative. We have seen that exact time derivatives in the Lagrangian do not contribute to the Euler-Lagrange equation.

5.0.4 Example

A particle moves in one dimension with Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - V(x),$$

and we are interested in motion near an equilibrium point, where $V'(0) = 0$. We can always choose our origin to be the equilibrium point. Write $x = \epsilon x^{(1)}$, with $0 < \epsilon \ll 1$, so

$$L = \frac{\epsilon^2}{2}m\dot{x^{(1)}}^2 - V(0) - \epsilon V'(0)x^1 - \frac{\epsilon^2}{2}V''(0)(x^{(1)})^2 + \mathrm{O}(\epsilon^3).$$

Here the constant term $V(0)$ is irrelevant to the motion, and the $\mathrm{O}(\epsilon)$ term vanishes, so that the leading term is:

$$L^2 = \frac{1}{2}m\dot{x^1}^2 - \frac{1}{2}V''(x^0)(x^1)^2.$$

Thus the $(1 \times 1\ !)$ matrices A, B and C are here

$$A = m \tag{5.11}$$

$$B = 0 \tag{5.12}$$

$$C = V''(0) \tag{5.13}$$

and the linearised equation of motion about the equilibrium point is

$$m\ddot{x^1} = -V''(x^0)x^1.$$

We see that the solutions $x^1(t)$ of this are all bounded if $V''(x^0) > 0$, and that there are unbounded solutions if $V''(x^0) \leq 0$.

Specifically, the solution of the linearised Euler-Lagrange equation is:

$$x^1 = X^{(1)}\cos(\omega t) + \frac{\dot{X}^{(1)}}{\omega}\sin(\omega t),$$

with the arbitrary constants depending on the initial conditions:

$$X^{(1)} = x^{(1)}(0),$$

$$\dot{X}^{(1)} = \dot{x}^{(1)}(0),$$

and where the frequency is given by

$$\omega^2 = C/A.$$

If $C < 0$, the solution is written instead in terms of exponentials or hyperbolic functions, all of which grow as $t \to \pm\infty$. In the marginal case $C = 0$, the solution is linear in time, so any non-stationary solution is unbounded.

We will be able to generalise this result to systems with many degrees of freedom.

5.0.5 Example - the compound pendulum

Recall we had found the Lagrangian of this system to be:

$$L = \frac{m_1}{2} l_1^2 \dot{\theta_1}^2 + \frac{m_2}{2}(l_1^2 \dot{\theta_1}^2 + 2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 + l_2^2 \dot{\theta_2}^2) + m_1 g l_1 \cos(\theta_1) + m_2 g (l_1 \cos(\theta_1) + l_2 \cos(\theta_2)).$$

Let us set $l_1 = l_2 = l$, $m_1 = m_2 = m$, for simplicity.

$$L = \frac{m}{2} l^2 \dot{\theta_1}^2 + \frac{m}{2}(l^2 \dot{\theta_1}^2 + 2 l^2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 + l^2 \dot{\theta_2}^2) + 2mgl \cos(\theta_1) + mgl \cos(\theta_2).$$

An equilibrium position is found where $\theta_1 = \theta_2 = 0$. There are others - what are they?

Now we put $\theta_1 = \epsilon q_1$, $\theta_2 = \epsilon(q_2 - q_1)$ again with $0 < \epsilon \ll 1$, and we find

$$L^{(2)} = \frac{m}{2} l^2 \dot{q_1}^2 + \frac{ml^2}{2} \dot{q}_2^2 - \frac{mgl}{2} q_1^2 - \frac{mgl}{2}(q_1^2 + (q_2 - q_1)^2).$$

This choice of coordinates q_1, q_2 has the effect of making A diagonal. We will see a general way of doing this below. Reading off the coefficients we find the (2×2) matrices A, B and C to be:

$$A = ml^2 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad (5.14)$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (5.15)$$

$$C = mgl \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}. \quad (5.16)$$

We can see that in general, as in these two cases, wherever L is an even function of the velocity components, the matrix B must always vanish, since $\dot{\mathbf{q}}^T B \mathbf{q}$ is an odd function of the velocity vector.

This term may well be non-zero, though, where L contains terms linear in the velocity components. The most important such cases giving non-zero B are where there is a magnetic term $\dot{\mathbf{x}} \cdot \mathbf{A}$ in the Lagrangian, or where we consider motion in a rotating frame, giving Coriolis terms.

5.0.6 Example - a spherical pendulum in a rotating frame

A spherical pendulum with mass m, and length l is viewed in axes rotating about a vertical axis through the point of support, with angular velocity ω.

The kinetic energy is

$$\begin{aligned} T &= \frac{m}{2}((\dot{x}+\omega y)^2 + (\dot{y}-\omega x)^2 + \dot{z}^2) && (5.17)\\ &= \frac{m}{2}((\dot{x}^2+\dot{y}^2+\dot{z}^2) + 2\omega(\dot{x}y - \dot{y}x) + \omega^2(x^2+y^2)), && (5.18)\end{aligned}$$

in spherical polars, where the azimuthal angle ϕ is measured from an axis rotating with angular velocity ω. The potential energy is

$$V = mgz.$$

Hence the Lagrangian is:

$$L = \frac{m}{2}((\dot{x}^2+\dot{y}^2+\dot{z}^2) + 2\omega(\dot{x}y - \dot{y}x) + \omega^2(x^2+y^2)) - mgz.$$

Now the constraint is solved to give:

$$z = -\sqrt{l^2 - x^2 - y^2},$$

and

$$\dot{z} = \frac{x\dot{x} + y\dot{y}}{\sqrt{l^2 - x^2 - y^2}}.$$

Expanding about the equilibrium $(x, y, z) = (0, 0, -l) + \epsilon(x^{(1)}, y^{(1)}, 0) + \mathrm{O}(\epsilon^2)$ we get

$$L^{(2)} = \frac{m}{2}((\dot{x^{(1)}}^2 + \dot{y^{(1)}}^2) + 2\omega(\dot{x^{(1)}}y^{(1)} - \dot{y^{(1)}}x^{(1)}) + \omega^2((x^{(1)})^2 + (y^{(1)})^2)) - \frac{mg}{2l}(x^{(1)})^2 + (y^{(1)})^2.$$

We can now read off A, B and C:

$$\begin{aligned} A &= \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, && (5.19)\\ B &= \begin{pmatrix} 0 & m\omega \\ -m\omega & 0 \end{pmatrix}, && (5.20)\\ C &= \begin{pmatrix} mgl & 0 \\ 0 & mgl \end{pmatrix}. && (5.21)\end{aligned}$$

Exercise Repeat this calculation for the other equilibrium $(x, y, z) = (0, 0, -l)$.

5.1 The linearised Euler-Lagrange equation

In all these cases, the Euler-Lagrange equation is linear, with constant coefficients, of the form:

$$A\ddot{\mathbf{x}}^{(1)} = (B^T - B)\dot{\mathbf{x}}^{(1)} - C\mathbf{x}^{(1)}.$$

It is possible to solve this in the usual way substituting

$$\mathbf{x} = \mathbf{x_0}\exp(i\Omega t),$$

where we make no particular assumptions about the possibly complex numbers Ω. We thus get a kind of quadratic eigenvalue problem:

$$(A\Omega^2 + i(B^T - B)\Omega - C)\mathbf{x}_0 = \mathbf{0},$$

so that Ω must solve the characteristic equation:

$$\Delta(\Omega) = \det(A\Omega^2 + i(B^T - B)\Omega - C) = 0.$$

The motion will only be bounded for all t if *all* the $2N$ roots of this equation Ω_i are real. Now it is straightforward to see that this polynomial $\Delta(\Omega)$ has real coefficients if A, B and C are real, (compare the transposed and complex conjugate problems, noting that A and B are symmetric, and $B^T - B$ is antisymmetric) so all the roots Ω must either be real, or else must come in complex conjugate pairs.

For the cases where B vanishes, this question is much easier to treat. We will only consider this time-reversible case below. This clearly includes all Lagrangians of the form $L = T - V$, with T being a positive definite quadratic form in the velocities, and V being independent of them. Such a Lagrangian is known sometimes as a 'natural mechanical system'.

In the case $B = 0$, the above characteristic equation reduces to:

$$\det(A\Omega^2 - C) = 0.$$

Now we know that both A and C are symmetric, and that the kinetic energy matrix A is positive definite.

The problem would be easy if both A and C were diagonal - then the possible roots Ω_i^2 are just given by the ratios of the diagonal terms $\Omega_i^2 = C_{ii}/A_{ii}$. We will construct a transformation which diagonalises A, then C, and which hence reduces the general case to this simpler one. We will see that the eigenvalues Ω_i^2 are all real, and hence that the eigenfrequencies Ω_i are either real or imaginary. We start with the Lagrangian of the linearised equation, dropping the superscripts $^{(1)}$ and $^{(2)}$ on the coordinates and Lagrangian:

$$L = \frac{1}{2}(\dot{\mathbf{q}})^T A\dot{\mathbf{q}} - \frac{1}{2}(\mathbf{q})^T C\mathbf{q}.$$

We are free to transform $\mathbf{q}$ by any constant linear invertible matrix U, say:

$$\mathbf{q} = U\tilde{\mathbf{q}},$$

without changing the form of L, which becomes:

$$\tilde{L} = \frac{1}{2}(\dot{\tilde{\mathbf{q}}})^T U^T A U \dot{\tilde{\mathbf{q}}} - \frac{1}{2}(\tilde{\mathbf{q}})^T U^T C U \tilde{\mathbf{q}}.$$

Then A is replaced by $\tilde{A} = U^T A U$, C by $\tilde{C} = U^T C U$, which are still both symmetric, and the Lagrangian has the same form as before:

$$\tilde{L} = \frac{1}{2}(\dot{\tilde{\mathbf{q}}})^T \tilde{A} \dot{\tilde{\mathbf{q}}} - \frac{1}{2}(\tilde{\mathbf{q}})^T \tilde{C} \tilde{\mathbf{q}}.$$

It is simplest to break up the transformation we need into three separate steps, first simplifying A and then C:

1. We know A is a symmetic, positive definite matrix, so all its eigenvectors can all be chosen to be orthogonal, and we can choose them to have unit modulus. Then the matrix U_1 whose columns are these eigenvectors is orthogonal, and $\tilde{A} = U_1^T A U_1$ is a diagonal matrix, whose diagonal terms are the eigenvalues of A. All these terms are positive, for A is positive definite. We can thus write $\tilde{A} = \tilde{D}^2$. D is a real diagonal nonsingular matrix. Thus our Lagrangian is now:

   $$\tilde{L} = \frac{1}{2}\dot{\tilde{\mathbf{q}}}^T \tilde{D}^2 \dot{\tilde{\mathbf{q}}} - \frac{1}{2}\tilde{\mathbf{q}}^T \tilde{C} \tilde{\mathbf{q}}.$$

   It is sometimes easier to use a non-orthogonal U_1 for this stage; provided the transformed $\tilde{A} = U_1^T A U_1$ is a diagonal matrix, it does not matter. The choice of an orthogonal U_1 is just for convenience, to prove A can be diagonalised.

2. Now let us transform coordinates again:

   $$\hat{\mathbf{q}} = D\tilde{\mathbf{q}}$$

   Then the Lagrangian becomes:

   $$\hat{L} = \frac{1}{2}\dot{\hat{\mathbf{q}}}^T \dot{\hat{\mathbf{q}}} - \frac{1}{2}\hat{\mathbf{q}}^T D^{-1} \tilde{C} D^{-1} \hat{\mathbf{q}}.$$

   Let us write $\hat{C} = D^{-1}\tilde{C}D^{-1}$, so the Lagrangian is

   $$\hat{L} = \frac{1}{2}\dot{\hat{\mathbf{q}}}^T \dot{\hat{\mathbf{q}}} - \frac{1}{2}\hat{\mathbf{q}}^T \hat{C} \hat{\mathbf{q}}.$$

   Hence the matrix A has been eliminated, and the matrix C has been replaced by the new symmetric matrix $\hat{C}$.

