The initial value problem for reductions of the Benney equations

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February 2, 2000

Abstract We consider a family of N-parameter reductions of Benney's equations, introduced in [1] as a generalisation of the dispersionless Lax equations. Using Geogdzhaev's method [2], we solve the initial value problem for the reduced system. This construction is carried out explicitly for the reduction associated with an elliptic curve.

1 Introduction

There has been much interest [3], [4], [5] in nonlinear integrable Hamiltonian systems of hydrodynamic type, that is, of the form

$$\frac{\partial u_i}{\partial t} + v^i_j \frac{\partial u^j}{\partial x} = 0,$$

which possess many conserved densities $H(u)$, independent of the derivatives of the u_i, or equivalently, are expressible in Riemann invariant form

$$\frac{\partial \lambda_i}{\partial t} + \mu^i(\lambda) \frac{\partial \lambda_i}{\partial x} = 0, \tag{1}$$

where λ_i are the *Riemann invariants* and μ^i are the *characteristic speeds*. Tsarev [5] has shown how the general solution of these may be constructed; if

$$x - \mu^i(\lambda) t = w^i(\lambda), \tag{2}$$

and the w^i satisfy the over-determined linear system

$$\frac{\frac{\partial w^i}{\partial \lambda_j}}{w^i - w^j} = \frac{\frac{\partial \mu^i}{\partial \lambda_j}}{\mu^i - \mu^j}, \quad i \neq j,$$

then solving the N equations (2) for the λ_i gives an implicit solution of equation (1). One hierarchy which can be viewed as underlying many of these is the Benney hierarchy [6]. The simplest of these is a nonlinear Vlasov equation [7]

$$\frac{\partial f}{\partial t_2} + p\frac{\partial f}{\partial x} - \frac{\partial A_0}{\partial x}\frac{\partial f}{\partial p} = 0, \quad f = f(x,p,t). \tag{3}$$

This is formally completely integrable in the sense of possessing infinitely many conserved densities polynomial in the moments $A_n = \int_{-\infty}^{\infty} p^n f dp$, a Hamiltonian structure of Lie-Poisson type [8], [9], and hence infinitely many commuting flows. However the initial value problem has not been solved in general.

It is possible to reduce the hierarchy to a simpler system, in which only N of the moments A_n are independent–this was discussed in [10], [11]. Each of these reduced systems, with N independent variables, can be written in terms of N Riemann invariants; one can therefore construct their general solution in the form (2). In this paper we discuss how the general construction of these reductions can also be used to solve the initial value problem for the reduced equations. We obtain here an implicit solution (18) in Tsarev's hodograph form, but depending explicitly on the initial data. Such initial value problems have been solved in some special cases; in [2], [14] Geogdzhaev solved the dispersionless KdV and the Zakharov reduction of the Benney hierarchy, while in [13] Kodama solved the dispersionless Toda and Gibbons and Kodama [1] solved the dispersionless Lax equations.

The outline of this paper is as follows: In §2 we recall the Benney hierarchy and its reductions. In §3 we construct a canonical transformation to new coordinates in which the equations of motion are trivial, and hence solve the initial value problem for the N-reduction. In §4 this construction is worked out in detail for a special case, which can be parametrised by elliptic functions, and it is solved in §5. Further work is discussed in §6.

2 The Benney Hierarchy

The well known Benney's equations were derived from the equations of an incompressible perfect fluid by Benney in 1973 [6]. This system can be represented by the Vlasov equation, equation (3); if the moments A_n are defined by:

$$A_n = \int_{-\infty}^{\infty} p^n f dp,$$

then we recover the moment equations:

$$\frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial x} + nA_{n-1}\frac{\partial A_0}{\partial x} = 0, \quad n = 0, 1, \cdots.$$

The equations of motion have the Lie-Poisson structure

$$\frac{\partial f}{\partial t} + \{\frac{\delta H}{\delta f}, f\}_{p,x} = 0, \tag{4}$$

where $\{,\}_{p,x}$ is the canonical Poisson bracket with respect to p and x. For Benney's equations, the Hamiltonian is $H = \frac{1}{2}H_2 = \frac{1}{2}(A_2 + A_0^2)$, so that $\frac{\delta H}{\delta f} = \frac{1}{2}p^2 + A_0$. It was shown in [6] that the system has infinitely many conservation laws and hence it is formally a completely integrable system. A generating function $p(\lambda)$ for the conserved densities can be constructed as follows [8] [9]. Consider

$$\lambda_R(p) = p + P\!\!\int \frac{f(x,p',t)}{p-p'}dp', \tag{5}$$

where $\mathrm{P}\!\!\int$ denotes the Cauchy principal value of the integral. It has the asymptotic series (provided all the A_n exist)

$$\lambda_R = p + \frac{A_0}{p} + \frac{A_1}{p^2} + \cdots \tag{6}$$

as $|p| \to \infty$. λ_R satisfies [7] [8]

$$\frac{\partial \lambda_R}{\partial t_2} + p\frac{\partial \lambda_R}{\partial x} - \frac{\partial A_0}{\partial x}\frac{\partial \lambda_R}{\partial p} = 0. \tag{7}$$

Note that equation (7) has the same characteristics as (3). By holding λ_R constant in the above, we obtain

$$\frac{\partial p}{\partial t_2} + \frac{\partial}{\partial x}(\frac{1}{2}p^2 + A_0) = 0.$$

The inverse of series (6) is an asymptotic series for $p(\lambda_R)$,

$$p(\lambda_R) = \lambda_R - \frac{H_0}{\lambda_R} - \frac{H_1}{\lambda_R^2} - \frac{H_2}{\lambda_R^3} - \cdots.$$

