EE2 Mathematics : Vector Calculus

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These notes are not identical word-for-word with my lectures which will be given on a BB/WB. Some of these notes may contain more examples than the corresponding lecture while in other cases the lecture may contain more detailed working. I will **NOT** be handing out copies of these notes – **you are therefore advised to attend lectures and take your own**.

1. The material in them is dependent upon the Vector Algebra you were taught at A-level and your 1st year. A summary of what you need to revise lies in **Handout 1 : "Things you need to recall about Vector Algebra"** which is also §1 of this document.

2. Further handouts are :

   (a) **Handout 2 :** "The role of grad, div and curl in vector calculus" summarizes most of the material in §3.

   (b) **Handout 3 :** "Changing the order in double integration" is incorporated in §5.5.

   (c) **Handout 4 :** "Green's, Divergence & Stokes' Theorems plus Maxwell's Equations" summarizes the material in §6, §7 and §8.

   Technically, while Maxwell's Equations themselves are not in the syllabus, three of the four of them arise naturally out of the Divergence & Stokes' Theorems and they connect all the subsequent material with that given from lectures on e/m theory given in your own Department.

[1] Do not confuse me with Dr J. Gibbons who is also in the Mathematics Dept.

Contents

1 Revision: Things you need to recall about Vector Algebra

Notation: $\boldsymbol{a} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}} \equiv (a_1, a_2, a_3)$.

1. The *magnitude or length of a vector* $\boldsymbol{a}$ is

$$|\boldsymbol{a}| = a = \left(a_1^2 + a_2^2 + a_3^2\right)^{1/2}. \tag{1.1}$$

2. The *scalar (dot) product* of two vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ & $\boldsymbol{b} = (b_1, b_2, b_3)$ is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \,. \tag{1.2}$$

Since $\boldsymbol{a} \cdot \boldsymbol{b} = ab\cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, assuming neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ are null.

3. The *vector (cross) product*[2] between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{1.3}$$

Recall that $\boldsymbol{a} \times \boldsymbol{b}$ can also be expressed as

$$\boldsymbol{a} \times \boldsymbol{b} = (ab\sin\theta)\,\hat{\boldsymbol{n}} \tag{1.4}$$

where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ in a direction determined by the right hand rule. If $\boldsymbol{a} \times \boldsymbol{b} = 0$ then $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if neither vector is null.

4. The *scalar triple product* between three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \begin{matrix} & \boldsymbol{a} & \\ \nearrow & & \searrow \\ \boldsymbol{c} & \longleftarrow & \boldsymbol{b} \end{matrix} \qquad \text{Cyclic Rule: clockwise +ve} \tag{1.5}$$

According to the cyclic rule

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \tag{1.6}$$

One consequence is that if **any two of the three vectors are equal (or parallel)** then their scalar product is zero: e.g. $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{a}) = \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{a}) = 0$. If $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = 0$ and no pair of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are parallel then the three vectors must be coplanar.

5. The *vector triple product* between three vectors is

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}) \,. \tag{1.7}$$

The placement of the brackets on the LHS is important: the RHS is a vector that lies in the same plane as $\boldsymbol{b}$ and $\boldsymbol{c}$ whereas $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{b}(\boldsymbol{c} \cdot \boldsymbol{a}) - \boldsymbol{a}(\boldsymbol{c} \cdot \boldsymbol{b})$ lies in the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus, $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c}$ without brackets is a meaningless statement!

[2] It is acceptable to use the notation $\boldsymbol{a} \wedge \boldsymbol{b}$ as an alternative to $\boldsymbol{a} \times \boldsymbol{b}$.

2 Scalar and Vector Fields

(L1) Our first aim is to step up from single variable calculus – that is, dealing with functions of one variable – to functions of two, three or even four variables. The physics of electro-magnetic (e/m) fields requires us to deal with the three co-ordinates of space $(x,\, y,\, z)$ and also time t. There are two different types of functions of the four variables :

1. A **scalar field** [3] is written as
$$\psi = \psi(x, y, z, t)\,. \tag{2.1}$$
Note that one cannot plot ψ as a graph in the conventional sense as ψ takes values at *every point in space and time*. A good example of a scalar field is the temperature of the air in a room. If the box-shape of a room is thought of as a co-ordinate system with the origin in one corner, then every point in that room can be labelled by a co-ordinate (x, y, z). If the room is poorly air-conditioned the temperature in different parts may vary widely : moving a thermometer around will measure the variation in temperature from point to point (spatially) and also in time (temporally). Another example of a scalar field is the concentration of salt or a dye dissolved in a fluid.
2. A **vector field** $\boldsymbol{B}(x, y, z, t)$ must have components $(B_1,\, B_2,\, B_3)$ in terms of the three unit vectors $(\hat{\boldsymbol{i}},\, \hat{\boldsymbol{j}},\, \hat{\boldsymbol{k}})$
$$\boldsymbol{B} = \hat{\boldsymbol{i}}B_1(x, y, z, t) + \hat{\boldsymbol{j}}B_2(x, y, z, t) + \hat{\boldsymbol{k}}B_3(x, y, z, t)\,. \tag{2.2}$$
These components can each be functions of (x, y, z, t). Three physical examples of vector fields are :
   (a) An **electric field :**
$$\boldsymbol{E}(x, y, z, t) = \hat{\boldsymbol{i}}E_1(x, y, z, t) + \hat{\boldsymbol{j}}E_2(x, y, z, t) + \hat{\boldsymbol{k}}E_3(x, y, z, t)\,, \tag{2.3}$$
   (b) A **magnetic field :**
$$\boldsymbol{H}(x, y, z, t) = \hat{\boldsymbol{i}}H_1(x, y, z, t) + \hat{\boldsymbol{j}}H_2(x, y, z, t) + \hat{\boldsymbol{k}}H_3(x, y, z, t)\,, \tag{2.4}$$
   (c) The **velocity field** $\boldsymbol{u}(x, y, z, t)$ in a fluid.

   A classic illustration of a three-dimensional vector field in action is the e/m signal received by a mobile phone which can be received anywhere in space.

3 The vector operators : grad, div and curl

3.1 Definition of the gradient operator ∇

The gradient operator (grad) is denoted by the symbol ∇ and is defined as
$$\nabla = \hat{\boldsymbol{i}}\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}\,. \tag{3.1}$$

[3] The convention is to use Greek letters for scalar fields and bold Roman for vector fields.

As such it is a vector form of partial differentiation because it has spatial partial derivatives in each of the three directions. On its right, ∇ can operate on a scalar field $\psi(x, y, z)$

$$\nabla\psi = \hat{\boldsymbol{i}}\frac{\partial\psi}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial\psi}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial\psi}{\partial z}\,. \tag{3.2}$$

Note that while ψ is a scalar field, $\nabla\psi$ itself is a vector.

- **Example 1) :** With $\psi = \frac{1}{3}\left(x^3 + y^3 + z^3\right)$

$$\nabla\psi = \hat{\boldsymbol{i}}x^2 + \hat{\boldsymbol{j}}y^2 + \hat{\boldsymbol{k}}z^2. \tag{3.3}$$

  As explained above, the RHS is a vector whereas ψ is a scalar.

- **Example 2) :** With $\psi = xyz$ the vector $\nabla\psi$ is

$$\nabla\psi = \hat{\boldsymbol{i}}yz + \hat{\boldsymbol{j}}xz + \hat{\boldsymbol{k}}xy\,. \tag{3.4}$$

End of L1

3.2 Definition of the divergence of a vector field div $\boldsymbol{B}$

L2 Because vector algebra allows two forms of multiplication (the scalar and vector products) there are two ways of operating ∇ on a vector

$$\boldsymbol{B} = \hat{\boldsymbol{i}}B_1(x, y, z, t) + \hat{\boldsymbol{j}}B_2(x, y, z, t) + \hat{\boldsymbol{k}}B_3(x, y, z, t)\,. \tag{3.5}$$

The first is through the scalar or dot product

$$\text{div}\,\boldsymbol{B} \;=\; \nabla\cdot\boldsymbol{B} = \left(\hat{\boldsymbol{i}}\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\boldsymbol{i}}B_1 + \hat{\boldsymbol{j}}B_2 + \hat{\boldsymbol{k}}B_3\right). \tag{3.6}$$

Recalling that $\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{i}} = \hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{j}} = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}} = 1$ but $\hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{j}} = \hat{\boldsymbol{i}}\cdot\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{j}} = 0$, the result is

$$\text{div}\,\boldsymbol{B} = \nabla\cdot\boldsymbol{B} = \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\,. \tag{3.7}$$

Note that div $\boldsymbol{B}$ is a scalar because div is formed through the dot product. The best physical explanation that can be given is that div $\boldsymbol{B}$ is a measure of the compression or expansion of a vector field through the 3 faces of a cube.

- If div $\boldsymbol{B} = 0$ the the vector field $\boldsymbol{B}$ is incompressible ;
- If div $\boldsymbol{B} > 0$ the the vector field $\boldsymbol{B}$ is expanding ;
- If div $\boldsymbol{B} < 0$ the the vector field $\boldsymbol{B}$ is compressing .

Example 1 : Let $\boldsymbol{r}$ be the straight line vector $\boldsymbol{r} = \hat{\boldsymbol{i}}x + \hat{\boldsymbol{j}}y + \hat{\boldsymbol{k}}z$ then

$$\text{div}\,\boldsymbol{r} = 1 + 1 + 1 = 3 \tag{3.8}$$

Example 2 : Let $\boldsymbol{B} = \hat{\boldsymbol{i}}x^2 + \hat{\boldsymbol{j}}y^2 + \hat{\boldsymbol{k}}z^2$ then

$$\text{div}\,\boldsymbol{B} = 2x + 2y + 2z \tag{3.9}$$

Note : the usual rule in vector algebra that $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$ (that is, $\boldsymbol{a}$ and $\boldsymbol{b}$ commute) doesn't hold when one of them is an operator. Thus

$$\boldsymbol{B}\cdot\nabla = B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_3\frac{\partial}{\partial z} \neq \nabla\cdot\boldsymbol{B} \tag{3.10}$$

3.3 Definition of the curl of a vector field curl $\boldsymbol{B}$

The alternative in vector multiplication is to use ∇ in a cross product with a vector $\boldsymbol{B}$:

$$\text{curl}\,\boldsymbol{B} = \nabla\times\boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ B_1 & B_2 & B_3 \end{vmatrix}\,. \tag{3.11}$$

The best physical explanation that can be given is to visualize in colour the intensity of $\boldsymbol{B}$ then curl $\boldsymbol{B}$ is a measure of the curvature in the field lines of $\boldsymbol{B}$.

