Conformal maps and reductions of the Benney equations

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February 23, 1999

Abstract

We consider Benney's equations, and their reductions to systems with finitely many Riemann invariants. The equations describing these reductions were given in [5] and a construction of a class of their solutions was briefly described there. Here we discuss the properties of these equations in more detail, and investigate the relationship between these and Loewner's [11] theory of conformal mappings of slit domains. A dense family of explicit solutions is constructed.

1 Introduction

Benney's equations [1], are, in their original form:

$$\begin{cases} u_t + uu_x - \left(\int_0^y u_x\,dy\right) u_y + h_x = 0, \\ h_t + \dfrac{\partial}{\partial x}\left(\int_0^h u\,dy\right) = 0. \end{cases} \tag{1}$$

*S.P.Tsarev was partially supported by INTAS grant 96-0770 and Russian Presidential grant 96-15-96834. His visit to Imperial College was supported by EPSRC on grant GR/M36106.

They describe long waves on a shallow inviscid, incompressible fluid. Here $u(x,y,t)$ is the horizontal component of the fluid velocity, while $h(x,t)$ is the height of the free surface above the flat rigid bottom $y=0$. The equations were derived on the assumption that the horizontal length scales are much greater than the depth of the fluid.

Benney showed that if moment variables A^n are defined by:

$$A^n(x,t) = \int_0^h u^n(x,y,t)\,dy, \qquad n=0,1,2,\ldots,$$

they satisfy an infinite autonomous system, known as Benney's moment equations:

$$A^n_t + A^{n+1}_x + nA^{n-1}A^0_x = 0, \qquad n=0,1,2,\ldots. \tag{2}$$

Another system which gives rise to the same system of moment equations is a Vlasov equation:

$$f_t + pf_x - A^0_x f_p = 0. \tag{3}$$

Here $f(x,p,t)$ is a distribution function in the (x,p)-plane. It will not necessarily be positive, but for the transformation to (1) to make sense, it must be of definite sign. The moments A^n are here defined by:

$$A^n(x,t) = \int_{-\infty}^{\infty} p^n f(x,p,t)\,dp.$$

Benney showed that the system has infinitely many conserved densities, polynomial in the moments A^n. One construction of these, due to Kupershmidt and Manin [2], uses the generating function:

$$\lambda(x,q,t) = q + \sum_{n=0}^{\infty} A^n/q^{n+1}. \tag{4}$$

This can be understood as the asymptotic series as $q\to\infty$ of either of the two singular integrals:

$$= q + \int_0^h 1/(q-u)\,dy \tag{5}$$

$$= q + \int_{-\infty}^{\infty} \frac{f}{q-p}\,dp. \tag{6}$$

For definiteness, we take the principal value of these integrals, (5) and (6). Later we will choose a different definition, where the path of integration is indented, to make the integral analytic in the upper half q-plane. If we calculate the first derivatives of λ with respect to x, q and t, and use the equation of motion (1), (2), or (3) then, allowing q to depend on x and t, we find:

$$\lambda_t + q\lambda_x = \frac{\partial \lambda}{\partial q}(q_t + qq_x + A^0_x).$$

Then, if q is held constant, λ satisfies a Vlasov equation of the same form as (3),

$$\lambda_t + q\lambda_x - A^0_x \frac{\partial \lambda}{\partial q} = 0, \tag{7}$$

while if λ is held constant, then q satisfies the conservation equation

$$q_t + (q^2/2 + A^0)_x = 0 \tag{8}$$

so if we invert the formal series (4), in the form

$$q(x, \lambda, t) = \lambda - \sum_{n=0}^{\infty} H^n/\lambda^{n+1}, \tag{9}$$

we see that each H^n is a polynomial in the moments, and is a conserved density of (2). Further, if $\partial\lambda/\partial q$ vanishes for some q_i, the corresponding value $\lambda_i = \lambda(q_i)$ is a Riemann invariant, with characteristic speed q_i:

$$(\lambda_i)_t + q_i(\lambda_i)_x = 0. \tag{10}$$

2 The reduction problem

An interesting problem, first studied in [5] is to find and classify those solutions of (2) in which only finitely many of the moments A^n are independent of one another. Many such reductions are known. One well known example is the Zakharov reduction [3], which can be derived as the dispersionless limit of the vector NLS. It is given by:

$$A^n = \sum_{i=1}^{N} h_i(u_i)^n,$$

for if $h_i(x,t)$, $u_i(x,t)$, $i = 1, \ldots, N$, satisfy the reduced equations of motion

$$\begin{cases} (h_i)_t + (u_i h_i)_x = 0, \\ (u_i)_t + u_i(u_i)_x + A^0_x = 0, \end{cases}$$

then the A^n all satisfy (2). Hence the A^n are all expressible in terms of $2N$ independent dynamical variables. Another reduction is the dispersionless $(N-1)$-parameter Lax reduction [7]; here the A^n are given by:

$$\lambda = q + \sum_{n=0}^{\infty} A^n/q^{n+1} = (q^N + \sum_{n=0}^{N-2} N U_n q^{N-n-2})^{1/N}. \tag{11}$$

so the moments A^i are all polynomials in the coefficients U_i. Benney's equations then reduce to the set of $N-1$ equations

$$(U_n)_t + (U_{n+1})_x - (N-n-1)(U_0)_x U_{n-1} = 0, \quad 0 \leq n < N-2, \tag{12}$$

where U_{N-1} is taken to be zero. Both these reductions are Hamiltonian systems of hydrodynamic type, and, as seen above, they can be diagonalised - written in terms of Riemann invariants. Hence they can be solved by the hodograph transformation [4].

To discuss the general properties of such reductions, we consider the consistency conditions which they must satisfy. We may suppose, without loss of generality, that the first N moments are independent variables, while the higher moments are functions of them (cf. [5]):

$$A^k = a^k(A^0, ..., A^{N-1}), \quad k \geq N.$$

The equations of motion for $(A^0, ..., A^{N-1})$ then become:

$$\begin{aligned} -A^j_t &= A^{j+1}_x + jA^{j-1}A^0_x, \quad j \leq N-2, \\ -A^{N-1}_t &= \sum_{i=0}^{N-1} \frac{\partial a^N}{\partial A^i} A^i_x + (N-1)A^{N-2}A^0_x, \end{aligned} \tag{13}$$

while each higher moment $(a^N, ...)$ must satisfy the overdetermined system:

$$\begin{aligned} -a^k_t &= \sum_{j=0}^{N-2} \frac{\partial a^k}{\partial A^j}(A^{j+1}_x + jA^{j-1}A^0_x) + \frac{\partial a^k}{\partial A^{N-1}} \left(\sum_{i=0}^{N-1} \frac{\partial a^N}{\partial A^i} A^i_x + (N-1)A^{N-2}A^0_x \right) \\ &= \sum_{j=0}^{N-1} \frac{\partial a^{k+1}}{\partial A^j} A^j_x + ka^{k-1}A^0_x \end{aligned}$$

Hence we find, on comparing coefficients of $\partial A^j/\partial x$, the system

$$\begin{aligned} \frac{\partial a^{k+1}}{\partial A^j} &= \frac{\partial a^k}{\partial A^{N-1}} \frac{\partial a^N}{\partial A^j} + \frac{\partial a^k}{\partial A^{j-1}} \quad 1 \leq j \leq N-1, \\ \frac{\partial a^{k+1}}{\partial A^0} &= \sum_{i=0}^{N-1} iA^{i-1} \frac{\partial a^k}{\partial A^i} + \frac{\partial a^k}{\partial A^{N-1}} \frac{\partial a^N}{\partial A^0} - ka^{k-1}. \end{aligned} \tag{14}$$

The compatibility condition for these with $k = N$, gives a system $\mathcal{S}$ of $N(N-1)/2$ nonlinear second order equations for the single unknown $a^N(A^0, ..., A^{N-1})$. It can be shown by induction that if $\mathcal{S}$ is satisfied then the analogous compatibility conditions for a^k in (14) with $k > N$, are satisfied too.