We thus see that the H_n are conserved densities of the hierarchy. Any of the H_n could be used as the Hamiltonian in (4). We then define the Benney hierarchy as the commuting family of evolution equations

$$\frac{\partial f}{\partial t_n} + \frac{1}{n}\{\frac{\delta H_n}{\delta f}, f\} = 0. \tag{8}$$

It is convenient to denote $h_n = \frac{1}{n}\frac{\delta H_n}{\delta f}$, which can be shown to be the unique polynomial in p such that

$$h_n(p) - \frac{\lambda_R^n}{n} = O(\frac{1}{p})$$

as $|p| \to \infty$. λ_R satisfies an analogous equation to (8)

$$\frac{\partial \lambda_R}{\partial t_n} + \{h_n, \lambda_R\} = 0. \tag{9}$$

If we consider a point $(\hat{p}, \hat{\lambda}_R)$, say, such that $\frac{\partial \lambda_R}{\partial p}|_{p=\hat{p}} = 0$, equation (9) becomes,

$$\frac{\partial \hat{\lambda}_R}{\partial t_n} + \mu_n(\hat{p})\frac{\partial \hat{\lambda}_R}{\partial x} = 0. \tag{10}$$

Thus, $\hat{\lambda}_R$ is a Riemann invariant with characteristic speed $\mu_n = \frac{\partial h_n}{\partial p}$.

We are now able to construct a family of distribution functions $f(p, x, t)$ which depend only on N Riemann invariants $\hat{\lambda}_i(x, t)$. If in (5), instead of taking a principal value integral along the real p-axis, we integrate along an indented contour Λ, which passes below the point p, a new function $\lambda_+(p)$ can be defined:

$$\lambda_+(p) = p + \int_\Lambda \frac{f(x, p', t)}{p - p'} dp', \tag{11}$$

λ_+ has the same asymptotics as λ_R but possesses an analytic continuation in $\Im m(p) > 0$: if f satisfies a Hölder condition, then the Plemelj formulae for λ_+ give

$$\lambda_+ = \lambda_R - i\pi f$$

as $\Im m(p) \to 0+$ [12]. Since f and λ are advected unchanged along the same characteristics, it follows that if at any time, in any region of the x-p plane, the relation: $f = F(\lambda)$ holds, then this relation will continue to hold as the region is carried around the plane. We now suppose that each line $x =$ constant passes through N disjoint such intervals I_j, with $f = F_j(\lambda_R)$ in the j-th region, and $f = 0$ otherwise. Then, (we have suppressed the x, t dependence) equation (11) becomes:

$$\lambda_+(p) = p + \sum_{j=1}^{N} \int_\Lambda \frac{F_j(\lambda_R(p'))}{p - p'} dp'. \tag{12}$$

It is clear that the new function $\lambda_+(p)$ is analytic in the upper half $p-$ plane, and on I_j it takes the value $\lambda_R - i\pi F_j(\lambda_R)$. On the boundary, therefore, (12) maybe interpreted as a nonlinear singular integral equation for $\lambda_+(p)$.

A family of solutions of this equation depending on N parameters $\hat{\lambda}_i(x,t)$ was constructed in [1] and discussed in more detail in [10], [11]. We summarise this construction here: consider the upper half λ plane Γ_+, and, N fixed Jordan arcs c_i in Γ_+, which intersect the real axis at points λ_i^0. An arbitrary point $\hat{\lambda}_i$ is taken on each c_i. A function $p(\lambda_+, \hat{\lambda}_i)$ is then constructed with the following properties,

(a) It has branch points at $\hat{\lambda}_i$. The branch cuts γ_i are taken to run from $\hat{\lambda}_i$ back to λ_i^0 along c_i.

(b) $p(\lambda_+, \hat{\lambda}_i)$ is real on the real λ_+–axis and on both sides of each γ_i.

(c) $p(\lambda_+, \hat{\lambda}_i)$ is analytic away from the cuts, and bounded on them.

(d) As $|\lambda| \to \infty$, with $\Im m(\lambda_+) \geq 0$, $p(\lambda_+, \hat{\lambda}_i)$ has the expansion

$$p(\lambda_+, \hat{\lambda}_i) \sim \lambda_+ + O(\frac{1}{\lambda_+}).$$

It is important that the cut half plane must be simply connected, so we require that these curves do not intersect one another. Note that the cuts γ_i are given by the relation

$$\Im m(\lambda_+) = -\pi F_i(\Re e(\lambda_+)),$$

We see that the F_i must all be continuous and negative. We then let the function $p(\lambda_+, \hat{\lambda}_i)$ depend on x, t only through the N independent variables $\hat{\lambda}_i(x,t)$. At each turning point $\hat{p}_k$, we have

$$\lambda_+ = \hat{\lambda}_k + \frac{1}{2}\frac{\partial^2 \lambda}{\partial p^2}|_{p=\hat{p}_k}\ (p - \hat{p}_k)^2 + O((p - \hat{p}_k)^3),$$

and hence if $\frac{\partial^2 \lambda}{\partial p^2}|_{p=\hat{p}_k} \neq 0$, we obtain:

$$p = \hat{p}_k + O((\lambda - \hat{\lambda}_k)^{\frac{1}{2}}). \tag{13}$$

At each of the N points $\hat{\lambda}_i$, we then get the generalisation of (10)

$$\frac{\partial \hat{\lambda}_i}{\partial t_n} + \mu_n(\hat{p}_i)\frac{\partial \hat{\lambda}_i}{\partial x} = 0. \tag{14}$$

Hence we have the following:

Theorem 2.1 *The Benney hierarchy*

$$\frac{\partial \lambda_+}{\partial t_n} + \{h_n, \lambda_+\} = 0,$$

restricted to the solutions of

$$\lambda_+(p) = p + \sum_{j=1}^{N} \int_\Lambda \frac{F_j(\lambda_R(p'))}{p - p'} dp',$$

constructed above, reduces to a diagonalisable Hamiltonian system of hydrodynamic type.