3. Now we may perform another orthogonal transformation:

   $$\hat{\mathbf{q}} = U_2 \bar{\mathbf{q}},$$

The Lagrangian is now

$$\bar{L} = \frac{1}{2}\dot{\bar{\mathbf{q}}}^T U_2^T U_2 \dot{\bar{\mathbf{q}}} - \frac{1}{2}\bar{\mathbf{q}}^T U_2^T \hat{C} U_2 \bar{\mathbf{q}}.$$

if U_2 is orthogonal, then $U_2^T U_2 = I$, the identity; hence the kinetic energy term is unchanged in form, being just a sum of squares of the $\dot{\bar{q}}_i$. Note that U_2 must be orthogonal if the simplified form of the kinetic term is to be preserved. The potential energy is also identical in form, except that $\hat{C}$ has been replaced by $\bar{C} = U_2^T \hat{C} U_2$, giving

$$\bar{L} = \frac{1}{2}\dot{\bar{\mathbf{q}}}^T \dot{\bar{\mathbf{q}}} - \frac{1}{2}\bar{\mathbf{q}}^T \bar{C} \bar{\mathbf{q}}.$$

Since $\hat{C}$ is symmetric, we may choose U_2 to be the matrix whose columns are its orthonormal eigenvectors. Then $\bar{C}$ is the diagonal matrix of the eigenvalues of $\hat{C}$, and we know that these are all real, for real symmetric $\hat{C}$. The Lagrangian has been reduced to:

$$\bar{L} = \sum_{i=1}^{N} \left(\frac{1}{2}\dot{\bar{q}}_i{}^2 - \frac{1}{2}\bar{C}_{ii}\bar{q}_i{}^2\right).$$

This is the Lagrangian for N uncoupled harmonic oscillators; the Euler-Lagrange equations separate to:

$$\ddot{\bar{q}}_i = -\bar{C}_{ii}\bar{q}_i.$$

The eigenfrequencies are $\omega_i = \sqrt{\bar{C}_{ii}}$. We see that they are real if the eigenvalues $\bar{C}_{ii}$ of C are positive; that is, if C is a positive definite matrix. This corresponds to the equilibrium point being a local *maximum* of L, (*minimum* of V) with respect to the q_i. Note that negative eigenvalues corrispond to imaginary eigenfrequencies - that is to growing or decaying real exponentials. In this case the equilibrium is *unstable.*

Now the eigenvectors $\hat{\mathbf{x}}_i$ of $\hat{C}$ with eigenvalue ω_i^2 are mutually orthogonal, so long as $\omega_i^2 \neq \omega_j^2$; and if the latter condition does not hold, we can choose the eigenvectors to be orthogonal in any case. We can also normalise the eigenvectors:

$$\hat{\mathbf{x}}_i^T \hat{\mathbf{x}}_j = \delta_{ij}.$$

In the original coordinates, $\mathbf{x}_i = U_1 D^{-1} \hat{\mathbf{x}}_i$, this result reads:

$$\mathbf{x}_i^T A \mathbf{x}_j = \delta_{ij}.$$

Further, the result

$$\hat{\mathbf{x}}_i^T \hat{C} \hat{\mathbf{x}}_j = \omega_i^2 \delta_{ij}$$

translates back to give:

$$\mathbf{x}_i^T C \mathbf{x}_j = \omega_i^2 \delta_{ij}.$$

These vectors $\mathbf{x}_i$, each with its $\exp(\pm i\omega_i t)$ time dependence, are a basis of solutions for the original linearised system. They are called the **normal modes** of the linearised system. Specifically, the solution of the linearised Euler-Lagrange equation is:

$$\mathbf{x} = \sum_{i=1}^{D} \mathbf{x}_i (a_i \cos(\omega_i t) + b_i \sin(\omega t)),$$

with the $2N$ arbitrary constants depending on the initial conditions. Using the orthogonality of the normal mode eigenvectors with respect to the matrix A, if we dot this solution with $\mathbf{x}_j^T$ we get:

$$\begin{aligned} \mathbf{x}_j^T A \mathbf{x} &= \sum_{i=1}^{D} \mathbf{x}_j^T A \mathbf{x}_i (a_i \cos(\omega_i t) + b_i \sin(\omega_i t)) \\ &= (a_j \cos(\omega_j t) + b_j \sin(\omega_j t)), \end{aligned}$$

so that $a_j = \mathbf{x}_j^T A \mathbf{x}|_{t=0}$, and similarly, $\omega_j b_j = \mathbf{x}_j^T A \dot{\mathbf{x}}|_{t=0}$.

Finally we can consider what happens if a system with an equilibrium depends on a parameter s in such a way that for some value $s = 0$, say, an eigenfrequency of the equilibrium changes sign. This corresponds to loss of stability of the equilibrium; to study the system near this point it is usually best to consider higher order terms in the expansion of the Lagrangian, for the corresponding term in $L^{(2)}$ vanishes. The study of such changes in the qualitative behaviour of a dynamical system is known as **bifurcation theory**. We will see some simple examples in the problems.

5.2 Example - the compound pendulum

We saw the Lagrangian of the linearised compound pendulum with equal masses m and equal lengths l was:

$$L = \frac{1}{2}(\dot{\mathbf{q}})^T A \dot{\mathbf{q}} - \frac{1}{2}(\mathbf{q})^T C \mathbf{q},$$

where $\mathbf{q} = (\theta_1, \theta_1 + \theta_2)^T$, and:

$$\begin{aligned} A &= ml^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ B &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ C &= mgl \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

In these coordinates, A is diagonal, so we go to step 2. We have

$$D = l\sqrt{m} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so with $\tilde{\mathbf{q}} = D\mathbf{q}$, we get:

$$\begin{aligned} \tilde{A} &= D^{-1} A D^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \tilde{B} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ \tilde{C} &= D^{-1} C D^{-1} = \frac{g}{l} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

Now, for step 3, it remains to diagonalise $\tilde{C}$. The characteristic equation is

$$\det \begin{pmatrix} \frac{3g}{l} - \omega^2 & -\frac{g}{l} \\ -\frac{g}{l} & \frac{g}{l} - \omega^2 \end{pmatrix} = 0.$$

This is

$$\frac{2g^2}{l^2} - \frac{4g}{l}\omega^2 + \omega^4 = 0,$$

giving the two eigenfrequencies $\omega_\pm$:

$$\omega_+^2 = \frac{g}{l}(2 + \sqrt{2}),$$

$$\omega_-^2 = \frac{g}{l}(2 - \sqrt{2}).$$

We note that for this equilibrium, both eigenfrequencies are real. This is because this equilibrium point is a minimum of the potential, so C is positive definite.

For the high-frequency mode with frequency ω_+, the unnormalised eigenvector is

$$\tilde{\mathbf{q}}_+ = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix},$$

while for the low-frequency mode, with frequency ω_1, the unnormalised eigenvector is

$$\tilde{\mathbf{q}}_- = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$$

Finally, undoing step 1 by returning to the (θ_1, θ_2) coordinates,

$$\boldsymbol{\theta}_+ = \tfrac{1}{l\sqrt{m}} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix},$$

and

$$\boldsymbol{\theta}_- = \tfrac{1}{l\sqrt{m}} \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix},$$

We note that the two bobs move in opposite directions in the high-frequency mode, the same direction in the low frequency mode.

5.2.1 The linearised solution

Taking real linear combinations of the 2 normal modes, the solution is easily found to be

$$\begin{aligned}\boldsymbol{\theta}^{(1)}(t) = a \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \exp(i\omega_+ t) + a^* \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \exp(-i\omega_+ t) \\ +b \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix} \exp(i\omega_- t) + b^* \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix} \exp(-i\omega_- t) \quad (3)\end{aligned}$$

Here $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary complex constants.

Note that here the ratio of the eigenfrequencies is

$$\boldsymbol{\omega_+/\omega_- = \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}},$$

which is the *irrational* number $\boldsymbol{(1+\sqrt{2})}$, so the solution $\boldsymbol{\theta}(t)$is a *non-periodic* function of time.

In general, if the N eigenfrequencies are not rationally related, as is usually the case, then the general solution will not be periodic as the periods of the different modes have no common multiple. However, if the period ratios of the different modes are all rational, the motion will repeat after any common multiple of the different periods.

5.3 Example - a linear triatomic molecule

Three atoms of masses m, M and m are arranged along a straight line with coordinates x_1, x_2, x_3. They are coupled by a potential $V(x_3-x_2)+V(x_2-x_1)$, so the Lagrangian is:

$$L = \frac{1}{2}(m\dot{x}_1^2 + M\dot{x}_2^2 + m\dot{x}_3^2) - V(x_3 - x_2) - V(x_2 - x_1).$$

We suppose the equilibrium spacing of the atoms is a, so we require $V'(a) = 0$. The linearised Lagrangian is then:

$$L^{(2)} = \frac{1}{2}(m\dot{x^{(1)}}_1^2 + M\dot{x^{(1)}}_2^2 + m\dot{x^{(1)}}_3^2) - \frac{V''(a)}{2}((x_3^{(1)} - x_2^{(1)})^2 + (x_2^{(1)} - x_1^{(1)})^2).$$

We can thus read off A and C:

$$A = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}, \tag{5.22}$$

$$C = V''(a)\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \tag{5.23}$$

With this choice of coordinates, A is already diagonal, so we proceed to stage 2. We write $A = D^2$,

$$D = \begin{pmatrix} \sqrt{m} & 0 & 0 \\ 0 & \sqrt{M} & 0 \\ 0 & 0 & \sqrt{m} \end{pmatrix},$$

and set $\tilde{\mathbf{x}} = D\mathbf{x}^{(\mathbf{1})}$. Then the new Lagrangian $\tilde{L}$ is given by

$$\tilde{L} = \frac{1}{2}\dot{\tilde{\mathbf{x}}}^T\dot{\tilde{\mathbf{x}}} - \frac{1}{2}\tilde{\mathbf{x}}^T D^{-1}CD^{-1}\tilde{\mathbf{x}}.$$

Thus

$$\tilde{C} = V''(a)\begin{pmatrix} \frac{1}{m} & -\frac{1}{\sqrt{mM}} & 0 \\ -\frac{1}{\sqrt{mM}} & \frac{2}{M} & -\frac{1}{\sqrt{mM}} \\ 0 & -\frac{1}{\sqrt{mM}} & \frac{1}{m} \end{pmatrix}.$$

The squared eigenfrequencies are the eigenvalues of $\tilde{C}$ which is clearly symmetric. The characteristic equation is

$$(\frac{V''}{m} - \omega^2)^2(\frac{2V''}{M} - \omega^2) - \frac{2V''^2}{mM}(\frac{V''}{m} - \omega^2) = 0.$$

Hence $\omega_1^2 = 0$, $\omega_2^2 = \frac{V''}{m}$ and $\omega_3^2 = V''(\frac{1}{m} + \frac{2}{M})$.

- The (unnormalised) eigenvector $\tilde{\mathbf{x}}_1$ is seen to be:

$$\tilde{\mathbf{x}}_1 = \begin{pmatrix} \sqrt{m} \\ \sqrt{M} \\ \sqrt{m} \end{pmatrix},$$

so the corresponding $\mathbf{x^{(1)}}_1$ in the original coordinates is

$$\mathbf{x^{(1)}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

which corresponds to translations of the whole system without changing the relative positions. Such symmetries always correspond to vanishing eigenfrequencies, though not conversely.

- The (unnormalised) eigenvector $\tilde{\mathbf{x}}_2$ is seen to be:

$$\tilde{\mathbf{x}}_2 = \begin{pmatrix} \sqrt{m} \\ 0 \\ -\sqrt{m} \end{pmatrix},$$

so the corresponding $\mathbf{x^{(1)}}_2$ is

$$\mathbf{x^{(1)}}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

where the two masses m move in opposite directions and the other mass M is stationary.

- The (unnormalised) eigenvector $\tilde{\mathbf{x}}_3$ is seen to be:

$$\tilde{\mathbf{x}}_3 = \begin{pmatrix} \frac{1}{\sqrt{m}} \\ -\frac{2}{\sqrt{M}} \\ \frac{1}{\sqrt{m}} \end{pmatrix},$$

so the corresponding $\mathbf{x^{(1)}}$ is

$$\mathbf{x^{(1)}}_3 = \begin{pmatrix} \frac{1}{m} \\ -\frac{2}{M} \\ \frac{1}{m} \end{pmatrix},$$

where the two masses m move in the same direction and the central mass M moves in the opposite direction to them.

We calculate that $|\tilde{\mathbf{x}}_1|^2 = 2m + M$, $|\tilde{\mathbf{x}}_2|^2 = 2m$ and $|\tilde{\mathbf{x}}_3|^2 = 2/m + 4/M$.

