EE2 Mathematics : Fourier and Laplace Transforms

J. D. Gibbon

(Professor J. D Gibbon[1], Dept of Mathematics)

j.d.gibbon@ic.ac.uk **http://www2.imperial.ac.uk/∼jdg**

These notes are not identical word-for-word with my lectures which will be given on a BB/WB. Some of these notes may contain more examples than the corresponding lecture while in other cases the lecture may contain more detailed working. I will not be handing out copies of these notes – **you are therefore advised to attend lectures and take your own**.

1. The material in them is dependent upon the material on complex variables in the second part of this course.
2. Handouts are :
   (a) **Handout No 5** on Fourier Transforms and a list of functions ;
   (b) **Handout No 6** on Laplace Transforms.

Contents

---

[1] Do not confuse me with Dr J. Gibbons who is also in the Mathematics Dept.

1 Fourier Transforms

1.1 Introduction

There are three definitions of the Fourier Transform (FT) of a function $f(t)$ – see Appendix A. The one used here, which is consistent with that used in your own Department, is[2]

$$\overline{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt\,. \tag{1.1}$$

where the **frequency** ω is real. Another common notation is to write $F(\omega)$ or $\mathcal{F}(\omega)$ for $\overline{f}(\omega)$. Given the **spectrum** $\overline{f}(\omega)$ the function $f(t)$ can be recovered through the **inverse transform**

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega) e^{i\omega t}\, d\omega\,. \tag{1.2}$$

Note the factor of $1/2\pi$ in the coefficient. The interplay between the function of time $f(t)$ (or a sampled time series) and the FT $\overline{f}(\omega)$ is subtle. Clearly, it is possible that functions $f(t)$ could be chosen for which the integral (1.1) is infinite – which means that this transform does not exist. There are two conditions that must be satisfied for the FT to exist :

(i) $f(t)$ must be absolutely integrable : that is

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty\,. \tag{1.3}$$

(ii) If $f(t)$ has discontinuities then it must be finite at these.

The following is a list of common functions :

1. The sign-function
$$\text{sgn}(t) = \begin{cases} -1 & t \le 0 \\ +1 & t \ge 0 \end{cases} \tag{1.4}$$
2. The triangle or tent function :
$$\Lambda(t) = \begin{cases} 1-t & 0 \le t \le 1 \\ 1+t & -1 \le t \le 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$
3. The rectangle function :
$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$
4. The filtering function :
$$\text{sinc}\,(t) = \frac{\sin(t/2)}{t/2}\,. \tag{1.7}$$
The $\frac{1}{2}$-factor is unusual but is the natural definition for this definition of the FT.

---

[2] The overbar notation $\overline{f}$ should not be confused with complex conjugate.

5. The Heaviside step function :

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t \leq 0 \end{cases} \tag{1.8}$$

6. The error function :

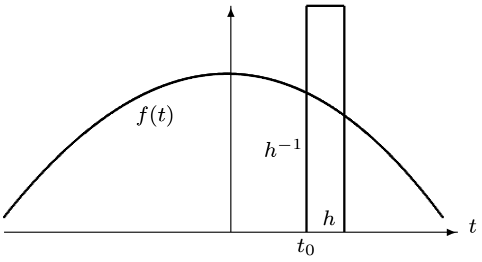
$$\mathsf{erf}\,(t) = \frac{1}{\sqrt{\pi}} \int_{-t}^{t} e^{-x^2}\,dx = \frac{2}{\sqrt{\pi}} \int_{0}^{t} e^{-x^2}\,dx\,. \tag{1.9}$$

7. The normalized autocorrelation function :

$$\gamma(t) = \frac{\int_{-\infty}^{\infty} f^*(u)\,f(t-u)\,du}{\int_{-\infty}^{\infty} |f(u)|^2\,du}\,. \tag{1.10}$$

1.2 The Dirac δ-function

Observing the list of functions in the previous section, it is clear that there is one missing. How can a spike be represented? For instance, it is intuitive that the spectrum $\overline{f}(\omega)$ of a single sine-wave $f(t) = \sin\omega_0 t$ should be a spike at ω_0 but the condition of absolute integrability (1.3) is not satisfied because of the infinite range of the integral. How can such an *improper function* be represented? One way of formaizing a spiky function is to introduce the **Dirac Delta function** $\delta(t-t_0)$ by considering the properties of a box of unit area under a limiting process, as in the figure below :



A box of unit area : width h & height h^{-1} at a point t_0 on the t-axis which limits to a spike as $h \to 0$ but retains unit area. The curve $f(t)$ is some other function : The product of the two is non-zero only within the range of the box.

From the picture we represent $\delta(t-t_0)$ as

$$\delta(t-t_0) = \lim_{h\to 0} \begin{cases} h^{-1} & t_0 \leq t \leq t_0 + h \\ 0 & \text{otherwise} \end{cases} \tag{1.11}$$

with the property of unit area

$$\text{Area} = \int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1\,. \tag{1.12}$$

In the limit $h \to 0$ the δ-function[3] acts as a 'spike' at t_0 : of course it is not a proper function at all but it possesses a the powerful property

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt &= \sum_{i=1}^{N} f(t_i)\delta(t_i - t_0)\,\Delta t_i \\ &= \lim_{h\to 0}\left[f(t_0)h^{-1}\,h\right] \\ &= f(t_0)\,. \end{aligned} \tag{1.13}$$

To express this in words, when multiplied on a function $f(t)$ and integrated, the δ-function simply picks out the value of $f(t)$ at the point of the spike t_0. This result can be expressed in a more general way :

$$\boxed{\int_{-\infty}^{\infty} f(t')\delta(t'-t)\,dt' = f(t)}\,. \tag{1.14}$$

This will be used many times in future sections.