In this paper, we use the formal integrability of the Benney hierarchy to solve the initial value problem of this reduced hierarchy explicitly. This has been done for a few special cases [2], [14], [13], [1], but the construction is generalised here for arbitrary Jordan arcs c_i.

3 The Inverse Scattering Problem

3.1 A Canonical Transformation

We suppose, for convenience, that the initial data $\hat{\lambda}_i(x)|_{t=0}$ satisfy

$$\frac{\partial \hat{\lambda}_i}{\partial x} \neq 0, \quad \forall x,$$

and as $x \to -\infty$, we require that $\hat{\lambda}_i(x)\,|_{t=0} \to \lambda_0^i$ sufficiently rapidly that

$$\int_{-\infty}^{x} (p(\lambda, x') - \lambda)\, dx$$

converges. We then seek a mapping from the slit half plane Γ_+ to the upper half p-plane, which is the inverse of the analytic function $\lambda_+(p)$ given by equation (12).[1]

In order to solve the reduced equation of motion (14), we perform a transformation from the canonical variables (x, p) to the pair (η, λ), and a new set of Hamiltonians $k_n(\eta, \lambda)$. The k_n will shown to be independent of η, so that the characteristics of the transformed equations become fixed straight lines. Since $\hat{\lambda}_i \to \lambda_i^0$ sufficiently fast as $x \to -\infty$, it follows that $|p - \lambda| \to 0$

[1]for brevity, we will use $\hat{\boldsymbol{\lambda}}$ to denote the set $\{\hat{\lambda}_i,\ i = 1, \cdots, N\}$ and λ for λ_+ hereafter.

rapidly for $\Im m(p) > 0$. We may thus construct a generating function for the transformation, following [2], [14], we have,

$$S(x, \lambda, t) = \int_{-\infty}^{x} (p(\lambda, \hat{\boldsymbol{\lambda}}) - \lambda) dx' + \lambda x. \tag{15}$$

We recall that p depends on x and t only through the finitely many Riemann invariants $\hat{\boldsymbol{\lambda}}$. We then have

$$\begin{aligned} p &= \frac{\partial S}{\partial x}, \quad \eta = \frac{\partial S}{\partial \lambda} \\ k_n &= \frac{\partial S}{\partial t_n} + h_n. \end{aligned}$$

On integrating (13) and changing the integration variable from x to $\hat{\lambda}_k$, we see that near the branch point $\hat{\lambda}_k(x)$, S has the behaviour:

$$S = S_k + O((\lambda - \hat{\lambda}_k)^{\frac{3}{2}}), \tag{16}$$

where S_k is non-singular. We note that η is bounded at the branch points. On differentiating (15) with respect to t_n, using the equation of motion $\frac{\partial p}{\partial t_n} + \frac{\partial}{\partial x}(h_n) = 0$. and integrating with respect to x, we get:

$$\frac{\partial S}{\partial t_n} = -h_n(\frac{\partial S}{\partial x}, x) + \frac{\lambda^n}{n},$$

so that $k_n = \frac{\lambda^n}{n}$. With this Hamiltonian, Hamilton's equations are

$$\begin{aligned} \frac{d\lambda}{dt_n} &= 0 \\ \frac{d\eta}{dt_n} &= \lambda^{n-1}, \end{aligned}$$

and hence the characteristics are the lines $\lambda =$ constant, as required.

3.2 The Inverse Transformation

To invert this transformation, we need to reconstruct $S(x, \lambda)$ from its asymptotics at ∞ and its discontinuities on the curves c_i. At some point λ_i on the curve c_i, either $p(\lambda,\hat{\boldsymbol{\lambda}})$ is analytic for all x, or there exists some x_i^* such that $\lambda_i = \hat{\lambda}_i(x_i^*)$. For $x > x_i^*$, $p(\lambda,\hat{\boldsymbol{\lambda}})$ is real, thus the imaginary part of S is independent of x. We know that $S \sim \lambda x + O(\frac{1}{\lambda})$ as $|\lambda| \to \infty$, and we therefore consider the function Ξ defined by

$$\Xi(x, \lambda) = S(x, \lambda) - \frac{\lambda^n}{n} t_n - p(\lambda, \hat{\boldsymbol{\lambda}}) x + h_n(p(\lambda, \hat{\boldsymbol{\lambda}})) t_n,$$

which has asymptotics $O(\frac{1}{\lambda})$ as $|\lambda| \to \infty$. Further we define

$$\Omega(x,\lambda) = S(x,\lambda) - \frac{\lambda^n}{n} t_n, \tag{17}$$

Theorem 3.1 *The solution of the reduced hierarchy*

$$\frac{\partial \hat{\lambda}_i}{\partial t_n} + \mu_n(\hat{p}_i) \frac{\partial \hat{\lambda}_i}{\partial x} = 0$$

is given by the hodograph equations

$$x - \mu_n(\hat{p}_i) t_n = -\frac{1}{\pi} \sum_j P \int_{\gamma_j} \frac{d\{\Im m(\Omega(x,\lambda))\}}{p(\lambda, \hat{\boldsymbol{\lambda}}) - \hat{p}_i}.$$