So

$$\mathbf{x^{(1)}}_1^T A\mathbf{x^{(1)}}_1 = 2m + M, \tag{5.24}$$

$$\mathbf{x^{(1)}}_2^T A\mathbf{x^{(1)}}_2 = 2m \tag{5.25}$$

$$\mathbf{x^{(1)}}_3^T A\mathbf{x^{(1)}}_3 = 2/m + 4/M. \tag{5.26}$$

.

We can verify directly that

$$\mathbf{x^{(1)}}_1^T A\mathbf{x^{(1)}}_1 = (2m + M)\omega_1^1, \tag{5.27}$$

$$\mathbf{x^{(1)}}_2^T A\mathbf{x^{(1)}}_2 = 2m\omega_2^2 \tag{5.28}$$

$$\mathbf{x^{(1)}}_3^T A\mathbf{x^{(1)}}_3 = (2/m + 4/M)\omega_3^2. \tag{5.29}$$

.

It is important to realise that the discrete reflection symmetry of the linearised Lagrangian - it is unchanged if $x_1^{(1)}$ and $x_3^{(1)}$ are exchanged - makes the spectrum easier to calculate, as each normal mode (or in general, eigenspace) must be mapped into itself by any discrete symmetry. This approach makes it possible to work out huge problems such as the normal modes of the 'Buckyball' C_{60} molecule, which apparently would involve diagonalising a (180×180) matrix (there are 3 degrees of freedom for each atom), by only diagonalising blocks no larger than (5×5) instead.

5.4 The harmonic chain

The method even extends to systems with infinitely many degrees of freedom, such as a chain of masses m each joined to its 2 neighbours by springs of strength k. The Lagrangian is

$$L = \sum_{n=-\infty}^{\infty} (\frac{m}{2}\dot{x}_i^2 - \frac{k}{2}(x_{i+1} - x_i)^2),$$

which is already quadratic. The matrix $A = mI$, while $C = k(-\Delta + 2I - \Delta^T)$, where $\Delta_{ij} = \delta i, (j + 1)$ is the matrix with ones immediately below the diagonal, and zeroes everywhere else. The eigenvalue problem

$$C\mathbf{x} = \lambda\mathbf{x}$$

reads

$$k(-x_{n+1} + 2x_n - x_{n-1}) = \lambda x_{n+1},$$

and this is a constant coefficient linear difference equation. The bounded eigenfunctions are $x_n = z^n$, with $|z| = 1$, and $k(2 - z - z^{-1}) = \lambda^i$. This is the basis for the discrete Fourier transform. The eigenfrequencies are thus

$$\omega^2 = \frac{k}{m}(2 - z - z^{-1}) = \frac{4k}{m}\sin^2(\theta),$$

with $z = \exp(i\theta/2)$. Note that there is a *continuous* spectrum of eigenfunctions.

5.4.1 Exercises

- What if the masses are close together $\epsilon \ll 1$ apart, and $m/\epsilon = \rho$, $k = \epsilon\kappa$? How does the system behave for waves of wavelength $O(1)$?
- What if the chain consists of two different sorts of atom, masses m and M, alternating along the line, still with nearest neighbour interactions?

5.5 Problems

1. *Coupled pendula* Two identical simple pendula of length l and mass $m$oscillate in the (x, z) plane as shown:

   They are coupled by a spring joining the two masses, whose elastic energy is $\frac{k}{2}(x_1 - x_2)^2$, where x_1 and x_2 are the horizontal displacements of the masses from their equilibrium. Find the Lagrangian for the system and hence find the Lagrangian for the linearised motion where $|x_i| \ll l$. Assume gravity acts in the negative z direction.

   The system is started with $x_1 = 0$, $\dot{x}_1 = v$, $x_2 = \dot{x}_2 = 0$. Describe the motion of the two pendula. What happens in the limit $0 < k \ll 1$?
   Solution *The Lagrangian is*

$$L = \frac{m}{2}(\dot{x}_1^2 + \dot{z}_1^2 + \dot{x}_2^2 + \dot{z}_2^2) - mg(z_1 + z_2) - \frac{k}{2}(x_1 - x_2)^2.$$

Here $z_i = -\sqrt{l^2 - x_i^2}$, *and* $\dot{z}_i = x_i \dot{x}_i / z_i$. *Then expanding for small* x_i, *retaining only quadratic terms,*

$$L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{mg}{2l}(x_1^2 + x_2^2) - \frac{k}{2}(x_1 - x_2)^2.$$

Thus

$$A = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{mg}{l} + k & -k \\ -k & \frac{mg}{l} + k \end{pmatrix}.$$

The characteristic equation for the eigenfrequencies ω_i *is thus:*

$$\begin{vmatrix} \frac{g}{l} + \frac{k}{m} - \omega^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{l} + \frac{k}{m} - \omega^2 \end{vmatrix} = 0.$$

Hence $\omega_1^2 = \frac{g}{l}$ *or* $\omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$. *The unnormalised eigenvectors are* $(1, 1)^T$ *and* $(1, -1)^T$, *both being symmetrical, one with the pendula moving in* *phase, in which the spring never stretches, the other with them exactly out of phase so the extension of the spring is* $(x_1 - x_2) = 2x_1$. *The solution with the given initial data is:*

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \frac{v}{2\omega_1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin(\omega_1 t) + \frac{v}{2\omega_2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin(\omega_2 t) \\ &= \frac{v}{2}\begin{pmatrix} \sin(\omega_1 t)/\omega_1 + \sin(\omega_2 t)/\omega_2 \\ \sin(\omega_1 t)/\omega_1 - \sin(\omega_2 t)/\omega_2 \end{pmatrix}.\end{aligned}$$

Here initially pendulum one is set swinging, but after a time $T = \pi/(\omega_1 - \omega_2)$, *the second pendulum is moving and the first is stationary. For very small* k, *the frequencies are almost identical and the time-scale* T *is correspondingly very long. The pendula are almost uncoupled.*

2. Two particles of mass m, are joined together and to two fixed points distance $3a$ apart by springs of unstretched length a and spring constants k, $2k$ and k, as in the diagram.

   They are free to move along the line between the two points. Calculate the normal modes of the system.

Solution *The Lagrangian is*

$$L = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{k}{2}x_1^2 - k(x_1 - x_2)^2 - \frac{k}{2}x_2^2.$$

Thus

$$A = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 3k & -2k \\ -2k & 3k \end{pmatrix}.$$

and the characteristic equation is

$$5\frac{k^2}{m^2} - 6\frac{k}{m}\omega^2 + \omega^4 = 0.$$

Hence $\omega_1 = \sqrt{\frac{k}{m}}$*, with eigenvector* $(1,1)$ *and* $\omega_2 = \sqrt{\frac{5k}{m}}$*, with eigenvector* $(1,-1)$*. Again the eigenvectors are invariant under the discrete symmetry of the system which exchanges the values of* x_1 *and* x_2*.*

3. ***** A mass m hangs from a spring with spring constant k. The top end of the spring is attached to the free end of a rope which passes around a light wheel of radius a, whose axis is horizontal. A mass $2m$ is fixed to the circumference of the wheel. Find the equilibria, and the normal modes about each equilibrium, with their characteristic frequencies. Which equilibrium is stable?

Solution *Take* z *to be the vertical coordinate of the mass* m*, measured downwards, and* θ *to be the rotation of the wheel anticlockwise; say* $\theta = 0$ *when the mass* $2m$ *is at the lowest point. The extension of the spring is then* $(z - a\theta - l_0)$*,* l_0 *being a constant, the value of* $(z - a\theta)$ *when the spring has zero tension - its precise value turns out to be irrelevant. The Lagrangian is then:*

$$L = \frac{m}{2}\dot{z}^2 + ma^2\dot{\theta}^2 + mgz - \frac{k}{2}(z - a\theta - l_0)^2 + 2mga\cos(\theta).$$

The equilibrium configurations are given by the two equations:

$$\frac{\partial L}{\partial z} = mg - k(z - a\theta - l_0) = 0,$$

$$\frac{\partial L}{\partial \theta} = ka(z - a\theta - l_0) - 2mga\sin(\theta).$$

Hence two equlibria are

$$\theta = \frac{\pi}{6}, \quad z = \frac{a\pi}{6} + l_0 + \frac{mg}{k},$$

and

$$\theta = \frac{5\pi}{6}, \quad z = \frac{5a\pi}{6} + l_0 + \frac{mg}{k}.$$

The other solutions differ from one or other of these by a full rotation of the wheel.

Expanding L *about these equilibria we get*

$$L^{(2)} = \frac{1}{2}(\dot{z}^{(1)}, \dot{\theta}^{(1)}) A \begin{pmatrix} \dot{z}^{(1)} \\ \dot{\theta}^{(1)} \end{pmatrix} - \frac{1}{2}(z^{(1)}, \theta^{(1)}) C \begin{pmatrix} z^{(1)} \\ \theta^{(1)} \end{pmatrix}.$$

Here

$$A = \begin{pmatrix} m & 0 \\ 0 & 2ma^2 \end{pmatrix},$$

$$C = \begin{pmatrix} k & -ka \\ -ka & ka^2 + 2mga\cos(\theta) \end{pmatrix}.$$

The squared eigenfrequencies are thus the eigenvalues of

$$\hat{C} = \begin{pmatrix} \frac{k}{m} & -\frac{k}{m\sqrt{2}} \\ -\frac{k}{m\sqrt{2}} & \frac{k}{2m} + \frac{g}{a}\cos(\theta) \end{pmatrix}.$$

Here $\cos(\theta) = \sigma\frac{\sqrt{3}}{2}$, *with the sign* $\sigma = +1$ *for the first equilibrium,* $\sigma = -1$ *for the other. Note that* l_0 *has dropped out from this expression.*

The characteristic equation is

$$\omega^4 - \omega^2(\frac{3k}{m} + \sigma\frac{g\sqrt{3}}{2a}) + \sigma\frac{kg\sqrt{3}}{2ma} = 0.$$

With $\sigma = 1$, *the two roots for* ω^2 *are real and positive,*

$$\omega^2 = \frac{3k}{2m} + \frac{g\sqrt{3}}{4a}) \pm \sqrt{\frac{9k^2}{4m^2} - \frac{kg\sqrt{3}}{4ma} + \frac{3g^2}{16a^2}},$$

so the equilibrium is stable. It is a minimum of the potential. With $\sigma = -1$, *one of the two roots for* ω^2 *is negative,*

$$\omega^2 = \frac{3k}{2m} - \frac{g\sqrt{3}}{4a}) \pm \sqrt{\frac{9k^2}{4m^2} + \frac{3kg\sqrt{3}}{4ma} + \frac{3g^2}{16a^2}},$$

so the equilibrium is unstable. It corresponds to a saddle point of the potential.

The unnormalised eigenvectors of $\hat{C}$ are

$$\begin{pmatrix} \frac{k}{\sqrt{2}} \\ k - \omega^2 m \end{pmatrix},$$

so those of the original problem are:

$$\begin{pmatrix} \frac{k}{\sqrt{2}} \\ \frac{k-m\omega^2}{\sqrt{2}} \end{pmatrix}.$$

4. A guitar string has Lagrangian

$$\int_0^L \frac{\sigma}{2}(\frac{\partial y}{\partial t})^2 - T\sqrt{1 + (\frac{\partial y}{\partial x})^2} dx$$

where the mass density of the string is σ, the tension is T, and the length of the string is L. Show that as in the finite-dimensional case, the linearised Euler-Lagrange equation is the same as the Euler-Lagrange equation for the quadratic approximation to the Lagrangian $L^{(2)}$. By looking for exponential time-dependence in the linearised equation, find the normal modes.

