M3M3 Partial Differential Equations

Solutions to problem sheet 3/4

1* (i) Show that the second order linear differential operators L and M, defined in some domain $\Omega \subset \mathbb{R}^n$, and given by

$$L\phi = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial \phi}{\partial x_i} + c\phi \tag{1}$$

$$M\phi = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} a_{ij}\phi - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} b_i \phi + c\phi \tag{2}$$

where a_{ij}, b_i, and c are differentiable functions of $\mathbf{x}$, are formally adjoint, in the sense that:

$$\langle u, Lv\rangle - \langle Mu, v\rangle = Q \tag{3}$$

where Q is some expression involving only terms evaluated on $\partial\Omega$.

(ii) Show that if L is self-adjoint, that is $L = M$, then

$$L\phi = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j}\phi + c\phi \tag{4}$$

where a_{ij} is symmetric. Find also the general expression for differential operators, of the form (1), to be skew-adjoint, that is satisfying $L = -M$.

Solution (i) *Since $\frac{\partial^2\phi}{\partial x_i \partial x_j}$ is symmetric, the antisymmetric part of a_{ij} is irrelevant. Take $a_{ij} = a_{ji}$.*

M, the adjoint of L, and L itself must satisfy:

$$uLv - vMu = \sum_{i=1}^{n} w_i,$$

for some local expressions w_i. When we integrate over the volume Ω, this divergence then integrates to a local expression on the boundary $\partial\Omega$. We find here:

$$uLv - vMu = \tag{5}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(ua_{ij}\partial_i\partial_j v - v\partial_i\partial_j a_{ij} v) \tag{6}$$

$$+u\sum_{i=1}^{n} b_i\partial_i v + v\sum_{i=1}^{n}\partial_i b_i u = \tag{7}$$

$$\sum_{i=1}^{n}\partial_i(\sum_{j=1}^{n}(ua_{ij}\partial_j v - v\partial_j a_{ij} u) + b_i uv), \tag{8}$$

as required.

(ii) *Expanding* Mu *and equating with* Lu, *for self-adjointness, we find the coefficient of* u *gives*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 a_{ij}}{\partial x_i \partial x_j} = \sum_{i=1}^{n}\frac{\partial b_i}{\partial x_i}$$

and the coefficient of $\partial_i u$ *gives*

$$\sum_{i=1}^{n}\frac{\partial(a_{ij}+a_{ji})}{\partial x_i} = 2b_i.$$

Since a_{ij} *is symmetric,* $b_i = \sum_{j=1}^{n}\frac{\partial a_{ij}}{\partial x_j}$. *Expression (4) follows.*

2i *Unseen and **unexamined** - Green's functions for hyperbolic equations*

Show that if the Riemann (or Riemann-Green) function $v(x, y; \xi, \eta)$ for a hyperbolic partial differential operator L, with adjoint $L^\dagger$,

$$L = \frac{\partial^2}{\partial x \partial y} + a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} + c(x,y) \tag{9}$$

is defined by

$$L^\dagger v = 0, \quad \xi > x, \quad \text{and} \quad \eta > y, \tag{10}$$
$$v_y = av, \quad x = \xi, \tag{11}$$
$$v_x = bv, \quad y = \eta \tag{12}$$
$$v = 0, \quad \xi < x \quad \text{or} \quad \eta < y \tag{13}$$
$$v(\xi, \eta) = 1 \tag{14}$$

then, for all x, y

$$L^\dagger v = \delta(x - \xi)\delta(y - \eta). \tag{15}$$

Hint: write $v(x, y) = w(x, y)H(\xi - x)H(\eta - y)$, where H is the Heaviside function, whose derivative is the δ-function, and w is a smooth function.

Solution 2(i) *Substitute $v = w(x, y)H(\xi - x)H(\eta - y)$ into the adjoint pde*

$$Mv = \delta(x - \xi)\delta(y - \eta)$$

. This gives terms in $H(\xi - x)H(\eta - y)$, implying that w and hence v satisfy the pde if $\xi > x$ and $\eta > y$. The conditions on $\xi = x$ and $\eta = y$ come from matching the coefficients of $\delta(\xi - x)H(\eta - y)$ and $H(\xi - x)\delta(\eta - y)$ respectively. The coefficient of $\delta(\xi - x)\delta(\eta - y)$ gives the condition at the singular point $\xi = x, \eta = y$.

