Reductions of the Benney Equations

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Abstract

The reductions of the Benney moment equations to systems of finitely many partial differential equations are discussed. Several families of these are constructed explicitly. These must all satisfy a compatibility condition, and the reduced equations are diagonalisable and semi-Hamiltonian. By imposing a further constraint, of scaling and Galilean invariance, the compatibility condition is reduced to a system of algebraic equations, whose solutions are described. A more general family of reductions is then constructed explicitly.

1 Introduction

Benney's moment equations, [1,2], are:

$$\frac{\partial A^n}{\partial t} + \frac{\partial A^{n+1}}{\partial x} + nA^{n-1}\frac{\partial A^0}{\partial x} = 0. \tag{1}$$

They admit many different reductions, in which the infinitely many moments A^n are expressible in terms of only finitely many variables. These are quite diverse in their properties. One of the best understood is the Zakharov reduction [3], which can be derived as the dispersionless limit of the vector NLS. It is

$$A^n = \sum_{i=1}^{N} h_i u_i^n, \tag{2}$$

for if the variables h_i, u_i satisfy the equations of motion

$$\frac{\partial h_i}{\partial t} + \frac{\partial (u_i h_i)}{\partial x} = 0, \tag{3}$$

$$\frac{\partial u_i}{\partial t} + u_i\frac{\partial u_i}{\partial x} + \frac{\partial A^0}{\partial x} = 0, \tag{4}$$

then the A^n all satisfy (1). Here the A^n are all expressible in terms of $2N$ independent dynamical variables, and further, the reduced system is Galilean invariant and scale invariant. It is also

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Hamiltonian with three different structures. Most important, it is diagonalisable. If, following [2], we define the generating function $\Lambda(p)$ by

$$\Lambda = p + \sum_{n=0}^{\infty} A^n/p^{n+1} = p + \sum_{i=0}^{N} \frac{h_i}{p - u_i}, \tag{5}$$

then each turning point

$$\frac{\partial \Lambda}{\partial p}(p_i) = 0,$$

$$\Lambda(p_i) = \lambda_i$$

of this function corresponds to a Riemann invariant λ^i with characteristic speed p_i. That is, equations (3,4) are equivalent to the system:

$$\frac{\partial \lambda^i}{\partial t} + p_i \frac{\partial \lambda^i}{\partial x} = 0. \tag{6}$$

In [4] it was shown how any diagonalisable Hamiltonian or semi-Hamiltonian system may be solved exactly by the hodograph transformation; this has been carried out in detail here by Geogdzhaev [5]. A degenerate case of the Zakharov reduction can be derived by letting some of the u_i approach the same limit, so that $\Lambda(p)$ has multiple poles.

Another important class of reductions consists of the dispersionless Lax equations, which were solved in [6]; here the A^n are given by:

$$p + \sum_{n=0}^{\infty} A^n/p^{n+1} = (p^N + \sum_{n=0}^{N-2} U^n p^{N-n-2})^{1/N}. \tag{7}$$

These are also Hamiltonian and diagonalisable, but are not Galilean invariant. A further class of reductions, important in the numerical investigation of Vlasov equations, is the so-called ‘waterbag’ model, where the variables A^n are the moments of a piecewise constant distribution function $f(x,p)$. This gives:

$$p + \sum_{n=0}^{\infty} A^n/p^{n+1} = p + \sum_{i=1}^{N} \rho_i \ \ln(\frac{p - u_i^+}{p - u_i^-}), \tag{8}$$

so that the A^n depend on $2N$ dynamical variables u_i^+ and u_i^-, as well as the N parameters ρ_i. The resulting system is Galilean invariant but not invariant under the scaling

$$A^n \to \alpha^{n+2} A^n. \tag{9}$$

Studying these different reductions individually cannot answer the obvious general questions which arise, such as whether the reduced equations are always diagonalisable or Hamiltonian, how many such reductions exist, or which of them are invariant under Galilean or scaling transformations. It makes more sense to study such reductions directly. This may be done in two ways; either by trying to construct much more general classes of reductions, or in general terms, by considering the consistency condition which all such reductions must satisfy.