Solution *The quadratic approximation to the Lagrangian is:*

$$L^{(2)} = \int_0^L \frac{\sigma}{2}(\frac{\partial y}{\partial t})^2 - \frac{T}{2}(\frac{\partial y}{\partial x})^2 dx,$$

and its EL equation is:

$$\sigma y_{tt} = T y_{xx}.$$

This is the same as the linearisation of the EL equation of the full Lagrangian. The normal modes have dependence $y = \exp(-i\omega t)u(x)$, so the eigenfunction $u(x)$ satisfies

$$u_{xx} = -\frac{\sigma\omega^2}{T}u.$$

The appropriate boundary conditions for a guitar string are $u(0) = u(L) = 0$, so the normal modes are:

$$u_n = \sin(n\pi x/L),$$

for any positive integer n, and the corresponding eigenfrequencies are then

$$\omega_n = \sqrt{\frac{T}{\sigma}}\frac{n\pi}{L}.$$

Chapter 6

Rigid bodies

6.1 Kinematics

A rigid body is a system with (very) many particles, satisfying very many constraints, that all *the interparticle distances, angles etc., are constant. Let us consider two particles with positions $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$, and separation $\boldsymbol{v} = \boldsymbol{x_i} - \boldsymbol{x_j}$. The constraint for these two particles is then $|\boldsymbol{v}|^2 =$ constant. Now the time derivative $\dot{\boldsymbol{v}}$, of any vector $\boldsymbol{v}$ of constant length, is perpendicular to it, $\dot{\boldsymbol{v}} \cdot \boldsymbol{v} = 0$.*

We now choose 3 such vectors, fixed in the body, $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, $\boldsymbol{v_3}$, which we choose to form a right-handed orthonormal frame; that is:

$$\boldsymbol{v_i} \cdot \boldsymbol{v_j} = \delta_{ij},$$

with

$$\boldsymbol{v_1} \wedge \boldsymbol{v_2} = \boldsymbol{v_3}.$$

These vectors can be used as the **columns** *of an orthogonal matrix*

$$V = (\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}).$$

Any other vector $\boldsymbol{u}$ fixed in the body, such as the vector from one point to another, will have constant components U_i with respect to this frame:

$$\boldsymbol{u} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{3}} U_i \boldsymbol{v_i},$$

$$U_i = \boldsymbol{v_i} \cdot \boldsymbol{u} = \boldsymbol{v_i^T} \boldsymbol{u}.$$

That is, if $\boldsymbol{u}$ has components, with respect to a basis fixed in space, u_i

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{v_1^T} \\ \boldsymbol{v_2^T} \\ \boldsymbol{v_3^T} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

Then we may write this as a matrix equation:

$$\boldsymbol{U} = V^T \boldsymbol{u},$$

and the inverse map is (since V is orthogonal):

$$u = VU$$

It is therefore sufficient to study the motion of the orthonormal frame V. Below, we will consistently use upper-case bold letters to denote vectors in the body *frame, and the corresponding lower-case letters for the same vectors in terms of the* space *frame.*

V is orthogonal, so that:

$$VV^T = I.$$

Differentiating this relation with respect to time, we find:

$$\dot{V}V^T + V\dot{V}^T = 0.$$

That is, $\dot{V}V^T \equiv \dot{V}V^{-1}$ is antisymmetric.

Similarly we find

$$V^T\dot{V} + \dot{V}^T V = 0.$$

Thus, $V^T\dot{V} \equiv V^{-1}\dot{V}$ is also antisymmetric.

Let us write:

$$\dot{V}V^{-1} = \omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Now let us diferentiate the time dependent components **in space $\boldsymbol{u}$** *of a vector $\boldsymbol{U}$ which is fixed in the body.*

$$\dot{\boldsymbol{u}} = \dot{V}\boldsymbol{U}$$

$$= \dot{V}V^T\boldsymbol{U}.$$

Thus

$$\dot{\boldsymbol{u}} = \omega \boldsymbol{u}$$

so that the first component, for instance, reads:

$$\dot{u}_1 = -\omega_3 u_2 + \omega_2 u_3,$$

and similarly for the other components - if we construct a vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, we can rewrite all of these as

$$\dot{\boldsymbol{u}} = \omega \boldsymbol{u} = \boldsymbol{\omega} \wedge \boldsymbol{u}.$$

Any vector fixed in the body will evolve in the same way, in particular the separation between two points in the body $\boldsymbol{x_i} - \boldsymbol{x_j}$ does so. If we label one such point $\boldsymbol{x_0}$ for reference:

$$\dot{\boldsymbol{x}}_{\boldsymbol{i}} = \boldsymbol{\omega} \wedge (\boldsymbol{x_i} - \boldsymbol{x_0}) + \dot{\boldsymbol{x}}_{\boldsymbol{0}}.$$

The quantity $\boldsymbol{\omega}$ *is called the* **angular velocity in space***; the unit vector* $\boldsymbol{\omega}/|\boldsymbol{\omega}|$ *is called the* **axis of rotation**. *If* $\boldsymbol{\omega}$ *is time-dependent, as will generally be the case, it is safer to call* $\boldsymbol{\omega}/|\boldsymbol{\omega}|$ *the* instantaneous *axis of rotation. The quantity* $\dot{\boldsymbol{x}}_{\mathbf{0}}$ *is the velocity of the reference particle - this depends on our choice of* $\boldsymbol{x}_{\mathbf{0}}$ *unless* $\boldsymbol{\omega} = \mathbf{0}$.

We recall that $V^{-1}\dot{V}$ *is also antisymmetric; we will write*

$$V^{-1}\dot{V} = \Omega = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

The vector $\mathbf{\Omega}= (\Omega_1, \Omega_2, \Omega_3)$ *is known as the* **angular velocity in the body**. *To understand the relation between* $\mathbf{\Omega}$ *and* $\boldsymbol{\omega}$*, we may verify, for instance*

$$\Omega_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \Omega \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} V^T \dot{V} V^T V \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \boldsymbol{v}_{\mathbf{3}}^{\boldsymbol{T}} \omega \boldsymbol{v}_{\mathbf{2}}$$

$$= \boldsymbol{v}_{\mathbf{3}} \cdot (\boldsymbol{\omega} \wedge \boldsymbol{v}_{\mathbf{2}})$$

$$= \boldsymbol{\omega} \cdot (\boldsymbol{v}_{\mathbf{2}} \wedge \boldsymbol{v}_{\mathbf{3}}) = \boldsymbol{v}_{\mathbf{1}} \cdot \boldsymbol{\omega},$$

and similarly for the other components. Thus we should consider $\mathbf{\Omega}$ *as the collection of components of* $\boldsymbol{\omega}$*, with respect to the rotating frame.*

$$\mathbf{\Omega} = \begin{pmatrix} \boldsymbol{v}_{\mathbf{1}} \cdot \boldsymbol{\omega} \\ \boldsymbol{v}_{\mathbf{2}} \cdot \boldsymbol{\omega} \\ \boldsymbol{v}_{\mathbf{3}} \cdot \boldsymbol{\omega} \end{pmatrix} = V^T \boldsymbol{\omega}.$$

That is $\boldsymbol{\omega}$ *and* $\mathbf{\Omega}$ *transform between the two frames in the same way as vectors.*

6.2 The Kinetic energy

The kinetic energy is

$$T = \sum_{i=1}^N \frac{m_i}{2} |\dot{\boldsymbol{x}}_{\boldsymbol{i}}|^{\mathbf{2}}$$

$$= \sum_{i=1}^N \frac{m_i}{2} (|\boldsymbol{\omega} \wedge (\boldsymbol{x}_{\boldsymbol{i}} - \boldsymbol{x}_{\mathbf{0}}) + \dot{\boldsymbol{x}}_{\mathbf{0}}|^{\mathbf{2}}$$

$$= \sum_{i=1}^N \frac{m_i}{2} (|\boldsymbol{\omega} \wedge (\boldsymbol{x}_{\boldsymbol{i}} - \boldsymbol{x}_{\mathbf{0}})|^2 + 2\boldsymbol{\omega} \wedge (\boldsymbol{x}_{\boldsymbol{i}} - \boldsymbol{x}_{\mathbf{0}}) \cdot \dot{\boldsymbol{x}}_{\mathbf{0}} + |\dot{\boldsymbol{x}}_{\mathbf{0}}|^{\mathbf{2}}).$$

Here the sum over all the particles could easily be replaced by an integral over the body, weighted by a mass density $\rho((\boldsymbol{x}_{\boldsymbol{i}} - \boldsymbol{x}_{\mathbf{0}}))$:

$$T = \int \frac{\rho(x)}{2} (|\boldsymbol{\omega} \wedge (\boldsymbol{x}_{\boldsymbol{i}} - \boldsymbol{x}_{\mathbf{0}})|^{\mathbf{2}} + \mathbf{2}(\boldsymbol{\omega} \wedge (\boldsymbol{x}_{\boldsymbol{i}} - \boldsymbol{x}_{\mathbf{0}})) \cdot \dot{\boldsymbol{x}}_{\mathbf{0}} + |\dot{\boldsymbol{x}}_{\mathbf{0}}|^{\mathbf{2}}) \mathrm{d}^3\mathbf{x}.$$

This will make sense in many applications. If we allow $\rho(\boldsymbol{x})$to be a sum of δ-functions, then this formula reduces to the previous one.

There are two easy ways in which this expression can be simplified. One is where the reference point $\boldsymbol{x_0}$ *is fixed in space. Then both terms involving* $\dot{\boldsymbol{x}}_\mathbf{0}$ *will vanish, giving*

$$T = \sum_{i=1}^{N} \frac{m_i}{2} |\boldsymbol{\omega} \wedge (\boldsymbol{x_i} - \boldsymbol{x_0})|^2,$$

a quadratic expression in $\boldsymbol{\omega}$.

Alternatively we may choose $\boldsymbol{x_0}$ *to be the centre of mass of the body. Then the vector* $\sum m_i(\boldsymbol{x_i} - \boldsymbol{x_0})$ *will vanish, the velocity of this point of the body is* $\boldsymbol{v_0} = \dot{\boldsymbol{x}}_\mathbf{0}$, *and we can expand* $\boldsymbol{T}$ *as follows:*

$$T = \sum_{i=1}^{N} \frac{m_i}{2} (|\boldsymbol{\omega} \wedge (\boldsymbol{x} - \boldsymbol{x_0}) + \boldsymbol{v_0})|^2$$

$$= \sum_{i=1}^{N} \frac{m_i}{2} (|\boldsymbol{\omega} \wedge (\boldsymbol{x} - \boldsymbol{x_0})|^2 + 2(\boldsymbol{\omega} \wedge ((\boldsymbol{x} - \boldsymbol{x_0})) \cdot \boldsymbol{v_0} + |\boldsymbol{v_0}|^2)$$

$$= \sum_{i=1}^{N} \frac{m_i}{2} (|\boldsymbol{\omega} \wedge (\boldsymbol{x} - \boldsymbol{x_0})|^2 + |\boldsymbol{v_0}|^2).$$

The crucial point here is that the linear terms in $\boldsymbol{x} - \boldsymbol{x_0}$ *all vanish.*

T *can thus be decomposed into the sum of a term quadratic in* $\boldsymbol{\omega}$, *the rotational kinetic energy, and another term, quadratic in* $\boldsymbol{v_0}$, *the translational kinetic energy. The translational term is:*

$$T_{trans} = \sum_{i=1}^{N} \frac{m_i}{2} |\boldsymbol{v_0}|^2 = \frac{M}{2} |\boldsymbol{v_0}|^2.$$

where M *is the total mass of the body.*

We write the rotational term as:

$$T_{rot} = \sum_{i=1}^{N} \frac{m_i}{2} (|\boldsymbol{\omega} \wedge (\boldsymbol{x} - \boldsymbol{x_0})|^2 = \frac{1}{2} \boldsymbol{\omega}^T I \boldsymbol{\omega}.$$

Here I *is a symmetric* 3×3 *matrix, called the* **inertia tensor**. *Expanding the vector product and rewriting in terms of scalar products, using the standard identity:*

$$|\mathbf{a} \wedge \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2,$$

we find

$$I_{ij} = \int \rho(\mathbf{x}) (|\boldsymbol{x} - \boldsymbol{x_0}|^2 \delta_{ij} - (\boldsymbol{x} - \boldsymbol{x_0})_{\boldsymbol{i}} (\boldsymbol{x} - \boldsymbol{x_0})_{\boldsymbol{j}}) \mathrm{d}^3 \boldsymbol{x}.$$

Exercise *Check this.*

Since I is real and symmetric, its eigenvectors may be taken to be orthogonal; they must be orthogonal if the eigenvalues are distinct. These eigenvectors are called the **principal axes of inertia**, *and the eigenvalues are called the* **principal moments of inertia**. *These eigenvalues are always positive, as the kinetic energy is positive. The principal axes are a natural choice for an orthonormal frame fixed in the body.*

We may also define an inertia tensor, with the same formula, about any *point* $\boldsymbol{x_0}$, *whether or not it is the centre of mass. Note that in such cases the* *linear term in* $(\boldsymbol{x} - \boldsymbol{x_0})$ *will not necessarily vanish; however it also involves the velocity* $\boldsymbol{v_0}$, *so the term will also vanish if* $\boldsymbol{x_0}$ *is a fixed point in space.*

This is important in applications, e.g. if an axle does not pass through the centre of mass of a wheel. In balancing a wheel, the centre of mass must be aligned with the axle, and the axis of inertia with the axis of rotation. Otherwise the wheel will tend to wobble.