Example : The Shannon sampling function (see the non-examinable extra material on the Shannon sampling Theorem in Appendix B) is a sum of δ-functions whose spikes occur at fixed times t_n :

$$III(t) = \sum_{n=-\infty}^{\infty} \delta(t-t_n)\,. \tag{1.15}$$

Its product with a signal $f(t)$ samples the signal only at discrete points t_n and so the area under the sampled signal is

$$\int_{-\infty}^{\infty} f(t)III(t)\,dt = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\delta(t-t_n)\,dt = \sum_{n=-\infty}^{\infty} f(t_n)\,. \tag{1.16}$$

1.3 Integral representation of the δ-function

The definitions of the the inverse FT $f(t)$ in (1.2) and the FT $\overline{f}(\omega)$ in (1.1) can be put together to give the *Dirichlet integral*

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t')e^{-i\omega t'}dt'\right)e^{i\omega t}\,d\omega \\ &= \int_{-\infty}^{\infty} f(t')\underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')}d\omega\right)}_{\delta(t-t')}\,dt' \end{aligned} \tag{1.17}$$

where the order of integration has been exchanged. As the underbrace in (1.17) shows, comparison with (1.14) gives

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')}d\omega\,. \tag{1.18}$$

---

[3] Another way of defining a δ-function is to take a Gaussian curve of half-width h in the limit $h \to 0$.

The reverse process gives

$$\begin{aligned}\overline{f}(\omega) &= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega')e^{i\omega' t}d\omega'\right)e^{-i\omega t}dt \\ &= \int_{-\infty}^{\infty} f(\omega')\underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega-\omega')t}dt\right)}_{\delta(\omega-\omega')}d\omega' \qquad (1.19)\end{aligned}$$

where the order of integration has been exchanged. As the underbrace in (1.19) shows, comparison with (1.14) gives[4]

$$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega-\omega')t}dt\,. \qquad (1.20)$$

Thus the ‘integral representation’ of the δ-function is :

$$\delta(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm i\Omega\tau}d\Omega\,, \qquad \text{or} \qquad \delta(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm i\Omega\tau}d\tau\,. \qquad (1.21)$$

Every student’s first reaction is to evaluate one of the integrals in (1.21)

$$\begin{aligned}\delta(\tau) &= \frac{1}{2\pi}\lim_{a\to\infty}\int_{-a}^{a} e^{\pm i\Omega\tau}d\Omega \\ &= \lim_{a\to\infty}\left(\frac{\sin a\tau}{\pi\tau}\right)\,, \qquad (1.22)\end{aligned}$$

but then it is observed that as a increases the oscillations become faster so the limit does not formally exist. As a function it has no meaning, but nevertheless, the two *integral representations* in (1.21) are extremely useful.

Example : Consider a function of time with one frequency – an exact sine-wave – in the form

$$f(t) = f_0\,e^{i\omega_0 t} \qquad (1.23)$$

and so

$$\overline{f}(\omega) = f_0\int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)t}\,dt = 2\pi f_0\,\delta(\omega-\omega_0)\,. \qquad (1.24)$$

Thus the spectrum is just a single frequency – a spike at $\omega=\omega_0$. The inverse transform is

$$f(t) = \frac{f_0}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{i\omega t}\,d\omega = f_0\,e^{i\omega_0 t}\,. \qquad (1.25)$$

---

[4] Either sign ± in the exponent can be chosen : the δ-function is the same either way ; $\delta(t-t_0)$ or $\delta(t_0-t)$.

1.4 Parseval's Theorem and its generalization

Theorem 1 *Given two (complex) functions of time $f(t)$ and $g(t)$*

$$\int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega)\, \overline{g}^*(\omega)\, d\omega\,. \tag{1.26}$$

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\overline{f}(\omega)|^2 d\omega\,. \tag{1.27}$$

The physical interpretation of (1.27) is that energy in time-space equals energy in spectral-space, as it must.

Proof: Firstly take the LHS of (1.26) and write $f(t)$ and $g^*(t)$ as inverse FTs:

$$\begin{aligned}\int_{-\infty}^{\infty} f(t) g^*(t)\, dt &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\overline{f}(\omega) e^{i\omega t} d\omega\right)\left(\int_{-\infty}^{\infty}\overline{g}^*(\omega') e^{-i\omega' t} d\omega'\right) dt \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\overline{f}(\omega)\left\{\int_{-\infty}^{\infty}\overline{g}^*(\omega')\left(\int_{-\infty}^{\infty} e^{i(\omega-\omega')t} dt\right) d\omega'\right\} d\omega\end{aligned} \tag{1.28}$$

Now use the integral representation

$$\int_{-\infty}^{\infty} e^{i(\omega-\omega')t} dt = 2\pi\delta(\omega-\omega') \tag{1.29}$$

to re-write (1.28) as

$$\begin{aligned}\int_{-\infty}^{\infty} f(t) g^*(t)\, dt &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\overline{f}(\omega)\left\{\int_{-\infty}^{\infty}\overline{g}^*(\omega')\left(2\pi\delta(\omega-\omega')\right) d\omega'\right\} d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}(\omega)\overline{g}^*(\omega) d\omega\,,\end{aligned} \tag{1.30}$$

which is the advertised result. (1.27) follows immediately by writing $g = f$. □

1.5 The Fourier Convolution Theorem

Every transform – Fourier, Laplace, Mellin, & Hankel – has a convolution theorem which involves a convolution product between two functions $f(t)$ and $g(t)$. The (Fourier) convolution is defined as[5]

$$f(t) \star g(t) = \int_{-\infty}^{\infty} f(t') g(t-t')\, dt'\,. \tag{1.31}$$

The delay $t - t'$ may be put in either function, to show this, write $\tau = t - t'$. Then

$$f(t) \star g(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau)\, d\tau\,. \tag{1.32}$$

---

[5] The $\star$ convolution product should not be confused with complex conjugate.