Proof. Since p and h_n are real on the cuts, the cuts grow monotonically in x, and $\Im m(\Omega) = \Im m(S) - \frac{\Im m(\lambda^n)}{n} t_n$, it follows that $\Im m(\Omega)$ is independent of time on the cuts. It may thus be determined once and for all, from the initial data $\hat{\lambda}_i(x, 0)$. Further, we note that $\Im m(\Omega) = \Im m(\Xi)$. If we take a curve L (see *fig. 1*) to be our contour, then by Cauchy's theorem, for λ' in $\Gamma_+ - \cup_{i-1}^N (\gamma_i)$ we have;

$$\Xi(x, \lambda') = \frac{1}{2\pi i} \oint_L Q(\lambda, \lambda')\, \Xi(x, \lambda)\, d\lambda,$$

where $Q(\lambda, \lambda')$ denotes the Cauchy kernel:

$$(p(\lambda, \hat{\boldsymbol{\lambda}}) - p(\lambda', \hat{\boldsymbol{\lambda}}))^{-1} \frac{\partial p(\lambda, \hat{\boldsymbol{\lambda}})}{\partial \lambda}.$$

Splitting L into a large semi-circle γ_c, the real axis $\tilde{\gamma}$ and the cuts γ_i, we get,

$$\Xi(x, \lambda') = \frac{1}{2\pi i} \int_{\tilde{\gamma} + \sum_j \gamma_j + \gamma_c} Q(\lambda, \lambda')\, \Xi(x, \lambda)\, d\lambda.$$

Now since $\Xi(x, \lambda)$ and $(p(\lambda, \hat{\boldsymbol{\lambda}}) - p(\lambda', \hat{\boldsymbol{\lambda}}))^{-1}$ are both $O(\frac{1}{\lambda})$ as $|\lambda| \to \infty$, the integral over γ_c must vanish. Thus

$$\Xi(x, \lambda') = \frac{1}{2\pi i} \int_{\tilde{\gamma} + \sum_j \gamma_j} Q(\lambda, \lambda')\, \Xi(x, \lambda)\, d\lambda.$$

Since p and h_n are real on the real axis $\tilde{\gamma}$ and on the cuts γ_i

$$\begin{aligned} \Xi(x, \lambda') = & \frac{1}{2\pi i} \int_{\tilde{\gamma} + \sum_j \gamma_j} Q(\lambda, \lambda')\, \Re e(\Xi(x, \lambda))\, d\lambda \\ & + \frac{1}{2\pi} \sum_j \int_{\gamma_j} Q(\lambda, \lambda')\, \Im m(\Xi(x, \lambda))\, d\lambda. \end{aligned}$$

(note that $\Re e(\Xi(x,\lambda)) = \Re e(\Omega(x,\lambda)) - p(\lambda,\hat{\boldsymbol{\lambda}})x + h_n(p(\lambda,\hat{\boldsymbol{\lambda}}),x)t_n.$) Now we let $\lambda' \to \gamma_j$, so that $p(\lambda',\hat{\boldsymbol{\lambda}})$ is real. Thus, if we indent γ_j to avoid the singularity at λ',

$$\begin{aligned}\Re e(\Xi(x,\lambda')) + i\Im m(\Xi(x,\lambda')) &= \frac{1}{2\pi i}\int_{\tilde{\gamma}} Q(\lambda,\lambda')\,\Re e(\Xi(x,\lambda))\,d\lambda \\ + \quad \frac{1}{2\pi i}\sum_j P\int_{\gamma_j} & Q(\lambda,\lambda')\,\Re e(\Xi(x,\lambda))\,d\lambda + \frac{1}{2}\Re e(\Xi(x,\lambda')) \\ + \quad \frac{1}{2\pi}\sum_j P\int_{\gamma_j} & Q(\lambda,\lambda')\,\Im m(\Xi(x,\lambda))\,d\lambda + \frac{i}{2}\Im m(\Xi(x,\lambda')).\end{aligned}$$

Collecting real parts on both sides, we have the Plemelj formula:

$$\Re e(\Xi(x,\lambda')) = \frac{1}{\pi}\sum_j P\int_{\gamma_j} Q(\lambda,\lambda')\,\Im m(\Xi(x,\lambda))\,d\lambda$$

Now we differentiate this with respect to $p(\lambda',\hat{\boldsymbol{\lambda}})$. We denote the speeds $\frac{\partial h_n(p)}{\partial p}$ by $\mu_n(p)$. Recalling that μ_n and p are real on the cuts as we indicated before, we see that

$$\frac{\partial\,\Re e(\Xi(x,\lambda'))}{\partial p(\lambda',\hat{\boldsymbol{\lambda}})} = \frac{1}{\pi}\sum_j P\int_{\gamma_j} \tilde{Q}(\lambda,\lambda')\,d\{\Im m(\Xi(x,\lambda))\}.$$

Here $\tilde{Q}(\lambda,\lambda') = (p(\lambda,\hat{\boldsymbol{\lambda}}) - p(\lambda',\hat{\boldsymbol{\lambda}}))^{-1}$. This is just,

$$\frac{\partial\lambda'}{\partial p}\frac{\partial\,\Re e(\Omega(x,\lambda'))}{\partial\lambda'} - \{x - \mu_n(p)t_n\} = \frac{1}{\pi}\sum_j P\int_{\gamma_j} \tilde{Q}(\lambda,\lambda')\,d\{\Im m(\Xi(x,\lambda))\}.$$