Exercise *A body of mass* M *has inertia tensor* I_0 *about its centre of mass,* $\mathbf{x}_0$. *What is its inertia tensor about the origin?*

Exercise *A body of mass* M *has inertia tensor*

$$I_0 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

about the origin with respect to the $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ *axes. What is the inertia tensor with respect to axes* $((\boldsymbol{i} - \boldsymbol{j})/\sqrt{\mathbf{2}}, (\boldsymbol{i} + \boldsymbol{j})/\sqrt{\mathbf{2}}, \boldsymbol{k})$? *If the new axes are given by 3 orthonormal vectors* $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$, *what is the inertia tensor in this frame?* Hint: construct an orthogonal matrix from $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$. *We will need to transform vectors and tensors frequently between, for example, a frame fixed in space, such as* $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, *and a frame fixed in a rotating body - the principal axes of inertia are often a very convenient choice.*

Exercise *A rigid body consists of a single point particle of mass* m *at the point* (x, y, z). *What is its inertia tensor about the origin?*

6.3 Angular momentum

We have seen previously that if a Lagrangian system has rotational symmetry, there is a conserved quantity corresponding to this by Noether's theorem, the angular momentum. If an N*-body system has rigid constraints maintained by forces acting along the lines between the bodies, we may see this directly:*

$$m_i \tfrac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{x_i} = \textstyle\sum_{j=1}^{N} \boldsymbol{f_{ij}},$$

we see that

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t} \textstyle\sum_{i=1}^{N} m_i \boldsymbol{x_i} \wedge \dot{\boldsymbol{x}}_{\boldsymbol{i}} \\ &= \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{x_i} \wedge \boldsymbol{f_{ij}} \\ &= \textstyle\sum_{i=1}^{N} \sum_{j=i+1}^{N} (\boldsymbol{x_i} - \boldsymbol{x_j}) \wedge \boldsymbol{f_{ij}} = \mathbf{0}. \end{aligned}$$

where in the last line we used Newton's 3rd law $\boldsymbol{f_{ij}} + \boldsymbol{f_{ji}} = \mathbf{0}$.*This result holds in particular for a Lagrangian system with the constraint terms* $\lambda_{ij}|\boldsymbol{x_i} - \boldsymbol{x_j}|$, *as then*

$$\boldsymbol{f_{ij}} = \lambda_{ij}(\boldsymbol{x_i} - \boldsymbol{x_j})/|\boldsymbol{x_i} - \boldsymbol{x_j}|.$$

Now this vector conserved quantity, the angular momentum,

$$\boldsymbol{j} = \textstyle\sum_{i=1}^{N} m_i \boldsymbol{x_i} \wedge \dot{\boldsymbol{x}}_{\boldsymbol{i}}$$

is constant in a fixed (space) frame of reference. For definiteness we will call $\boldsymbol{j}$ *the* **angular momentum in space**. *Below we will need to work out the components in a frame* $(\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3})$ *fixed in the body; if we write*

$$J_i = \boldsymbol{v_i} \cdot \boldsymbol{j},$$

the vector $\boldsymbol{J} = (J_1, J_2, J_3)$ *is called* **the angular momentum in the body**.

If the body is subjected to external forces f_i *on the i-th particle, we generalise this result on the constancy of* $\boldsymbol{j}$ *to*

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{j} &= \textstyle\sum_{i=1}^{N} \boldsymbol{x_i} \wedge \boldsymbol{f_i} \\ &= \textstyle\sum_{i=1}^{N} ((\boldsymbol{x_i} - \boldsymbol{x_0}) + \boldsymbol{x_0}) \wedge \boldsymbol{f_i} \\ &= \textstyle\sum_{i=1}^{N} \boldsymbol{x_0} \wedge \boldsymbol{f_i} + \sum_{i=1}^{N} (\boldsymbol{x_i} - \boldsymbol{x_0}) \wedge \boldsymbol{f_i} \\ &= \boldsymbol{x_0} \wedge \boldsymbol{f} + \boldsymbol{g}. \end{aligned}$$

Here the first term is the vector product of the (arbitrary) position vector $\boldsymbol{x_0}$ *with the total force* $\boldsymbol{f}$ *on the body, while the second term*

$$\boldsymbol{g} = \textstyle\sum_{i=1}^{N} (\boldsymbol{x_i} - \boldsymbol{x_0}) \wedge \boldsymbol{f_i}$$

is called the total **couple** *exerted by these forces about the point* $\boldsymbol{x_0}$. *It will be convenient in applications to choose* $\boldsymbol{x_0}$ *in some particular convenient way - it could be a fixed point of the body, or the origin, or the centre of mass of the body.*

If a body of mass M *is rotating with angular velocity* $\boldsymbol{\omega}$ *and its centre of mass* $\boldsymbol{x_0}$ *is moving with velocity* $\boldsymbol{v_0}$, *the velocity* $\boldsymbol{v}$ *at a point* $\boldsymbol{x}$ *is:*

$$\boldsymbol{v} = \boldsymbol{v_0} + \boldsymbol{\omega} \wedge (\boldsymbol{x} - \boldsymbol{x_0}),$$

so we find

$$\begin{aligned} \boldsymbol{j} &= \textstyle\sum_{i=1}^{N} m_i \boldsymbol{x_i} \wedge (\boldsymbol{v_0} + \boldsymbol{\omega} \wedge (\boldsymbol{x} - \boldsymbol{x_0})) \\ &= \textstyle\sum_{i=1}^{N} m_i (\boldsymbol{x_0} + (\boldsymbol{x_i} - \boldsymbol{x_0})) \wedge (\boldsymbol{v_0} + \boldsymbol{\omega} \wedge (\boldsymbol{x} - \boldsymbol{x_0})) \\ &= M\boldsymbol{x_0} \wedge \boldsymbol{v_0} + \textstyle\sum_{i=1}^{N} m_i (\boldsymbol{x_i} - \boldsymbol{x_0}) \wedge (\boldsymbol{\omega} \wedge (\boldsymbol{x_i} - \boldsymbol{x_0})). \end{aligned}$$

Here, as before, the terms linear in $\sum_{i=1}^{N} m_i(\boldsymbol{x_i} - \boldsymbol{x_0})$ *all vanish if* $\boldsymbol{x_0}$ *is the centre of mass. The first term is the angular momentum of a point mass* M *with velocity* $\boldsymbol{v_0}$ *at the point* $\boldsymbol{x_0}$. *The second term is linear in* $\boldsymbol{\omega}$; *using the identity*

$$\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{a}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{a}) - \boldsymbol{a}(\boldsymbol{a} \cdot \boldsymbol{b})$$

we may rewrite:

$$\begin{aligned} \boldsymbol{j} &= \textstyle\sum_{i=1}^{N} m_i (\boldsymbol{x_i} - \boldsymbol{X}) \wedge (\boldsymbol{\omega} \wedge (\boldsymbol{x} - \boldsymbol{X})) \\ &= \textstyle\sum_{i=1}^{N} m_i \boldsymbol{\omega} ((\boldsymbol{x_i} - \boldsymbol{X}) \cdot (\boldsymbol{x} - \boldsymbol{X})) - (\boldsymbol{x_i} - \boldsymbol{X})(\boldsymbol{\omega} \cdot (\boldsymbol{x} - \boldsymbol{X})), \end{aligned}$$

so that, in vector form, we have:

$$\boldsymbol{j} = I\boldsymbol{\omega}.$$

The inertia tensor acts on the angular velocity vector, giving the angular momentum vector. This will not in general be parallel to the angular velocity, unless this is parallel to a principal axis of inertia.

6.4 Euler's equations

Let us consider a body with inertia tensor I, subject to zero net force $\boldsymbol{f}$, but with a couple $\boldsymbol{g}$ about its centre of mass. Then, in a non-rotating frame with origin at the centre of mass,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{j} = \boldsymbol{g},$$

and

$$\boldsymbol{j} = I\boldsymbol{\omega}.$$

The difficulty with this equation is that I depends on the orientation of the body which is changing. It is much easier to treat in the frame which is corotating with the body - if the eigenvalues of I are distinct, then the normalised eigenvector matrix $V = (\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3})$ defines a suitable frame, in which I is particularly simple, and constant. Then the angular momentum in the body is given by:

$$\boldsymbol{J} = V^T\boldsymbol{j} = \begin{pmatrix} \boldsymbol{v_1}\cdot\boldsymbol{j} \\ \boldsymbol{v_2}\cdot\boldsymbol{j} \\ \boldsymbol{v_3}\cdot\boldsymbol{j} \end{pmatrix}.$$

Thus, if we differentiate,

$$\begin{aligned} \dot{\boldsymbol{J}} &= V^T\dot{\boldsymbol{j}} + \dot{V}^T\boldsymbol{j} = V^T\boldsymbol{g} + \dot{V}^T V V^T\boldsymbol{j} \\ &= V^T\boldsymbol{g} + \Omega^T V^T\boldsymbol{j} \\ &\boldsymbol{G} - \boldsymbol{\Omega}\wedge\boldsymbol{J}. \end{aligned}$$

Here $\boldsymbol{G}$ denotes the couple expressed in terms of the body frame, $\boldsymbol{G} = \boldsymbol{V^T g}$,and we have defined, as above the antisymmetric 3×3 matrix

$$\Omega = V^T\dot{V} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

and the vector $\boldsymbol{\Omega}= (\Omega_1, \Omega_2, \Omega_3)^T$. The matrix Ω acts on vectors in the same way as the vector product with the vector $\boldsymbol{\Omega}$:

$$\Omega\boldsymbol{v} = \boldsymbol{\Omega}\wedge\boldsymbol{v},$$

for any vector $\boldsymbol{v}$. The term $-\boldsymbol{\Omega}\wedge\boldsymbol{J}$ can be understood by considering the motion in the space frame with zero couple; then $\dot{\boldsymbol{j}} = \boldsymbol{0}$. In a frame rotating with angular velocity $\boldsymbol{\omega}$ in space, $\boldsymbol{\Omega}$ in that frame, the angular momentum must rotate backwards - with angular velocity $-\boldsymbol{\Omega}$ in the body frame, to remain fixed in the space frame.

Since the components of $\boldsymbol{J}$ and of $\boldsymbol{\Omega}$ are related by the constant, diagonal inertia tensor,

$$\boldsymbol{J} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}\boldsymbol{\Omega} = \begin{pmatrix} I_1\Omega_1 \\ I_2\Omega_2 \\ I_3\Omega_3 \end{pmatrix},$$

the equation of motion

$$\dot{\boldsymbol{J}} + \boldsymbol{\Omega} \wedge \boldsymbol{J} = \boldsymbol{G}$$

is expanded as

$$\begin{pmatrix} I_1 \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \\ I_2 \frac{\mathrm{d}\Omega_2}{\mathrm{d}t} \\ I_3 \frac{\mathrm{d}\Omega_3}{\mathrm{d}t} \end{pmatrix} + \begin{pmatrix} (I_3 - I_2)\Omega_2\Omega_3 \\ (I_1 - I_3)\Omega_3\Omega_1 \\ (I_2 - I_1)\Omega_1\Omega_2 \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}.$$

These are known as **Euler's equations** *for a rotating body.*

6.4.1 The Euler Top

Let us consider a body which rotates freely in the absence of an applied couple. Then Euler's equations are:

$$\begin{pmatrix} I_1 \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \\ I_2 \frac{\mathrm{d}\Omega_2}{\mathrm{d}t} \\ I_3 \frac{\mathrm{d}\Omega_3}{\mathrm{d}t} \end{pmatrix} + \begin{pmatrix} (I_3 - I_2)\Omega_2\Omega_3 \\ (I_1 - I_3)\Omega_3\Omega_1 \\ (I_2 - I_1)\Omega_1\Omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We may solve this almost immediately once we identify 2 conserved quantities. One is the energy:

$$E = \frac{1}{2}\sum_{i=1}^{3} I_i\Omega_i^2,$$

while the other is

$$|\boldsymbol{J}|^{\mathbf{2}} = \textstyle\sum_{i=1}^{3} I_i^2\Omega_i^2,$$

Since $\boldsymbol{J}$ is only rotated with angular velocity $-\boldsymbol{\Omega}$, its modulus cannot change.