For convenience let us introduce the notation

$$\mathcal{F}[f(t)] \equiv \overline{f}(\omega)\,. \tag{1.33}$$

Theorem 2 *(**Fourier convolution theorem**) The transform of the convolution product in time is the product of the transforms in frequency:*

$$\mathcal{F}[f(t) \star g(t)] = \overline{f}(\omega)\,\overline{g}(\omega) \qquad \textit{or} \qquad f(t) \star g(t) = \mathcal{F}^{-1}\left[\overline{f}(\omega)\,\overline{g}(\omega)\right]. \tag{1.34}$$

Conversely, 2π times the transform of the product in time is the convolution product of the transforms in frequency:

$$2\pi\mathcal{F}[f(t)\,g(t)] = \overline{f}(\omega) \star \overline{g}(\omega) \qquad \textit{or} \qquad 2\pi f(t)\,g(t) = \mathcal{F}^{-1}\left[\overline{f}(\omega) \star \overline{g}(\omega)\right]. \tag{1.35}$$

Proof of (1.34):

$$\mathcal{F}[f(t) \star g(t)] = \int_{-\infty}^{\infty} e^{-i\omega t}\left(\int_{-\infty}^{\infty} f(t')g(t-t')\,dt'\right)dt\,. \tag{1.36}$$

Writing $\tau = t - t'$ and reversing the order of integration[6], (1.36) becomes

$$\begin{aligned} \mathcal{F}[f(t) \star g(t)] &= \int_{-\infty}^{\infty} f(t')\left(\int_{-\infty}^{\infty} e^{-i\omega(t'+\tau)}g(\tau)\,d\tau\right)dt' \\ &= \left(\int_{-\infty}^{\infty} f(t')e^{-i\omega t'}dt'\right)\left(\int_{-\infty}^{\infty} g(\tau)e^{-i\omega\tau}d\tau\right) \\ &= \overline{f}(\omega)\,\overline{g}(\omega)\,. \end{aligned} \tag{1.37}$$

Proof of (1.35):

$$\frac{1}{2\pi}\mathcal{F}^{-1}\left[\overline{f}(\omega) \star \overline{g}(\omega)\right] = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} e^{i\omega t}\left(\int_{-\infty}^{\infty} \overline{f}(\omega')\overline{g}(\omega-\omega')\,d\omega'\right)d\omega\,. \tag{1.38}$$

Writing $\Omega = \omega - \omega'$ and reversing the order of integration the RHS becomes $f(t)\,g(t)$. ☐

1.6 Examples of Fourier Transforms

1. As in §1.1, the rectangle function $\Pi(t)$ is defined as

$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.39}$$

---

[6] The $t - t'$-plane is infinite in all four directions.

Therefore

$$\begin{aligned}\overline{\Pi}(\omega) &= \int_{-\infty}^{\infty} \Pi(t)e^{-i\omega t}\,dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i\omega t}dt \\ &= \frac{e^{\frac{1}{2}i\omega t} - e^{-\frac{1}{2}i\omega}}{i\omega} = \frac{\sin\frac{1}{2}\omega}{\frac{1}{2}\omega} = \mathsf{sinc}\,\omega\,. \end{aligned} \tag{1.40}$$

The inverse is a little trickier :

$$\begin{aligned}\Pi(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,\mathsf{sinc}\,\omega\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\left(\frac{e^{\frac{1}{2}i\omega} - e^{-\frac{1}{2}i\omega}}{i\omega}\right)d\omega \\ &= \frac{1}{2i\pi}\left(I_1 - I_2\right) \end{aligned} \tag{1.41}$$

where

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{ip_1\omega}}{\omega}d\omega \qquad p_1 = t + \tfrac{1}{2} \tag{1.42}$$

and

$$I_2 = \int_{-\infty}^{\infty} \frac{e^{ip_2\omega}}{\omega}d\omega \qquad p_2 = t - \tfrac{1}{2} \tag{1.43}$$

When p_1 and p_2 have the same sign then $I_1 = I_2$; that is when $t > \frac{1}{2}$ and $t < -\frac{1}{2}$, in which case $\Pi(t) = 0$ through cancellation in (1.41). In the range $-\frac{1}{2} < t < \frac{1}{2}$ I_1 and I_2 have opposite signs where $I_1 = i\pi$ (see Complex Variable notes on integration when a pole is on the real axis) but $I_2 = -I_1$. Altogether we have the correct result

$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.44}$$

2. As in §1.1, the tent function $\Lambda(t)$ is defined as

$$\Lambda(t) = \begin{cases} 1 - t & 0 \le t \le 1 \\ 1 + t & -1 \le t \le 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.45}$$

Thus we have

$$\overline{\Lambda}(\omega) = \int_{-1}^{0} (1+t)e^{-i\omega t}dt + \int_{0}^{1} (1-t)e^{-i\omega t}dt \tag{1.46}$$

Now we know that

$$\begin{aligned}\int_a^b t\,e^{-i\omega t}dt &= \frac{i}{\omega}\int_a^b t\,d\big[e^{-i\omega t}\big] \\ &= \frac{i}{\omega}\left(\big[t\,e^{-i\omega t}\big]_a^b - \int_a^b e^{-i\omega t}dt\right) \\ &= \frac{i}{\omega}\left[t\,e^{-i\omega t} - \frac{i}{\omega}e^{-i\omega t}\right]_a^b , \end{aligned} \tag{1.47}$$

and

$$\int_a^b e^{-i\omega t}dt = \frac{i}{\omega}\left[e^{-i\omega t}\right]_a^b . \tag{1.48}$$