As we let λ' approach $\hat{\lambda}_i$, $\frac{\partial\lambda'}{\partial p} \to 0$. Further by using equation (13) and (16), we find

$$\Omega = (nonsingular) + O(\lambda - \hat{\lambda}_i)^{\frac{3}{2}}$$

and also $\frac{\partial\lambda}{\partial p} = O(\lambda - \hat{\lambda}_i)^{\frac{1}{2}}$ near $\hat{\lambda}_i$, so that $\frac{\partial}{\partial\lambda'}\Re e(\Omega(x,\lambda'))$ is bounded there, thus $\frac{\partial}{\partial p}\Re e(\Omega(x,\lambda'))$ must vanish at the branch point $\hat{\lambda}_i$. Consequently, we obtain finally, the stated result:

$$x - \mu_n(\hat{p}_i)t_n = -\frac{1}{\pi}\sum_j P\int_{\gamma_j} \frac{d\{\Im m(\Omega(x,\lambda))\}}{p(\lambda,\hat{\boldsymbol{\lambda}}) - \hat{p}_i}. \tag{18}$$

□

Formally, this is the same as Tsarev's hodograph solution (2). In Tsarev's formula the w^i are the characteristic speeds corresponding to *any* symmetry of the system. For Benney's equations, any such w^i can be expressed in the form of the right hand side of (18). To see this, we write the right hand side of (18) as

$$-\frac{1}{\pi}\sum_j P\int_{\gamma_j} \frac{1}{p(\lambda,\hat{\boldsymbol{\lambda}})-\hat{p}_i}\frac{\partial\Omega}{\partial\lambda}\,d\lambda = \frac{1}{\pi}\sum_j P\int_{\gamma_j}\frac{\partial Q(\lambda,\hat{\boldsymbol{\lambda}})}{\partial\hat{p}_i}\Omega\,d\lambda.$$

The expression $Q(\lambda,\hat{\boldsymbol{\lambda}})$ has the expansion for large λ:

$$\frac{1}{\lambda}+\sum_1^\infty m\frac{h_m(\hat{p}_i)}{\lambda^{m+1}}$$

and so its $\hat{p}_i$ derivative has the expansion

$$\sum_1^\infty m\frac{\mu_m(\hat{p}_i)}{\lambda^{m+1}},$$

where the μ_m are the characteristic velocities of the different flows of the hierarchy. We can thus think of the right hand side of (18) as a generating formula for all such characteristic velocities. When Ω is constructed from the initial data as above, this construction gives the required solution of the initial value problem.

4 Elliptic Reductions

One of the simplest examples of this type of reduction is the elliptic case. We consider two regions Γ_1 and Γ_2, where Γ_1 is the upper half of the complex p-plane, with six points marked on the real axis $p_1<\hat{p}_1<p_2<p_3<\hat{p}_2<p_4$; Γ_2 is the upper half complex λ-plane with two vertical slits $\Re e(\lambda)=\lambda_1^0$ and $\Re e(\lambda)=\lambda_2^0$, stretching from λ_1^0 to $\hat{\lambda}_1$and from λ_2^0 to $\hat{\lambda}_2$ (see *fig. 2a, 2b*). We construct the unique conformal map between these two regions satisfying $\lambda=p+O(\frac{1}{p})$ as $p\to\infty$. This is of Schwarz-Christoffel type [15]:

$$\lambda(p)=p+\int_{-\infty}^{p}\{\varphi(p')-1\}\,dp'; \tag{19}$$

where $\varphi(p)$ is

$$\frac{\prod_{i=1}^2(p-\hat{p}_i)}{\sqrt{\prod_{i=1}^4(p-p_i)}}=\frac{p^2-\alpha p-\beta}{\sqrt{\prod_{i=1}^4(p-p_i)}}.$$

This form guarantees that the interior angles at the vertices $\lambda(p_1)$, $\lambda(p_2)$, $\lambda(p_3)$ and $\lambda(p_4)$ are $\frac{\pi}{2}$, and at $\hat{\lambda}_1$ and $\hat{\lambda}_2$ they are 2π. We impose four side conditions, firstly that the pairs $\lambda(p_1)$ and $\lambda(p_2)$ and $\lambda(p_3)$ and $\lambda(p_4)$ should coincide, and secondly that they take some prescribed *constant* real value:

$$\begin{aligned}\lambda(p_1) &= \lambda(p_2) = \lambda_1^0, \\ \lambda(p_3) &= \lambda(p_4) = \lambda_2^0.\end{aligned}$$

We can replace one of these conditions by the condition that the residue at infinity of the integrand $\varphi(p)$ be zero. This can be satisfied by setting α to be

$$\alpha = \sum_{i=1}^{4} \frac{p_i}{2}. \tag{20}$$

Then the map depends on only two parameters, which we take to be $\Im m(\hat{\lambda}_1)$ and $\Im m(\hat{\lambda}_2)$. The differential

$$\frac{\prod_{i=1}^{2}(p-\hat{p}_i)}{\sqrt{\prod_{i=1}^{4}(p-p_i)}}\, dp$$

is an Abelian differential of second kind on the elliptic Riemann surface

$$r_1^2(p) = \prod_{1}^{4}(p-p_i).$$

This can be constructed as two copies of the complex p-plane; the sheets are joined along the two intervals $[p_1, p_2]$ and $[p_3, p_4]$. The basis of cycles on this surface consists of the a–cycle which we take to be a closed loop on one of these sheets, circling the cut $[p_1, p_2]$ in the positive sense, and the b–cycle which passes from the interval $[p_3, p_4]$ to the interval $[p_1, p_2]$ on the upper sheet, crosses the branch cut and returns to $[p_3, p_4]$ on the lower sheet.