The simplest such system is found by setting $N = 2$; on denoting $x = A_0$, $y = A_1$ and $z = a^2 + (A^0)^2/2$, we find that z satisfies the Monge–Ampère equation:

$$z_{xx} + z_y z_{xy} - z_x z_{yy} + 1 = 0. \tag{15}$$

If two new variables, the characteristic speeds u and v, are defined as

$$(-z_y \pm \sqrt{z_y^2 + 4z_x})/2,$$

then this equation is transformed to an inhomogeneous hydrodynamic type system:

$$\begin{cases} u_x = v u_y - \frac{1}{u-v} \\ v_x = u v_y + \frac{1}{u-v}. \end{cases} \tag{16}$$

This has one obvious hydrodynamic type conserved density $(u+v)$, together with several involving x, y and z explicitly. There is also a first-order integral $(u-v)(u_y^2 - v_y^2)$; which does not fit into this pattern.

This system was also derived in another context, in [9, 10] where the conditions for two quadratic Hamiltonians to be in involution were studied.

It can be shown, using a general result of Haantjes [8] that the reduced equations of motion (13) corresponding to any solution of $\mathcal{S}$ can be diagonalised (in the domain where the reduced system is hyperbolic), so it is reasonable to regard the moments A^n as functions of the N Riemann invariants λ_i, each satisfying the equation (10). The Vlasov equation (3) is then satisfied if

$$(p - q_i)\frac{\partial f}{\partial \lambda_i} - \frac{\partial A^0}{\partial \lambda_i} f_p = 0,$$

and thus, on dividing by f_p, we get:

$$\frac{\partial p}{\partial \lambda_i} = -\frac{\frac{\partial A^0}{\partial \lambda_i}}{p - q_i}. \tag{17}$$

The consistency conditions for these equations are:

$$\begin{aligned} \frac{\partial^2 A^0}{\partial \lambda_i \partial \lambda_j} &= \frac{2\frac{\partial A^0}{\partial \lambda_i}\frac{\partial A^0}{\partial \lambda_j}}{(q_i - q_j)^2}, \\ \frac{\partial q_j}{\partial \lambda_i} &= \frac{\frac{\partial A^0}{\partial \lambda_i}}{(q_i - q_j)}. \end{aligned} \tag{18}$$

for each pair $i \neq j$. The solutions of this system are parametrised by $2N$ functions of a single variable. Half of these are inessential, corresponding to the freedom to reparametrise each λ_i separately without changing the form of (21), but the other N functions distinguish essentially different reductions.

We can study the case $N = 2$ in more detail after a change of variables, using the conserved densities A^0 and A^1 as dependent variables. The equations of motion (2) give:

$$\frac{\partial A^1}{\partial \lambda_i} = q_i \frac{\partial A^0}{\partial \lambda_i}.$$

Then, on substituting for q_i, (17) becomes:

$$\frac{\partial p}{\partial \lambda_i} = \left(\frac{\partial A^0}{\partial \lambda_i}\right)^2 \Big/ \left(p\frac{\partial A^0}{\partial \lambda_i} - \frac{\partial A^1}{\partial \lambda_i}\right),$$

and (18) then becomes:

$$\begin{aligned} \frac{\partial^2 A^0}{\partial \lambda_1 \partial \lambda_2} &= \frac{2\left(\frac{\partial A^0}{\partial \lambda_1}\right)^3 \left(\frac{\partial A^0}{\partial \lambda_2}\right)^3}{\left(\frac{\partial A^1}{\partial \lambda_1}\frac{\partial A^0}{\partial \lambda_2} - \frac{\partial A^1}{\partial \lambda_2}\frac{\partial A^0}{\partial \lambda_1}\right)^2}, \\ \frac{\partial^2 A^1}{\partial \lambda_1 \partial \lambda_2} &= \left(\frac{\partial A^0}{\partial \lambda_1}\right)^2 \left(\frac{\partial A^0}{\partial \lambda_2}\right)^2 \frac{\left(\frac{\partial A^1}{\partial \lambda_1}\frac{\partial A^0}{\partial \lambda_2} + \frac{\partial A^1}{\partial \lambda_2}\frac{\partial A^0}{\partial \lambda_1}\right)}{\left(\frac{\partial A^1}{\partial \lambda_1}\frac{\partial A^0}{\partial \lambda_2} - \frac{\partial A^1}{\partial \lambda_2}\frac{\partial A^0}{\partial \lambda_1}\right)^2}. \end{aligned} \tag{19}$$

We believe this form will be particularly convenient for investigating the open question of whether the equation (15) is Darboux integrable.