Qualitative dynamics

The level set of the energy is an ellipsoid

$$\sum_{i=1}^{3} I_i\Omega_i^2 = K_1,$$

and the level set of the second constant is another ellipsoid

$$\sum_{i=1}^{3} I_i^2\Omega_i^2 = K_2.$$

These will typically intersect in sets of dimension $(3-2) = 1$ a union of curves, which must be closed. We want to investigate what these look like - they are the orbits of the system. For definiteness let us suppose $I_1 > I_2 > I_3$. We see that

$$I_1 \geq \frac{K_2}{K_1} \geq I_3$$

If the upper bound is attained

$$I_1 = \frac{K_2}{K_1}$$

then $\mathbf{\Omega}$ *points along the minor axis of the energy ellipsoid* $(1, 0, 0)^T$*, while if the lower bound is attained,*

$$\frac{K_2}{K_1} = I_3$$

then $\mathbf{\Omega}$ *points along the major axis of the energy ellipsoid* $(0, 0, 1)^T$*. In these cases the closed curves degenerate to the ends of the axes. For*

$$I_1 > \frac{K_2}{K_1} > I_2$$

t0he curves are small loops around the ends of the minor axis, getting larger as the ratio drops towards I_2*. Similarly for*

$$I_2 > \frac{K_2}{K_1} > I_3$$

the curves are small loops around the ends of the major axis, getting larger as the ratio drops towards I_2*. In these cases the motion is confined to a simple closed curve, and the angular momentum in the body precesses around it with some finite period.*

The most interesting case is where

$$\frac{K_2}{K_1} = I_2$$

which is satisfied in particular when $\mathbf{\Omega}$ *points along the 'mean' axis of the energy ellipsoid* $(0, 1, 0)^T$*. Expanding* $K_2 = I_2 K_1$ *we get:*

$$I_1(I_1 - I_2)\Omega_1^2 = I_3(I_2 - I_3)\Omega_3^2,$$

so the set we want is the intersection of the ellipsoid with the union of two planes:

$$\Omega_1 = \pm\Omega_3 \sqrt{\frac{I_3(I_2 - I_3)}{I_1(I_1 - I_2)}}.$$

Linearised dynamics

We can look at the linearised motion near the $(1, 0, 0)^T$ *axis; let* $0 < \epsilon \ll 1$*, and*

$$\mathbf{\Omega} = \begin{pmatrix} \Omega_1^{(0)} \\ 0 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} \Omega_1^{(1)} \\ \Omega_2^{(1)} \\ \Omega_3^{(1)} \end{pmatrix},$$

and Euler's equations become to first order

$$I_1\frac{\mathrm{d}\Omega_1^{(1)}}{\mathrm{d}t} = 0,$$

$$0I_2\frac{\mathrm{d}\Omega_2^{(1)}}{\mathrm{d}t} + (I_1 - I_3)\Omega_1^{(0)}\Omega_3^{(1)} = 0,$$

$$I_3\frac{\mathrm{d}\Omega_3^{(1)}}{\mathrm{d}t} + (I_2 - I_1)\Omega_1^{(0)}\Omega_2^{(1)} = 0,$$

Since $I_1 > I_2 > I_3$, *it follows that the motion of* $\boldsymbol{\omega}^{(\mathbf{1})}$ *is oscillatory, going like* $\exp(ipt)$, *with*

$$p^2 = \frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}(\Omega_1^{(0)})^2 > 0.$$

Similarly we can show that motion about the third axis is stable. In either of these cases if the body starts spinning about an axis near the major or minor axis, it remains spinning about a nearby axis for all t.

However motions started near the second axis are linearly unstable, and perturbations will grow or decay like $\exp(\pm\lambda t)$, *with*

$$\lambda^2 = \frac{(I_1 - I_2)(I_2 - I_3)}{I_1 I_3}(\Omega_2^{(0)})^2.$$

Thus motions started near the mean axis of inertia do not remain there - if the motion starts near the positive mean axis, it will be spinning about an axis near the negative mean axis some finite (possibly long) time later.

DEMONSTRATION

6.4.2 The heavy symmetric top

THIS SECTION WAS OMITTED FROM LECTURES *A top of mass* M *has fixed point at the origin, and centre of mass* $(0, 0, h)$ *with respect to the principal axes, fixed in the body. It is subject to a gravitational field* $(0, 0, -g)$ *which is fixed in space. In the body frame, this gives a time-dependent force of constant modulus,* $\boldsymbol{F}$. *The couple in the body frame is then:*

$$\boldsymbol{G} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \wedge \boldsymbol{F}$$

Since $\boldsymbol{F}$ *is a vector in the body frame which is fixed in space, it satisfies:*

$$\begin{pmatrix} \dot{F}_1 \\ \dot{F}_2 \\ \dot{F}_3 \end{pmatrix} = -\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} \wedge \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix},$$

while Euler's equations are:

$$\begin{pmatrix} I_1\frac{\mathrm{d}\Omega_1}{\mathrm{d}t} \\ I_2\frac{\mathrm{d}\Omega_2}{\mathrm{d}t} \\ I_3\frac{\mathrm{d}\Omega_3}{\mathrm{d}t} \end{pmatrix} + \begin{pmatrix} (I_3 - I_2)\Omega_2\Omega_3 \\ (I_1 - I_3)\Omega_3\Omega_1 \\ (I_2 - I_1)\Omega_1\Omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \wedge \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} -hF_2 \\ hF_1 \\ 0 \end{pmatrix}.$$

We will study a special case of this system using another approach below; here we only note that if $I_1 = I_2$ these equations simplify. It is then possible to find enough conserved quantities to solve the system explicitly.

Exercise *How many can you find? Most other cases of these equations are not exactly solvable, and the motion is chaotic.*

6.5 The Euler Angles

A different approach to the description of a rotating body is to choose a set of 3 coordinates which parametrise our configuration space - the set of orthonormal frames. Euler described the following way of doing this. The key idea is that an orthonormal frame in R^2 can be written as

$$\begin{pmatrix} \boldsymbol{i'} \\ \boldsymbol{j'} \end{pmatrix} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix},$$

so the unit vector $\boldsymbol{i}$along the $x-$axis is rotated anticlockwise by an angle ϕ into $\boldsymbol{v_1}$, and the $\boldsymbol{y}$-axis is similarly rotated into $\boldsymbol{v_2}$.

We can extend this to construct a 1-parameter family of orthonormal frames in R^3 just by adding an unchanged z-axis:

$$\begin{pmatrix} \boldsymbol{i'} \\ \boldsymbol{j'} \\ \boldsymbol{k} \end{pmatrix} = V_1 = \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

This matrix describes a rotation by an angle $\boldsymbol{\phi}$ about the $\boldsymbol{z}$-axis. We could similarly rotate by an angle $\boldsymbol{\theta}$ about the $\boldsymbol{y}$-axis:

$$V_2 = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}.$$

Now if we compose these two transformations, we can arrange for the original axes $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ to be rotated first about the $\boldsymbol{k}$ axis to $(\boldsymbol{i'}, \boldsymbol{j'}, \boldsymbol{k})$, then about $\boldsymbol{j'}$, to new axes

$$\begin{pmatrix} \boldsymbol{i''} \\ \boldsymbol{j'} \\ \boldsymbol{k'} \end{pmatrix} = V_2 \begin{pmatrix} \boldsymbol{i'} \\ \boldsymbol{j'} \\ \boldsymbol{k} \end{pmatrix} = V_2 V_1$$

$$= \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{pmatrix}$$

We see that k' can be pointed anywhere on the unit sphere by suitably chosen (θ, ϕ), as the bottom row of this matrix corresponds to taking spherical polars on *the unit sphere. Note however that given* $\boldsymbol{k'}$*, the angles* (θ, ϕ) *are not determined uniquely, particularly if* $\theta = n\pi$*, when* ϕ *is undetermined - what is the longitude at the North pole?*

To complete the construction, we now rotate again by an angle ψ *about this* $\boldsymbol{k'}$ *axis. This is given by a matrix*

$$V_3 = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which takes the axes $(\boldsymbol{i''}, \boldsymbol{j'}, \boldsymbol{k'})$ *to new axes* $(\boldsymbol{i'''}, \boldsymbol{j''}, \boldsymbol{k'})$ *given by:*

$$\boldsymbol{V} = \begin{pmatrix} \boldsymbol{i'''} \\ \boldsymbol{j''} \\ \boldsymbol{k'} \end{pmatrix} = V_3 \begin{pmatrix} \boldsymbol{i''} \\ \boldsymbol{j'} \\ \boldsymbol{k'} \end{pmatrix} = V_2 V_1$$

$$= \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\psi)\cos(\theta)\cos(\phi) - \sin(\psi)\sin(\phi) & \cos(\psi)\cos(\theta)\sin(\phi) + \sin(\psi)\cos(\phi) & -\cos(\psi)\sin(\theta) \\ -\sin(\psi)\cos(\theta)\cos(\phi) - \cos(\psi)\sin(\phi) & -\sin(\psi)\cos(\theta)\sin(\phi) + \cos(\psi)\cos(\phi) & \sin(\psi)\sin(\theta) \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \end{pmatrix}.$$

The exact formula here is less important that the way it was constructed. This is the most general possible orthogonal 3×3 *matrix with determinant +1. The parameters* (ϕ, θ, ψ) *are caled the* **Euler angles***.*

Let us suppose that a body is rotating, and the axes $(\boldsymbol{i'''}, \boldsymbol{j''}, \boldsymbol{k'})$ *are fixed in it; then the angular velocity in the body frame is given by:*

$$\Omega = \dot{V} V^T,$$

which will be linear in $(\dot{\phi}, \dot{\theta}, \dot{\psi})$*. Transforming this antisymmetric matrix into a vector, as above, we get*

$$\boldsymbol{\Omega} = \dot{\phi}\boldsymbol{k} + \dot{\theta}\boldsymbol{j'} + \dot{\psi}\boldsymbol{k'}.$$

We can then calculate the kinetic and potential energy of a body in terms of the Euler angles and their derivatives.

6.6 The symmetric top

6.6.1 The symmetric rotator

A rigid body of mass $\boldsymbol{M}$ *is free to rotate about any axis through a fixed point. Its inertia tensor about the fixed point is, in the principal axes,*

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.$$

That is, the first and second principal moments are equal. This will be true for instance if the body is a homogeneous solid of revolution about the third axis.

The angular velocity,

$$\boldsymbol{\Omega} = \dot{\phi}\boldsymbol{k} + \dot{\theta}\boldsymbol{j'} + \dot{\psi}\boldsymbol{k'},$$

needs to be projected onto the 'body' axes $(\boldsymbol{i'''}, \boldsymbol{j''}, \boldsymbol{k'})$

$$\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Omega} \cdot \boldsymbol{i'''} \\ \boldsymbol{\Omega} \cdot \boldsymbol{j''} \\ \boldsymbol{\Omega} \cdot \boldsymbol{k'} \end{pmatrix}$$

$$= \dot{\phi} \begin{pmatrix} -\cos(\psi)\sin(\theta) \\ \sin(\psi)\sin(\theta) \\ \cos(\theta) \end{pmatrix} + \dot{\theta} \begin{pmatrix} \sin(\psi) \\ \cos(\psi) \\ 0 \end{pmatrix} + \dot{\psi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Hence the kinetic energy is found to be

$$\frac{I_1}{2}(-\dot{\phi}\cos(\psi)\sin(\theta)+\dot{\theta}\sin(\psi))^2+\frac{I_1}{2}(\dot{\phi}\sin(\psi)\sin(\theta)+\dot{\theta}\cos(\psi))^2+\frac{I_3}{2}(\dot{\phi}\cos(\theta)+\dot{\psi})^2$$

$$= \frac{I_1}{2}(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2) + \frac{I_3}{2}(\dot{\psi} + \cos(\theta)\dot{\phi})^2.$$

This is the Lagrangian of the symmetric rotator, or equivalently, the kinetic energy of the symmetric top.

6.6.2 The symmetric top

The symmetric top is a symmetric rotator whose centre of mass is on its symmetry axis, and which is subject to a uniform gravitational field, acting in the negative z*-direction in the space frame. The centre of mass of the body is assumed to lie on the third principal axis (the symmetry axis) through the fixed point, a distance* h *from it.*

PICTURE

The potential energy is then, in terms of Euler angles,

$$M\boldsymbol{k'} \cdot \boldsymbol{k} = Mgh\cos(\theta)$$

Hence the Lagrangian is:

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}) + \frac{I_3}{2}(\dot{\psi} + \cos(\theta)\dot{\phi})^2 - Mgh\cos(\theta).$$

6.6.3 Symmetries and conserved quantities

We will be able to solve this system exactly. There are two symmetries - the system is invariant under rotations in both ϕ and ψ - rotations about the $\boldsymbol{k}$ and $\boldsymbol{k''}$ axes. That is, rotations about a vertical axis in space, or rotations about the 3rd principal axis (symmetry axis) of the body. The corresponding Noether integrals, the conjugate momenta to these two angles, are:

$$p_\phi = I_1 \sin^2(\theta)\dot{\phi} + I_3(\dot{\psi} + \cos(\theta)\dot{\phi})\cos(\theta),$$

and

$$p_\psi = I_3(\dot{\psi} + \cos(\theta)\dot{\phi}).$$

These will have constant value throughout the evolution of the system. We will treat them as constants when solving for the evolution of θ.