In the next three sections we study the consistency condition, and hence show that the reductions to N variables are parameterised by N functions of one variable; in section 5 such a family of reductions is constructed explicitly, by deforming the Lax reduction.

2 The Compatibility Condition

Let us consider the most general case in which all the moments A^i are functions of only N independent variables u^j. If the A^i satisfy (1) it is straightforward to show that the mapping $(u^1,...,u^N) \to (A^0,...,A^{N-1})$ is nondegenerate. Hence we may without loss of generality set $u^1 = A^0,...,u^n = A^{N-1}$. The first N moments are the independent variables, while the higher moments are functions of them:

$$A^k = a^k(A^0,...,A^{N-1}), \quad k \geq N. \tag{10}$$

The equations of motion for $(A^0,...,A^{N-1})$ then become:

$$-\frac{\partial A^j}{\partial t} = \frac{\partial A^{j+1}}{\partial x} + jA^{j-1}\frac{\partial A^0}{\partial x}, \quad j \leq N-2 \tag{11}$$

$$-\frac{\partial A^{N-1}}{\partial t} = \frac{\partial a^N}{\partial A^i}\frac{\partial A^i}{\partial x} + (N-1)A^{N-2}\frac{\partial A^0}{\partial x}, \tag{12}$$

while each higher moment $(a^N,....)$ must satisfy the overdetermined system:

$$-\frac{\partial a^k}{\partial t} = \sum_{j=0}^{N-2}\frac{\partial a^k}{\partial A^j}(\frac{\partial A^{j+1}}{\partial x} + jA^{j-1}\frac{\partial A^0}{\partial x}) + \frac{\partial a^k}{\partial A^{N-1}}(\sum_{i=0}^{N-1}\frac{\partial a^N}{\partial A^i}\frac{\partial A^i}{\partial x} + (N-1)A^{N-2}\frac{\partial A^0}{\partial x})$$

$$= \sum_{j=0}^{N-1}\frac{\partial a^{k+1}}{\partial A^j}\frac{\partial A^j}{\partial x} + ka^{k-1}\frac{\partial A^0}{\partial x} \tag{13}$$

Hence we find, on comparing coefficients of $\partial A^j/\partial x$, the system

$$\frac{\partial a^{k+1}}{\partial A^j} = \frac{\partial a^k}{\partial A^{N-1}}\frac{\partial a^N}{\partial A^j} + \frac{\partial a^k}{\partial A^{j-1}} \quad 1 \leq j \leq N-1$$

$$\frac{\partial a^{k+1}}{\partial A^0} = \sum_{i=0}^{N-1} iA^{i-1}\frac{\partial a^k}{\partial A^i} + \frac{\partial a^k}{\partial A^{N-1}}\frac{\partial a^N}{\partial A^0} - ka^{k-1}. \tag{14}$$

The compatibility condition for these with $k = N$, gives a system S of $N(N-1)/2$ nonlinear second order equations for the single unknown $a^N(A^0,...,A^{N-1})$. It can be shown by induction that if S is satisfied then the analogous compatibility conditions for a^k with $k > N$, are satisfied too. Taking $N = 2$ we get the simplest such system; on denoting $x = A_0$, $y = A_1$ and $z = a_2 - a_0^2/2$, we get

$$z_{xx} - z_y z_{xy} + z_x z_{yy} - 1 = 0. \tag{15}$$

If the variables u and v are defined as $(z_y \pm \sqrt{z_y^2 - 4z_x})/2$, then this equation becomes:

$$u_x = vu_y - \frac{1}{u-v}$$

$$v_x = uv_y + \frac{1}{u-v}. \tag{16}$$

This has one obvious hydrodynamic type conserved density $(u+v)$, one involving first derivatives $(u-v)(u_y^2-v_y^2)$, together with several involving x, y and z explicitly. A simple form of S for general N will be given in the next section.