Example 1) : Take the vector denoting a straight line from the origin to a point (x, y, z) denoted by $\boldsymbol{r} = \hat{\boldsymbol{i}}x + \hat{\boldsymbol{j}}y + \hat{\boldsymbol{k}}z$. Then

$$\text{curl}\,\boldsymbol{r} = \nabla\times\boldsymbol{r} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = 0\,. \tag{3.12}$$

Example 2) : Choose $\boldsymbol{B} = \frac{1}{2}\left(\hat{\boldsymbol{i}}x^2 + \hat{\boldsymbol{j}}y^2 + \hat{\boldsymbol{k}}z^2\right)$ then

$$\text{curl}\,\boldsymbol{B} = \nabla\times\boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ \frac{1}{2}x^2 & \frac{1}{2}y^2 & \frac{1}{2}z^2 \end{vmatrix} = 0\,. \tag{3.13}$$

Example 3) : Take the vector denoted by $\boldsymbol{B} = \hat{\boldsymbol{i}}y^2z^2 + \hat{\boldsymbol{j}}x^2z^2 + \hat{\boldsymbol{k}}x^2y^2$. Then

$$\text{curl}\,\boldsymbol{B} = \nabla\times\boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ y^2z^2 & x^2z^2 & x^2y^2 \end{vmatrix} = 2x^2\hat{\boldsymbol{i}}(y-z) - 2y^2\hat{\boldsymbol{j}}(x-z) + 2z^2\hat{\boldsymbol{k}}(x-y)\,. \tag{3.14}$$

Example 4) : For the curl of a two-dimensional vector $\boldsymbol{B} = \hat{\boldsymbol{\imath}} B_1(x,y) + \hat{\boldsymbol{\jmath}} B_2(x,y)$ we have

$$\text{curl}\, \boldsymbol{B} = \nabla \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ B_1(x,y) & B_2(x,y) & 0 \end{vmatrix} = \hat{\boldsymbol{k}} \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} \right), \tag{3.15}$$

which points in the vertical direction only because there are no $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components. End of L2

3.4 Five vector identities

L3 There are 5 useful vector identities (see hand out No 2). The proofs of 1), 4) and 5) are obvious : for No 1) use the product rule. 2) and 3) can be proved with a little effort.

1. The gradient of the product of two scalars ψ and ϕ

$$\nabla(\phi\psi) = \psi \nabla \phi + \phi \nabla \psi \,. \tag{3.16}$$

2. The divergence of the product of a scalar ψ with a vector $\boldsymbol{b}$

$$\text{div}\,(\psi \boldsymbol{B}) = \psi\, \text{div}\, \boldsymbol{B} + (\nabla \psi) \cdot \boldsymbol{B} \,. \tag{3.17}$$

3. The curl of the product of a scalar ψ with a vector $\boldsymbol{B}$

$$\text{curl}\,(\psi \boldsymbol{B}) = \psi\, \text{curl}\, \boldsymbol{B} + (\nabla \psi) \times \boldsymbol{B} \,. \tag{3.18}$$

4. The curl of the gradient of any scalar ψ

$$\text{curl}\,(\nabla \psi) = \nabla \times \nabla \psi = 0 \,, \tag{3.19}$$

   because the cross product $\nabla \times \nabla$ is zero.

5. The divergence of the curl of any vector $\boldsymbol{B}$

$$\text{div}\,(\text{curl}\, \boldsymbol{B}) = \nabla \cdot (\nabla \times \boldsymbol{B}) = 0 \,. \tag{3.20}$$

   The cyclic rule for the scalar triple product in (3.20) shows that this is zero for all vectors $\boldsymbol{B}$ because two vectors (∇) in the triple are the same.

3.5 Irrotational and solenoidal vector fields

Consider identities 4) & 5) above (see also Handout 2 "The role of grad, div & curl ...")

$$\text{curl}\,(\nabla \phi) = \nabla \times \nabla \phi = 0 \,, \tag{3.21}$$

$$\text{div}\,(\text{curl}\, \boldsymbol{B}) = 0 \,. \tag{3.22}$$

(3.21) says that if any vector $\boldsymbol{B}(x,y,z)$ can be written as the gradient of a scalar $\phi(x,y,z)$ (which can't always be done)

$$\boldsymbol{B} = \nabla \phi \tag{3.23}$$

then automatically curl $B = 0$. Such vector fields are called irrotational vector fields. It is equally true that for a given field $\boldsymbol{B}$, then if it is found that curl $\boldsymbol{B} = 0$, then we can write [4]

$$\boldsymbol{B} = \pm\nabla\phi \tag{3.24}$$

ϕ is called the "scalar potential". Note that not every vector field has a corresponding scalar potential, but only those that are curl-free.

Likewise we now turn to (3.22): vector fields $\boldsymbol{B}$ for which div $\boldsymbol{B} = 0$ are called **solenoidal**, in which case $\boldsymbol{B}$ can be written as

$$\boldsymbol{B} = \text{curl}\,\boldsymbol{A} \tag{3.25}$$

where the vector A is called a "vector potential". Note that only vectors those that are div-free have a corresponding vector potential [5].

Example: The Newtonian gravitational force between masses m and M (with gravitational constant G) is

$$\boldsymbol{F} = -GmM\frac{\boldsymbol{r}}{r^3}\,, \tag{3.26}$$

where $\boldsymbol{r} = \hat{\boldsymbol{i}}x + \hat{\boldsymbol{j}}y + \hat{\boldsymbol{k}}z$ and $r^2 = x^2 + y^2 + z^2$.

1. Let us first calculate curl $\boldsymbol{F}$

$$\text{curl}\,\boldsymbol{F} = -GmM\,\text{curl}\,(\psi\boldsymbol{r})\,, \qquad \text{where} \qquad \psi = r^{-3}\,. \tag{3.27}$$

The 3rd in the list of vector identities gives

$$\text{curl}\,(\psi\boldsymbol{r}) = \psi\,\text{curl}\,\boldsymbol{r} + (\nabla\psi)\times\boldsymbol{r} \tag{3.28}$$

and we already know that curl $\boldsymbol{r} = 0$. It remains to calculate $\nabla\psi$:

$$\nabla\psi = \nabla\left\{\left(x^2+y^2+z^2\right)^{-3/2}\right\} = -\frac{3(\hat{\boldsymbol{i}}x + \hat{\boldsymbol{j}}y + \hat{\boldsymbol{k}}z)}{(x^2+y^2+z^2)^{5/2}} = -\frac{3\boldsymbol{r}}{r^5}\,. \tag{3.29}$$

Thus, from (3.26),

$$\text{curl}\,\boldsymbol{F} = -GmM\left(0 - \frac{3\boldsymbol{r}}{2r^5}\times\boldsymbol{r}\right) = 0\,. \tag{3.30}$$

Thus the Newton gravitational force field is curl-free, which is why a gravitational potential exists. We can write [6]

$$\boldsymbol{F} = -\nabla\phi\,. \tag{3.31}$$

One can find ϕ by inspection: it turns out that $\phi = -GmM\,1/r$.

[4] Whether we use $+$ or $-$ in (3.24) depends on convention: in e/m theory normally uses a minus sign whereas fluid dynamics uses a plus sign.

[5] The last lecture on Maxwell's Equations stresses that all magnetic fields in the universe are div-free as no magnetic monopoles have yet been found: thus all magnetic fields are solendoidal vector fields.

[6] In this case a negative sign is adopted on the scalar potential.

2. Now let us calculate div $\boldsymbol{F}$

$$\text{div}\,\boldsymbol{F} = -GmM\,\text{div}\,(\psi\boldsymbol{r}) \qquad \text{where} \qquad \psi = r^{-3}\,. \tag{3.32}$$

The 2nd in the list of vector identities gives

$$\text{div}\,(\psi\boldsymbol{r}) = \psi\,\text{div}\,\boldsymbol{r} + (\nabla\psi)\cdot\boldsymbol{r} \tag{3.33}$$

and we already know that div $\boldsymbol{r} = 3$ and we have already calculated $\nabla\psi$ in (3.29).

$$\text{div}\,\boldsymbol{F} = -GmM\left(\frac{3}{r^3} - \frac{3\boldsymbol{r}}{r^5}\cdot\boldsymbol{r}\right) = 0\,. \tag{3.34}$$

Thus the Newton gravitational force field is also div-free, which is why a gravitational vector potential $\boldsymbol{A}$ also exists.

As a final remark it is noted that with $\boldsymbol{F}$ satisfying both $\boldsymbol{F} = -\nabla\phi$ and div $\boldsymbol{F} = 0$, then

$$\text{div}\,(\nabla\phi) = \nabla^2\phi = 0\,, \tag{3.35}$$

which is known as **Laplace's equation.** *End of L3.*

4 Line (path) integration

(L4) In single variable calculus the idea of the integral

$$\int_a^b f(x)\,dx = \sum_{i=1}^{N} f(x_i)\delta x_i \tag{4.1}$$

is a way of expressing the sum of values of the function $f(x_i)$ at points x_i multiplied by the area of small strips δx_i: correctly it is often expressed as the area under the curve $f(x)$. Pictorially the concept of an area sits very well in the plane with $y = f(x)$ plotted against x. **However, the idea of area under a curve has to be dropped when line integration is considered because we now wish to place our curve C in 3-space where a scalar field $\psi(x, y, z)$ or a vector field $\boldsymbol{F}(x, y, z)$ take values at every point in this space.** Instead, we consider a specified continuous curve C in 3-space – known as the path[7] of integration – and then work out methods for summing the values that either ψ or $\boldsymbol{F}$ take on that curve. **It is essential to realize that the curve C sitting in 3-space and the scalar/vector fields ψ or $\boldsymbol{F}$ that take values at every point in this space are wholly independent quantities and must not be conflated.**

[7] The same idea arises in complex integration where, by convention, a closed C is known as a 'contour'.