Using these in (1.46)

$$\begin{aligned}
\overline{\Lambda}(\omega) &= \frac{i}{\omega}\left[\left(1-e^{i\omega t}\right)+\left(e^{-i\omega t}-1\right)\right] \\
&+ \frac{i}{\omega}\left[-\frac{i}{\omega}-\left(-e^{i\omega}-\frac{i}{\omega}e^{i\omega}\right)\right]-\frac{i}{\omega}\left[\left(e^{-i\omega}-\frac{i}{\omega}e^{-i\omega}\right)+\frac{i}{\omega}\right] \\
&= -\frac{i}{\omega}\left(e^{i\omega}-e^{-i\omega}\right)+\frac{2}{\omega^2}+\frac{i}{\omega}\left(e^{i\omega}-e^{-i\omega}\right)-\frac{1}{\omega^2}\left(e^{i\omega}+e^{-i\omega}\right) \\
&= \frac{2(1-\cos\omega)}{\omega^2} \\
&= \frac{4\sin^2\frac{1}{2}\omega}{\omega^2} = \mathsf{sinc}^2\omega\,.
\end{aligned} \tag{1.49}$$

3. The auto-correlation function definition from (1.10) is

$$\gamma(t) = \frac{\int_{-\infty}^{\infty} f^*(u)\, f(t-u)\, du}{\int_{-\infty}^{\infty} |f(u)|^2\, du} = \frac{f^*(t)\star f(t)}{\int_{-\infty}^{\infty} |f(u)|^2\, du}\,. \tag{1.50}$$

Using the Convolution Theorem, its FT is

$$\begin{aligned}
\overline{\gamma}(\omega) &= \frac{\mathcal{F}\left[f^*(t)\star f(t)\right]}{\int_{-\infty}^{\infty} |f(u)|^2\, du} \\
&= \frac{|\overline{f}(\omega)|^2}{\int_{-\infty}^{\infty} |f(u)|^2\, du}\,.
\end{aligned} \tag{1.51}$$

Notice that the denominator has been taken outside the integral because it is a number. Integrating the result w.r.t. ω it is found that

$$\int_{-\infty}^{\infty} \overline{\gamma}(\omega)\, d\omega = \frac{\int_{-\infty}^{\infty} |\overline{f}(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |f(u)|^2\, du} = 2\pi\,, \tag{1.52}$$

where Parseval's Theorem (1.27) has been used to obtain the last line.

4. For the Shannon sampling function in (1.15):

$$I\!I\!I(t) = \sum_{n=-\infty}^{\infty} \delta(t-t_n) \tag{1.53}$$

Then the FT of $I\!I\!I(t)$ is

$$\overline{I\!I\!I}(\omega) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega t}\delta(t-t_n)\, dt = \sum_{n=-\infty}^{\infty} e^{-i\omega t_n} \tag{1.54}$$

so the Convolution Theorem gives

$$\begin{aligned}\mathcal{F}\left[f(t)\star III(t)\right] &= \overline{f}(\omega)\overline{III}(\omega)\\ &= \sum_{n=-\infty}^{\infty}\overline{f}(\omega)e^{-i\omega t_n}\,,\end{aligned}\tag{1.55}$$

which is (correctly) the product of the transforms. Moreover, the FT of the ordinary product between $f(t)$ and $III(t)$ is

$$\begin{aligned}2\pi\mathcal{F}\left[f(t)III(t)\right] &= 2\pi\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}\delta(t-t_n)\,dt\\ &= 2\pi\sum_{n=-\infty}^{\infty}f(t_n)e^{-i\omega t_n}\,,\end{aligned}\tag{1.56}$$

which should be the convolution of $\overline{f}(\omega)$ and $III(\omega)$. To check this write

$$\begin{aligned}\overline{f}(\omega)\star III(\omega) &= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f}(\omega')e^{-i(\omega-\omega')t_n}\,d\omega'\\ &= 2\pi\sum_{n=-\infty}^{\infty}f(t_n)e^{-i\omega t_n}\end{aligned}\tag{1.57}$$

which agrees with (1.56). The convolution in time is

$$\begin{aligned}f(t)\star III(t) &= \int_{-\infty}^{\infty}f(t-t')III(t')\,dt'\\ &= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}f(t-t')\delta(t'-t_n)\,dt'\\ &= \sum_{n=-\infty}^{\infty}f(t-t_n)\end{aligned}\tag{1.58}$$

and so

$$\begin{aligned}\mathcal{F}\big[f(t)\star III(t)\big] &= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i\omega t}f(t-t_n)\,dt\\ &= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i\omega(\tau_n+t_n)}f(\tau_n)\,d\tau_n\\ &= \sum_{n=-\infty}^{\infty}e^{-i\omega t_n}\int_{-\infty}^{\infty}e^{-i\omega\tau_n}f(\tau_n)\,d\tau_n\\ &= \sum_{n=-\infty}^{\infty}e^{-i\omega t_n}\overline{f}(\omega)\,,\end{aligned}\tag{1.59}$$

which is (1.55), the ordinary product of the transforms.

2 Laplace Transforms

2.1 Introduction

For a function $f(t)$ uniquely defined on $0 \le t \le \infty$, its Laplace transform (LT) is defined as

$$\mathcal{L}\left[f(t)\right] = \overline{f}(s) = \int_0^\infty e^{-st} f(t)\, dt\,, \tag{2.1}$$

where s may be complex. The LT may not exist if $f(t)$ becomes singular in $[0\,\infty]$. The LT is a **one-sided transform** in that it operates on $[0\,\infty]$ and not, like the FT, on $[-\infty\,\infty]$. For this reason, LTs are useful for initial value problems, such as circuit theory, where a function switches on at $t = 0$ and where $f(0)$ has been specified.

Because s is a complex variable the inverse transform

$$f(t) = \mathcal{L}^{-1}\left[\overline{f}(s)\right] = \oint_C e^{st}\, \overline{f}(s)\, ds \tag{2.2}$$

is more difficult to handle because the contour C is a tricky infinite rectangle in the right-hand-half of the s-plane. Referred to as 'Bromwich integrals' the evaluation of these is beyond our present course. To circumvent this difficulty we resort firstly to a **library of transforms** (see Handout 7) for the standard functions and secondly to ways of piecing combinations of these together for those not in the list.