Haantjes [7] derived the condition that a system of N partial differential equations of the form

$$\frac{\partial u^i}{\partial t}+v^i_j\frac{\partial u^j}{\partial x}=0 \tag{17}$$

should possess Riemann invariants. That is, that there should exist N functions λ^n, depending on the variables u^i, in which these equations are diagonalised:

$$\frac{\partial \lambda^n}{\partial t}+V^n\frac{\partial \lambda^n}{\partial t}=0, \tag{18}$$

where the V^n are the eigenvalues of the matrix v^i_j, called the characteristic speeds. The condition is the following. Define N^i_{jk} by

$$N^i_{jk}=v^s_j\frac{\partial v^i_k}{\partial u^s}-v^s_k\frac{\partial v^i_j}{\partial u^s}-v^i_s\left(\frac{\partial v^s_k}{\partial u^j}-\frac{\partial v^s_j}{\partial u^k}\right), \tag{19}$$

and then define H^i_{jk} by

$$H^i_{jk}=(N^i_{qp}v^q_k-N^q_{kp}v^i_q)v^p_j-v^i_p(N^p_{qj}v^q_k-N^q_{kj}v^p_q). \tag{20}$$

The system (17) can be diagonalised whenever H^i_{jk} vanishes identically, and the V^n are all real and distinct. In the case of system (11,12), the reduced Benney equations, the former condition holds precisely when S is satisfied. Hence any consistent reduction of Benney's equations is diagonalisable. Thus we can use the Riemann invariants and their characteristic speeds as the natural variables in which to discuss the problem further.

3 The Characteristic Form of the Consistency Condition

Suppose that each moment A^k is expressible as a function of N Riemann invariants λ^i, which satisfy the equation

$$\frac{\partial \lambda^i}{\partial t}+V^i\frac{\partial \lambda^i}{\partial x}=0. \tag{21}$$

The moment equations are then satisfied if

$$V^i\frac{\partial A^n}{\partial \lambda^i}=\frac{\partial A^{n+1}}{\partial \lambda^i}+nA^{n-1}\frac{\partial A^0}{\partial \lambda^i}. \tag{22}$$

Following [2], we consider the generating function $\Lambda(p)$ defined by

$$\Lambda=p+\sum_{n=0}^{\infty}A^n/p^{n+1}. \tag{23}$$

This satisfies the equation, equivalent to (22),

$$V^i\frac{\partial \Lambda}{\partial \lambda^i}=p\frac{\partial \Lambda}{\partial \lambda^i}-\frac{\partial A^0}{\partial \lambda^i}\frac{\partial \Lambda}{\partial p} \tag{24}$$

or, rearranging,

$$\frac{\partial \Lambda}{\partial \lambda^i} = \frac{\partial A^0}{\partial \lambda^i} \frac{\partial \Lambda}{\partial p} \frac{1}{p - V^i}. \tag{25}$$

Cross-differentiating, we derive consistency conditions, equivalent to S:

$$\frac{\partial^2 A^0}{\partial \lambda^i \partial \lambda^j} = \frac{\partial A^0}{\partial \lambda^i} \frac{\partial A^0}{\partial \lambda^j} \frac{1}{(V^i - V^j)^2} \tag{26}$$

and

$$\frac{\partial V^j}{\partial \lambda^i} = \frac{\partial A^0}{\partial \lambda^i} \frac{1}{(V^i - V^j)}, \tag{27}$$

for all $i \neq j$. The higher moments A^n, with $n > 0$ are then given recursively by solving (22). These equations are compatible, and their solutions are parameterised by $2N$ functions of a single variable. Half of these are inessential, corresponding to the freedom to reparameterise each λ^i separately without changing the form of (21), but the other N functions distinguish essentially different reductions. If the λ^i are replaced as independent variables by the moments $A^0, ...A^{N-1}$, then (27) may be taken into a form analogous to (16).