We emphasise that a and b intersect in only one point. The integral around the a-cycle vanishes:

$$\oint_a \varphi(p)\, dp = 0,$$

while the corresponding integral around the b–cycle does not. Note that the condition on the a–cycle can be satisfied by fixing β to be

$$\beta = \int_{p_1}^{p_2} \frac{(p^2-\alpha p)\, dp}{\sqrt{\prod_{i=1}^{4}(p-p_i)}} \Big/ \int_{p_1}^{p_2} \frac{dp}{\sqrt{\prod_{i=1}^{4}(p-p_i)}}. \tag{21}$$

The mapping (19) is calculated explicitly in the appendix. The solution is

$$\lambda(p) = \frac{1}{k}\{\gamma(\chi+\chi_0)+\gamma(\chi-\chi_0)\}+C,$$

where $\gamma(\chi) = -\zeta(\chi)+\frac{\zeta(\omega_1)}{\omega_1}\chi$, and χ, χ_0 and k are given by

$$\begin{aligned}
\wp(\chi) &= \frac{1}{(p_4-p)} - \sum_i \frac{1}{3(p_4-p_i)} \\
\wp(\chi_0) &= -\sum_i \frac{1}{3(p_4-p_i)} = \nu, \\
k &= \sqrt{-\frac{4}{\prod_{i=1}^{3}(p_4-p_i)}}.
\end{aligned}$$

Here ζ is the Weierstrass zeta function.

5 Solutions Of The Inverse Problem

The characteristic speeds of the elliptic reduction (19) are found to be:

$$\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\alpha-\sqrt{\alpha^2-4\beta}) \\ \frac{1}{2}(\alpha+\sqrt{\alpha^2-4\beta}) \end{pmatrix},$$

where α and β are defined by equation (20),(21):

$$\alpha = \sum_{i=1}^{4}\frac{p_i}{2}$$

and

$$\beta = \int_{p_1}^{p_2}\frac{(p^2-\alpha p)\,dp}{\sqrt{\prod_{i=1}^{4}(p-p_i)}} \Big/ \int_{p_1}^{p_2}\frac{dp}{\sqrt{\prod_{i=1}^{4}(p-p_i)}}.$$

The Riemann invariants are given by,

$$\begin{aligned}
\hat{\lambda}_1 &= \frac{1}{k}\{\gamma(\chi_1+\chi_0)+\gamma(\chi_1-\chi_0)\}+C, \\
\hat{\lambda}_2 &= \frac{1}{k}\{\gamma(\chi_2+\chi_0)+\gamma(\chi_2-\chi_0)\}+C,
\end{aligned}$$

where χ_1 and χ_2 are given by:

$$\begin{aligned}
\chi_1 &= \wp^{-1}\{\frac{1}{p_4-\hat{p}_1} - \sum_i \frac{1}{3(p_4-p_i)}\}, \\
\chi_2 &= \wp^{-1}\{\frac{1}{p_4-\hat{p}_2} - \sum_i \frac{1}{3(p_4-p_i)}\}.
\end{aligned}$$

Here the inverse Weierstrass function is defined to take values in the rectangle $0 \leq \Re e(\wp^{-1}) \leq \omega_1$ and $0 \leq \Im m(\wp^{-1}) \leq \omega_3$, and is therefore single valued in the upper half p–plane. If $p(\lambda, \hat{\boldsymbol{\lambda}})$ which is a single valued map from Γ_2 to Γ_1, is defined as the inverse function of $\lambda(p)$, given by (30), then we have, from the results of §2:

$$\Omega|_{t=0}= \int_{-\infty}^{x} (p(\lambda, \hat{\boldsymbol{\lambda}}) - \lambda)dx' + \lambda x.$$

The solution is then given by the hodograph formula (18),

$$x - \frac{1}{2}(\alpha \mp \sqrt{\alpha^2 - 4\beta})t_2 = -\frac{1}{\pi}\sum_j P\!\int_{\gamma_j} \frac{d(\Im m(\Omega(x,\lambda)))}{p(\lambda, \hat{\boldsymbol{\lambda}}) - (\alpha \mp \sqrt{\alpha^2 - 4\beta})}. \qquad (22)$$

We recall that the only independent variables here are $\Im m(\hat{\lambda}_1)$ and $\Im m(\hat{\lambda}_2)$, and all the other quantities are defined in terms of these. Hence this formula gives two equations for these two unknowns, which are indeed determined implicitly as functions of x and t, for t small enough.

6 Conclusions And Further Problems

The reductions of the Benney hierarchy to N Riemann invariants, constructed in [10], are integrable systems of hydrodynamic type. Their general solution is given by the hodograph formula. By considering the canonical transformation which maps the characteristics of the Benney equation to straight lines, and its inversion, we are able to obtain the precise solution of the initial value problem, equation (18). This can in principle be applied to any reduction of the class. We have carried it out specifically for the case of two slits given by (22), in which $\lambda(p)$ is found in terms of Weierstrass ζ functions. We may extend this approach to the case of N straight slits $\Re e(\lambda) = \lambda_i^o$; the map $\lambda(p)$ may be written down in Schwarz-Christoffel form as before, reducing to a hyperelliptic integral of genus $(N-1)$, but it is much harder to obtain explicit formulae in this case. Such a formula does exist however and the method of [17] may be useful here. However, this has not yet led to an effective result. It should be stressed that this hyperelliptic reduction of the Benney hierarchy is, despite some strong similarities, distinct from the Whitham equations discussed in [18] and elsewhere.