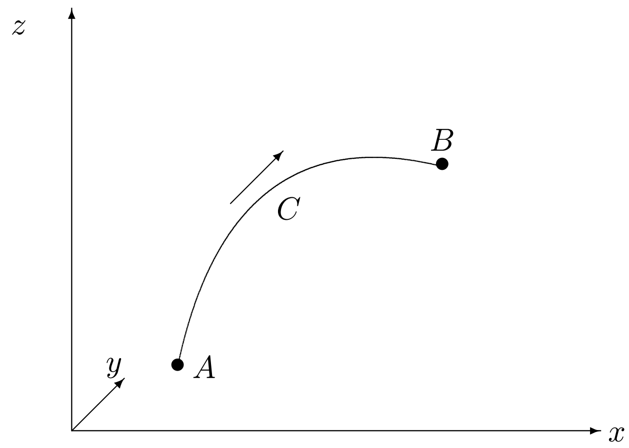


Figure 3.1 : *The curve C in 3-space starts at the point A and ends at B.*

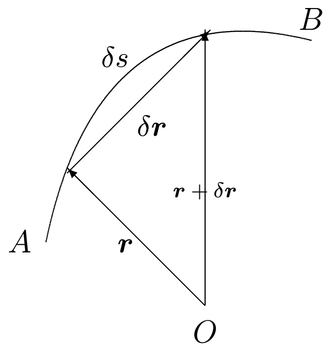


Figure 3.2 : *On a curve C, small elements of arc length δs and the chord $\delta \boldsymbol{r}$, where O is the origin.*

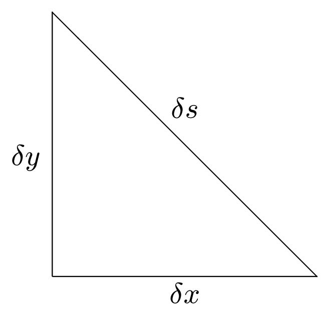


Figure 3.3 : *In 2-space we can use Pythagoras' Theorem to express δs in terms of δx and δy.*

Pythagoras' Theorem in 3-space (see Fig 3.3 for a 2-space version) express δs in terms of δx, δy and δz : $(\delta s)^2 = (\delta x)^2 + (\delta y)^2 + (\delta z)^2$. There are two types of line integral :

Type 1 : The first concerns the integration of a scalar field ψ along a path C

$$\int_C \psi(x, y, z)\, ds \tag{4.2}$$

Type 2 : The second concerns the integration of a vector field $\boldsymbol{F}$ along a path C

$$\int_C \boldsymbol{F}(x, y, z) \cdot d\boldsymbol{r} \tag{4.3}$$

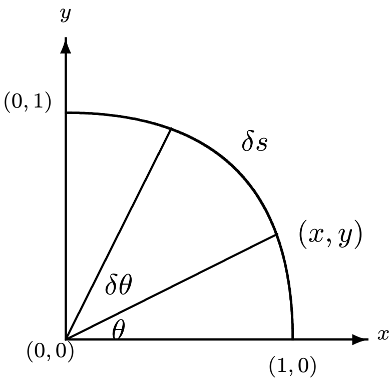
Remark : In either case, if the curve C is closed then we use the designations

$$\oint_C \psi(x, y, z)\, ds \qquad \text{and} \qquad \oint_C \boldsymbol{F}(x, y, z) \cdot d\boldsymbol{r} \tag{4.4}$$

4.1 Line integrals of Type 1 : $\int_C \psi(x, y, z)\, ds$

How to evaluate these integrals is best shown by a series of examples keeping in mind that, where possible, one should always draw a picture of the curve C :

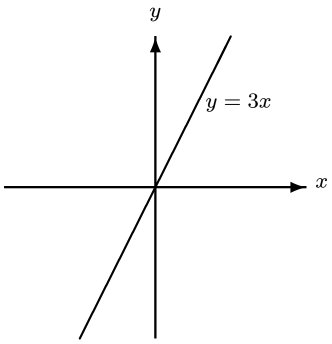
Example 1) : Show that $\int_C x^2 y\, ds = 1/3$ where C is the circular arc in the first quadrant of the unit circle.



C is the arc of the unit circle $x^2 + y^2 = 1$ represented in polars by $x = \cos\theta$ and $y = \sin\theta$ for $0 \le \theta \le \pi/2$. Thus the small element of arc length is $\delta s = 1.\delta\theta$.

$$\begin{aligned} \int_C x^2 y\, ds &= \int_0^{\pi/2} \cos^2\theta \sin\theta (d\theta) \\ &= 1/3\,. \end{aligned} \tag{4.5}$$

Example 2) : Show that $\int_C xy^3\, ds = 54\sqrt{10}/5$ where C is the line $y = 3x$ from $x = -1 \to 1$.

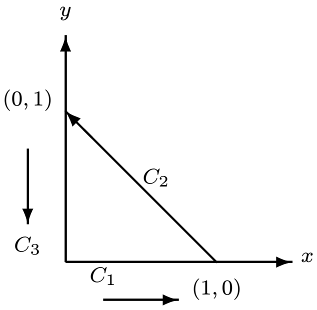


C is an element on the line $y = 3x$. Thus $\delta y = 3\delta x$ so

$$(\delta s)^2 = (\delta x)^2 + 9(\delta x)^2 = 10(\delta x)^2$$

$$\int_C xy^3\, ds \;=\; 27\sqrt{10} \int_{-1}^{1} x^4 dx = 54\sqrt{10}/5\,.$$

Example 3) : Show that $\oint_C x^2 y\, ds = -\sqrt{2}/12$ where C is the closed triangle in the figure.



On $C_1:\ y=0$ so $ds=dx$ and $\int_{C_1}=0$ (because $y=0$).
On $C_2:\ y=1-x$ so $dy=-dx$ and so $(ds)^2=2(ds)^2$.
On $C_3:\ x=0$ so $ds=dy$ and $\int_{C_3}=0$ (because $x=0$).

Therefore

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + \int_1^0 x^2(1-x)\sqrt{2}\,dx + 0 = -\sqrt{2}/12\,. \tag{4.6}$$

Note that we take the positive root of $(ds)^2 = 2(dx)^2$ but use the fact that following the arrows on C_2 the variable x goes from $1\to 0$.

Example 4): (see Sheet 2). Find $\int_C (x^2+y^2+z^2)\,ds$ where C is the helix

$$\boldsymbol{r} = \hat{\boldsymbol{\imath}}\cos\theta + \hat{\boldsymbol{\jmath}}\sin\theta + \hat{\boldsymbol{k}}\theta\,, \tag{4.7}$$

with one turn: that is θ running from $0\to 2\pi$.

From (4.7) we note that $x=\cos\theta$, $y=\sin\theta$ and $z=\theta$. Therefore $dx=-\sin\theta d\theta$, $dy=\cos\theta d\theta$ and $dz=d\theta$. Thus

$$(ds)^2 = \left(\sin^2\theta+\cos^2\theta+1\right)(d\theta)^2 = 2(d\theta)^2\,. \tag{4.8}$$

Therefore we can write the integral as

$$\int_C (x^2+y^2+z^2)\,ds = \sqrt{2}\int_0^{2\pi}\left(1+\theta^2\right)\,d\theta = 2\pi\sqrt{2}\left(1+4\pi^2/3\right)\,. \tag{4.9}$$

End of L4

4.2 Line integrals of Type 2: $\int_C \boldsymbol{F}(x,y,z)\cdot d\boldsymbol{r}$

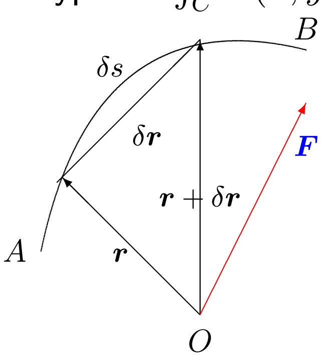


Figure 3.4: *A vector $\boldsymbol{F}$ and a curve C with the chord $\delta\boldsymbol{r}$: O is the origin.*

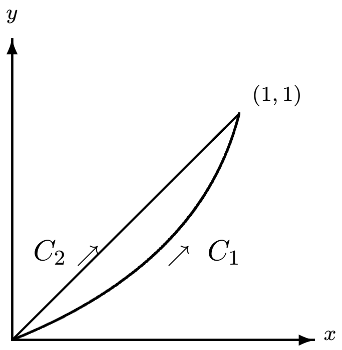
(L5)

1. Think of $\boldsymbol{F}$ as a force on a particle being drawn through the path of the curve C. Then the work done δW in pulling the particle along the curve with arc length δs and chord $\delta\boldsymbol{r}$ is $\delta W = \boldsymbol{F}\cdot\delta\boldsymbol{r}$. Thus the full work W is

$$W = \int_C \boldsymbol{F}\cdot d\boldsymbol{r}\,. \tag{4.10}$$

2. The second example revolves around taking $\boldsymbol{F}$ as an electric field $\boldsymbol{E}(x,y,z)$. Then the mathematical expression of Faraday's Law says that the electro-motive force on a particle of charge e travelling along C is precisely $e\int_C \boldsymbol{E}(x,y,z)\cdot d\boldsymbol{r}$.

Example 1 : Evaluate $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ given that $\boldsymbol{F} = \hat{\boldsymbol{i}}x^2y + \hat{\boldsymbol{j}}(x-z) + \hat{\boldsymbol{k}}xyz$ and the path C is the parabola $y = x^2$ in the plane $z = 2$ from $(0,0,2)\to(1,1,2)$.