A direct calculation confirms that the reduced equations (21) are semi-Hamiltonian; that is, the characteristic speeds satisfy

$$\frac{\partial}{\partial \lambda^k} \left(\frac{\partial V^i}{\partial \lambda^j} / (V^i - V^j)\right) = \frac{\partial}{\partial \lambda^j} \left(\frac{\partial V^i}{\partial \lambda^k} / (V^i - V^k)\right) \tag{28}$$

for i, j, k all distinct. The reduced equations are thus integrable by the generalised hodograph transformation [4]. However, (28) does not imply that they possess a local Hamiltonian structure. Even if they do, it may be distinct from the reduction of the Hamiltonian structure of the full system (1).

4 Homogeneity and Galilean Invariance

We may impose two further restrictions on these reduced systems. First, we require that the reduced system is Galilean invariant in the sense that:

$$\lambda^i \to \lambda^i + c \quad \Rightarrow V^i \to V^i + c, \quad A^0 \to A^0 \tag{29}$$

or, equivalently,

$$\delta V^i = 1 \tag{30}$$

$$\delta A^0 = 0, \tag{31}$$

where $\delta = \sum_{i=1}^N \partial / \partial \lambda^i$. Secondly, we require the functions A^0 and V^i to be homogeneous in the λ^i; for balance, A^0 should be of weight 2, and the V^i of weight 1. Thus

$$R V^i = V^i, \tag{32}$$

$$R \frac{\partial A^0}{\partial \lambda^i} = \frac{\partial A^0}{\partial \lambda^i}, \tag{33}$$

where $R = \sum_{i=1}^N \lambda^i \partial / \partial \lambda^i$. We may now substitute in (26,27), eliminating the second derivatives,

$$\frac{\partial A^0}{\partial \lambda^i} = R \frac{\partial A^0}{\partial \lambda^i} = \sum_{j=1, j \neq i}^N \lambda^j \frac{\partial^2 A^0}{\partial \lambda^i \partial \lambda^j} + \lambda^i \frac{\partial^2 A^0}{\partial {\lambda^i}^2} =$$

$$\sum_{j=1,j\neq i}^{N}\lambda^j\frac{\partial^2 A^0}{\partial\lambda^i\partial\lambda^j}+\lambda^i(\delta\frac{\partial A^0}{\partial\lambda^i}-\sum_{j=1,j\neq i}^{N}\frac{\partial^2 A^0}{\partial\lambda^i\partial\lambda^j})=$$

$$2\sum_{j=1,j\neq i}^{N}\frac{\lambda^j-\lambda^i}{(V^i-V^j)^2}\frac{\partial A^0}{\partial\lambda^i}\frac{\partial A^0}{\partial\lambda^j} \tag{34}$$

Hence, either $\partial A^0/\partial\lambda^i=0$ or

$$\sum_{j=1,j\neq i}^{N}2\frac{(\lambda^j-\lambda^i)}{(V^j-V^i)^2}\frac{\partial A^0}{\partial\lambda^j}=1 \tag{35}$$

Similarly we may obtain:

$$V^i=\lambda^i+\sum_{j=1,j\neq i}^{N}(\frac{\lambda^j-\lambda^i}{V^j-V^i})\frac{\partial A^0}{\partial\lambda^j} \tag{36}$$

Together these are a system of $2N$ algebraic equations for $2N$ unknowns, the V^i and $\partial A_0/\partial\lambda^i$. The solutions of this system are essentially unique. With two Riemann invariants, the only solution is the Zakharov reduction with one pair (u,h), the classical shallow water equations. With $2N$ invariants, one solution is the Zakharov system with N pairs (u^i,h^i); we may also obtain degenerations of these, with fewer independent variables, by letting $u^i\to u^j$. All these reductions are tri-Hamiltonian, and their metrics are of Egorov type. It would be interesting to know whether any other such reductions exist.