(ii) *Here v must satisfy*

$$v_{xy} - \partial_x \frac{v}{x+y} - \partial_y \frac{v}{x+y} = 0, \tag{16}$$
$$v_x = v/(x+y) \qquad y = \eta \tag{17}$$
$$v_y = v/(x+y) \qquad x = \xi \tag{18}$$
$$v(\xi, \eta) = 1. \tag{19}$$

Clearly, taking $v = (x+y)/(\xi+\eta)$ achieves this. Put $L = \partial_x\partial_y + 1/(x+y)(\partial_x + \partial_y)$, and $L^\dagger$ is its adjoint. Integrate

$$vLu - uL^\dagger v$$

over the triangle Δ bounded by $x = y$, $x = \xi$, $y = \eta$. The integrand vanishes, as $Lu = L^\dagger v = 0$. However, it can also be written as a divergence

$$\partial_x\left(\frac{1}{2}(v\partial_y u - u\partial_y v) + uv/(x+y)\right) + \partial_y\left(\frac{1}{2}(v\partial_x u - u\partial_x v) + uv/(x+y)\right) = \phi_x + \psi_y,$$

say. Thus we get

$$0 = \int\int_{\Delta} \phi_x + \psi_y dxdy = \int_{\partial\Delta} \phi dy - \psi dx.$$

Now on $x = y$, *we have* $u = 0$, $u_x = f(x)$, *and also* $u_y = -f(x)$, *for* $u_x dx + u_y dy = du = 0$. *Integrating anticlockwise, along* $x = y$, $\quad x = \xi$, *and* $y = \eta$, *we get, using the boundary conditions on* u *and on* v,

$$u(\xi, \eta) = \frac{2}{\xi + \eta} \int_{\xi}^{\eta} x f(x) dx.$$

3* Using the maximum property for harmonic functions, prove the uniqueness of the solution to the Dirichlet problem for Poisson's equation.

Solution 3 *The difference v between any two solutions of Poisson's equation with the same Dirichlet data is a harmonic function which vanishes at the boundary. Being harmonic, this function takes its maximum (and minimum) values on the boundary; hence it vanishes everywhere.(4 marks)*

4i* Show that the solution of Helmholtz' equation in 3 dimensions with Dirichlet boundary conditions:

$$\nabla^2 u + \lambda u = f(\mathbf{x}), \quad \mathbf{x} \in D \tag{20}$$

$$u = g(\mathbf{x}), \quad \mathbf{x} \in \partial D, \tag{21}$$

is unique provided $\lambda \leq 0$.

ii With $\lambda = k^2 > 0$, a constant, find the radially symmetric solution $u(r)$ of the Dirichlet BVP in the ball $0 < r < a$, which satisfies:

$$\nabla^2 u + k^2 u = r^{-2} \frac{d}{dr} r^2 \frac{du}{dr} + k^2 u = 0, \quad 0 < r < a, \tag{22}$$

$$u = \frac{1}{4\pi a}, \quad r = a, \tag{23}$$

$$u \simeq -\frac{1}{4\pi r} + O(1), \quad \text{as } r \to 0. \tag{24}$$

It will be helpful to put $u(r) = v(r)/r$ and to find the ode satisfied by $v(r)$. Hence show directly that the solution is not unique if $k = \pi/a$.

Solution 4 *The difference between 2 solutions with the same Dirichlet data satisfies*

$$\nabla^2 u + \lambda u = 0, \quad \mathbf{x} \in D \tag{25}$$

$$u = 0, \quad \mathbf{x} \in \partial D. \tag{26}$$

Multiply the pde by u, and integrate over D. There is a vanishing boundary term, together with

$$\int_D -|\nabla u|^2 + \lambda u^2 dV.$$

If u satisfies the pde, then this must vanish; however if $\lambda < 0$, and u is not identically zero, this is strictly negative. Hence u must be zero throughout D.