C_1 is along the curve $y = x^2$ in the plane $z = 2$ in which case $dz = 0$ and $dy = 2x\,dx$. C_2 is along the straight line $y = x$ in which case $dy = dx$.

$$\begin{aligned}\int_{C_1}\boldsymbol{F}\cdot d\boldsymbol{r} &= \int_{C_1}(F_1dx + F_2dy + F_3dz)\\ &= \int_{C_1}\big(x^2y\,dx + (x-z)\,dy + xyz\,dz\big)\\ &= \int_{C_1}\big(x^2y\,dx + (x-2)\,dy\big)\end{aligned} \tag{4.11}$$

Using the fact that $dy = 2x\,dx$ we have

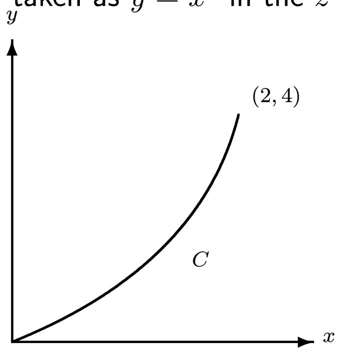
$$\begin{aligned} I &= \int_0^1 \big(x^4\,dx + (x-2)2x\,dx\big)\\ &= \int_0^1 \big(x^4 + 2x^2 - 4x\big)\,dx = -17/15\,.\end{aligned} \tag{4.12}$$

Finally, with the same integrand but along C_2 we have

$$\begin{aligned}\int_{C_2}\boldsymbol{F}\cdot d\boldsymbol{r} &= \int_0^1 \big(x^3\,dx + (x-2)\,dx\big)\\ &= 1/4 + 1/2 - 2 = -5/4\,.\end{aligned} \tag{4.13}$$

This example illustrates the point that with the same integrand and start/end points the value of an integral can differ when the route between these points is varied.

Example 2: Evaluate $I = \int_C (y^2 dx - 2x^2 dy)$ given that $\boldsymbol{F} = (y^2, -2x^2)$ with the path C taken as $y = x^2$ in the $z = 0$ plane from $(0,0,0) \to (2,4,0)$.



C is the curve $y = x^2$ in the plane $z = 0$, in which case $dz = 0$ and $dy = 2x\,dx$. The starting point has co-ordinates $(0,0,0)$ and the end point $(2,4,0)$.

$$\begin{aligned} I &= \int_C (y^2 dx - 2x^2 dy) \\ &= \int_0^2 \left(x^4 - 4x^3\right) dx = -48/5\,. \end{aligned} \tag{4.14}$$

Example 3: Given that $\boldsymbol{F} = \hat{\boldsymbol{\imath}}x - \hat{\boldsymbol{\jmath}}z + 2\hat{\boldsymbol{k}}y$, where C is the curve $z = y^4$ in the $x = 1$ plane, show that from $(1,0,0) \to (1,1,1)$ the value of the line integral is $7/5$.

On C we have $dz = 4y^3 dy$ and in the plane $x = 1$ we also have $dx = 0$. Therefore

$$\begin{aligned} \int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_C (x\,dx - z\,dy + 2y\,dz) \\ &= \int_0^1 \left(-y^4 + 8y^4\right) dy = 7/5\,. \end{aligned} \tag{4.15}$$

End of L5

4.3 Independence of path in line integrals of Type 2

(L6) Are there circumstances in which a line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, for a given $\boldsymbol{F}$, takes values which are independent of the path C?

To explore this question let us consider the case where $\boldsymbol{F} \cdot d\boldsymbol{r}$ is an *exact differential*: that is $\boldsymbol{F} \cdot d\boldsymbol{r} = -d\phi$ for some scalar[8] ϕ. For starting and end co-ordinates A and B of C we have

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_C d\phi = \phi[A] - \phi[B]\,. \tag{4.16}$$

This result is independent of the route or path taken between A and B. Thus we need to know what $\boldsymbol{F} \cdot d\boldsymbol{r} = -d\phi$ means. Firstly, from the chain rule

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = \nabla \phi \cdot d\boldsymbol{r}\,. \tag{4.17}$$

[8] As stated earlier, the choice of sign is by convention.

Hence, if $\boldsymbol{F}\cdot d\boldsymbol{r} = -d\phi$ we have $\boldsymbol{F} = -\nabla\phi$, **which means that F must be a curl-free vector field**

$$\text{curl}\,\boldsymbol{F} = 0\,, \tag{4.18}$$

where ϕ is the scalar potential. A curl-free $\boldsymbol{F}$ is also known as a **conservative** vector field.

Result : The integral $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ is independent of path only if curl $\boldsymbol{F} = 0$. Moreover, when C is closed then

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = 0\,. \tag{4.19}$$

Example 1 : Is the line integral $\int_C (2xy^2\,dx + 2x^2y\,dy)$ independent of path?

We can see that $\boldsymbol{F}$ is expressed as $\boldsymbol{F} = 2xy^2\hat{\boldsymbol{\imath}} + 2x^2y\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$. Then

$$\text{curl}\,\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ 2xy^2 & 2x^2y & 0 \end{vmatrix} = (4xy - 4xy)\hat{\boldsymbol{k}} = 0\,. \tag{4.20}$$

Thus the integral is independent of path and we should be able to calculate ϕ from $\boldsymbol{F} = -\nabla\phi$.

$$-\frac{\partial\phi}{\partial x} = 2xy^2\,, \qquad -\frac{\partial\phi}{\partial y} = 2x^2y\,, \qquad -\frac{\partial\phi}{\partial z} = 0\,. \tag{4.21}$$

Partial integration of the 1st equation gives $\phi = -x^2y^2 + A(y)$ where $A(y)$ is an arbitrary function of y only, whereas from the second $\phi = -x^2y^2 + B(x)$. Thus $A(y) = B(x) =$ const $= C$. Hence

$$\phi = -x^2y^2 + C\,. \tag{4.22}$$

Example 2 : Find the work done $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ by the force $\boldsymbol{F} = (yz,\, xz,\, xy)$ moving from $(1,1,1) \to (3,3,2)$.

Is this line integral independent of path? We check to see if curl $\boldsymbol{F} = 0$.

$$\text{curl}\,\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ yz & xz & xy \end{vmatrix} = 0\,. \tag{4.23}$$

Thus the integral is independent of path and ϕ must exist. In fact $-\phi_x = yz,\ -\phi_y = xz$ and $-\phi_z = xy$. Integrating the first gives $\phi = -xyz + A(y,z)$, the second gives $\phi = -xyz + B(x,z)$ and the third $\phi = -xyz + C(x,y)$. Thus $A = B = C =$ const and

$$\phi = -xyz + \text{const}\,. \tag{4.24}$$

$$W = -\int_{(1,1,1)}^{(3,3,2)} d\phi = \big[xyz\big]_{(1,1,1)}^{(3,3,2)} = 18 - 1 = 17\,. \tag{4.25}$$

Example 3 : Find $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ on every path between $(0,0,1)$ and $(1,\pi/4,\,2)$ where

$$\boldsymbol{F} = \left(2xyz^2,\ (x^2z^2 + z\cos yz),\ (2x^2yz + y\cos yz)\right). \tag{4.26}$$

We check to see if curl $\boldsymbol{F} = 0$.

$$\text{curl}\,\boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ 2xyz^2 & (x^2z^2 + z\cos yz) & (2x^2yz + y\cos yz) \end{vmatrix} = 0\,. \tag{4.27}$$

Thus the integral is independent of path and ϕ must exist. Then $\phi_x = -2xyz^2$, $\phi_y = -(x^2z^2 + z\cos yz)$ and $\phi_z = -(2x^2yz + y\cos yz)$. Integration of the first gives

$$\phi = -x^2yz^2 + A(y,z)\,, \tag{4.28}$$

of the second

$$\phi = -(x^2yz^2 + \sin yz) + B(x,z)\,, \tag{4.29}$$

and the third

$$\phi = -(x^2yz^2 + \sin yz) + C(x,y)\,. \tag{4.30}$$

The only way these are compatible is if $A(y,z) = \sin yz + c$ and $B = C = c =$ const. and so $\phi = -(x^2yz^2 + \sin yz) + c$. In consequence

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = -\left[\phi\right]_{(0,0,1)}^{(1,\pi/4,\,2)} = \pi + 1\,. \qquad \text{End of L6} \tag{4.31}$$

5 Double and multiple integration

(L7) Now we move on to yet another different concept of integration : that is, summation over an area instead of along a curve.

Consider the value of a scalar function $\psi(x_i, y_i)$ at the co-ordinate point (x_i, y_i) at the lower left hand corner of the square of area $\delta A_i = \delta x_i \delta y_i$. Then

$$\sum_{i=1}^{N}\sum_{j=1}^{M} \psi(x_i,y_i)\,\delta A_i \to \underbrace{\int\!\!\int_R \psi(x,y)\,dxdy}_{double\ integral} \qquad \text{as} \qquad \delta x \to 0,\ \delta y \to 0\,. \tag{5.1}$$

We say that the RHS is the "double integral of ψ over the region R". **Note : do not confuse this with the area of R itself, which is**

$$\text{Area of R} = \int\!\!\int_R dxdy\,. \tag{5.2}$$

5.1 How to evaluate a double integral

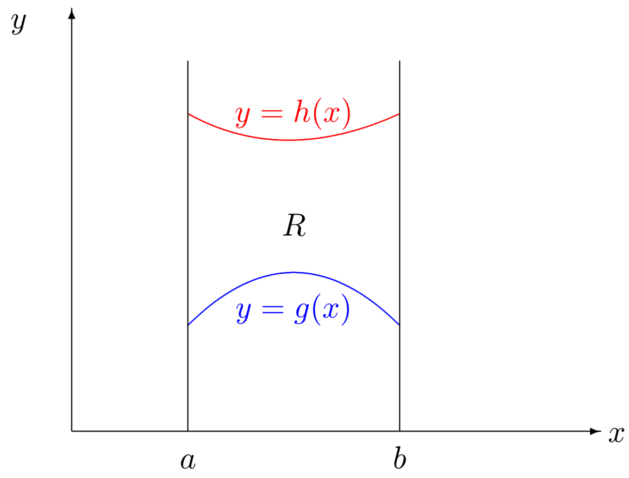


Figure 4.2 : *The region R is bounded between the upper curve $y = h(x)$, the lower curve $y = g(x)$ and the vertical lines $x = a$ and $x = b$.*

$$\int\int_R \psi(x,y)\,dxdy = \int_a^b \left\{ \int_{y=g(x)}^{y=h(x)} \psi(x,y)\,dy \right\} dx \tag{5.3}$$