2.2 Library of Laplace Transforms

1. **The constant function** $f(t) = 1$ **:**

$$\boxed{f(t) = 1\,; \qquad \overline{f}(s) = \frac{1}{s} \qquad \text{Re}\, s > 0} \tag{2.3}$$

Proof :

$$\overline{f}(s) = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}\,, \tag{2.4}$$

provided Re $s > 0$.

2. **The exponential-function** $f(t) = e^{at}$ **:**

$$\boxed{f(t) = \exp(at)\,; \qquad \overline{f}(s) = \frac{1}{s-a}\,; \qquad \text{Re}\, s > a} \tag{2.5}$$

Proof :

$$\overline{f}(s) = \int_0^\infty e^{-(s-a)t}\, dt = \left[\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}\,, \tag{2.6}$$

provided Re$(s-a) > 0$.

3. **The sine function:**

$$\boxed{f(t) = \sin(at)\,; \qquad \overline{f}(s) = \frac{a}{s^2+a^2}\,; \qquad \text{Re}\, s > 0} \tag{2.7}$$

Proof: Take both the sine and cosine functions in combination: $\cos at + i \sin at = e^{iat}$

$$\mathcal{L}\big(e^{iat}\big) = \int_0^\infty e^{-(s-ia)t}dt = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} \tag{2.8}$$

provided Re$(s) > 0$. Then the imaginary (real) part gives the result for sine (cosine).

4. **The cosine function**

$$\boxed{f(t) = \cos(at)\,; \qquad \overline{f}(s) = \frac{s}{s^2+a^2}\,; \qquad \text{Re}\, s > 0} \tag{2.9}$$

5. **The polynomial function** $f(t) = t^n$ **:**

$$\boxed{f(t) = t^n; \quad \overline{f}(s) = \frac{n!}{s^{n+1}}\,; \qquad (n \geq 0)\,; \qquad \text{Re}\, s > 0} \tag{2.10}$$

Proof: Define the LT as $\overline{f}(s) = I_n$ as

$$\begin{aligned} I_n &= \int_0^\infty e^{-st}t^n\,dt = -\frac{1}{s}\int_0^\infty t^n\,d\big[e^{-st}\big] \\ &= \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,dt = \frac{n}{s}I_{n-1} \end{aligned} \tag{2.11}$$

provided Re$(s) > 0$. With $n = 0$ and $\mathcal{L}[1] = s|^{-1}$ we obtain $I_1 = s^{-2}$ and end up with

$$\overline{f}(s) = I_n = \frac{n!}{s^{n+1}}\,. \tag{2.12}$$

6. **The Heaviside function:**

$$\boxed{f(t) = H(t-t_0)\,; \qquad \overline{f}(s) = \frac{\exp(-st_0)}{s}\,; \qquad \text{Re}\, s > 0} \tag{2.13}$$

Proof: For Re $s > 0$

$$\begin{aligned} \mathcal{L}\big[H(t-t_0)\big] &= \int_0^\infty e^{-st}H(t-t_0)\,dt \\ &= \int_{t_0}^\infty e^{-st}dt = \frac{e^{-st_0}}{s}\,. \end{aligned} \tag{2.14}$$

7. **The Dirac δ-function:**

$$\boxed{f(t) = \delta(t-t_0)\,; \qquad \overline{f}(s) = \exp(-st_0)\,; \qquad t_0 \geq 0} \tag{2.15}$$

Proof: t_0 needs to reside within the positive range of t

$$\int_0^\infty e^{-st}\delta(t-t_0)\,dt = \left\{ \begin{array}{cc} e^{-st_0} & t_0 \geq 0\,, \\ 0 & t_0 < 0\,. \end{array} \right. \tag{2.16}$$

8. **Shift theorem:**

$$\boxed{\mathcal{L}\,[\exp(at)f(t)] = \overline{f}(s-a)} \tag{2.17}$$

Proof: Provided Re$(s-a) > 0$

$$\begin{aligned}\mathcal{L}\,[\exp(at)f(t)] &= \int_0^\infty e^{-(s-a)t} f(t)\,dt \\ &= \overline{f}(s-a)\,. \end{aligned} \tag{2.18}$$

9. **Second shift theorem:**

$$\boxed{\mathcal{L}\,[H(t-a)f(t-a)] = \exp(-sa)\,\overline{f}(s)} \tag{2.19}$$

Proof: let $\tau = t-a$. Then

$$\begin{aligned}\mathcal{L}\,[H(t-a)f(t-a)] &= \int_0^\infty e^{-st} H(t-a)f(t-a)\,dt \\ &= e^{-sa}\int_{-a}^\infty e^{-s\tau} H(\tau)f(\tau)\,d\tau \\ &= e^{-sa}\int_0^\infty e^{-s\tau} f(\tau)\,d\tau = e^{-sa}\overline{f}(s)\,. \end{aligned} \tag{2.20}$$

10. **Convolution theorem:**

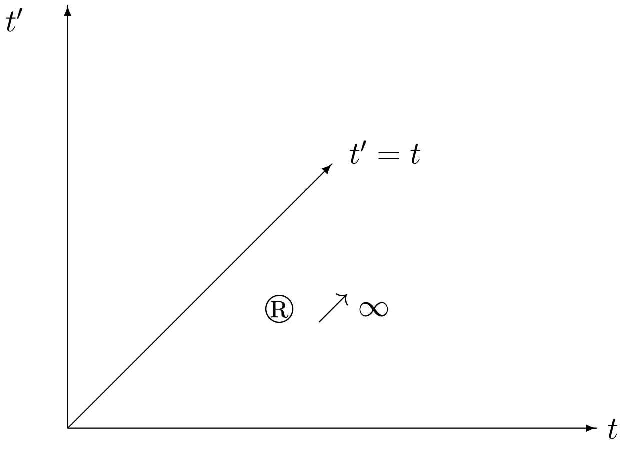
$$\boxed{\mathcal{L}\,\{f\star g\} = \overline{f}(s)\,\overline{g}(s)} \tag{2.21}$$

where the convolution between two functions $f(t)$ and $g(t)$ is defined as

$$f\star g = \int_0^t f(t')g(t-t')\,dt'\,. \tag{2.22}$$

Note that the convolution is over $[0,\,t]$ and not $[-\infty,\,\infty]$ as for the FT. he convolution integral on the RHS can also be written with f and g reversed: that is $\int_0^t f(t-t')g(t')\,dt'$.