For finite domains, this result is not the best possible; rather the solution is unique if $\lambda < \lambda_0$, the smallest eigenvalue of $-\nabla^2$ in D. So in the cube of side a, in 3 dimensions, $\lambda_0 = 3(\pi/a)^2$.

ii *In the ball of radius a, with $\lambda = k^2$, the radially symmetric solution satisfies:*

$$r^{-2} \frac{d}{dr} r^2 \frac{du}{dr} + k^2 u = 0, \quad 0 < r < a. \tag{27}$$

Put $u(r) = v(r)/r$. So

$$\frac{d^2 v}{dr^2} + k^2 v = 0, \quad 0 < r < a. \tag{28}$$

The b.c's are $v(0) = -1/(4\pi)$, so u is close to the free space Green's function for the Laplace equation, as $r \to 0$, and $v(a) = 1/(4\pi)$, so

$$v = -1/(4\pi) \cos(kr) + A \sin(kr) \tag{29}$$

with

$$v(a) = -1/(4\pi)\cos(ka) + A\sin(ka) = 1/(4\pi). \tag{30}$$

Thus

$$A = 1/(4\pi)\frac{\cos(ka)+1}{\sin(ka)}. \tag{31}$$

This obviously fails if $k = \pi/a$, when numerator and denominator both vanish; any value for A will do in this case.

5 Show how the method of images may be used to solve the Dirichlet problem for Laplace's equation in a two-dimensional wedge-shaped domain between two straight lines meeting at an angle α, for certain values of α.
Hint - *it is necessary to use multiple images.* What values of α can be treated in this way?

What image systems would be needed if instead we had Neumann conditions on one or both lines?

Hence solve Laplace's equation in the quarter-plane $x > 0$, $y > 0$, with Dirichlet conditions on the two axes and infinity:

$$u = 1 \quad \text{on } y = 0, \quad 0 < x < 1 \tag{32}$$

$$u = 0 \quad \text{otherwise.} \tag{33}$$

$$u \to 0 \quad \text{as } (x^2 + y^2) \to \infty. \tag{34}$$

Solution 5 *In polar coordinates, with the origin at the vertex, if the free-space Green's function is*

$$G_0(r, \theta, r', \theta')$$

we introduce images by repeated reflection in the two half-lines $\theta = 0$, $\theta = \alpha$. *These reflections are given by the two maps* $\theta \to -\theta$, *and* $\theta \to 2\alpha - \theta$ *respectively.*

These images give

$$G(r, \theta, r', \theta') = G_0(r, \theta, r', \theta') \tag{35}$$

$$-G_0(r, -\theta, r', \theta') - G_0(r, 2\alpha - \theta, r', \theta') \tag{36}$$

$$+G_0(r, 2\alpha + \theta, r', \theta') + G_0(r, -2\alpha + \theta, r', \theta') \ldots \tag{37}$$

The sum is periodic in θ *with period* 2α, *corresponding to a double reflection, but also periodic with period* 2π. *Hence, if we are to ensure that the system of images is finite, without images in the original wedge, we find* 2π *must be an even integer multiple of* α, $\alpha = \pi/n$ *say. (3 marks)*

Neumann problems can be treated in the same way, but the sum must then be even *under each reflection, so each term has a plus sign. (2 marks)*

In the quarter-plane, $n = 2$ *and we need 3 images. The Green's function we need here is*

$$G(x', y', z'; x, y, 0) = \frac{1}{4\pi}(\ln((x - x')^2 + (y - y')^2) - \ln((x + x')^2 + (y - y')^2) \tag{38}$$

$$- \ln((x - x')^2 + (y + y')^2) + \ln((x + x')^2 + (y + y')^2)) \tag{39}$$

The first term is the free-space Green's function, the second and third are its reflections under $x \to -x$ *and* $y \to -y$, *while the fourth term is the double reflection in both planes. The solution of the given problem is then*

$$u(x', y') = \int_0^1 \frac{\partial G}{\partial n}(x', y', x, 0) 1 dx, \tag{40}$$