The inner integral is a partial integral over y holding x constant. Thus the inner integral is a function of x

$$\int_{y=g(x)}^{y=h(x)} \psi(x,y)\,dy = P(x) \tag{5.4}$$

and so

$$\int\int_R \psi(x,y)\,dxdy = \int_a^b P(x)\,dx\,. \tag{5.5}$$

Moreover the area of R itself is

$$\text{Area of R} = \int_a^b \left\{ \int_{y=g(x)}^{y=h(x)} dy \right\} dx = \int_a^b \{h(x) - g(x)\}\,dx \tag{5.6}$$

5.2 Applications

1. **Area under a curve:** For a function of a single variable $y = f(x)$ between limits $x = a$ and $x = b$

$$\text{Area} = \int_a^b \left\{ \int_0^{f(x)} dy \right\} dx = \int_a^b f(x)\,dx\,. \tag{5.7}$$

2. **Volume under a surface :** A surface is 3-space may be expressed as $z = f(x,y)$

$$\begin{aligned} \text{Volume} &= \int\int\int_V dxdydz = \int_R \left\{ \int_0^{f(x,y)} dz \right\} dxdy \\ &= \int\int_R f(x,y)\,dxdy \end{aligned} \tag{5.8}$$

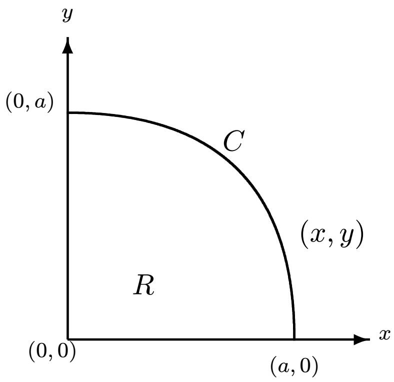
By this process, a 3-integral has been reduced to a double integral. A specific example would be the volume of an upper unit hemisphere $z = +\sqrt{1-(x^2+y^2)} \equiv f(x,y)$ centred at the origin.

3. **Mass of a solid body:** Let $\rho(x,y,z)$ be the variable density of the material in a solid body. Then the mass δM of a small volume $\delta V = \delta x \delta y \delta z$ is $\delta M = \rho\,\delta V$, End of L7

$$\text{Mass of body} = \int\int\int_V \rho(x,y,z)\,dV\,. \tag{5.9}$$

5.3 Examples of multiple integration

L8 **Example 1:** Consider the first quadrant of a circle of radius a. Show that:



(i) Area of R $= \pi a^2/4$

(ii) $\int\int_R xy\,dxdy = a^4/8$

(iii) $\int\int_R x^2y^2\,dxdy = \pi a^6/96$

i): The area of R is given by

$$A = \int_0^a \left\{\int_0^{\sqrt{a^2-x^2}} dy\right\} dx = \int_0^a \sqrt{a^2-x^2}\,dx\,. \tag{5.10}$$

Let $x = a\cos\theta$ and $y = a\sin\theta$ then $A = \frac{1}{2}a^2\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta = \pi a^2/4$.

ii):

$$\begin{aligned}\int\int_R xy\,dxdy &= \int_0^a x\left(\int_0^{\sqrt{a^2-x^2}} y\,dy\right)dx\\ &= \tfrac{1}{2}\int_0^a x(a^2-x^2)\,dx\\ &= \tfrac{1}{2}\left[\tfrac{1}{2}x^2a^2 - \tfrac{1}{4}x^4\right]_0^a = a^4/8\,.\end{aligned} \tag{5.11}$$

iii):

$$\begin{aligned}\int\int_R x^2y^2\,dxdy &= \int_0^a x^2\left(\int_0^{\sqrt{a^2-x^2}} y^2\,dy\right)dx\\ &= \tfrac{1}{3}\int_0^a x^2(a^2-x^2)^{3/2}\,dx\\ &= \tfrac{1}{3}a^6\int_0^{\pi/2}\cos^2\theta\sin^4\theta\,d\theta\\ &= \tfrac{1}{3}a^6\big(I_2 - I_3\big)\,.\end{aligned} \tag{5.12}$$

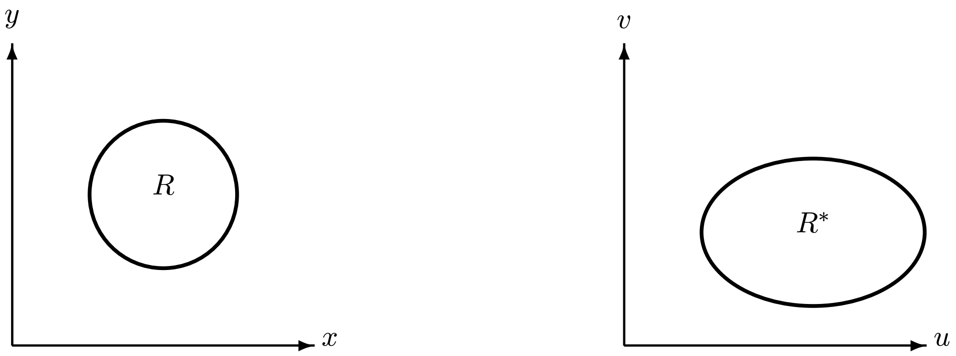
where $I_n = \int_0^{\pi/2} \sin^{2n}\theta\, d\theta$. Using a Reduction Formula method we find that

$$I_n = \left(\frac{2n-1}{2n}\right) I_{n-1}\,, \qquad I_0 = \pi/2\,. \tag{5.13}$$

Thus $I_2 = 3\pi/16$ and $I_2 = 15\pi/96$ and so from (5.12) the answer is $\pi a^6/96$.

5.4 Change of variable and the Jacobian

In the last example it might have been easier to have invoked the natural circular symmetry in the problem. Hence we must ask how $\delta A = \delta x \delta y$ would be expressed in polar co-ordinates. This suggests considering a more general co-ordinate change. In the two figures below we see a region R in the $x - y$ plane that is distorted into R^* in the plane of the new co-ordinates $u = u(x, y)$ and $v = v(x, y)$



The transformation relating the two small areas $\delta x \delta y$ and $\delta u \delta v$ is given here by (the modulus sign is a necessity):

Result 1:

$$dxdy = |J_{u,v}(x, y)|\; dudv\,, \tag{5.14}$$

where the Jacobian $J_{u,v}(x, y)$ is defined by

$$J_{u,v}(x, y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{5.15}$$

Note that

$$\frac{\partial x}{\partial u} \neq \left(\frac{\partial u}{\partial x}\right)^{-1} \tag{5.16}$$

because u_x is computed at $y = const$ whereas x_u is computed at $v = const$. However, luckily there is an inverse relationship between the two Jacobians

$$J_{x,y}(u, v) = [J_{u,v}(x, y)]^{-1} \tag{5.17}$$

$$dudv = |J_{x,y}(u, v)|\; dxdy \tag{5.18}$$

Either (5.14) and (5.18) can therefore be used at one's convenience.

Proof: (not examinable). Consider two sets of orthogonal unit vectors; $(\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}})$ in the $x-y$-plane, and $(\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}})$ in the $u-v$-plane. Keeping in mind that the component of $\delta\boldsymbol{x}$ along $\hat{\boldsymbol{u}}$ is $(\delta x/\delta u)\delta u$ (v is constant along $\hat{\boldsymbol{u}}$), in vectorial notation we can write

$$\delta\boldsymbol{x} = \hat{\boldsymbol{u}}\,\frac{\delta x}{\delta u}\delta u + \hat{\boldsymbol{v}}\,\frac{\delta x}{\delta v}\delta v \tag{5.19}$$

$$\delta\boldsymbol{y} = \hat{\boldsymbol{u}}\,\frac{\delta y}{\delta u}\delta u + \hat{\boldsymbol{v}}\,\frac{\delta y}{\delta v}\delta v \tag{5.20}$$

and so the cross-product is

$$\delta\boldsymbol{x}\times\delta\boldsymbol{y} = \left(\hat{\boldsymbol{u}}\,\frac{\delta x}{\delta u}\delta u + \hat{\boldsymbol{v}}\,\frac{\delta x}{\delta v}\delta v\right)\times\left(\hat{\boldsymbol{u}}\,\frac{\delta y}{\delta u}\delta u + \hat{\boldsymbol{v}}\,\frac{\delta y}{\delta v}\delta v\right), \tag{5.21}$$

Therefore

$$\begin{aligned} dx\,dy &= |\delta\boldsymbol{x}\times\delta\boldsymbol{y}| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}\right| |\hat{\boldsymbol{u}}\times\hat{\boldsymbol{v}}|\,dudv \\ &= |J_{u,v}(x,y)|\,dudv\,. \end{aligned} \tag{5.22}$$

The inverse relationship (5.18) can be proved in a similar manner □

Result 2:

$$\int\int_R f(x,y)\,dxdy = \int\int_{R^*} f(x(u,v),\,y(u,v))\,|J_{u,v}(x,y)|\,dudv\,. \tag{5.23}$$

Example 1: For polar co-ordinates $x = r\cos\theta$ and $y = r\sin\theta$ we take $u = r$ and $v = \theta$

$$J_{r,\theta}(x,y) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix} = r \tag{5.24}$$

Thus $dxdy = r\,drd\theta$.