3 A family of solutions

The following construction of some solutions of (18) was sketched in [5].

Since (3) and (7) have the same form, and in particular, the same characteristics, any relation such as $f(x, p, t) = F(\lambda(x, p, t))$ is preserved by the dynamics. The definition (6) for the generating function λ then becomes a nonlinear singular integral equation:

$$\lambda(q) = q + P\int_{-\infty}^{\infty} F(\lambda(p))\frac{dp}{q - p}. \tag{20}$$

It is useful here to define a function $\lambda_+(q)$, where instead of taking the principal value of the integral in (6), we indent the contour to pass below the point q, so that λ_+ then can be analytically continued throughout the upper half q-plane. On the real q-axis we have:

$$\lambda_+ = \lambda - i\pi f.$$

If $F \leq 0$ then we may describe some solutions of (20) in terms of a conformal mapping of a slit domain. The construction is to take the upper half complex plane Λ_+, and to draw some *fixed* Jordan arc $Im(\lambda_+) = -\pi F(Re(\lambda_+))$, from the real axis as far as some end point λ_0^*. Then we position a point λ^* on this arc (the position of this point depends on a parameter u_1) and make a slit along the arc from the real axes to λ^*.

The function $q(\lambda_+)$, satisfying the equation (8), is determined uniquely by three properties:

(i) q is real and continuous on the real λ_+-axis and on both sides of the slit;

(ii) it is analytic in the cut half plane Λ_+;

(iii) as $|\lambda_+| \to \infty$, with $Im(\lambda_+) > 0$, then $q = \lambda_+ + O(1/\lambda_+)$. The conserved densities H^n, and hence the moments A^n, are then obtained from the asymptotics of $q(\lambda_+)$ as $|\lambda_+| \to \infty$, by (9).

The equation of motion (8), expanded near the end point λ^* of the slit then shows that λ^* is a Riemann invariant, with characteristic speed $q(\lambda^*)$, for the derivative $\frac{\partial(\lambda_+)}{\partial q}$ vanishes there.

This construction may be generalised at once to the case of N non-intersecting slits along *fixed* paths given by $Im(\lambda_+) = -\pi F_j(Re(\lambda_+))$ each starting on the real λ_+-axis and ending in a branch point λ_j^*; these $\lambda_j^*(x,t)$ are the N Riemann invariants of the system, and their characteristic speeds are $q(\lambda_j^*)$. The solution of the system (18) thus can be reduced to the Riemann mapping problem for the half-plane with N slits.

4 The Löwner equations and the Bieberbach Conjecture

The constructions of the previous sections bear a remarkable resemblance with the celebrated Löwner (or Loewner) parametric method in the theory of extremal univalent conformal mappings [13]. The Löwner method proved to be fruitful in the final solution of the Bieberbach Conjecture (see an exposition of the history of this Conjecture in [14]). Let us recall the main landmarks: if an analytic function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{21}$$

(compare with (4)) is univalent in the unit disk $D = \{|z| < 1\}$ (that is it maps D injectively) then the Bieberbach Conjecture states that $|a_k| \leq k$. Originally proved by