Then, with

$$\frac{\partial G}{\partial n}|_{y=0} = \frac{y'}{\pi}\left(\frac{1}{(x - x')^2 + y'^2} - \frac{1}{(x + x')^2 + y'^2}\right) \tag{41}$$

we get

$$u(x',y') = \frac{1}{\pi}(\tan^{-1}(\frac{y'}{x'-1}) - 2\tan^{-1}(\frac{y'}{x'}) + \tan^{-1}(\frac{y'}{x'+1}))$$

It can easily be checked geometrically that this satisfies the boundary conditions. (4 marks)

6 Using the method of images, construct the Green's function of the Neumann problem for Laplace's equation in the half-space $D = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$. Hence solve

$$\nabla^2 u = 0, \quad \mathbf{x} \in D \tag{42}$$

$$\frac{\partial u}{\partial z} = 1 - x^2 - y^2, \quad x^2 + y^2 < 1, \quad z = 0 \tag{43}$$

$$\frac{\partial u}{\partial z} = 0, \quad x^2 + y^2 > 1 \quad z = 0, \tag{44}$$

and evaluate the resulting integral on the z-axis.

Solution 6 *The Neumann Green's function is:*

$$G(x, y, z; x', y', z') =$$

$$-\frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} + \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}\right)$$

which is even *in z. Integrate $u\nabla^2 G - G\nabla^2 u$ over the upper half-space $z > 0$, getting $u(x', y', z')$. By the divergence theorem, this is equal to the surface integral*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \quad dxdy$$

Now $\partial/\partial n = -\partial/\partial z$, and $\partial G/\partial z|_{z=0} = 0$. Thus

$$u(x', y', z') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y, 0; x', y', z')\frac{\partial u}{\partial z} \quad dxdy.$$

Hence

$$u(0, 0, z') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y, 0; 0, 0, z')\frac{\partial u}{\partial z} \quad dxdy.$$

$$= -\frac{1}{4\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} \frac{2}{\sqrt{r^2 + z'^2}} \quad rdrd\theta$$

$$= -\int_{r=0}^{1} \frac{1}{\sqrt{r^2 + z'^2}} \quad rdr$$

$$= -\int_{\rho=0}^{1} \frac{1}{2\sqrt{\rho + z'^2}} \quad d\rho$$

$$= -[\sqrt{\rho + z'^2}]_0^1 = -(\sqrt{1 + z'^2} - z').$$

This plainly has the correct z' derivative at z'=0.

7 Solve the heat equation

$$u_t = u_{xx} \tag{45}$$

with Neumann (insulating) boundary conditions $u_x = 0$ on the ends of the interval $[0, \pi]$, and initial condition

$$u(x, 0) = \delta(x - \pi/2), \tag{46}$$

in two different ways,
(i) in terms of Green's functions, using the method of images, and
(ii) in terms of a Fourier cosine series, by separation of variables.
Write these solutions in terms of two of the four theta functions, defined by:

$$\theta_4(s, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \cos(2ns), \tag{47}$$

and

$$\theta_2(s, q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \cos((2n + 1)s), \tag{48}$$

where q is chosen appropriately in each case. Hence, using the uniqueness theorem, derive the *Jacobi imaginary transformation* formula which relates θ_2 and θ_4 for different values of q. Such formulae are important in the theory of elliptic functions, as these may be written as quotients of theta functions.

See Lawden: Elliptic Functions and Applications, or other texts on elliptic functions, for further information.

Solution 7 This heat equn can be solved in 2 ways. (i)By separation of variables: the solution must be a sum of terms like $X(x)T(t)$, satisfying

$$T_t/T = X_{xx}/X = \text{constant}. \tag{49}$$

Now X must be $\cos(2nx)$, n integer, to satisfy the Neumann boundary conditions at $x = 0$ and $x = \pi$, and we get

$$u(x, t) = \sum_{n=0}^{\infty} a_n \cos(2nx) \exp(-4n^2 t). \tag{50}$$

At $t = 0$ we get

$$\delta(x) = \sum_{n=0}^{\infty} (a_n \cos(2nx)), \tag{51}$$

so on multiplying by $\cos(2nx)$ or 1, and integrating, we find $a_n \pi/2 = \cos(n\pi) = (-1)^n$, and $b_n = 0$, and $a_0 \pi = 1$.