Example 2: To calculate the volume of a sphere of radius a we note that $z = \pm\sqrt{a^2-x^2-y^2}$. Doubling up the two hemispheres we obtain

$$\text{Volume} = 2\int\int_R \sqrt{a^2-x^2-y^2}\,dxdy \tag{5.25}$$

where R is the circle $x^2+y^2=a^2$ in the $z=0$ plane. Using a change of variable

$$\begin{aligned} \text{Volume} &= 2\int\int_R \sqrt{a^2-x^2-y^2}\,dxdy \\ &= 2\int\int_R \sqrt{a^2-r^2}\,rdrd\theta \\ &= 2\int_0^a \sqrt{a^2-r^2}\,rdr\int_0^{2\pi} d\theta \\ &= 4\pi a^3/3\,. \end{aligned} \tag{5.26}$$

Example 3 : From the example in §5.3 over the quarter circle in the first quadrant it would be easier to compute in polars. Thus

$$\begin{aligned} \text{Area of R} &= \int\int_R dxdy = \int\int_R rdrd\theta \\ &= \int_0^a rdr \int_0^{\pi/2} d\theta = \pi a^2/4 \end{aligned} \tag{5.27}$$

$$\begin{aligned} \int\int_R xy\,dxdy &= \int\int_R r^3 \cos\theta \sin\theta drd\theta \\ &= \int_0^a r^3 dr \int_0^{\pi/2} \cos\theta\sin\theta d\theta \\ &= \tfrac{1}{4}a^4 \int_0^{\pi/2} \tfrac{1}{2}\sin 2\theta d\theta \\ &= (a^4/16)[-\cos 2\theta]_0^{\pi/2} = a^4/8\,. \end{aligned} \tag{5.28}$$

Likewise one may show that

$$\begin{aligned} \int\int_R x^2y^2\,dxdy &= \int\int_R r^5\cos^2\theta\sin^2\theta\,drd\theta \\ &= \int_0^a r^5 dr \int_0^{\pi/2} \cos^2\theta\sin^2\theta d\theta = \pi a^6/96\,. \end{aligned} \tag{5.29}$$

Example 4 : Show that $\int\int_R (x^2+y^2)\,dxdy = 8/3$ using $u = x+y$ and $v = x-y$. where R has corners at $(0,0)$, $(1,1)$, $(2,0)$ and $(1,-1)$ and rotates to a square with corners at $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$. From $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$ it is found that

$$J_{u,v}(x,y) = -\tfrac{1}{2} \tag{5.30}$$

$$\begin{aligned} I &= \tfrac{1}{4}\int\int_{R^*} 2(u^2+v^2)\,|-\tfrac{1}{2}|\;dudv \\ &= \tfrac{1}{4}\left\{\left[\tfrac{1}{3}u^3\right]_0^2 [v]_0^2 + \left[\tfrac{1}{3}v^3\right]_0^2 [u]_0^2\right\} = 8/3\,. \end{aligned} \tag{5.31}$$

End of L8

5.5 Changing the order in double integration

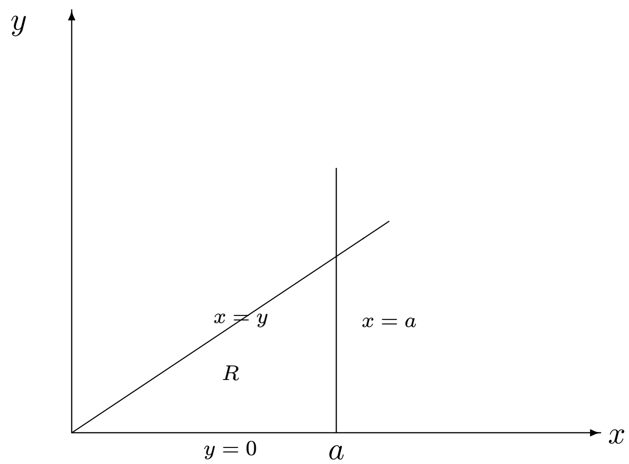
In a handout, not in lectures. An example is used to illustrate this : consider

$$I = \int_0^a \left(\int_y^a \frac{x^2\,dx}{x^2+y^2}\right) dy\,. \tag{5.32}$$

Performing the integration in this order is hard as it involves a term in $\tan^{-1}\left(\frac{a}{y}\right)$. To make evaluation easier we change the order of the x-integration and the y-integration; first, however, we must deduce what the area of integration is from the internal limits in (5.32). The internal integral is of the type

$$\int_{x=y}^{x=a} f(x,y)\,dx. \tag{5.33}$$

Being an integration over x means that we are summing horizontally so we have *left and right hand limits*. The *left limit* is $x = y$ and the *right limit* is $x = a$. This is shown in the drawing of the graph where the region of integration is labelled as R:



Now we want to perform the integration over R but in reverse order to that above: we integrate vertically first by reading off the *lower limit* as $y = 0$ and the *upper limit* as $y = x$. After integrating horizontally with limits $x = 0$ to $x = a$, this reads as

$$I = \int_0^a \left(\int_{y=0}^{y=x} \frac{x^2\,dy}{x^2+y^2} \right) dx. \tag{5.34}$$

The inside integral can be done with ease. x is treated as a constant because the integration is over y. Therefore define $y = x\theta$ where θ is the new variable. We find that

$$\int_0^x \frac{x^2\,dy}{x^2+y^2} = x\int_0^1 \frac{d\theta}{1+\theta^2} = \frac{x\pi}{4}. \tag{5.35}$$

Hence

$$\begin{aligned} I &= \frac{\pi}{4}\int_0^a x\,dx \\ &= \frac{\pi a^2}{8}. \end{aligned} \tag{5.36}$$

6 Green's Theorem in a plane

L9 Green's Theorem in a plane – which will be quoted at the bottom of an exam question where necessary – tells us how behaviour on the boundary of a close curve C through a closed line integral in two dimensions is related to the double integral over the region inside R.

Theorem 1 *Let R be a closed bounded region in the $x-y$ plane with a piecewise smooth boundary C. Let $P(x,y)$ and $Q(x,y)$ be arbitrary, continuous functions within R having continuous partial derivatives Q_x and P_y. Then*

$$\oint_C (Pdx + Qdy) = \int\int_R (Q_x - P_y)\, dxdy\,. \tag{6.1}$$

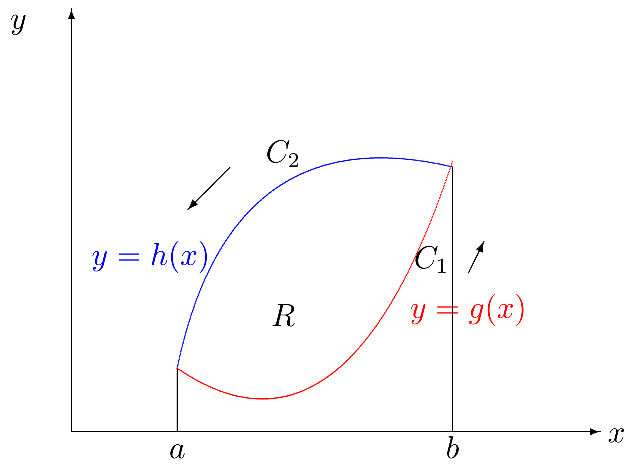


Figure: *In the $x-y$ plane the boundary curve C is made up from two counter-clockwise curves C_1 and C_2: R denotes the region inside.*

Proof: R is represented by the upper and lower boundaries (as in the Figure above)

$$g(x) \le y \le h(x) \tag{6.2}$$

and so

$$\begin{aligned}
\int\int_R \frac{\partial P}{\partial y}\, dxdy &= \int_a^b \left\{ \int_{y=g(x)}^{y=h(x)} \frac{\partial P}{\partial y}\, dy \right\} dx \\
&= \int_a^b \left\{ P\big(x,\, h(x)\big) - P\big(x,\, g(x)\big) \right\} dx \\
&= -\int_a^b P\big(x,\, g(x)\big)\, dx - \underbrace{\int_b^a P\big(x,\, h(x)\big)\, dx}_{\textit{switched limits \& sign}} \\
&= -\int_{C_1} P\big(x,y\big)\, dx - \int_{C_2} P\big(x,y\big)\, dx \\
&= -\oint_C P\big(x,y\big)\, dx\,.
\end{aligned} \tag{6.3}$$

The same method can be used the other way round to prove that (note the +ve sign)

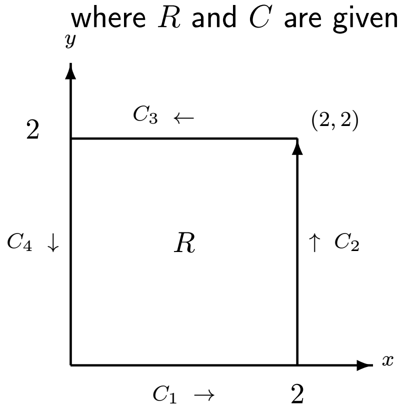
$$\int\int_R \frac{\partial Q}{\partial x}\,dxdy = \oint_C Q(x,y)\,dy \tag{6.4}$$

Both these results are true separately but can be pieced together to form the final result. ■

Example 1 : Use Green's Theorem to evaluate the line integral

$$\oint \left\{(x-y)\,dx - x^2 dy\right\}\ , \tag{6.5}$$

where R and C are given by



Using G.T. with $P = x - y$ and $Q = -x^2$ over the box-like region R we obtain

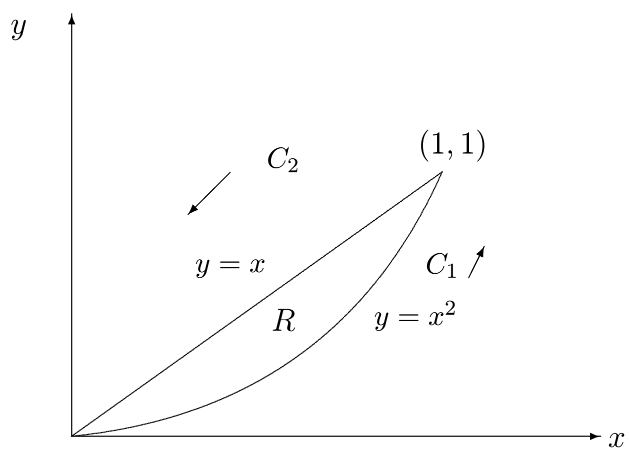
$$\begin{aligned}\oint \left\{(x-y)\,dx - x^2 dy\right\} &= \int\int_R (1-2x)\,dxdy \\ &= \int_0^2 dy \int_0^2 (1-2x)\,dx \\ &= -4\,. \end{aligned} \tag{6.6}$$