Bieberbach himself in 1916 for $k = 2$ and by Löwner [11] for $k = 3$, the problem stimulated a great deal of effort, and many more partial results were obtained. In 1984, the conjecture was completely proved by de Branges, who essentially used Löwner's method. This consisted in using one-parameter families of univalent mappings $f(z,t)$ satisfying the *Löwner ordinary differential equation*

$$\frac{\partial f(z,t)}{\partial t} = \frac{1}{t} \cdot \frac{1 + \kappa(t) f(z,t)}{1 - \kappa(t) f(z,t)} f(z,t) \tag{22}$$

with a continuous function $\kappa(t)$, $|\kappa(t)| \equiv 1$ and the initial data $f(z,0) \equiv z$. The solution of (22) maps the interior of the unit disk D onto $D \setminus S(t)$ where $S(t)$ is a slit (its shape is fixed and depends on the choice of the function $\kappa(t)$) growing from the boundary of D in the same way as our slits in the previous section. Actually our equation (17) for $N = 1$ is a form of (22) for the upper half plane! So the general case (17) with $N > 1$ describes a family of "generalized multi-parameter Löwner slit mappings" as explained in the previous section. The main complication for the case of several parameters is the compatibility conditions (18) for the parametric multipliers $\frac{\partial A^0}{\partial \lambda_i}$ — analogues of the parametric multiplier $1/t$ in (22) — and "slit shape-descriptors" q_j — analogues of $\kappa(t)$.

A number of remarkable results for a similar problem of description of extremal bounded univalent functions have been obtained ([17], [16]). Such functions map D onto D with several (forked in general) slits. For our case this suggests that the mappings which will be constructed below in section 5 are extremals of the functionals H^n in (9) so the explicit algebraic reductions of section 5 should be considered "solitonic solutions" of (17)–(18); for the case $N = 2$ we obtain "solitons" for (15). These properties of our explicit solutions will be discussed in a subsequent publication.

5 A class of explicit algebraic mappings

In some cases the mapping described in Section 3 can be constructed in explicit terms; in particular, if the cuts are taken along the $N - 1$ rays R_j through the origin $arg(\lambda_+) = j\pi/N$, $1 \le j \le N - 1$, this construction gives the dispersionless Lax reduction (11) ([7]):

$$\begin{aligned} \lambda &= \prod_{i=1}^{N} (q - p_i)^{1/N}, \\ \sum_{i=1}^{N} p_i &= 0. \end{aligned} \tag{23}$$

More generally, the (positive) angles $\alpha_i\pi$ between these rays can be chosen arbitrarily, by putting

$$\begin{aligned} \lambda_+ &= \prod_{i=1}^{N}(q-p_i)^{\alpha_i}, \\ \sum_{i=1}^{N}\alpha_i &= 1, \qquad \sum_{i=1}^{N}\alpha_i p_i = 0. \end{aligned} \tag{24}$$

The characteristic speeds q_i are given, in this case, by the algebraic equation:

$$\frac{\partial \log(\lambda_+)}{\partial q} = \sum_{i=1}^{N}\frac{\alpha_i}{q-p_i} = 0, \tag{25}$$

which will have at most $N-1$ roots q_i, interlacing the zeroes p_i; the Riemann invariants are the critical values $\lambda_i^* = \lambda_+(q_i)$.

We show how, starting form (23), a more general explicit M-parametric reductions with M *curvilinear* slits may be constructed. Firstly we construct two such Lax mappings

$$\lambda^{(1)}(q) = \left(\int_0^q N(M+1)\prod_{i=1}^{N(M+1)-1}(q-v_i^{(1)})\,dq + v_{N(M+1)}^{(1)}\right)^{\frac{1}{N(M+1)}} = \left(P_{MN}(q, v_1^{(1)},\ldots,v_{N(M+1)}^{(1)})\right)^{\frac{1}{N(M+1)}}, \tag{26}$$

$$\lambda^{(2)}(q) = \left(\int_0^q N(M+1)\prod_{i=1}^{N(M+1)-1}(q-v_i^{(2)})\,dq + v_{N(M+1)}^{(2)}\right)^{\frac{1}{N(M+1)}} = \left(P_{MN}(q, v_1^{(2)},\ldots,v_{N(M+1)}^{(2)})\right)^{\frac{1}{N(M+1)}}, \tag{27}$$