Appendix

We recall that the integral under consideration is:

$$\lambda(p) = p + \int_{-\infty}^{p} \{\varphi(p') - 1\}\, dp', \quad \varphi(p) = \frac{p^2 - \alpha p - \beta}{\sqrt{\prod_{i=1}^{4}(p - p_i)}}.$$

Lemma 7.1 *$\varphi(p)\, dp$ is reduced by the transformation*

$$p = p_4 - \frac{1}{\wp(\chi) - \wp(\chi_0)}, \quad dp = \frac{\wp'(\chi)}{(\wp(\chi) - \wp(\chi_0))^2}\, d\chi \tag{23}$$

to

$$k\frac{\{a\wp(\chi)^2 + b\wp(\chi) + c\}}{(\wp(\chi) - \wp(\chi_0))^2}\, d\chi, \tag{24}$$

where a, b and c are polynomials in p_4 and

$$\begin{aligned} \wp(\chi) &= \frac{1}{(p_4 - p)} - \sum_i \frac{1}{3(p_4 - p_i)} \\ \wp(\chi_0) &= -\sum_i \frac{1}{3(p_4 - p_i)} = \nu, \quad k = \sqrt{\frac{4}{-\prod_{i=1}^{3}(p_4 - p_i)}}, \end{aligned} \tag{25}$$

here we have introduced the Weierstrass elliptic $\wp$ function [16] with half periods ω_i, $i = 1, 2$, satisfying $\Im m \frac{\omega_2}{\omega_1} > 0$, and $\wp'(\chi)$ is given by the standard expression:

$$\begin{aligned} \wp'^2(\chi) &= 4\wp^3(\chi) - g_2\, \wp(\chi) - g_3, \\ &= 4\prod(\wp - e_i). \end{aligned}$$

Proof. This can be achieved by substituting equation (23) into $\varphi(p)\, dp$, we reduce it to the following form:

$$\begin{aligned} &\varphi(p)\, dp = \\ &\quad k\left\{\frac{(p_4^2 - \alpha p_4 - \beta)\wp(\chi)^2}{\wp'(\chi)(\wp(\chi) - \wp(\chi_0))^2} + \frac{(\alpha - 2p_4 - 2\nu(p_4^2 - \alpha p_4 - \beta))\wp(\chi)}{\wp'(\chi)(\wp(\chi) - \wp(\chi_0))^2}\right. \\ &\quad \left. + \frac{\nu^2(p_4^2 - \alpha p_4 - \beta) - \nu(\alpha - 2p_4) + 1}{\wp'(\chi)(\wp(\chi) - \wp(\chi_0))^2}\right\} d\wp(\chi). \end{aligned}$$

Letting a, b, c denote the coefficients of $\wp^2(\chi)$, $\wp(\chi)$ and 1 respectively, we have the required equation (24). □

Since the residue at $p = \infty$ of $\varphi(p)\, dp$ is required to be zero, the following result must hold.

Proposition 7.1 *The residue of the expression*

$$k\,\frac{\{a\wp(\chi)^2 + b\wp(\chi) + c\}}{(\wp(\chi) - \wp(\chi_0))^2}\,d\chi \tag{26}$$

must vanish at $\chi = \pm\chi_0 \mod 2\omega_i$.

Proof. We note that $\wp(\chi)$ is an even function, and that as $\chi \to \pm\chi_0$, $p \to \infty$. We expand equation (26) near $\chi = \pm\chi_0$. The coefficient of $\frac{1}{\chi \pm \chi_0}$ is

$$\frac{k}{\wp'(\chi_0)^3}\Big\{b\wp'(\chi_0)^2 + 2a\wp(\chi_0)\wp'(\chi_0)^2 - (c + b\wp(\chi_0) + a\wp(\chi_0)^2)\wp''(\chi_0)\Big\}. \tag{27}$$

Since $(c + b\wp(\chi_0) + a\wp(\chi_0)^2) = 1$, which can be verified by a direct substitution, we simplify the residue (27) to,

$$\frac{k}{\wp'(\chi_0)^3}(b\wp'(\chi_0)^2 + 2a\wp(\chi_0)\wp'(\chi_0)^2 - \wp''(\chi_0)). \tag{28}$$

A straightforward calculation gives,

$$\begin{aligned}
\wp'(\chi_0)^2 &= k^2 = -\frac{4}{\prod_{i=1}^{3}(p_4 - p_i)}, \\
\wp''(\chi_0) &= \frac{1}{2}\wp'(\chi_0)^2(\sum_i -(p_4 - p_i)).
\end{aligned} \tag{29}$$

Substituting equations (25) and (29) into expression (28), the result follows. □

We can now do the integration with the aid of the following,

Lemma 7.2

$$\varphi_1 = k\frac{\{a\wp(\chi)^2 + b\wp(\chi) + c\}}{(\wp(\chi) - \wp(\chi_0))^2}$$

differs only by a constant from

$$\varphi_2 = k\frac{\{a\wp(\chi_0)^2 + b\wp(\chi_0) + c\}}{\wp'(\chi_0)^2}\{\wp(\chi + \chi_0) + \wp(\chi - \chi_0)\}.$$