Direct valuation gives $\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$ where C_1 is $y=0$; C_2 is $x=2$; C_3 is $y=2$ and C_4 is $x=0$. Thus

$$\oint_C = \int_0^2 x\,dx - 4\int_0^2 dy + \int_2^0 (x-2)\,dx + 0 = 2 - 8 + 2 = -4\,. \tag{6.7}$$

Example 2 : Using Green's Theorem over R in the diagram, show that

$$\oint \left\{y^3\,dx + (x^3 + 3xy^2)\,dy\right\} = 3/20 \tag{6.8}$$

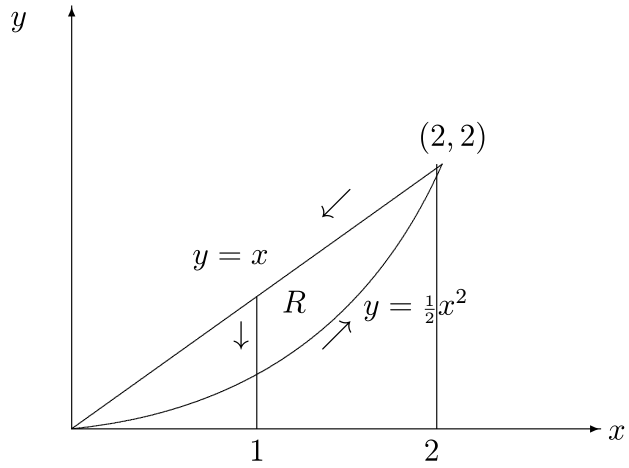


Using Green's Theorem

$$\oint \{y^3\,dx + (x^3 + 3xy^2)\,dy\} = 3\int\int_R x^2\,dxdy$$
$$= 3\int_0^1 x^2 \left(\int_{y=x^2}^{y=x} dy\right) dx$$
$$= 3\int_0^1 \left(x^3 - x^4\right)\,dx = 3/20\,. \quad \textit{End of L9} \qquad (6.9)$$

L10 **Example 3:** (Part of 2005 exam): With a suitable choice of P and Q and R as in the figure, show that

$$\tfrac{1}{2}\oint_C (x\,dy - y\,dx) = \int\int_R dxdy = 1/3\,. \qquad (6.10)$$



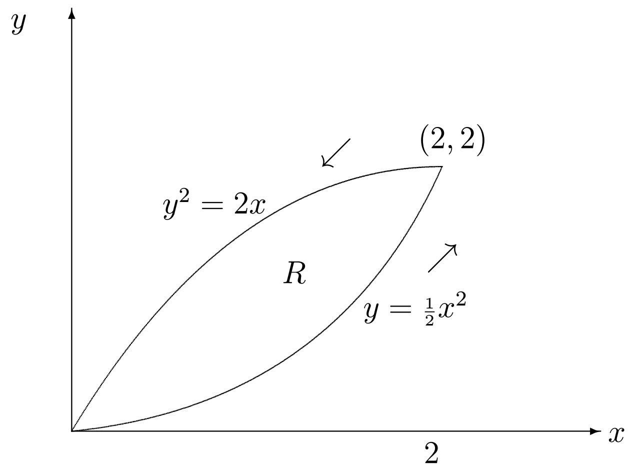
$$\int\int_R dxdy = \int_1^2 \left(\int_{\frac{1}{2}x^2}^{x} dy\right) dx = \int_1^2 \left(x - \tfrac{1}{2}x^2\right) dx = 1/3\,. \qquad (6.11)$$

Around the line integral we have

$$\oint = \tfrac{1}{2}\int_1^2 \left(x^2 - \tfrac{1}{2}x^2\right) dx + \tfrac{1}{2}\int_1^2 (xdx - xdx) + \tfrac{1}{2}\int_1^{1/2} dy$$
$$= 7/12 - 1/4 = 1/3\,. \qquad (6.12)$$

Example 4: (Part of 2004 exam): If $Q = x^2$ and $P = -y^2$ and R is as in the figure below, show that

$$\oint_C (x^2dy - y^2dx) = 2\int\int_R (x + y)\,dxdy = 24/5\,. \qquad (6.13)$$



The RHS is obvious in the sense that $Q_x - P_y = 2x + 2y$ and so

$$\begin{aligned} 2\int\int_R (x+y)\,dxdy &= 2\int_0^2 x\left(\int_{\frac{1}{2}x^2}^{\sqrt{2x}} dy\right)\,dx + 2\int_0^2 \left(\int_{\frac{1}{2}x^2}^{\sqrt{2x}} y\,dy\right)\,dx \\ &= 2\int_0^2 \left[x\left(\sqrt{2x} - \tfrac{1}{2}x^2\right) + \tfrac{1}{2}\left(2x - \tfrac{1}{4}x^4\right)\right]\,dx \\ &= 2\int_0^2 \left[\sqrt{2}x^{3/2} - \tfrac{1}{2}x^3 + x - \tfrac{1}{8}x^4\right]\,dx = 24/5\,. \end{aligned} \tag{6.14}$$

The sum of the three line integrals can be done to get the same answer.

7 $2D$ Divergence and Stokes' Theorems

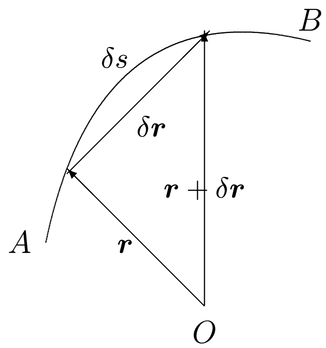


Figure 6.1 : *On a curve C, with starting and ending points A and B, small elements of arc length δs and the chord $\delta\boldsymbol{r}$, where O is the origin.*

The chord $\delta\boldsymbol{r}$ and the arc δs in the figure show us how to define a unit tangent vector $\hat{\boldsymbol{T}}$

$$\begin{aligned}\hat{\boldsymbol{T}} &= \lim_{\delta \boldsymbol{r}\to 0}\frac{\delta\boldsymbol{r}}{\delta s} = \frac{d\boldsymbol{r}}{ds}\\ &= \hat{\boldsymbol{i}}\frac{dx}{ds} + \hat{\boldsymbol{j}}\frac{dy}{ds}\,. \end{aligned} \tag{7.1}$$

The unit normal $\hat{\boldsymbol{n}}$ must be perpendicular to this: that is $\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{T}} = 0$, giving

$$\hat{\boldsymbol{n}} = \pm\left(\hat{\boldsymbol{i}}\frac{dy}{ds} - \hat{\boldsymbol{j}}\frac{dx}{ds}\right)\,, \tag{7.2}$$

where $\pm$ refer to inner and outer normals.

A) The Divergence Theorem: Define a $2D$ vector $\boldsymbol{u} = \hat{\boldsymbol{i}}Q - \hat{\boldsymbol{j}}P$. Then

$$\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = P\frac{dx}{ds} + Q\frac{dy}{ds} \qquad\qquad \mathrm{div}\,\boldsymbol{u} = Q_x - P_y \tag{7.3}$$

Thus Green's Theorem turns into a $2D$ version of **Divergence Theorem**

$$\int\!\!\int_R \mathrm{div}\,\boldsymbol{u}\,dxdy = \oint_C \boldsymbol{u}\cdot\hat{\boldsymbol{n}}\,ds \tag{7.4}$$

This line integral simply expresses the sum of the normal component of $\boldsymbol{u}$ around the boundary. If $\boldsymbol{u}$ is a solenoidal vector ($\mathrm{div}\,\boldsymbol{u} = 0$) then automatically $\oint_C \boldsymbol{u}\cdot\hat{\boldsymbol{n}}\,ds = 0$.

The $3D$ version[9] uses an arbitrary $3D$ vector field $\boldsymbol{u}(x,y,z)$ that lives in some finite, simply connected volume V whose surface is S: dA is some small element of area on the curved surface S

$$\int\!\!\int\!\!\int_V \mathrm{div}\,\boldsymbol{u}\,dV = \int\!\!\int_S \boldsymbol{u}\cdot\hat{\boldsymbol{n}}\,dA \tag{7.5}$$

End of L10

B) Stokes' Theorem: L11 Now define a $2D$ vector $\boldsymbol{v} = \hat{\boldsymbol{i}}P + \hat{\boldsymbol{j}}Q$. Then

$$\boldsymbol{v}\cdot d\boldsymbol{r} = (\boldsymbol{v}\cdot\hat{\boldsymbol{T}})\,ds = P\,dx + Q\,dy\,. \tag{7.6}$$

Moreover

$$\mathrm{curl}\,\boldsymbol{v} = \begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ \partial_x & \partial_y & \partial_z\\ P & Q & 0\end{vmatrix} = \hat{\boldsymbol{k}}\,(Q_x - P_y) \tag{7.7}$$

Thus Green's Theorem turns into a $2D$ version of **Stokes' Theorem**

$$\int\int_R (\hat{\boldsymbol{k}} \cdot \mathrm{curl}\,\boldsymbol{v})\, dxdy = \oint_C \boldsymbol{v} \cdot d\boldsymbol{r} \tag{7.8}$$

The line integral on the RHS is called the *circulation*. Note that if $\boldsymbol{v}$ is an irrotational vector then $\oint_C \boldsymbol{v} \cdot d\boldsymbol{r} = 0$ which means there is no circulation.