6.6.4 The Hamiltonian

The other canonical momentum coordinate is

$$p_\theta = I_1\dot{\theta}.$$

Hence the Hamiltonian is found to be:

$$H = \frac{1}{2I_1}(p_\theta^2 + \frac{(p_\phi - p_\psi \cos(\theta))^2)}{\sin^2(\theta)}) + \frac{p_\psi^2}{2I_3} + Mgh\cos(\theta).$$

6.6.5 Nutation - motion in θ

Now, if $H = e$, a constant, and p_ϕ and p_ψ are replaced by their constant values, then

$$p_\theta^2 = 2eI_1 - \frac{(p_\phi - p_\psi \cos(\theta))^2}{\sin^2(\theta)} - \frac{I_1 p_\psi^2}{I_3} - 2MghI_1\cos(\theta).$$

Writing $\cos(\theta) = x$, we get

$$p_\theta^2 = \frac{2eI_1(1-x^2) - (p_\phi - p_\psi x)^2 - \frac{I_1 p_\psi^2}{I_3}(1-x^2) - 2MghI_1(x - x^3)}{1-x^2}.$$

The numerator here is a cubic in x with real coefficients. If the constants e, p_ϕ, p_ψ are given values corresponding to real initial conditions, then this numerator will be non-negative somewhere in $-1 \le x \le 1$. However, at $x = \pm 1$, the numerator is clearly negative, being $-(p_\phi \mp p_\psi)^2$ there. Hence the cubic has two roots $x_\pm$ in this interval. Then $\cos(\theta)$ will oscillate between the two turning points $x_\pm$. This motion is called **nutation** *- 'nodding' in Latin.*

Since $\dot{x} = -\sin(\theta)\dot{(\theta)}$, and $\dot{(\theta)} = \frac{p_\theta}{I_1}$, we find:

$$I_1^2 \dot{x}^2 = 2eI_1(1-x^2) - (p_\phi - p_\psi x)^2 - \frac{I_1 p_\psi^2}{I_3}(1-x^2) - 2MghI_1(x - x^3),$$

which can be solved in terms of the Weierstrass elliptic function $\wp(u)$. This is the most fundamental elliptic function, and satisfies:

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

where g_2 and g_3 are arbitrary constants. Since $\dot{x}^2$ is cubic in x, we can write

$$x = \alpha\wp(\beta t) + \gamma,$$

for some definite constants α, β and γ.

6.6.6 Precession - motion in ϕ

The equation for the evolution of ψ is

$$\dot{\phi} = \frac{(p_\phi - p_\psi x)}{I_1(1-x^2)}$$

This is singular at $\theta = 0$, $\theta = \pi$, where ϕ is undefined. It also has a simple zero at

$$x_0 = \frac{p_\phi}{p_\psi},$$

which may or may not lie between the two turning points of x, $x_\pm$.

If it does not, then ϕ will be monotonic in t. The axis of the top moves steadily round the vertical.

The motion is more interesting when the zero x_0 lies between the two turning points $x_- < x_0 < x_+$, for then $\dot{\phi}$ is negative for large $x > x_0$, and positive for $x < x_0$. The axis moves in loops:

PICTURE

There is a transitional case when $x_0 = x_+$; in this case the loops shrink to cusps, and the loops swing downwards from these. Then $\dot{\phi}$ vanishes at $x = x_+$, but is otherwise always positive.

PICTURE

It is instructive to consider why it is that the other case $x_0 = x_-$ cannot occur for real initial conditions; at a cusp the kinetic energy is a minimum, so the potential energy must be a maximum. The cusp must be at the top of the loop.

DEMONSTRATION

6.7 Problems 5

1. Triangular Lamina *OAB is a uniform triangular plate (lamina) lying in the $x - y$-plane, whose vertices are at $O = (0,0,0)$, $A = (a,0,0)$ and $B = (0,b,0)$. It has total mass M. Show that the inertia tensor about the origin O is*

$$I' = \begin{pmatrix} \frac{1}{6}Mb^2 & -\frac{1}{12}Mab & 0 \\ -\frac{1}{12}Mab & \frac{1}{6}Ma^2 & 0 \\ 0 & 0 & \frac{1}{6}M(a^2+b^2) \end{pmatrix}.$$

   Find the centre of mass G of the lamina, and calculate directly the inertia tensor about G. Hence verify the 'Parallel axis theorem' in this case, that the inertia tensor about O is given by the sum of the inertia tensor about G and the inertia tensor about O of a point mass M at G.

2. *Wobbling plate Show that for a uniform disc of mass m and radius a in the $x - y$ plane, the principal moments of inertia about the centre are $\frac{ma^2}{4}$, $\frac{ma^2}{4}$ and $\frac{ma^2}{2}$. Suppose the disc spins about its symmetry axis, and is then slightly perturbed. Show that it makes two small wobbles for every revolution of the disc.*

3. *Control of a satellite A satellite is equipped with attitude jets which can*

exert a specified couple,

$$\mathbf{G} = G_1(t)\mathbf{e}_1 + G_2(t)\mathbf{e}_2 + G_3(t)\mathbf{e}_3$$

to control its orientation. Here the vectors $\mathbf{e}_i$ are the principal axes of the inertia tensor, whose principal moments are I_1 and $I_2 = I_3$. Write down Euler's equations for the satellite.

Initially the satellite spins about its axis of symmetry with constant angular velocity $\Omega\mathbf{e}_1$, but a couple $\mathbf{G} = (0, G_2(t), 0)$ is applied. Here

$$\begin{aligned} G_2(t) &= G \quad \textit{for} 0 \leq t \leq t_0, \\ &= 0 \quad \textit{otherwise.} \end{aligned}$$

Here G is a constant. Show that for $t > t_0$, the angular momentum $\mathbf{J}$ in the body describes a circle on the sphere $|\mathbf{J}| =$ constant, and that the axis $\mathbf{e}_1$ points towards the centre of this circle. Hence show that in a frame of reference fixed in space, the axis $\mathbf{e}_1$ moves around a circle on the unit sphere. What vector quantity points towards the centre of the circle? What is the orientation of the angular velocity vector? Show that the angle α between $\mathbf{J}$ and $\mathbf{e}_1$ is given by

$$\cos(\alpha) = \frac{I_1}{\sqrt{I_1^2 + 2\frac{G^2}{\omega^2\Delta^2}(1 - \cos(\Omega\Delta t_0))}},$$

where $\Delta = \frac{I_1 - I_2}{I_2}$.

4. *The symmetric top The Lagrangian for a heavy symmetric top pivoted at a point on its axis of symmetry is, as in the notes:*

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}) + \frac{I_3}{2}(\dot{\psi} + \cos(\theta)\dot{\phi})^2 - Mgh\cos(\theta).$$

Here $I_1 = I_2 \neq I_3$ are the principal moments, and θ, ϕ and ψ are the Euler angles. Here M is the mass of the top and l is the distance of the pivot from the centre of mass and g is the acceleration due to gravity.

Describe all solutions in which θ is constant;

   (a) *If $\theta = 0$, and $\dot{\phi} + \dot{\psi} = \omega$, show that this is a 'relative equilibrium' - that is a configuration in which the motion is steady, though the system is non-stationary. When is this configuration, where the top spins vertically on its axis, stable?*

   (b) *find the value of θ corresponding to given $\dot{\phi}$ and $\dot{\psi}$, (which should now be taken as independent). Are these 'relative equilibria' stable to small perturbations in θ?*

It may be helpful to write the Hamiltonian in the form

$$H = \frac{p_\theta^2}{2I_1} + U(\theta).$$

Chapter 7

Courseworks, Examples and Solutions

7.1 Buckling in a strut

A straight uniform horizontal bar of length L is subjected to a horizontal compression force F. It deforms, so that at a position a distance s along the bar, the line of the bar is rotated by an angle $\theta(s)$.

The elastic energy in the bar is known to be

$$E = \int_0^L \frac{k}{2}(\frac{\mathrm{d}\theta}{\mathrm{d}s})^2 \mathrm{d}s.$$

The work done by the force is

$$W = \int_0^L F(1 - \cos(\theta))\mathrm{d}s.$$

The bar takes up a configuration which minimises the total energy $E - W$.

1. *Write down the Euler-Lagrange equation satisfied by $\theta(s)$.*

Solution *Write* $E - W = \int_0^L f ds$*, and the Euler-Lagrange equation is:*

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial f}{\partial \theta_s} = \frac{\partial f}{\partial \theta},$$

which is, explicitly,

$$k\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = -F\sin(\theta).$$

(3 marks)

2. *Find the conserved quantity which follows from the fact that the problem has no explicit* s*-dependence. Sketch the solutions* $\theta(s)$ *if* $|\theta| < \pi$*. (*Hint - compare with a more familiar mechanical system, satisfying an analogous equation*). What are the corresponding shapes of the bar?*

   Solution *The quantity required is*

$$Q = \theta_s\frac{\partial f}{\partial \theta_s} - f$$

$$= \frac{k}{2}\theta_s^2 + F(\cos(\theta) - 1).$$

   (3 marks) This system has the same Euler-Lagrange equation as the simple pendulum; it is well known that for small amplitude $\max|\theta| < \pi$ *the solutions of this are oscillatory.*

   (4 marks)

3. *Linearise the equation of motion, assuming that* $|\theta| << 1$*. If the bar is clamped at both ends so that*

$$\theta(0) = \theta(L) = 0,$$

   show that a non-zero solution exists for $L\sqrt{F/k} = \pi$*, but not for any smaller value of* L*.*

   Solution *The linearised equation is:*

$$k\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = -F\theta.$$

(3 marks)

With the given boundary condition at $s = 0$*, we get*

$$\theta = A\sin(\sqrt{\frac{F}{k}}s),$$

where A *is an arbitrary constant. This can not satisfy the condition at* $s = L$*, except for* $A = 0$*, for* $\sqrt{\frac{F}{k}}L < \pi$*. (3 marks)*

4. *If the boundary condition is replaced by*

$$\theta(0) = \frac{\mathrm{d}\theta}{\mathrm{d}s}(L) = 0,$$

find the critical value of L *below which the bar does not buckle.*

Solution *As above,*

$$\theta = A\sin(\sqrt{\frac{F}{k}}s),$$

but the condition at $s = L$ *is now* $\theta_s|_{s=L} = 0$*, giving*

$$A\cos(\sqrt{\frac{F}{k}}L) = 0,$$

and now A *can only be non-zero if the* cos *vanishes. This does not happen for* $\sqrt{\frac{F}{k}}L < \pi/2$*. Hence the critical* L *is:*

$$L = \sqrt{\frac{k}{F}}\frac{\pi}{2}.$$

(4 marks)

7.2 A charged particle in an electromagnetic field

1. *The Lagrangian for a particle moving in three dimensions in a time-independent magnetic field with vector potential* $\mathbf{A}(\mathbf{x})$ *is given by:*

$$L = \frac{m}{2}|\dot{\mathbf{x}}|^2 + e\mathbf{A}(\mathbf{x})\cdot\dot{\mathbf{x}}. \tag{7.1}$$

Write L *as the sum of two homogeneous functions of the* $\dot{\mathbf{x}}$*,* L_2 *and* L_1*, say, of degrees 2 and 1 respectively. The energy is given, as usual by:*

$$E = \sum_{i=1}^{3}\dot{x}_i\frac{\partial L}{\partial \dot{x}_i} - L;$$

evaluate this using Euler's theorem.

Hence show that $|\dot{\mathbf{x}}| =$ *constant, so the magnetic field does no work. What can you say about the acceleration* $\ddot{\mathbf{x}}$*?*

2. *Calculate the Euler-Lagrange equations for the Lagrangian (7.1), and verify* directly *that* $|\dot{\mathbf{x}}| =$*constant.*

   Now repeat the calculation for the more general Lagrangian, which also includes a scalar potential $\phi(\mathbf{x})$,

   $$L = \frac{m}{2}|\dot{\mathbf{x}}|^2 + e\mathbf{A}(\mathbf{x}).\dot{\mathbf{x}} + e\phi(\mathbf{x}).$$

   Find the Euler-Lagrange equations for this Lagrangian.