Proof: The LT of the convolution product in (2.22) is written down and then the order of the integrals is exchanged, as in the figure, using $\tau = t - t'$



The region of integration Ⓡ can be read from the figure: the t'-integration is taken in the vertical direction to cover Ⓡ but to cover this in the reverse order, the t-integration is taken in the horizontal direction.

$$\begin{aligned}
\mathcal{L}\big(f\star g\big) &= \int_0^\infty e^{-st}\left(\int_0^t f(t')g(t-t')\,dt'\right)dt \\
&= \int_0^\infty \left(\int_{t=t'}^{t=\infty} e^{-st}g(t-t')\,dt\right) f(t')\,dt' \\
&= \int_0^\infty e^{-st'}\left(\int_{\tau=0}^{\tau=\infty} e^{-s\tau}g(\tau)\,d\tau\right) f(t')\,dt' \\
&= \overline{f}(s)\,\overline{g}(s)\,. \qquad (2.23)
\end{aligned}$$

11. **Integral:**

$$\boxed{\mathcal{L}\left(\int_0^t f(t')\,dt'\right) = \frac{\overline{f}(s)}{s}} \qquad (2.24)$$

Proof: The integral in (2.24) is a convolution product between $f(t)$ and $g(t) = 1$. Thus $\overline{g}(s) = 1/s$, giving the result from (2.23).

12. **Derivative:**

$$\boxed{\mathcal{L}\,[\dot{f}(t)] = s\overline{f}(s) - f(0)} \qquad (2.25)$$

Proof: Noting that $f(0)$ means $f(t=0)$

$$\begin{aligned}
\mathcal{L}[\dot{f}] &= \int_0^\infty e^{-st}\dot{f}\,dt \\
&= \int_0^\infty e^{-st}\,df = \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f\,dt \\
&= s\overline{f}(s) - f(0) \qquad (2.26)
\end{aligned}$$

provided Re $s > 0$.

13. **Second derivative:** Noting that $\dot{f}(0)$ means $\dot{f}(t=0)$

$$\boxed{\mathcal{L}\,[\ddot{f}(t)] = s^2\overline{f}(s) - sf(0) - \dot{f}(0)} \qquad (2.27)$$

Proof:

$$\begin{aligned}
\mathcal{L}\,[\ddot{f}] &= \int_0^\infty e^{-st}\ddot{f}\,dt \\
&= \int_0^\infty e^{-st}\,d\dot{f} = \left[e^{-st}\dot{f}(t)\right]_0^\infty + s\int_0^\infty e^{-st}\dot{f}\,dt \\
&= s\mathcal{L}[\dot{f}] - \dot{f}(0) \\
&= s^2\overline{f}(s) - sf(0) - \dot{f}(0)\,, \qquad (2.28)
\end{aligned}$$

provided Re $s > 0$.

2.3 Using the Convolution Theorem to find inverses

If we are given an inverse LT as a function $\overline{F}(s)$ which is too complicated to appear in the Library above but can be split into composite functions $\overline{F}(s) = \overline{f}(s)\,\overline{g}(s)$ where $\overline{f}(s)$ and $\overline{g}(s)$ do belong to the Library, then the Convolution Theorem allows us to write

$$F(t) = \mathcal{L}^{-1}\left(\overline{f}(s)\,\overline{g}(s)\right) = f(t)\star g(t)\,. \tag{2.29}$$

Example 1: Find $\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right]$. We identify $f(s) = s^{-1}$ and $g(s) = (s^2+1)^{-1}$. The Library tell us that $f(t) = 1$ and $g(t) = \sin t$. Thus

$$F(t) = 1\star\sin t = \int_0^t \sin t'\,dt' = 1-\cos t\,. \tag{2.30}$$

Example 2: Find $\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$. Identify

$$\overline{f}(s) = \frac{s}{s^2+a^2} \qquad\qquad \overline{g}(s) = \frac{1}{s^2+a^2}\,. \tag{2.31}$$

The Library tell us that $f(t) = \cos at$ and $g(t) = a^{-1}\sin at$, and so

$$F(t) = a^{-1}\sin at\star\cos at = a^{-1}\int_0^t \sin(at')\cos a(t-t')\,dt'\,. \tag{2.32}$$

Using $\sin(A+B)+\sin(A-B) = 2\sin A\cos B$ we find

$$\sin(at')\cos a(t-t') = \tfrac{1}{2}\left[\sin at + \sin a(2t'-t)\right] \tag{2.33}$$

and so from (2.32)

$$\begin{aligned} F(t) &= \frac{1}{2a}\int_0^t \left[\sin(at)+\sin a(2t'-t)\right]\,dt' \\ &= \frac{1}{2a}\left[t\sin at - \frac{1}{2a}\big\{\cos at-\cos at\big\}\right] \\ &= \frac{t}{2a}\sin at\,. \end{aligned} \tag{2.34}$$

Example 3: Find $\mathcal{L}^{-1}\left[\frac{a^2}{(s^2+a^2)^2}\right]$. Identify $F(s) = |\overline{f}(s)|^2$ where

$$\overline{f}(s) = \frac{a}{s^2+a^2} \qquad\qquad \overline{g}(s) = \overline{f}(s)\,. \tag{2.35}$$

The Library tell us that $f(t) = g(t) = \sin at$ so $\sin at$ is convolved with itself

$$\begin{aligned} F(t) &= \sin at\star\sin at \\ &= \int_0^t \sin at'\sin a(t-t')\,dt' \\ &= \frac{1}{2a}\left[\sin at - at\cos at\right]\,. \end{aligned} \tag{2.36}$$

having used the trig-identity $\cos(A-B)-\cos(A+B) = 2\sin A\sin B$.