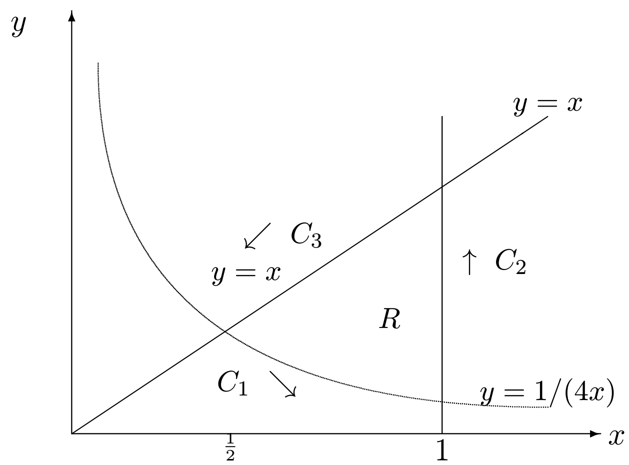
The 3D-version of Stokes' Theorem for an arbitrary $3D$ vector field $\boldsymbol{v}(x, y, z)$ in a volume V is given by

$$\oint_C \boldsymbol{v} \cdot d\boldsymbol{r} = \int\int_S \hat{\boldsymbol{n}} \cdot \mathrm{curl}\,\boldsymbol{v}\, dA\,. \tag{7.9}$$

$\hat{\boldsymbol{n}}$ is the unit normal vector to the surface S of V. C is a closed circuit circumscribed on the surface of the volume and S is the surface of the 'cap' above C.

Example 1: (part of 2002 exam) If $\boldsymbol{v} = \hat{\boldsymbol{i}}y^2 + \hat{\boldsymbol{j}}x^2$ and R is as in the figure below, by evaluating the line integral, show that

$$\oint_C \boldsymbol{v} \cdot d\boldsymbol{r} = 5/48\,. \tag{7.10}$$



Firstly

$$\oint \boldsymbol{v} \cdot d\boldsymbol{r} = \oint_C \left(y^2 dx + x^2 dy\right). \tag{7.11}$$

(i) On C_1 we have $y = 1/(4x)$ and so $dy = -dx/(4x^2)$. Therefore

$$\int_{C_1} \boldsymbol{v} \cdot d\boldsymbol{r} = \int_{\frac{1}{2}}^{1} \left(\frac{1}{16x^2} - \frac{1}{4}\right) dx = -\frac{1}{16}\,. \tag{7.12}$$

(ii) On C_2 we have $x = 1$ and so $dx = 0$. Therefore

$$\int_{C_2} \boldsymbol{v} \cdot d\boldsymbol{r} = \int_{\frac{1}{4}}^{1} dy = \frac{3}{4}\,. \tag{7.13}$$

(iii) On C_3 we have $y = x$ and so $dy = dx$. Therefore

$$\int_{C_3} \boldsymbol{v} \cdot d\boldsymbol{r} = 2\int_1^{\frac{1}{2}} x^2\, dx = -\frac{7}{12}\,. \tag{7.14}$$

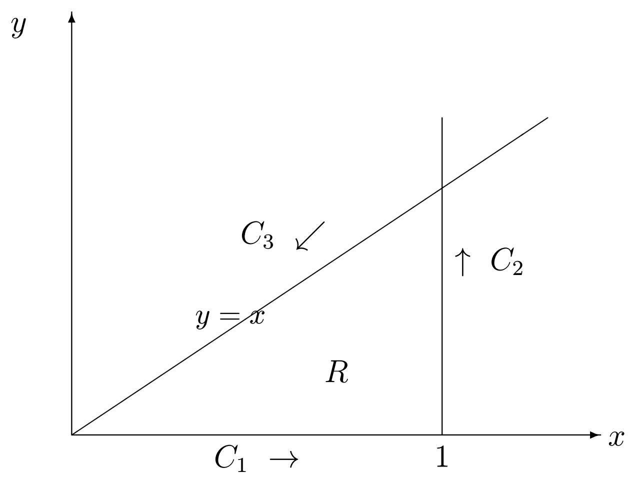
Summing these three results gives $-\frac{1}{16} + \frac{3}{4} - \frac{7}{12} = \frac{5}{48}$.

Example 2: (part of 2003 exam) If

$$\boldsymbol{u} = \hat{\boldsymbol{i}}\frac{x^2 y}{1+y^2} + \hat{\boldsymbol{j}}\left[x\ln(1+y^2)\right] \tag{7.15}$$

and R is as in the figure below, show that

$$\int\!\!\int_R \text{div}\,\boldsymbol{u}\, dxdy = 2\ln 2 - 1\,. \tag{7.16}$$



Firstly we calculate div $\boldsymbol{u}$

$$\text{div}\,\boldsymbol{u} = \frac{4xy}{1+y^2} \tag{7.17}$$

and so

$$\begin{aligned}
\int\!\!\int_R \text{div}\,\boldsymbol{u}\, dxdy &= 4\int\!\!\int_R \frac{xy}{1+y^2}\, dxdy \\
&= 4\int_0^1 x\left(\int_{y=0}^{y=x}\int \frac{y\, dy}{1+y^2}\right) dx \\
&= 4\int_0^1 x\left[\tfrac{1}{2}\ln(1+y^2)\right]_0^x\, dx \\
&= 2\int_0^1 x\,\ln(1+x^2)\, dx \\
&= 2\ln 2 - 1
\end{aligned} \tag{7.18}$$

End of L11

8 Maxwell's Equations

L12 Maxwell's equations are technically not in my syllabus but I give this lecture to tie everything together in a physical context.

Consider a $3D$ volume with a closed circuit C drawn on its surface. Let's use the $3D$ versions of the Divergence (7.5) and Stokes' Theorems (7.9) to derive some relationships between various electro-magnetic variables.

1. Using the charge density ρ, the total charge within the volume V must be equal to surface area integral of the electric flux density $\boldsymbol{\mathcal{D}}$ through the surface S (recall that $\boldsymbol{\mathcal{D}} = \epsilon \boldsymbol{E}$ where $\boldsymbol{E}$ is the electric field).

$$\int\int\int_V \rho \, dV = \int\int_S \boldsymbol{\mathcal{D}} \cdot \hat{\boldsymbol{n}} \, dA \,. \tag{8.1}$$

   Using a 3D version of the Divergence Theorem (7.5) above on the RHS of (8.1)

$$\int\int\int_V \rho \, dV = \int\int\int_V \text{div} \, \boldsymbol{\mathcal{D}} \, dV \,. \tag{8.2}$$

   Hence we have the first of Maxwell's equations

$$\boxed{\text{div} \, \boldsymbol{\mathcal{D}} = \rho} \tag{8.3}$$

   This is also known as Gauss's Law.

2. Following the above in the same manner for the magnetic flux density $\boldsymbol{B}$ (recall that $\boldsymbol{B} = \mu \boldsymbol{H}$ where $\boldsymbol{H}$ is the magnetic field) but noting that there are no magnetic sources (so $\rho_{mag} = 0$), we have the 2nd of Maxwell's equations

$$\boxed{\text{div} \, \boldsymbol{B} = 0} \tag{8.4}$$

3. Faraday's Law says that the rate of change of magnetic flux linking a circuit C is proportional to the electromotive force (in the negative sense). Mathematically this is expressed as

$$\frac{d}{dt} \int\int_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}} \, dA = -\oint_C \boldsymbol{E} \cdot d\boldsymbol{r} \tag{8.5}$$

   Using a 3D version of Stokes' Theorem (7.9) on the RHS of (8.4), and taking the time derivative through the surface integral (thereby making it a partial derivative) we have the 3rd of Maxwell's equations

$$\boxed{\text{curl} \, \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0} \tag{8.6}$$

4. Ampère's (Biot-Savart) Law expressed mathematically (the line integral of the magnetic field around a circuit C is equal to the current enclosed) is

$$\int\int_S \boldsymbol{J}\cdot\hat{\boldsymbol{n}}\,dA = \oint_C \boldsymbol{H}\cdot d\boldsymbol{r}\,, \tag{8.7}$$

where $\boldsymbol{J}$ is the current density and $\boldsymbol{H}$ is the magnetic field. Using 3D-Stokes' Theorem (7.9) on the RHS we find that we have curl $\boldsymbol{H} = \boldsymbol{J}$ and therefore div $\boldsymbol{J} = 0$, which is inconsistent with the first three of Maxwell's equations. Why? The continuity equation for the total charge is

$$\frac{d}{dt}\int\int\int_V \rho\,dV = -\int\int_S \boldsymbol{J}\cdot\hat{\boldsymbol{n}}\,dA\,. \tag{8.8}$$

Using the Divergence Theorem on the LHS we obtain

$$\text{div}\,\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0\,. \tag{8.9}$$

If div $\boldsymbol{J} = 0$ then ρ would have to be independent of t. To get round this problem we use Gauss's Law $\rho = \text{div}\,\boldsymbol{\mathcal{D}}$ expressed above to get

$$\text{div}\left\{\boldsymbol{J} + \frac{\partial\boldsymbol{\mathcal{D}}}{\partial t}\right\} = 0 \tag{8.10}$$

and so this motivates us to replace $\boldsymbol{J}$ in curl $\boldsymbol{H} = \boldsymbol{J}$ by $\boldsymbol{J} + \partial\boldsymbol{\mathcal{D}}/\partial t$ giving the 4th of Maxwell's equations

$$\boxed{\text{curl}\,\boldsymbol{H} = \boldsymbol{J} + \frac{\partial\boldsymbol{\mathcal{D}}}{\partial t}} \tag{8.11}$$

5. We finally note that because div $\boldsymbol{B} = 0$ then $\boldsymbol{B}$ is a solenoidal vector field: there must exist a vector potential $\boldsymbol{A}$ such that $\boldsymbol{B} = \text{curl}\,\boldsymbol{A}$. Using this in the 3rd of Maxwell's equations, we have

$$\text{curl}\left[\boldsymbol{E} + \frac{\partial\boldsymbol{A}}{\partial t}\right] = 0\,. \tag{8.12}$$

This means that there must also exist a scalar potential ϕ that satisfies

$$\boldsymbol{E} = -\frac{\partial\boldsymbol{A}}{\partial t} - \nabla\phi\,. \tag{8.13}$$

End of L12 and the Vector Calculus part of EE2 Maths.