   Show that these can be written as

   $$m\ddot{\mathbf{x}} = e(\dot{\mathbf{x}} \wedge \mathbf{B} + \mathbf{E}),$$

   and find explicit expressions for the **magnetic field B** *and the* **electric field E**.

3. *Calculate the effect on the motion if the vector potential* $\mathbf{A}$*, is changed by adding to it the gradient of a scalar* $\psi(\mathbf{x})$*:*

   $$\mathbf{A}' = \mathbf{A} + \nabla\psi(\mathbf{x}).$$

   Find the the difference ΔL *between the two Lagrangians* L' *and* L*, given by (7.1) using the vector potentials* $\mathbf{A}'$ *and* $\mathbf{A}$ *respectively; find the Euler-Lagrange equation corresponding to* ΔL*, and explain your result.*

4. *Consider the simple case* $\mathbf{A} = (0, 0, x)$*,* $\phi = x$*. Identify the symmetries of the Lagrangian, and derive the corresponding conserved quantities using Noether's theorem. Find and then solve the Euler-Lagrange equations, and describe the motion of the particle.*

7.2.1 Solution

1. *The Lagrangian is given by:*

   $$L = L_1 + L_2$$
   $$L_2 = \frac{m}{2}|\dot{\mathbf{x}}|^2 L_1 = e\mathbf{A}(\mathbf{x})\cdot\dot{\mathbf{x}}.$$

   The energy is given, as usual by:

   $$E = \sum_{i=1}^{3} \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L$$
   $$= (2L_2 + L_1) - (L_2 + L_1)$$
   $$= L_2;$$

   using Euler's theorem. Now L *is independent of* t*, so* $\dot{E} = 0$*. Hence* $|\dot{\mathbf{x}}|^2 =$ *constant, so the magnetic field does no work. Differentiating this, we find the acceleration* $\ddot{\mathbf{x}}$ *must satisfy* $\ddot{\mathbf{x}}\cdot\dot{\mathbf{x}} = 0$*. It is perpendicular to the velocity. (3 marks)*

2. *To calculate the Euler-Lagrange equations for the Lagrangian (7.1), we write L in components:*

$$\begin{aligned} L &= \frac{m}{2}|\dot{\mathbf{x}}|^2 + e\mathbf{A}(\mathbf{x}).\dot{\mathbf{x}} \\ &= \frac{m}{2}\sum_{i=1}^{3}\dot{x}_i^2 + eA_i(x_1,x_2,x_3)\dot{x}_i. \end{aligned}$$

Hence the momentum is given by the sum of a kinetic and potential term:

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + eA_i(x_1,x_2,x_3),$$

and

$$\frac{\partial L}{\partial x_i} = \sum_{j=1}^{3} e\frac{\partial A_j}{\partial x_i}\dot{x}_j,$$

changing the dummy index. Now the total t-derivative of the momentum is:

$$m\ddot{x}_i + e\sum_{j=1}^{3}\frac{\partial A_i}{\partial x_j}\dot{x}_j,$$

so that the equation of motion becomes:

$$m\ddot{x}_i = e\sum_{j=1}^{3}(\frac{\partial A_j}{\partial x_i}\dot{x}_j - \frac{\partial A_i}{\partial x_j}\dot{x}_j \tag{7.2}$$

that is,

$$m\ddot{\mathbf{x}} = e\dot{\mathbf{x}} \wedge (\nabla \wedge \mathbf{A}). \tag{7.3}$$

so the acceleration is clearly perpendicular to the velocity.

(4 marks)

Now repeating the calculation for the more general Lagrangian,

$$L = \frac{m}{2}|\dot{\mathbf{x}}|^2 + e\mathbf{A}(\mathbf{x}).\dot{\mathbf{x}} + e\phi(\mathbf{x}).$$

The Euler-Lagrange equation for this Lagrangian is the same as before, only with an additional term:

$$m\ddot{\mathbf{x}} = e(\dot{\mathbf{x}} \wedge (\nabla \wedge \mathbf{A}) + \nabla\phi) \tag{7.4}$$

These can be written as

$$m\ddot{\mathbf{x}} = e(\dot{\mathbf{x}} \wedge \mathbf{B} + \mathbf{E}),$$

where the magnetic field $\mathbf{B} = \nabla \wedge \mathbf{A}$ and the electric field $\mathbf{E} = \nabla\phi$.

(2 marks)

If the vector potential $\mathbf{A}$*, is changed by adding to it the gradient of a scalar* $\psi(\mathbf{x})$*:*

$$\mathbf{A}' = \mathbf{A} + \nabla\psi(\mathbf{x}).$$

The difference ΔL *between the two Lagrangians* L' *and* L*, given by (7.1) using the vector potentials* $\mathbf{A}'$ *and* $\mathbf{A}$ *respectively is*

$$\Delta L = e\nabla\psi\cdot\dot{\mathbf{x}}.$$

Now the only term in the Euler-Lagrange equation here is proportional to:

$$\nabla \wedge (\nabla\psi),$$

which vanishes identically.

(2 marks)

Any Lagrangian which can be written as an exact time derivative, as this can, leads to an action integral depending only on the end points:

$$\int_{t_1}^{t_2} \Delta L \mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}(e\psi)\mathrm{d}t = e\psi|_{t_1}^{t_2}.$$

This cannot change if the path is varied keeping the end points fixed. So the Euler-Lagrange equation must be trivial.

3. *In the case* $\mathbf{A} = (0, 0, x_1)$*,* $\phi = x_1$*, some more obvious symmetries of the Lagrangian, with their Noether integrals, are*

   - L *is independent of* x_2*, so* $\mathrm{p}_2 = m\dot{x}_2$ *is conserved.*
   - L *is independent of* x_3*, so* $\mathrm{p}_3 = m\dot{x}_3 + ex_1$ *is conserved.*
   - L *is independent of* t*, so* $E = \frac{m}{2}|\dot{\mathbf{x}}|^2 - ex_1$ *is conserved.*

   (2 marks)

   There is one other, rather less straightforward:

   - *If* $x_1 \to x_1 + \epsilon$*, then*

     $$L \to L + \epsilon(e\dot{x}_3 + e) = L + \epsilon\frac{\mathrm{d}}{\mathrm{d}t}(ex_3 + et)$$

     so x_1*-translation is also a symmetry. Using Noether's theorem, this gives the conserved quantity*

     $$I = m\dot{x}_1 - ex_3 - et.$$

   (1 mark)

   The Euler-Lagrange equations are

$$\begin{aligned} m\ddot{\mathbf{x}} &= e(\dot{\mathbf{x}} \wedge (\nabla \wedge \mathbf{A}) + \nabla\phi) \\ &= e((\dot{x}_1, \dot{x}_2, \dot{x}_3) \wedge (0, -1, 0) + e(1, 0, 0), \\ \text{that is:} & \\ m(\ddot{x}_1, \ddot{x}_2, \ddot{x}_3) &= e(1 + \dot{x}_3, 0, -\dot{x}_1). \end{aligned}$$

(2 marks)

Combining the first and third components:

$$m(\ddot{x}_1 - i\ddot{x}_3) = ie(\dot{x}_1 - i\dot{x}_3) + e,$$

so that

$$\dot{x}_1 - i\dot{x}_3 = V\exp(iet/m) + i,$$

giving for some arbitrary complex constants R, X,

$$x_1 - ix_3 = R\exp(iet/m) + it + X.$$

The real and imaginary parts of this give x_1 *and* x_3. *Also we get, for arbitrary real constants* y_0, v_0,

$$x_2 = y_0 + v_0 t$$

Thus the motion consists of a circular motion with frequency e/m *in the* (x_1, x_3)*-plane, superimposed with a steady drift in the direction* $(0, v_0, -1)$.

(3 marks)

7.3 Angular momentum - worked exercise.

1. *Consider the two functions* L_1, L_2 *for a system with three degrees of freedom:*

$$L_1 = p_2 q_3 - p_3 q_2,$$
$$L_2 = p_3 q_1 - p_1 q_3.$$

2. *Show directly that they both Poisson commute with the Hamiltonian for a particle in a central potential:*

$$H = \frac{|\mathbf{p}|^2}{2m} + V(|\mathbf{q}|).$$

3. *Show that* L_3 *defined by* $L_3 = -\{L_1, L_2\}$ *is not identically zero and verify directly that* $\{L_3, H\} = 0$. *Verify directly that* L_1, L_2 *and* L_3 *satisfy the Jacobi identity.*

4. *Show that* L_3 *and* $K = L_1^2 + L_2^2 + L_3^2$ *do Poisson commute.*

5. *Find Hamilton's equations, when the Hamiltonian is* $\omega_1 L_1 + \omega_2 + \omega_3 L_3$, *and you may assume* ω *is a unit vector. What is the geometrical meaning of these equations? That is, what is the corresponding symmetry of* H*?*

7.3.1 Solution

1. *Direct calculation:*

$$\begin{aligned}\{L_1, H\} = \frac{\partial L_1}{\partial \mathbf{q}}.\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial L_1}{\partial \mathbf{p}}.\frac{\partial H}{\partial \mathbf{q}} &= \\ p_2\frac{\partial H}{\partial p_3} - p_3\frac{\partial H}{\partial p_2} + q_2\frac{\partial H}{\partial q_3} - q_3\frac{\partial H}{\partial q_2} &= \\ p_2\frac{p_3}{m} - p_3\frac{p_2}{m} + (q_2\frac{q_3}{|\mathbf{q}|} - q_3\frac{q_2}{|\mathbf{q}|})V'(|\mathbf{q}|) & \\ &= 0.\end{aligned}$$

   Also $\{L_2, H\} = 0$, *similarly.*

2. *Direct calculation:*

$$\begin{aligned}\{L_1, L_2\} = \frac{\partial L_1}{\partial \mathbf{q}}.\frac{\partial L_2}{\partial \mathbf{p}} - \frac{\partial L_1}{\partial \mathbf{p}}.\frac{\partial L_2}{\partial \mathbf{q}} &= \\ p_2\frac{\partial L_2}{\partial p_3} - p_3\frac{\partial L_2}{\partial p_2} + q_2\frac{\partial L_2}{\partial q_3} - q_3\frac{\partial L_2}{\partial q_2} &= \\ p_2q_1 - p_3.0 + q_2(-p_1) - q_3.0 &= \\ -(p_1q_2 - p_2q_3)&,\end{aligned}$$

   so that

$$L_3 = p_1q_2 - p_2q_1.$$

   Similarly to $\{L_2, H\} = 0$, *we get* $\{L_3, H\} = 0$. *By cyclic permutation of indices, we get all the Poisson brackets:*

$$\{L_1, L_2\} = -L_3$$

$$\{L_2, L_3\} = -L_1$$

$$\{L_3, L_1\} = -L_2$$

   Then, for instance,

$$\{\{L_1, L_2\}, L_1\} = -\{L_3, L_1\} = L_2,$$

$$\{\{L_1, L_1\}, L_2\} = -\{0, L_2\} = 0,$$

$$\{\{L_2, L_1\}, L_1\} = \{L_3, L_1\} = -L_2,$$

   which add to zero as required. Other cases are similar or easier.

3. *We see, by the chain rule,*

$$\{L_3, K\} = 2L_1\{L_3, L_1\} + 2L_2\{L_3, L_2\} = -2L_1L_2 + 2L_2L_1 = 0.$$

4. The Hamiltonian is $\omega.(\mathbf{p} \wedge \mathbf{q})$.

Thus Hamilton's equations, with parameter θ *are*

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{q} = \mathbf{q} \wedge \omega,$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{p} = \mathbf{p} \wedge \omega,$$

which describe rigid rotation about the axis ω*, by an angle* θ*, if* $|\omega| = 1$.

7.4 Normal modes - coursework

A mass m *hangs vertically from a spring with spring constant* k*. The top end of the spring is attached to the free end of a rope which passes around a light wheel of radius* a*, which is free to rotate about its horizontal axis. A mass* $2m$ *is fixed to the circumference of the wheel. The first mass is free to move in the vertical direction.*

Find the equilibria, and the normal modes about each equilibrium, with their characteristic frequencies. Which equilibrium is stable?