with polynomial P_{MN}; the constants $v_s^{(1)}$, $v_s^{(2)}$ will be chosen later; $\sum_{s=1}^{N(M+1)-1} v_s^{(k)} = 0$. Obviously the corresponding characteristic velocities of the reduced Benney systems (10) are $v_s^{(1)}$, $v_s^{(2)}$ and the Riemann invariants are $\lambda_s^{(1)} = \lambda^{(1)}(v_s^{(1)})$, $\lambda_s^{(2)} = \lambda^{(2)}(v_s^{(2)})$; $arg(\lambda_s^{(1)}) = arg(\lambda_s^{(2)}) = s\pi/N(M+1)$, $1 \le s \le N(M+1)-1$; only the moduli of $\lambda_s^{(1)}$ and $\lambda_s^{(2)}$ differ. Now we choose $v_s^{(1)}$, $v_s^{(2)}$ in such a way that all $|\lambda_s^{(1)}| = |\lambda_s^{(2)}|$ **except** $|\lambda_N^{(1)}| > |\lambda_N^{(2)}|$, $|\lambda_{2N}^{(1)}| > |\lambda_{2N}^{(2)}|$, …, $|\lambda_{MN}^{(1)}| > |\lambda_{MN}^{(2)}|$. This is achieved by choosing algebraic functions $v_s(u_1,\ldots,u_{N(M+1)})$ — solutions of the following polynomial system

$$\begin{cases} |\lambda_s|^{N(M+1)} = |\lambda_s(v_s)|^{N(M+1)} = P_{MN}(v_s, v_1, v_2, \dots, v_{N(M+1)}) = c_s, \\ \qquad\qquad \text{for } 1 \le s \le N(M+1)-1,\ s \ne N, 2N, \dots, MN; \\ |\lambda_{kN}|^{N(M+1)} = |\lambda_{kN}(v_{kN})|^{N(M+1)} = \\ \qquad\qquad P_{MN}(v_{kN}, v_1, v_2, \dots, v_{N(M+1)}) = u_k, \text{ for } \ k = 1, 2, \dots, M; \\ \sum\limits_{s=1}^{N(M+1)-1} v_s = 0; \end{cases} \tag{28}$$

then we fix c_s for both $\lambda^{(1)}$, $\lambda^{(2)}$, fix $u_k^{(2)}$, and choose some variable parameters $u_k^{(1)}$, $u_k^{(1)} > u_k^{(2)}$. Then $v_s^{(1)} = v_s(u_k^{(1)})$, $v_s^{(2)} = v_s(u_k^{(2)})$ will guarantee $|\lambda_s^{(1)}| = |\lambda_s^{(2)}|$ except $|\lambda_{kN}^{(1)}| > |\lambda_{kN}^{(2)}|$.

The composite mapping

$$\widehat{\lambda} = \left(\lambda^{(2)}\right)^{-1} \circ \lambda^{(1)} \tag{29}$$

is correctly defined on the upper half plane Λ_+ mapping it onto Λ_+ with M *curvilinear* slits — images of the "upper" segments of the rays R_{kN} of $\lambda^{(1)}$ which are longer than the rays of $\lambda^{(2)}$. Varying $u_k^{(1)}$ and retaining $u_k^{(2)}$ fixed (as well as c_s) we get the desired M-parametric slit mapping $\widehat{\lambda}(u_1^{(2)}, \dots, u_M^{(2)})$; other rays R_s, $s \ne N, 2N, \dots$ disappear after $(\lambda^{(2)})^{-1}$. Provided N may be chosen arbitrarily big we obtain for any given M a family of explicit algebraic mappings with M curvilinear slits whose shapes depend on arbitrary many additional parameters c_s, $u_k^{(1)}$. In the next Section we show that these M-parameter solutions of (14) form a locally dense subset of the set of *all* (hyperbolic) M-parametric reductions of (2).