Thus

$$u(x, t) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \cos(2n)x) \exp(-4n^2 t) = \frac{1}{\pi} \theta_4(x, t). \tag{52}$$

(ii) Alternatively, using Green's functions and the method of images, letting each image of the δ-function evolve into a copy of the free-space Green's function centred at $(2n+1)\pi/2$, we have:

$$\begin{aligned}
u(x,t) &= \sum_{n=-\infty}^{\infty} \exp(-(x-(2n+1)\pi/2)^2/4t)/\sqrt{4\pi t} \\
&= \sum_{n=-\infty}^{\infty} \exp(-x^2+(2n+1)\pi x-(2n+1)^2\pi^2/4)/4t)/\sqrt{4\pi t} \\
&= \exp(-x^2/4t) \sum_{n=-\infty}^{\infty} \exp(\frac{(2n+1)x\pi}{4t}) \exp(-(n+\frac{1}{2})^2\pi^2/t))/\sqrt{4\pi t} \\
&= \exp(-x^2/4t) \quad \sum_{n=-\infty}^{\infty} \cosh(\frac{(2n+1)x\pi}{4t}) \exp(-(n+\frac{1}{2})^2\pi^2/t))/\sqrt{4\pi t} \\
&= \exp(-x^2/4t)/\sqrt{4\pi t} \quad \theta_2(\frac{-ix\pi}{4t}, \frac{\pi^2}{4t}),
\end{aligned}$$

where we have used the evenness of cosh to symmetrise the sum in the last step. Since these 2 expressions solve the same equation with the same boundary

conditions, they must be equal by the uniqueness theorem. This gives the transformation formula required, which relates values of θ_3 for large and small values of its second argument.

8* Solve

$$u_t = u_{xx} \tag{53}$$

i on the line with the initial condition:

$$u(x,0) = \frac{1}{2a}, \qquad |x| < a, \tag{54}$$

$$u(x,0) = 0, \qquad |x| > a, \tag{55}$$

and describe the limit $a \to 0$;

ii and, on the half-line $x > 0$, with the boundary and initial conditions:

$$u(x,0) = 0, \tag{56}$$

$$u(0,t) = 1. \tag{57}$$

Solution 8* (i) As in notes, using the Green's function $G(x,t) = \frac{1}{\sqrt{4\pi t}}\exp(-x^2/(4t))$, we get

$$u(x,t) = \tag{58}$$

$$\int_{-\infty}^{\infty} u(x',0)G(x-x',t)dx' = \tag{59}$$

$$\frac{1}{\sqrt{4\pi t}}\int_{-a}^{a}\frac{1}{2a}\exp(-(x-x')^2/4t)dx' = \tag{60}$$

$$\frac{1}{2a}\frac{1}{\sqrt{4\pi t}}\int_{-(x+a)}^{(a-x)}\exp(-(x')^2/4t)dx' = \tag{61}$$

$$\frac{1}{2a\sqrt{\pi}}(\mathrm{erf}((a-x)/\sqrt{(4t)}) - \mathrm{erf}(-(a+x)/\sqrt{(4t)})), \tag{62}$$

where $\mathrm{erf}(x) = 2/\sqrt{\pi}\int_0^x \exp(-t^2)dt$. As $a \to 0$, $u(x,t) \to G(x,t)$.