2.4 Examples involving partial fractions and the Shift theorem

Example 1: Find $f(t)$ when

$$\overline{f}(s) = \frac{6s^2 + 10s + 2}{s(s^2 + 3s + 2)}\,. \tag{2.37}$$

Noting that $s^2 + 3s + 2 = (s + 1)(s + 2)$ (2.37) can be split by Partial Fractions (PFs) into

$$\overline{f}(s) = \frac{6s^2 + 10s + 2}{s(s + 1)(s + 2)} = \frac{1}{s} + \frac{2}{s + 1} + \frac{3}{s + 2}\,. \tag{2.38}$$

Thus, using the Library

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{2}{s + 1} + \frac{3}{s + 2}\right) \\ &= 1 + 2e^{-t} + 3e^{-2t}\,. \end{aligned} \tag{2.39}$$

Example 2: Find $f(t)$ when

$$\overline{f}(s) = \frac{2}{s(s - 2)}\,. \tag{2.40}$$

in which case

$$\overline{f}(s) = -\frac{1}{s} + \frac{1}{s - 2}\,, \tag{2.41}$$

and so

$$f(t) = -1 + e^{2t}\,. \tag{2.42}$$

Example 3: Find $f(t)$ when $\overline{f}(s) = (s - 1)^{-4}$. From the Library,

$$\mathcal{L}\left[t^3\right] = \frac{3!}{s^4} \tag{2.43}$$

therefore $\mathcal{L}^{-1}\left[s^{-4}\right] = t^3/6$. With the application of the Shift Theorem with $a = 1$ we have

$$\mathcal{L}^{-1}\left[(s - 1)^{-4}\right] = \frac{1}{6}t^3e^t\,. \tag{2.44}$$

2.5 Solving ODEs using Laplace Transforms

Many textbook methods are given to solve 2nd order ODEs of the type

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x = f(t)\,, \tag{2.45}$$

but only the LT-method can handle those cases when the forcing function is not smooth. Examples might be voltage inputs of the square wave or saw-tooth type. To approach this using LTs, the transform is taken of (2.45)

$$\left(s^2\overline{x}(s) - sx_0 - \dot{x}_0\right) + \alpha\left(s\overline{x}(s) - x_0\right) + \omega_0^2\,\overline{x}(s) = \overline{f}(s)\,. \tag{2.46}$$

where $x_0 = x(0)$ and $\dot{x}_0 = \dot{x}(0)$. This re-organizes into

$$\left(s^2 + \alpha s + \omega_0^2\right)\overline{x}(s) = \overline{f}(s) + (s+\alpha)x_0 + \dot{x}_0\,. \tag{2.47}$$

Note that the final expression for $\overline{x}(s)$ divides conveniently into two parts corresponding to the Complementary Function and the Particular Integral

$$\overline{x}(s) = \underbrace{\frac{\overline{f}(s)}{s^2+\alpha s+\omega_0^2}}_{P.I.} + \underbrace{\frac{(s+\alpha)x_0+\dot{x}_0}{s^2+\alpha s+\omega_0^2}}_{C.F.} \tag{2.48}$$

The initial conditions appear in x_0 and $\dot{x}_0$ as part of the Complementary Function. How to take the inverse depends on whether the denominator has real or complex roots. These we consider by example.

Example 1: Solve $\ddot{x} + \dot{x} - 2x = e^t$ with $x_0 = 3$ and $\dot{x}_0 = 0$.
(2.47) becomes

$$\left(s^2+s-2\right)\overline{x}(s) = \frac{1}{s-1} + 3(s+1)\,. \tag{2.49}$$

Noting that $s^2+s-2 = (s-1)(s+2)$ we have

$$\overline{x}(s) = \underbrace{\frac{1}{(s-1)^2(s+2)}}_{P.I.} + \underbrace{\frac{3(s+1)}{(s-1)(s+2)}}_{C.F.} \tag{2.50}$$

Using PFs

$$\overline{x}(s) = \frac{1}{3(s-1)^2} + \frac{17}{9(s-1)} + \frac{10}{9(s+2)} \tag{2.51}$$

and so the Library gives us

$$x(t) = \frac{1}{3}te^t + \frac{17}{9}e^t + \frac{10}{9}e^{-2t}\,. \tag{2.52}$$

Example 2: Solve $\ddot{x} + 16x = \sin 2t$ with $x_0 = 0$ and $\dot{x}_0 = 1$.
(2.47) becomes

$$\left(s^2+16\right)\overline{x}(s) = 1 + \frac{2}{s^2+4}\,, \tag{2.53}$$

and so

$$\begin{aligned} \overline{x}(s) &= \frac{1}{s^2+16} + \frac{2}{(s^2+4)(s^2+16)} \\ &= \frac{5}{6(s^2+16)} + \frac{1}{6(s^2+4)} \\ &= \frac{5}{24}\left(\frac{4}{s^2+4^2}\right) + \frac{1}{12}\left(\frac{2}{s^2+2^2}\right)\,. \end{aligned} \tag{2.54}$$

Therefore, from the Library

$$x(t) = \frac{5}{24}\sin 4t + \frac{1}{12}\sin 2t\,. \tag{2.55}$$

Example 3 (real roots): Solve $\ddot{x} + 3\dot{x} + 2x = f(t)$ with $x_0 = 1$ and $\dot{x}_0 = -2$. In this example $f(t)$ has not been specified although it must be assumed that its LT exists.