6 Local density of the family (29) of explicit algebraic solutions

First of all we choose some star-shaped domain $\mathcal{D}$ in the upper half plane given in polar coordinates by some positive continuous piecewise-analytic function $\rho(\theta)$, $\mathcal{D} = \{z : |z| < \rho(arg(z)), Im(z) > 0\}$. Let N increase and M be fixed; we set $u_k^{(2)} = \rho(\frac{kN\pi}{N(M+1)})$, $c_s = \rho(\frac{s\pi}{N(M+1)})$ in (28) in order to obtain the mapping $\lambda^{(2)}$ which converges to a mapping $\widetilde{\lambda}^{(2)}$ of the upper half plane Λ_+ onto $\Lambda_+ \setminus \mathcal{D}$ due to the classical Caratheodory kernel theorem [12, Ch. II, § 5], [13, Ch. 3, § 3.1]. For $\lambda^{(1)}$ we have the same c_k in (28) and M free parameters $u_k^{(1)} > \rho(\frac{kN\pi}{N(M+1)})$; $\lambda^{(1)}$ converges to a mapping $\widetilde{\lambda}^{(1)}$ of Λ_+ onto $\Lambda_+ \setminus (\mathcal{D} \cup R_N \cup \dots \cup R_{MN})$ where R_{kN} are rectilinear segments along the rays $arg(z) = kN\pi/N(M+1)$. Since the parts of the boundary of $\mathcal{D}$ between

R_{kN} and $R_{(k+1)N}$ may be chosen arbitrarily we may expect that the mapping $\widetilde{\lambda}(q) = \left(\widetilde{\lambda}^{(2)}\right)^{-1} \circ \widetilde{\lambda}^{(1)}$ will give an arbitrary M-slit mapping described in Section 3. Here we prove a weaker local version of such a density theorem, similar to the "completeness" result proved in [15] for some (2+1)-dimensional solitonic equations of geometric origin. Namely if we choose $\left(u^{(1)}_{kN}\right)^{N(M+1)} = \left(u^{(2)}_{kN}\right)^{N(M+1)} + \epsilon$ (so the resulting M slits are small enough) then for arbitrary K the first K coefficients in the expansions $z_k(\epsilon) = z^0_k + z^1_k\epsilon + z^2_k\epsilon^2 + \ldots$ of the equations describing the slits may be chosen arbitrary for N big enough.

Now one may reduce this local density problem for M slits to the case of 1 slit: choosing $\mathcal{D}$ to be the half-circle everywhere except near the rays R_{kN} where small perturbations of its shape will be taken we see that $\widetilde{\lambda}^{(2)}(q)$ may be represented as an appropriate composition of mappings $\widetilde{\lambda}^{(2)}_{(k)}$ which are constructed using $\mathcal{D}_{(k)}$ — half circles with only one perturbation near R_{kN}, $k = 1, \ldots, M$ and the mapping $\mu : \Lambda_+ \setminus \mathcal{D}_{(0)} \to \Lambda_+$; $\mathcal{D}_{(0)} = \{z : |z| < 1, Im(z) > 0\}$ is the (unperturbed) half-circle. Due to the Rado theorem [12, Ch. II, § 5, Th. 2], the Weierstrass uniform convergence theorem [12, Ch. I, § 1, Th. 1], and the standard Schwartz symmetry principle applied to each $\widetilde{\lambda}_{(k)}$ near another ray R_{mN} we conclude that a small (by value, not necessary by curvature) perturbation in $\mathcal{D}_{(k)}$ near R_{kN} will have infinitesimal influence on the m-th slit at the point of the boundary of $\mathcal{D}$ compared to the influence of the perturbation of $\mathcal{D}_{(m)}$ near R_{mN} for $m \neq k$.

Thus we need to prove the local density result for $M = 1$ only which is achieved with the standard technique of conformal mappings [12] since we have the necesary parametric freedom here (1 function of 1 variable — the variation of the boundary of $\mathcal{D}$).

7 Further questions

The family of hyperbolic reductions described above suggests a more detailed study of the case of non-hyperbolic reductions where we would not have fixed slits; such study should use the technique developed in [16] for univalent bounded mappings. Another question to be investigated in more detail concerns the characterisation of these mappings as extrema of the conserved densities.

There is also the question of the Hamiltonian and geometrical properties of these reductions of the Benney hierarchy, in particular a Riemannian metric, and how these relate to the corresponding N-parameter family of univalent functions.

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