Proof. Since rational combinations of elliptic functions are elliptic, so are φ_1 and φ_2. Now we look at the expansions for both expressions near $\chi = \chi_0$,

$$\varphi_1 \sim k\frac{a\wp(\chi_0)^2 + b\wp(\chi_0) + c}{\wp'(\chi_0)^2}(\frac{1}{(\chi - \chi_0)^2} + \theta_1(\chi)),$$

$$\varphi_2 \sim k\frac{a\wp(\chi_0)^2 + b\wp(\chi_0) + c}{\wp'(\chi_0)^2}(\frac{1}{(\chi - \chi_0)^2} + \theta_2(\chi)).$$

where θ_1 and θ_2 are analytic functions in χ near χ_0. Similarly, near $\chi = -\chi_0$, we have

$$\varphi_1 \sim k\frac{a\wp(-\chi_0)^2 + b\wp(-\chi_0) + c}{\wp'(-\chi_0)^2}(\frac{1}{(\chi + \chi_0)^2} + \tilde{\theta}_1(\chi)),$$
$$\varphi_2 \sim k\frac{a\wp(-\chi_0)^2 + b\wp(-\chi_0) + c}{\wp'(-\chi_0)^2}(\frac{1}{(\chi + \chi_0)^2} + \tilde{\theta}_2(\chi)).$$

where $\tilde{\theta}_1$ and $\tilde{\theta}_2$ denote analytic functions in χ near $-\chi_0$. Now consider the elliptic function $\varphi_3 = \varphi_1 - \varphi_2$, since φ_1 and φ_2 have the same leading terms in the expansion at the two singularities $\pm\chi_0$, and there are no other singularities, for the singularity at $\chi = 0$ is removable. It follows that φ_3 is holomorphic and elliptic and therefore must be a constant by Liouville's theorem. Thus, if we let this constant be c_1, $\varphi\, dp$ becomes

$$\varphi(p)\, dp = k\frac{\{a\wp(\chi_0)^2 + b\wp(\chi_0) + c\}}{\wp'(\chi_0)^2}\{\wp(\chi + \chi_0) + \wp(\chi - \chi_0) + c_1\}d\chi.$$

□

Since the derivative of the Weierstrass zeta function $\zeta(\chi)$ is $-\wp(\chi)$, we have

$$\begin{aligned} \lambda(p) &= -\frac{1}{k}\{\zeta(\chi + \chi_0) + \zeta(\chi - \chi_0) - 2\frac{\zeta(\omega_1)}{\omega_1}\chi\} + C \\ &= \frac{1}{k}\{\gamma(\chi + \chi_0) + \gamma(\chi - \chi_0)\} + C. \end{aligned} \tag{30}$$

We note that $\lambda(p)$ is periodic in χ with period $2\omega_1$, for the integral around the a-cycle vanishes, hence $c_1 = -2\frac{\zeta(\omega_1)}{\omega_1}$. The constant of integration C should be chosen so that $\lambda(p)$ has asymptotics $p + O(\frac{1}{p})$ near infinity. The function γ defined by $-\zeta(\chi) + \frac{\zeta(\omega_1)}{\omega_1}\chi$, is periodic with period $2\omega_1$. However it is not itself an elliptic function, for $2\omega_2$ is not a period:

$$\begin{aligned} \gamma(\chi + 2\omega_2) &= \gamma(\chi) + \frac{\pi i}{\omega_1} \\ &\neq \gamma(\chi). \end{aligned}$$

References

[1] J. Gibbons and Y. Kodama, *Solving Dispersionless Lax Equations*, In Pro. Singular limits of dispersive waves, Nato ASI Adv. Sci. Inst. Ser. B: Phys., Plenum press, New York, **320**, (1994), 61.

[2] V. V. Geogjaev [V. V. Geogdzhaev], *The quasiclassical limit of the inverse scattering problem method*, In Pro. Singular limits of dispersive waves, Nato ASI Adv. Sci. Inst. Ser. B: Phys., Plenum press, New York, **320**, (1994), 53.

[3] B. A. Dubrovin and S. P. Novikov, *Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonians*, Russian Math. Surveys, **44**, (1989), 35.

[4] E.V. Ferapontov and A. P. Fordy, *Separable Hamiltonians and integrable systems of hydrodynamic type*,J. Geom. Phys. **21**, (1997), 169

[5] S. P. Tsarev, *On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type* , Soviet Math. Dokl. **31**, (1985), 488.

[6] D. J. Benney, *Some properties of long non-linear waves*, Stud. Appl. Math. **52**, (1973), 45.

[7] J. Gibbons, *Collisionless Boltzmann equations and integrable moments equations*, Physica. D **3**, (1981),503

[8] Kupershmidt and Manin, *Long wave equation with a free surface I. Conservation laws and solutions*, Funct. Anal. Appl. **11**, (1977), 31

[9] Kupershmidt and Manin, *Long wave equation with a free surface II. The Hamiltonian structure and the higher equations*, Funct. Anal. Appl.**12**, (1978), 25.

[10] J. Gibbons and S. P. Tsarev, *Reductions of The Benney equations*, Phys. Lett., **211A**, (1996),19.

[11] J. Gibbons and S. P. Tsarev, *Conformal maps and reductions of the Benney equations*, Phys. Lett., **258A**, (1999),263.

[12] P. Henrici, Applied and Computational Complex Analysis, John Weily and Sons, 1974

[13] Y. Kodama, *Solution of the dispersionless Toda equation*, Phy. Lett. A, **147**,(1990),477

[14] V. V. Geogdzaev, *Solution of the Benney equations by the inverse scattering method*, Theor. Mat. Fiz. **73**, (1987), 255.

[15] A. I. Markushevich. The Theory of Analytic Functions: a brief course. MIR Publisher Moscow, 1983.

[16] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, CUP, 1996.

[17] J. C. Eilbeck, V. Z. Enolskii and D. V. Leykin, *On the Kleinian construction of Abelian functions of canonical algeraic curves*, Proc. of the Conference SIDE III, 1998.

[18] H. Flaschka, M. G. Forest and D. W. Mclaughlin, *Multiphase averaging and the inverse spectral solution of the Korteweg-de Vries equation*, Comm. On Pure and Appl. Maths, **XXXIII**, (1980), 739.