5 A Deformation of the Lax Reductions

If the A^n are the moments with respect to p of some distribution function $f(x,p)$, then the generating function $\Lambda(p)$, analogous to (5) may be defined by

$$\Lambda=p+P\int_{-\infty}^{\infty}f(x,p')\frac{dp'}{p-p'}. \tag{37}$$

Here, for definiteness, the principal value of the integral has been taken. Alternatively, the contour can be indented so that p' passes below p, giving a function Λ_+ analytic in the upper half p-plane; its boundary value on the axis is then $\Lambda-i\pi f$. It is straightforward to show that if $f(x,p)$ is advected along the characteristics

$$\frac{dx}{dt}=p \tag{38}$$

$$\frac{dp}{dt}=-\frac{\partial A^0}{\partial x}, \tag{39}$$

then the moments A^n satisfy (1), and Λ is advected along the same characteristics. It follows that any relation such as $f=F(\Lambda)$ is preserved by the dynamics. The generating function Λ will then satisfy the nonlinear singular integral equation:

$$\Lambda=p+P\int_{-\infty}^{\infty}F(\Lambda(p'))\frac{dp'}{p-p'}. \tag{40}$$

If $F \leq 0$ then we may describe the solutions of this equation in terms of a conformal mapping. The construction is to take the upper half of the complex Λ_+-plane, and to cut it, along the curve $Im(\Lambda_+) = -\pi F(Re(\Lambda_+))$, from the real axis as far as $\lambda - i\pi F(\lambda)$, where the parameter λ of the branch point depends on x and t.

We may now define a function $p(\Lambda_+)$ uniquely by three properties:

(i) p is real and continuous on the real Λ_+-axis and on the cut;

(ii) it is analytic in the cut half Λ_+-plane;

(iii) as $|\Lambda_+| \to \infty$, with $Im(\Lambda_+) > 0$, then $p = \Lambda_+ + O(1/\Lambda_+)$. The moments A^n then are obtained from the asymptotics of $\Lambda_+(p)$ as $p \to \infty$:

$$\Lambda_+ = p + \sum_{n=0}^{\infty} A^n/p^{n+1}. \tag{41}$$

We may similarly construct other functions H_n with asymptotics $H_n = \Lambda_+^n/n + O(1/\Lambda_+)$. The equation of motion then takes the simple form

$$\frac{\partial p}{\partial t} + \frac{\partial H_2}{\partial x} = 0. \tag{42}$$

This equation may be expanded near infinity, to give the conservation laws, or near the branch point, to see that λ is a Riemann invariant, with characteristic speed $p(\lambda)$.

This construction may be generalised to the case of N non-intersecting cuts given by $Im(\Lambda_+) = -\pi F_j(Re(\Lambda_+))$, each starting on the real Λ_+-axis and ending in a branch point $Re(\Lambda_+) = \lambda^j$; these λ^j are the N Riemann invariants of the system. If the cuts are taken along the N rays through the origin $arg(\Lambda_+) = j\pi/(N+1)$, this construction gives the dispersionless Lax reduction. We have shown above that the most general reduction with N Riemann invariants is similarly parameterised by N functions of a single variable, but it is easy to see that the waterbag and Zakharov reductions do not fit directly into the scheme described here, which forces the even numbered moments A_{2n} to be negative. It seems likely that the Riemann surface used by Geogdzhaev in [5] can be deformed to give a similar large family of reductions.

6 Conclusions

The most important open question concerning these systems is whether the compatibility conditions (31,32), and their analogues for other integrable moment hierarchies, can themselves be regarded as integrable systems. This is related to the problem of finding the most general reduction of the type described in Section 5; of the reductions described in the introduction, only the dispersionless Lax equations belong to this family. Other problems include classifying those reductions of the Benney hierarchy which are Galilean or scaling invariant, but not both. For $n = 2$, the only Galilean invariant solutions are the waterbag reductions, with their limit, the Zakharov reduction. Looking for homogeneous solutions of (15), however, leads to a nonlinear o.d.e. which we have not been able to solve; it does not have the Painlevé property. One intriguing connection has recently been discovered by Ferapontov and Fordy [8,9] who have related the system (26,27) to isometries of Stäckel metrics, and have hence found some solutions. This approach needs further study.

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