(ii) Here we need $G(x,x',t) = \frac{1}{\sqrt{4\pi t}}(\exp(-(x-x')^2/(4t)) - (\exp(-(x+x')^2/(4t))$, so that:

$$u(x,t) = \int_0^{\infty} u(x',0)G(x-x',t)dx' - \int_0^t u(0,t')\frac{\partial G(x,t-t')}{\partial x} = \tag{63}$$

$$\int_0^t \frac{1}{\sqrt{4\pi}}\frac{x}{(t-t')^{3/2}}\exp(-\frac{x^2}{4(t-t')})dt'. \tag{64}$$

Now put $\tau = x/(2\sqrt{t-t'})$; the result becomes:

$$\int_{\tau=x/(2\sqrt{t})}^{\infty}\frac{2}{\sqrt{\pi}}\exp(-\tau^2)d\tau,$$

that is

$$u = (1 - \mathrm{erf}(\frac{x}{2\sqrt{t}})).$$

9 Consider the nonlinear diffusion equation

$$u_t = u^n u_{xx}, \tag{65}$$

in the domain $t > 0$, $0 < x < 1$, with the initial and boundary conditions:

$$u(x, 0) = f(x), \tag{66}$$
$$u(0, t) = u(1, t) = 0. \tag{67}$$

If n is a positive integer, and $f(x)$ is square-integrable, show that if

$$E = \int_0^1 u^2 dx, \tag{68}$$

then

$$\frac{dE}{dt} \leq 0, \tag{69}$$

provided that n is even. Discuss why the problem may not be well-posed for odd n; show that $E(t)$ can increase in this case. Discuss the generalisation to more than one space dimension.

Solution 9 Multiply the equation by $2u$, so that:

$$(u^2)_t = 2u^{n+1} u_{xx} \tag{70}$$

Hence on integrating by parts, over the interval $0 < x < 1$,

$$\frac{dE}{dt} = -2(n+1) \int_0^1 u^n u_x^2 dx. \tag{71}$$

If n is even, the rhs is negative, and E is a decreasing function of time. However the sign of the integrand on the right is not definite for odd n. In the latter case, E may increase if u is somewhere negative; indeed, linearising about some negative constant u_0, with $u = u_0 + \epsilon u_1$, where $0 < \epsilon \ll 1$, we see that $u_1$satisfies a *backwards* heat equation, which is ill-posed. We would expect solutions to grow without bound in this case, in any region where u is negative. Analogous results can be obtained using the divergence theorem in more than one space dimension.

10 *Self-similar solutions* Find m, n such that the ansatz $u(x,t) = t^m f(xt^n)$ satisfies Burger's equation:

$$u_t + uu_x = u_{xx}. \tag{72}$$

Find the ordinary differential equation satisfied by f, and hence solve Burger's equation with

$$u(0,t) = -2/(\pi t)^{1/2} \tag{73}$$
$$u \to 0, \qquad x \to \infty. \tag{74}$$

Solution 10 Put $\xi = xt^n, \quad u = t^m f(xt^n)$, in the equation; we get:

$$mt^{m-1}f + nxt^{m+n-1}f' + t^{2m+n}ff' = t^{m+2n}f''. \tag{75}$$

Rearranging,

$$mf + n\xi f' + t^{m+n+1}ff' = t^{2n+1}f''. \tag{76}$$

Equate coefficients of t to get $n = m = -1/2$.

$$u(x,t) = f(x/t^{1/2})/t^{1/2}. \tag{77}$$

Then

$$f'' - ff' + 1/2\xi f' + 1/2f = 0 \tag{78}$$

Integrating, with $f \to 0$ as $x \to \infty$:

$$f' - f^2/2 + 1/2\xi f = 0. \tag{79}$$

Put $f = -2\psi'/\psi$ so that

$$\psi'' + 1/2\xi\psi' = 0, \tag{80}$$

giving $\psi' = \exp(-\xi^2/4)$. The constant of integration is irrelevant here. Hence

$$\psi = \sqrt{\pi}\mathrm{erf}(\xi/2) + A. \tag{81}$$

Thus we get

$$f = -2\frac{\exp(-\xi^2/4)}{\sqrt{\pi}\mathrm{erf}(\xi/2) + A}. \tag{82}$$

At $\xi = 0$, this reduces, with the condition on $x = 0$, to $f = -2/A = -2/\sqrt{\pi}$. Finally, we obtain

$$u = -\frac{2}{\sqrt{t}}\frac{\exp(-x^2/(4t))}{\sqrt{\pi}\mathrm{erf}(\xi/2) + \sqrt{\pi}}. \tag{83}$$