We obtain

$$\overline{x}(s) = \frac{\overline{f}(s)}{s^2+3s+2} + \frac{x_0(s+3)+\dot{x}_0}{s^2+3s+2} \tag{2.56}$$

so from (2.48) with the specified initial conditions

$$\overline{x}(s) \;=\; \frac{\overline{f}(s)}{(s+1)(s+2)} + \frac{1}{s+2}\,. \tag{2.57}$$

Using PFs we find

$$\begin{aligned}\overline{x}(s) &= \frac{\overline{f}(s)}{s+1} - \frac{\overline{f}(s)}{s+2} + \frac{1}{s+2}\\ &\equiv \overline{f}(s)\overline{g}_1(s) - \overline{f}(s)\overline{g}_2(s) + \overline{g}_2(s)\end{aligned} \tag{2.58}$$

where $\overline{g}_1(s) = (s+1)^{-1}$ and $\overline{g}_2(s) = (s+2)^{-1}$. From these definitions it is clear that $g_1(t) = e^{-t}$ and $g_2(t) = e^{-2t}$. From the Convolution Theorem we have

$$x(t) = \underbrace{\int_0^t \left[e^{-(t-t')} - e^{-2(t-t')}\right] f(t')\,dt'}_{P.I.} + \underbrace{e^{-2t}}_{C.F.}\,. \tag{2.59}$$

The power of the LT-method can be seen here in that it solves, in principle, an ODE with any forcing, provided $\overline{f}(s)$ exists.

Example 4 (complex roots): Solve $\ddot{x} + 2\dot{x} + 2x = f(t)$ with $x_0 = 1$ and $\dot{x}_0 = 0$. In this example $f(t)$ has not been specified although it must be assumed that its LT exists.

From (2.48) the next step comes out to be

$$\overline{x}(s) \;=\; \frac{\overline{f}(s)}{(s+1)^2+1} + \frac{s+2}{(s+1)^2+1} \tag{2.60}$$

where it has been noted that $s^2 + 2s + 2$ does not have real roots. Now define

$$\overline{g}_1(s) = \frac{1}{(s+1)^2+1} \qquad\qquad \overline{g}_2(s) = \frac{s+1}{(s+1)^2+1}\,. \tag{2.61}$$

Therefore $\overline{x}(s)$ can be re-expressed as

$$\overline{x}(s) = \overline{f}(s)\overline{g}_1(s) + \overline{g}_2(s) + \overline{g}_1(s)\,. \tag{2.62}$$

Inverse transforms can be found from the Shift Theorem and the Library

$$g_1(t) = e^{-t}\sin t \qquad\qquad g_2(t) = e^{-t}\cos t\,. \tag{2.63}$$

The Convolution Theorem gives the final result

$$x(t) = \int_0^t f(t-t')e^{-t'}\sin t'\,dt' + e^{-t}\left[\cos t + \sin t\right]\,. \tag{2.64}$$

A Appendix : Three definitions of the Fourier Transform

1. **Definition 1:** (used in these notes)

$$\overline{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt\,, \tag{A.1}$$

with the inverse Fourier transform written as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega)e^{i\omega t}\,d\omega\,. \tag{A.2}$$

2. **Definition 2:** This definition is sometimes used in signal processing :

$$\overline{f}(s) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ist}\,dt\,, \tag{A.3}$$

with the inverse Fourier transform written as

$$f(t) = \int_{-\infty}^{\infty} \overline{f}(s)e^{2\pi ist}\,ds\,. \tag{A.4}$$

Hence s acts like the frequency with $\omega = 2\pi s$.

3. **Definition 3:** This next definition is used more in mathematical physics because of the symmetry in the coefficients of both the transform and its inverse :

$$\overline{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt\,, \tag{A.5}$$

with the inverse Fourier transform written as

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f}(\omega)e^{i\omega t}\,d\omega\,. \tag{A.6}$$

B Appendix : The Nyquist-Shannon Sampling Theorem

Not examinable : this result belongs to your own Signals Processing course. Let $\omega_s = 2\pi/T$ be the sampling rate of a band-width-limited signal which is centred around zero with bandwidth $[-\omega_{max},\, \omega_{max}]$.

Theorem 3 *When sampling a signal, the sampling frequency must be greater than twice the bandwidth in order to reconstruct the signal perfectly from the sampled version.*

Proof : Let $f(t)$ be a continuous signal and let $\mathit{III}(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT)$. Then we consider

$$f^{(s)}(t) = f(t)\mathit{III}(t) \tag{B.1}$$

with $\omega_s = 2\pi/T$. Then

$$\begin{aligned}
\overline{f^{(s)}}(\omega) &= \mathcal{F}\left[f(t)\sum_{n=-\infty}^{n=\infty}\delta(t-nT)\right] \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{n=\infty}\overline{f}(\omega)\star e^{-i\omega nT} \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{n=\infty}\int_{-\infty}^{\infty}\overline{f}(\omega')e^{-i(\omega'-\omega)nT}\,d\omega' \\
&= \sum_{n=-\infty}^{n=\infty} f(nT)\,e^{i\omega nT} \\
&= \sum_{n=-\infty}^{n=\infty}\overline{f}(\omega-n\omega_s) \qquad \text{(B.2)}
\end{aligned}$$

where the Poisson summation formula has been used in the last step. The signal bandwidth is $2\omega_{max}$ so in order for a replicated $\overline{f}(\omega)$, shifted by ω_s, not to overlap then the condition $\omega_s > 2\omega_{max}$ must hold. If ω_s is not large enough then overlap occurs with aliasing. ☐